

ROBUSTLY N-EXPANSIVE SURFACE DIFFEOMORPHISMS

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ABSTRACT. We give sufficient conditions for a diffeomorphism of a compact surface to be robustly N -expansive and cw-expansive in the C^r -topology. We give examples on the genus two surface showing that they need not to be Anosov diffeomorphisms. The examples are axiom A diffeomorphisms with tangencies at wandering points.

1. INTRODUCTION

Let M be a smooth compact manifold without boundary and consider a C^1 -diffeomorphism $f: M \rightarrow M$. We say that f is *expansive* if there is a positive constant δ such that if $x, y \in M$ and $x \neq y$ then there is $n \in \mathbb{Z}$ such that $\text{dist}(f^n(x)f^n(y)) > \delta$, where dist is a metric induced by a Finsler $\|\cdot\|$ on the tangent bundle TM . We say that f is *C^r -robustly expansive* if it is in the interior of the set of expansive C^r -diffeomorphisms. In [15] Mañé proved that f is C^1 -robustly expansive if and only if it is a *quasi-Anosov* diffeomorphism, i.e., for every tangent vector $v \in TM$, $v \neq 0$, the set $\{\|df^n(v)\|\}_{n \in \mathbb{Z}}$ is unbounded. Also, he proved that f is quasi-Anosov if and only if it satisfies Smale's Axiom A and the *quasi-transversality* condition of stable and unstable manifolds: $T_x W^s(x) \cap T_x W^u(x) = 0$ for all $x \in M$. If M is a compact surface then every quasi-Anosov diffeomorphism is Anosov. In higher dimensional manifolds there are examples of quasi-Anosov diffeomorphisms not being Anosov, see for example [9]. Obviously, every quasi-Anosov C^r -diffeomorphism is C^r -robustly expansive. To our best knowledge it is unknown whether the converse is true for $r \geq 2$.

The results of [15] were extended in several directions. In [3] Lipschitz perturbations of expansive homeomorphisms with respect to a hyperbolic metric were considered. There it is shown that quasi-Anosov diffeomorphisms are robustly expansive even allowing Lipschitz perturbations. In [19] it is shown that a vector field is C^1 -robustly expansive in the sense of Bowen and Walters [7] if and only if it is a quasi-Anosov vector field. In [2] this result is proved for kinematic expansive flows. For vector fields with singular (equilibrium) points Komuro [13] introduced a definition called k^* -expansivity. He proved that the Lorenz attractor is k^* -expansive, consequently, we have a robustly k^* -expansive attractor. The question of determining whether a compact boundaryless three-dimensional manifold admits a C^1 -robustly k^* -expansive vector field with singular points seems to be an open problem.

In the discrete-time case the definition of expansivity has several variations. Let us start mentioning the weakest one that will be considered in this paper. We say

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that $f: M \rightarrow M$ is a *cw-expansive* diffeomorphism [11] if there is $\delta > 0$ such that if $C \subset M$ is a non-trivial (not a singleton) connected set then there is $n \in \mathbb{Z}$ such that $\text{diam}(f^n(C)) > \delta$, where $\text{diam}(C) = \sup_{x,y \in C} \text{dist}(x,y)$. In [22] it is proved that every C^1 -robustly cw-expansive diffeomorphism is quasi-Anosov. In this paper, Section 5, we show that there are C^2 -robustly cw-expansive surface diffeomorphisms that are not quasi-Anosov. For this purpose we introduce the notion Q^r -Anosov diffeomorphism. The idea is to control the order of the tangencies of stable and unstable manifolds as will be explained in Section 3.

In [16,17] (see also [18]) Morales introduced other forms of expansivity that will be explained now. The first one says that f is *N-expansive* [17], for a positive integer N , if there is $\delta > 0$ such that if $\text{diam}(f^n(A)) < \delta$ for all $n \in \mathbb{Z}$ and some $A \subset M$ then $|A| \leq N$, where $|A|$ stands for the cardinality of A . In this case we say that δ is an *N-expansivity constant*. In [14] examples are given on compact metric spaces showing that $N+1$ -expansivity does not imply N -expansivity for all $N \geq 1$. This extends previous results of [17]. The examples of Section 5 of the present paper are Axiom A diffeomorphisms of a compact surface exhibiting this phenomenon.

