APPROXIMATELY HOLOMORPHIC GEOMETRY FOR PROJECTIVE CR MANIFOLDS

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Abstract. For compact CR manifolds of hypersurface type which embed in complex projective space, we show for all k large enough the existence of linear systems of $\mathcal{O}(k)$, which when restricted to the CR manifold are generic in a suitable sense. In general these systems are constructed using approximately holomorphic geometry, but for strictly $\mathbb{C}$-convex hypersurfaces generic degree one pencils are obtained via dual geometry. In particular known results about the differential topological type of strictly $\mathbb{C}$-convex hypersurfaces are recovered.

Approximately holomorphic geometry can be applied to compact symplectic manifolds to obtain "generic" approximately holomorphic maps to complex projective spaces $\mathbb{CP}^m$ [4]. These maps can be understood as the analogs of generic linear systems for Hodge manifolds. The analogy is valid when their construction only involves studying (pseudo-holomorphic) 1 and 2-jets [13, 3]; it breaks down for higher order jets due to the difficulty of developing a theory of normal forms in the absence of an integrable almost complex structure. Moreover, in case the manifold is Hodge and according to section 7 in [12], it is possible to adjust the constructions of approximately holomorphic geometry so that the outcome is holomorphic. This gives a new construction of generic linear systems.

Applications of approximately holomorphic theory to symplectic manifolds include the construction of plenty submanifolds with control on their topology [12, 2, 27], construction of symplectic invariants [5], and a proof -via Lefschetz pencil structures [13] and making little use of elliptic theory- of the existence of symplectic curves realizing the canonical class of a 4-dimensional symplectic manifold [14].

Recall that a 2-calibrated manifold is a triple $(M^{2n+1}, D^{2n}, \omega)$, where D is a codimension one distribution, and ω a closed 2-form maximally non-degenerate over D. The most relevant classes of 2-calibrated manifolds are related to the two extreme behaviors of the distribution: when D is integrable we speak about 2-calibrated foliations, and when D is maximally non-integrable 2-calibrated structures include contact structures.

Approximately holomorphic geometry can be applied to compact 2-calibrated manifolds to obtain "generic" approximately holomorphic maps to complex projective spaces [24]. Interesting applications only involve the study of (pseudo-holomorphic) 1-jets, and they include the existence of contact submanifolds whose Poincaré dual realizes any determinantal class [24], the construction of open book decompositions compatible with a contact structure [16, 15], and a description of the leaf spaces of 2-calibrated foliations [25].

When the auxiliary compatible almost complex structure on D needed to develop the approximately holomorphic theory is integrable, what we have is a CR manifold (of hypersurface type). We are interested in analyzing under which conditions the

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approximately holomorphic techniques can be refined to yield CR constructions, and which are are the applications that can be obtained.

The structure of the paper is as follows: in section 1 we recall the definitions and results concerning CR geometry needed to state the main results of this paper. These results have to do with the existence of generic linear systems for *projective CR manifolds* (definition 5), together with some topological implications.

For complex projective manifolds generic linear systems are constructed using dual geometry. In section 2 we outline the difficulties to develop a dual geometry for arbitrary projective CR manifolds. However, it is possible to use this approach for the so called *strict* $\mathbb{C}$*-convex hypersurfaces of* $\mathbb{C}^N$ (definition 9). This is explored in subsection 2.1, where the corresponding rank one generic linear systems (Lefschetz pencils) are seen to be CR analogs of the simplest Morse functions for spheres (i.e. with two critical points); in particular the known results about the topology of strict $\mathbb{C}$-convex hypersurfaces are recovered.

Finally, in section 3 we sketch how to adapt the constructions of approximately holomorphic geometry for projective CR manifolds, thus proving the results stated in section 1.

1. Definitions and statements of the main results

Definition 1. *A CR manifold (always of hypersurface type for us) is a triple* (M^{2n+1}, D^{2n}, J)*, such that either of the sub-bundles* $D^{*1,0}, D^{*0,1}$ *of the complexification of* D *are involutive.*

Equivalently, (i) the Nijenhuis tensor of J *has to vanish, and (ii) for all* X, Y *local sections of* D*, the linear combination* $[X, JY] + [JX, Y]$ *is required to be a section of* D*.*

All our CR manifolds will be closed and oriented unless otherwise stated.

Definition 2. *The Levi form of a CR manifold is the symmetric bilinear form*

$$\begin{array}{ccc} D \times D & \to & TM/D \cong \underline{\mathbb{R}} \\ (U, V) & \longrightarrow & [U, JV]/\sim \ , \end{array}$$

where we consider the class of the above bracket in the quotient real line bundle TM/D*.*

The Levi form keeps track of the behavior of the distribution D. Its vanishing is equivalent to D integrating into a foliation $\mathcal{F}$, in which case we speak of a Levi-flat CR manifold. The opposite case is that of a CR manifold with strictly positive (resp. negative) Levi form. In particular, the distribution D of such CR manifolds is a contact distribution and they are called strictly pseudo-convex (resp. pseudo-concave).

Definition 3. *Let* (M, D, J) *be a CR manifold and* (M', D', J') *either a CR manifold or a complex manifold. A map* $\phi \colon M \to M'$ *is CR if* $\phi_* D \subset D'$ *and* $\phi_* \circ J = J' \circ \phi_*$*.*

Definition 4. *A CR vector bundle is a complex vector bundle* $\pi \colon E \to (M, D, J)$ *defined by CR transition maps.*

In order to be able to find CR sections of a CR line bundle $L \to M$ (or actually of its tensor powers), a natural condition to impose is that of positivity along D. In the Levi-flat case and if the leaves are compact (so from the differential viewpoint the foliated manifold is a mapping torus), the complex holomorphic sections over the leaves are expected to fit into CR sections.

At least for Levi-flat CR manifolds, one might try to produce CR sections perturbing approximately holomorphic sections (see definition 13, and section 3 for more details) into CR ones, by solving the corresponding $\bar{\partial}$-problem [30].

There are two technical complications to develop an intrinsic approximately holomorphic theory for a compact CR manifold with a positive CR line bundle: firstly appropriate concentrated CR sections have to be constructed. Secondly the perturbations that have to be used to solve the corresponding (estimated) transversality problems, are not small multiples of the concentrated sections but rather deformations which are leafwise constant for certain local Levi-flat CR structure approximating the initial CR structure. Thus in the Levi-flat case we would stay within the class of CR sections, but for a general CR manifold we would not.

Hence, further requirements need to be imposed so that these difficulties can be overcome.

Definition 5. *A CR manifold (M, D, J) is projective if it admits a CR embedding into some $\mathbb{CP}^N$. It is called embeddable if it admits a CR embedding into some $\mathbb{C}^N$.*

Given a projective CR manifold $j\colon (M, D, J) \hookrightarrow \mathbb{CP}^N$, the CR sections we are interested in will be restrictions of suitable holomorphic sections of $\mathcal{O}(k) \to \mathbb{CP}^N$, where $\mathcal{O}(k)$ is the k-th tensor power of the hyperplane line bundle $\mathcal{O}(1)$, and its holomorphic sections are identified with degree k homogeneous polynomials.

A large number of projective CR manifolds are provided by the following embedding theorems.

Theorem 1. *(Boutet de Monvel* [8]*) Any compact strictly pseudo-convex (resp. pseudo-concave) oriented CR manifold of dimension bigger or equal than five, admits a CR embedding into some $\mathbb{C}^N$, and hence it is projective.*

A recent result of