From a *probabilistic* viewpoint expansivity can be defined as follows. For $\delta > 0$, $x \in M$ and a diffeomorphism $f: M \rightarrow M$ consider the set

$$\Gamma_\delta(x) = \{y \in M : \text{dist}(f^n(x), f^n(y)) \leq \delta \text{ for all } n \in \mathbb{Z}\}.$$

Given a Borel probability measure μ on M we say that f is *μ -expansive* [16] if there is $\delta > 0$ such that for all $x \in M$ it holds that $\mu(\Gamma_\delta(x)) = 0$. We say that f is *measure-expansive* if it is μ -expansive for every non-atomic Borel probability measure μ . Recall that μ is non-atomic if $\mu(\{x\}) = 0$ for all $x \in M$. Also, we say that f is *countably-expansive* if there is $\delta > 0$ such that for all $x \in M$ the set $\Gamma_\delta(x)$ is countable. In [4] it is shown that, in the general context of compact metric spaces, countably-expansivity is equivalent to measure-expansivity. In Table 1 we summarize these definitions. The implications indicated by the arrows are easy to prove.

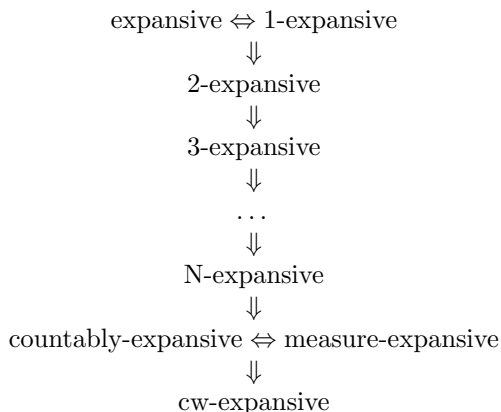


TABLE 1. Hierarchy of some generalizations of expansivity.

As we said before, in [22] it is shown that C^1 -robustly cw-expansive diffeomorphisms are quasi-Anosov. Therefore, in the C^1 -category, all the definitions of Table

1 coincide in the robust sense. The purpose of the present paper is to investigate robust expansivity and its generalizations in the C^r -topology for $r \geq 2$. Let us explain our results while describing the contents of the article. In Section 2 we recall some known results while introducing the notion of Ω -*expansivity*, i.e., expansivity in the non-wandering set. We prove that C^1 -robust Ω -expansivity is equivalent with Ω -stability. In Section 3 we introduce Q^r -Anosov C^r -diffeomorphisms of compact surfaces. In Corollary 3.8 we show that Q^r -Anosov diffeomorphisms are C^r -robustly r -expansive (i.e., N -expansive with $N = r$). In Section 4 we investigate the converse of Corollary 3.8. We show that every periodic point of a C^r -robustly cw-expansive diffeomorphism on a compact surface is hyperbolic. Also, we prove that if f is an Axiom A diffeomorphism without cycles and C^r -robustly cw-expansive then f is Q^r -Anosov. In Section 5 we prove that for each $r \geq 2$ there is a C^r -robustly r -expansive surface diffeomorphism that is not $(r - 1)$ -expansive.

I am in debt with J. Brum and R. Potrie for conversations related with the examples presented in Section 5. Some of the results of this paper are part of my Thesis made under the guidance of M. J. Pacifico and J. L. Vieitez.

2. OMEGA-EXPANSIVITY

Let M be a smooth compact manifold without boundary. In this section the dimension of M will be assumed to be greater than one. Given a C^1 -diffeomorphism $f: M \rightarrow M$ define $\text{Per}(f)$ as the set of periodic points of f and the non-wandering set $\Omega(f)$ as the set of those $x \in M$ satisfying: for all $\varepsilon > 0$ there is $n \geq 1$ such that $B_\varepsilon(x) \cap f^n(B_\varepsilon(x)) \neq \emptyset$. Recall that f satisfies Smale's *Axiom A* if $\text{clos}(\text{Per}(f)) = \Omega(f)$ and $\Omega(f)$ is hyperbolic. A compact invariant set $\Lambda \subset M$ is *hyperbolic* if the tangent bundle over Λ splits as $T_\Lambda M = E^s \oplus E^u$ the sum of two sub-bundles invariant by df and there are $c > 0$ and $\lambda \in (0, 1)$ such that:

- (1) if $v \in E^s$ then $\|df^n(v)\| \leq c\lambda^n\|v\|$ for all $n \geq 0$ and
- (2) if $v \in E^u$ then $\|df^n(v)\| \leq c\lambda^n\|v\|$ for all $n \leq 0$.

Definition 2.1. We say that f is Ω -*expansive* if $f: \Omega(f) \rightarrow \Omega(f)$ is expansive. We say that f is C^r -*robustly Ω -expansive* if there is a C^r -neighborhood U of f such that every $g \in U$ is Ω -expansive.

Remark 2.2. *Trivially, if $\Omega(f)$ is a finite set then f is Ω -expansive. Also, if $\Omega(f)$ is a hyperbolic set then f is Ω -expansive.*

A C^1 -diffeomorphism $f: M \rightarrow M$ is Ω -*stable* if there is a C^1 -neighborhood U of f such that for all $g \in U$ there is a homeomorphism $h: \Omega(f) \rightarrow \Omega(g)$ such that $h \circ f = g \circ h$. We say that f is a C^r -*star diffeomorphism* if there is a C^r -neighborhood U of f such that every periodic orbit of every $g \in U$ is a hyperbolic set. If f satisfies the axiom A then $\Omega(f)$ decomposes into a finite disjoint union basic sets $\Omega(f) = \Lambda_1 \cup \dots \cup \Lambda_l$. A collection $\Lambda_{i_1}, \dots, \Lambda_{i_k}$ is called a *cycle* if there exist points $a_j \notin \Omega(f)$, for $j = 1, \dots, k$, such that $\alpha(a_j) \subset \Lambda_{i_j}$ and $\omega(a_j) \subset \Lambda_{i_{j+1}}$ (with $k + 1 \equiv 1$). We say that f *has not cycles* (or satisfies the no cycle condition) if there are not cycles among the basic sets of $\Omega(f)$. See for example [21, 24] for more on this subject.

From [1, 8, 10, 25] we know that the following statements are equivalent in the C^1 -topology:

- (1) f satisfies axiom A and has not cycles,
- (2) f is Ω -stable,

(3) f is a star diffeomorphism.

We add another equivalent statement, with a *simple* proof based on deep results, that will be used in the next sections.

Proposition 2.3. *A diffeomorphism is C^1 -robustly Ω -expansive if and only if it is Ω -stable.*

Proof. In order to prove the direct part suppose that f is C^1 -robustly Ω -expansive. If a periodic point of f is not hyperbolic then, by [8, Lemma 1.1], we find a small C^1 -perturbation of f with an arc of periodic points. This contradicts the C^1 -robust expansivity of f because every expansive homeomorphism of a compact metric space has at most a countable set of periodic points. This proves that f is a star diffeomorphism.

If f is Ω -stable then f satisfies Smale's axiom A. Therefore $\Omega(f)$ is hyperbolic and consequently $f: \Omega(f) \rightarrow \Omega(f)$ is expansive. Since f is Ω -stable we have that f is robustly Ω -expansive. \square

3. Q^r -ANOSOV DIFFEOMORPHISMS

In this section we assume that $M = S$ is a compact surface, i.e., $\dim(M) = 2$. The *stable set* of $x \in S$ is

$$W_f^s(x) = \{y \in S : \lim_{n \rightarrow +\infty} \text{dist}(f^n(x), f^n(y)) = 0\}.$$

The *unstable set* is defined by $W_f^u(x) = W_{f^{-1}}^s(x)$. Assume that f is Ω -stable, E^s, E^u are one-dimensional and define $I = [-1, 1]$. We denote by $\text{Emb}^r(I, S)$ the space of C^r -embeddings of I in S with the C^r -topology. Let us recall the following fundamental result for future reference.

Theorem 3.1 (Stable manifold theorem). *Let $\Lambda \subset S$ be a hyperbolic set of a C^r -diffeomorphism f of a compact surface S . Then, for all $x \in \Lambda$, $W_f^s(x)$ is an injectively immersed C^r -submanifold. Also the map $x \mapsto W_f^s(x)$ is continuous: there is a continuous function $\Phi: \Lambda \rightarrow \text{Emb}^r(I, S)$ such that for each $x \in \Lambda$ it holds that the image of $\Phi(x)$ is a neighborhood of x in $W_f^s(x)$. Finally, these stable manifolds also depend continuously on the diffeomorphisms f , in the sense that nearby diffeomorphisms yield nearby mappings Φ .*

Proof. See [20] Appendix 1. \square

Given $x \in S$ we can take $\delta_1, \delta_2 > 0$, a C^r -coordinate chart $\varphi: U \subset S \rightarrow [-\delta_1, \delta_1] \times [\delta_2, \delta_2]$ such that $\varphi(x) = (0, 0)$ and two C^r functions $g^s, g^u: [-\delta_1, \delta_1] \rightarrow [\delta_2, \delta_2]$ such that the graph of g^s and g^u are the local expressions of the local stable and the local unstable manifold of x , respectively. If the degree r Taylor polynomials of g^s and g^u at 0 coincide we say that there is an r -tangency at x .

Definition 3.2. Given $r \geq 1$, a C^r -diffeomorphism $f: S \rightarrow S$ is Q^r -Anosov if it is axiom A, has no cycles and there are no r -tangencies.

Remark 3.3. *For $r = 1$ we have that Q^1 -Anosov is quasi-Anosov, and in fact, given that S is two-dimensional, it is Anosov. For $r = 2$ we are requiring that if there is a tangency of a stable and an unstable manifold it is a quadratic one.*

We will show that Q^r -Anosov diffeomorphisms form an open set of N -expansive diffeomorphisms. Several results from [24] will be used. For $\delta > 0$ define

$$\begin{aligned} W_\delta^s(x, f) &= \{y \in S : \text{dist}(f^n(x), f^n(y)) \leq \delta \forall n \geq 0\}, \\ W_\delta^u(x, f) &= \{y \in S : \text{dist}(f^n(x), f^n(y)) \leq \delta \forall n \leq 0\}. \end{aligned}$$

Theorem 3.4. *In the C^r -topology the set of Q^r -Anosov diffeomorphisms of a compact surface is a C^r -open set.*

Proof. We know that the set of axiom A C^r -diffeomorphisms without cycles form an open set U in the C^r -topology. Let g_k be a sequence in U converging to $f \in U$. Assume that g_k is not Q^r -Anosov for all $k \geq 0$. In order to finish the proof it is sufficient to show that f is not Q^r -Anosov. Since $g_k \in U$ and it is not Q^r -Anosov, there is $x_k \in S$ with an r -tangency for g_k .

By [24, Proposition 8.11] we know that $\Omega(f)$ has a local product structure, then, we can apply [24, Proposition 8.22] to conclude that $\Omega(f)$ is uniformly locally maximal, that is, there are neighborhoods $U_1 \subset S$ of $\Omega(f)$ and U_2 a C^r -neighborhood of f such that $\Omega(g) = \bigcap_{n \in \mathbb{Z}} g^n(U_1)$ for all $g \in U_2$. Consider the compact set $K = S \setminus U_1$. We have that for all $x \notin \Omega(g)$, with $g \in U_2$, there is $n \in \mathbb{Z}$ such that $g^n(x) \in K$. Notice that if g_k has an r -tangency at x_k then every point in its orbit by g_k has an r -tangency too. Therefore we can assume that $x_k \in K$ for all $k \geq 1$. Since K is compact we can suppose that $x_k \rightarrow x \in K$. By [24, Proposition 9.1] we can take $y_k, z_k \in \Omega(g_k)$ such that $x_k \in W_{g_k}^s(y_k) \cap W_{g_k}^u(z_k)$. Suppose that $y_k \rightarrow y$ and $z_k \rightarrow z$ with $y, z \in \Omega(f)$ [24, Theorem 8.3]. By Theorem 3.1 we know that for some $\delta > 0$ the local manifolds $W_\delta^s(y_k, g_k)$ and $W_\delta^u(z_k, g_k)$ converges in the C^r -topology to $W_\delta^s(y, f)$ and $W_\delta^u(z, f)$ respectively. Since K is compact and disjoint from $\Omega(f)$ there is $m > 0$ such that

$$x_k \in g_k^{-m}(W_\delta^s(y_k, g_k)) \cap g_k^m(W_\delta^u(z_k, g_k))$$

for all $k \geq 1$. Then, taking limit $k \rightarrow \infty$ we find an r -tangency at x for f . Therefore f is not Q^r -Anosov. This proves that the set of Q^r -Anosov C^r -diffeomorphisms is an open set in the C^r -topology. \square

Definition 3.5. We say that a C^r -diffeomorphism f is C^r -robustly N -expansive if there is a C^r -neighborhood of f such that every diffeomorphism in this neighborhood is N -expansive.

The following is an elementary result from Analysis.

Lemma 3.6. *If $g: \mathbb{R} \rightarrow \mathbb{R}$ is a C^r functions with $r+1$ roots in the interval $[a, b] \subset \mathbb{R}$ then $g^{(n)}$ has $r+1-n$ roots in $[a, b]$ for all $n = 1, 2, \dots, r$ where $g^{(n)}$ stands for the n^{th} derivative of g .*

Proof. It follows by induction in n using the Rolle's theorem. \square

Recall that r -expansivity means N -expansivity with $N = r$.

Theorem 3.7. *Every Q^r -Anosov diffeomorphism of a compact surface is r -expansive. Moreover, if f is Q^r -Anosov then there are a C^r neighborhood \mathbb{U} of f and $\delta > 0$ such that δ is an r -expansive constant for every $g \in \mathbb{U}$.*

Proof. Let $f: S \rightarrow S$ be a Q^r -Anosov diffeomorphism. We can take a neighborhood U of $\Omega(f)$, a C^r -neighborhood \mathbb{U}' of f and $\delta' > 0$ such that:

- (1) $\Omega(g)$ is expansive with expansivity constant δ' for all $g \in \mathbb{U}'$ and

(2) $\Omega(g) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ for all $g \in \mathcal{U}'$.

Therefore, we have to show that there is $\delta > 0$ such that if $X \cap \Omega(f) = \emptyset$ and $\text{diam}(f^n(X)) < \delta$ for all $n \in \mathbb{Z}$ then $|X| \leq r$. Arguing by contradiction assume that there are g_n converging to f in the C^r -topology and two sequences s_n and u_n of arcs in S such that s_n is stable for g_n , u_n is unstable for g_n , $|s_n \cap u_n| > r$ and $\text{diam}(s_n), \text{diam}(u_n) \rightarrow 0$ as $n \rightarrow +\infty$. Considering the compact set K of the proof of Theorem 3.4 we can assume that $s_n, u_n \subset K$. Take $x \in K$ an accumulation point of s_n . By Lemma 3.6 and the arguments in the proof of Theorem 3.4 we have that there is an r -tangency at x for f . This contradiction finishes the proof. \square

From Theorems 3.4 and 3.7 we deduce:

Corollary 3.8. *Every Q^r -Anosov diffeomorphism of a compact surface is C^r -robustly r -expansive with uniform r -expansivity constant on a C^r -neighborhood.*

4. ROBUST CW-EXPANSIVITY

In this section we will prove that if f is C^r -robustly cw-expansive, $r \geq 1$, then its periodic points are hyperbolic. It is a first step in the direction of proving the converse of Corollary 3.8 (in case that this converse is true). A second step is done in Theorem 4.4 assuming that the diffeomorphism is Axiom A without cycles.

Lemma 4.1. *Consider a C^∞ manifold M of dimension $n \geq 1$, $p \in M$ and $U \subset M$ a neighborhood of p . Then there are $\varepsilon > 0$ and a one-parameter family of C^∞ diffeomorphisms $f_\mu: M \rightarrow M$, $|\mu - 1| < \varepsilon$, such that for all μ : $f_\mu(p) = p$, $f_\mu(x) = x$ for all $x \in M \setminus U$, $d_p f_\mu = \mu \text{Id}$. Moreover, for all $r \geq 0$ the function $\mu \mapsto f_\mu$ is continuous in the C^r -topology.*

Proof. Taking a local chart the problem is reduced to Euclidean \mathbb{R}^n . Then we will assume that $M = \mathbb{R}^n$, $p = 0$ and $\text{clos}(B_s(p)) \subset U$ for some $s > 0$. Consider a C^∞ function $\rho: \mathbb{R} \rightarrow [0, 1]$ such that $\rho(x) = 1$ if $x \leq s/2$ and $\rho(x) = 0$ if $x \geq s$. Define $f_\mu: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(1) \quad f_\mu(x) = x + (\mu - 1)\rho(\|x\|)x$$

where $\|\cdot\|$ denotes the Euclidean norm. If $\|x\| < s/2$ then $f_\mu(x) = x + (\mu - 1)x = \mu x$. Therefore, $d_p f_\mu = \mu \text{Id}$. The rest of the details are direct from (1). \square

A point $x \in M$ is *Lyapunov stable* for $f: M \rightarrow M$ if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}(y, x) < \delta$ then $\text{dist}(f^n(x), f^n(y)) < \varepsilon$ for all $n \geq 0$. In [12, Theorem 1.6] it is shown that cw-expansive homeomorphisms admits no stable points. This is done for a Peano continuum, as is our compact connected manifold M . This result was previously proved by Lewowicz and Hiraide for expansive homeomorphisms on compact manifolds. This is a key point in the following proof.

Proposition 4.2. *If f is a C^r -robustly cw-expansive diffeomorphism on a compact manifold M (arbitrary dimension) and $f^l(p) = p$ then $d_p f^l$ has at least one eigenvalue of modulus greater than 1 and at least one eigenvalue of modulus less than 1.*

Proof. Consider an open set $U \subset M$ containing the periodic point p and such that $f^i(p) \notin \text{clos}(U)$ for all $i = 1, \dots, l - 1$. Arguing by contradiction assume that the eigenvalues of $d_p f^l$ are smaller or equal than 1 in modulus (being the other case similar). Take from Lemma 4.1 a C^r -diffeomorphism f_μ of M fixing p and being the

identity outside U . In particular, f_μ is the identity in a neighborhood of the points $f(p), \dots, f^{l-1}(p)$. Assume that $\mu \in (0, 1)$ is close to 1. Define $g = f \circ f_\mu$. In this way p is a periodic point of g of period l , g is C^r -close to f and the eigenvalues of $d_p g^l$ are $\mu\lambda_1, \dots, \mu\lambda_j$ which have modulus (strictly) smaller than 1 (being $\lambda_1, \dots, \lambda_j$ the eigenvalues of $d_p f^l$). Then p is a hyperbolic sink for g , in particular it is Lyapunov stable. Since f is C^r -robustly cw-expansive, we can assume that g is cw-expansive, arriving to a contradiction with [12, Theorem 1.6]. \square

This proposition has the following direct consequence on two-dimensional manifolds:

Corollary 4.3. *Every C^r -robustly cw-expansive diffeomorphism on a compact surface is a C^r -star diffeomorphism.*

To our best knowledge it is not known whether for $r \geq 2$ every C^r -star diffeomorphism is Axiom A, even for M a compact surface. The next result is another partial result in the direction of proving the converse of Corollary 3.8.

Theorem 4.4. *Let $f: S \rightarrow S$ be a C^r -diffeomorphism Axiom A without cycles. If f is C^r -robustly cw-expansive then f is Q^r -Anosov.*

Proof. We will argue by contradiction assuming that f is not Q^r -Anosov. This implies that there is a wandering point $p \in S$ with an r -tangency. Take C^r local coordinates $\phi: I \times J \subset \mathbb{R}^2 \rightarrow S$ around p , where $I, J \subset \mathbb{R}$ are open intervals. Since p is a wandering point we can suppose that $f^n(\phi(I \times J)) \cap \phi(I \times J) = \emptyset$ for all $n \in \mathbb{Z}$, $n \neq 0$. Let $g_s, g_u: I \rightarrow J$ be C^r functions such that their graphs describe the local stable and local unstable manifold of p in coordinates. Since there is an r -tangency at p we can suppose that the Taylor polynomials of order r of g_s and g_u vanishes at 0.

Define the C^r diffeomorphism $h: I \times \mathbb{R} \rightarrow I \times \mathbb{R}$ as

$$h(x, y) = (x, g_s(x) - g_u(x) + y).$$

Let $\sigma: \mathbb{R} \rightarrow [0, 1]$ be a C^∞ function such that $\sigma(a) = 1$ for $a \leq 1/2$ and $\sigma(a) = 0$ for $a \geq 1$. For $\mu > 0$ define $j_\mu: I \times J \rightarrow I \times \mathbb{R}$ as

$$j_\mu(x, y) = \sigma(\sqrt{x^2 + y^2}/\mu)h(x, y).$$

Define $f_\mu: S \rightarrow S$ by

$$f_\mu(q) = \begin{cases} f \circ \phi \circ j_\mu \circ \phi^{-1}(q) & \text{if } q \in \phi(B_\mu(0, 0)) \\ f(q) & \text{in other case.} \end{cases}$$

Let us show that f_μ is not cw-expansive for $\mu > 0$ small. If $\sqrt{x^2 + y^2} < \mu/2$ then $j_\mu(x, y) = h(x, y)$. Then, we must note that $h(x, g_u(x)) = (x, g_s(x))$ and this means that h maps the graph of g_u into the graph of g_s . For f_μ this implies that there is an arc $\gamma = \{\phi(x, g_s(x)) : |x| < \mu/\sqrt{8}\} \subset S$ such that $\text{diam}(f^n(\gamma)) \rightarrow 0$ as $n \rightarrow \pm\infty$. Then, arbitrarily small subarcs of γ contradict the cw-expansivity of each f_μ for arbitrarily small cw-expansive constants.

Now we will show that f_μ is a C^r small perturbation f if μ is close to 0. By definition, they coincide for $q \notin \phi(B_\mu(0, 0))$. Therefore, the problem is reduced to show that j_μ is a C^r small perturbation of the identity in $I \times J$. Notice that

$j_\mu(x, y) - Id(x, y) = \sigma(\sqrt{x^2 + y^2}/\mu)h(x, y) = (0, g_s(x) - g_u(x))$. In order to conclude we will show that the map

$$l(x, y) = \sigma(\sqrt{x^2 + y^2}/\mu)(0, g_s(x) - g_u(x))$$

is C^r -close to $(x, y) \mapsto (0, 0)$. Because of the r -tangency at p we know that $R(x) = g_s(x) - g_u(x)$ satisfies $R(x)/x^r \rightarrow 0$ as $x \rightarrow 0$. This and L'Hospital's rule implies that $R^{(t)}(x)/x^{r-t} \rightarrow 0$ as $x \rightarrow 0$ for all $t = 0, 1, \dots, r$. As before, $R^{(t)}(x)$ denotes the t^{th} derivative of R at x . Define $\rho(x, y) = \sigma(\sqrt{x^2 + y^2})$ and let $K = \|\rho\|_{C^r}$. Given $\varepsilon > 0$ consider $\mu > 0$ such that

$$\left| \frac{R^{(t)}(x)}{x^{r-t}} \right| < \frac{\varepsilon}{r!rK}$$

for all $x \in (-\mu, \mu)$. For $\|(x, y)\| \geq \mu$ we have that $\rho(x/\mu, y/\mu) = 0$, so there is nothing to estimate. We will assume that $\|(x, y)\| \leq \mu$. Given two non-negative integers i, j such that $i + j \leq r$ we have that

$$\begin{aligned} \left| \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} [\rho(x/\mu, y/\mu)R(x)] \right| &= \left| \frac{\partial^i}{\partial x^i} \left[\frac{\partial^j \rho}{\partial y^j}(x/\mu, y/\mu) \frac{1}{\mu^j} R(x) \right] \right| \\ &\leq \frac{1}{\mu^j} \left| \sum_{l=0}^i i! \frac{\partial^l}{\partial x^l} \frac{\partial^j \rho}{\partial y^j}(x/\mu, y/\mu) \frac{1}{\mu^l} R^{(i-l)}(x) \right| \\ &\leq \frac{r!}{\mu^{j+l}} \|\rho\|_{C^r} \left| \sum_{l=0}^i R^{(i-l)}(x) \right| \\ &\leq r!K \left| \sum_{l=0}^i \frac{R^{(i-l)}(x)}{x^{j+l}} \right| \leq r!K \sum_{l=0}^i \left| \frac{R^{(i-l)}(x)}{x^{r-i+l}} \right| \leq \varepsilon. \end{aligned}$$

This proves that f_μ is a C^r -approximation of f that is not cw-expansive. This contradiction proves the theorem. \square

5. EXAMPLES OF N -EXPANSIVE DIFFEOMORPHISMS

In this section we present examples of C^r -robustly N -expansive surface diffeomorphisms that are not Anosov. They are variations of the 2-expansive homeomorphism presented in [5], that in turn, is based on the three-dimensional quasi-Anosov diffeomorphism given in [9].

Theorem 5.1. *For each $r \geq 2$ there is a C^r -robustly r -expansive surface diffeomorphism that is not $(r-1)$ -expansive.*

Proof. We start with the case of $r = 2$. It is essentially the example given in [5], we recall some details from this paper. Consider S_1 and S_2 two copies of the torus $\mathbb{R}^2/\mathbb{Z}^2$ and the C^∞ -diffeomorphisms $f_i: S_i \rightarrow S_i$, $i = 1, 2$, such that: 1) f_1 is a derived-from-Anosov as detailed in [21], 2) f_2 is conjugate to f_1^{-1} and 3) f_i has a fixed point p_i , where p_1 is a source and p_2 is a sink. Also assume that there are local charts $\varphi_i: D \rightarrow S_i$, $D = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$, such that

- (1) $\varphi_i(0) = p_i$,
- (2) the pull-back of the stable (unstable) foliation by φ_1 (φ_2) is the vertical (horizontal) foliation on D and
- (3) $\varphi_1^{-1} \circ f_1^{-1} \circ \varphi_1(x) = \varphi_2^{-1} \circ f_2 \circ \varphi_2(x) = x/4$ for all $x \in D$.

Let A be the annulus $\{x \in \mathbb{R}^2 : 1/2 \leq \|x\| \leq 2\}$. Consider the diffeomorphism $\psi: A \rightarrow A$ given by $\psi(x) = x/\|x\|^2$. Denote by \hat{D} the open disk $\{x \in \mathbb{R}^2 : \|x\| < 1/2\}$. On $[S_1 \setminus \varphi_1(\hat{D})] \cup [S_1 \setminus \varphi_2(\hat{D})]$ consider the equivalence relation generated by

$$\varphi_1(x) \sim \varphi_2 \circ \psi(x)$$

for all $x \in A$. Denote by \bar{x} the equivalence class of x . The surface

$$S = \frac{[S_1 \setminus \varphi_1(\hat{D})] \cup [S_1 \setminus \varphi_2(\hat{D})]}{\sim}$$

has genus two, we are considering the quotient topology on S . Define the C^∞ -diffeomorphism $f: S \rightarrow S$ by

$$f(\bar{x}) = \begin{cases} \overline{f_1(x)} & \text{if } x \in S_1 \setminus \varphi_1(\hat{D}) \\ \overline{f_2(x)} & \text{if } x \in S_2 \setminus \varphi_2(D) \end{cases}$$

We know that f is Axiom A because the non-wandering set consists of a hyperbolic repeller and a hyperbolic attractor. Also f has no cycles. The stable and unstable foliations in the annulus $\bar{A} = \varphi_1(A)$ looks like Figure 1. The tangen-

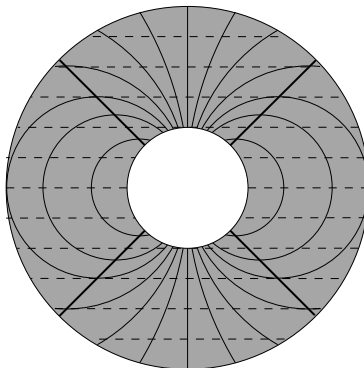


FIGURE 1. Foliations in the annulus \bar{A} . The circle arcs represent the unstable foliation after the inversion and the horizontal dot-lines are the stable foliation. The diagonal bold lines are the tangencies between stable and unstable manifolds.

cies are quadratic because in local charts stable manifolds are straight lines and unstable manifolds are circle arcs. Then, applying Theorem 3.7 we have that f is C^2 -robustly 2-expansive. It is not 1-expansive (i.e. expansive) because near the line of tangencies we find pairs of points contradicting expansivity (for every expansive constant).

For the case $r = 3$ we will change ψ in an open set U contained in A . Consider ψ such that the stable and the unstable foliations looks like in Figure 2. There are two curves that are topologically transverse but there is a tangency of order 2. Between these two curves there are points of non topologically transversality. The unstable arcs are modeled by the one parameter family

$$p_a(x) = x^3 + (a^2 - 1)x + 9a$$

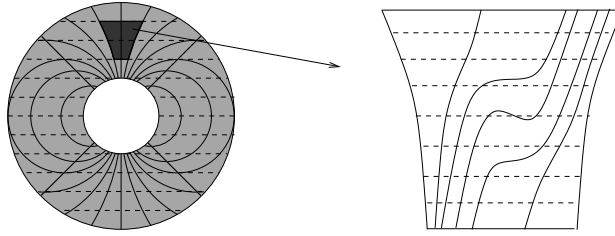


FIGURE 2. Dot-lines represents the stable foliation and curved lines are the unstable foliation. In this way f is not 2-expansive but it is C^3 -robustly 3-expansive.

for $x, a \in [-2, 2]$. The presence of the term $9a$ implies that $\partial p_a / \partial a > 0$ for all $x, a \in [-2, 2]$. If $|a| > 1$ then $p'_a(x) > 0$ for all x , so we have transversality. If $|a| = 1$ then $p_a(x)$ is a translation of x^3 , so there is a tangency of order 2 at $x = 0$. If $|a| < 1$ then $p_a(x)$ has a local maximum and local minimum that are close if $|a|$ is close to 1. Therefore, we see that f is not 2-expansive. It is C^3 -robustly 3-expansive because $p''_a(x) = 6 \neq 0$ for all $x, a \in [-2, 2]$.

For the case $r = 4$ we consider an open set $U \subset A$ containing a quadratic tangency as in Figure 3. The map ψ is changed in U in such a way that the unstable arcs corresponds to the curves on the right hand of the figure. They can be modeled with the polynomials

$$p_a(x) = x^4 + (a^2 - 1)x^2 + 16a.$$

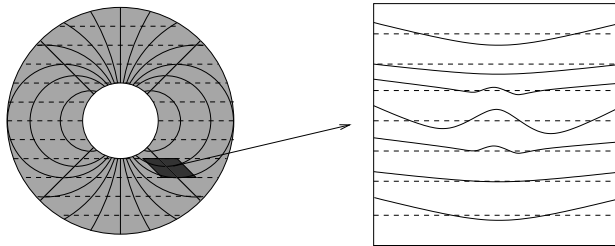


FIGURE 3. Stable and unstable foliations for a C^4 -robustly 4-expansive diffeomorphism that is not 3-expansive.

The general case $r \geq 5$ follows the same ideas. For r odd ψ is changed near a box of local product structure. For r even ψ is changed near a quadratic tangency. \square

In particular we have:

Corolary 5.2. *There are C^r -robustly cw-expansive surface diffeomorphisms that are not Anosov.*

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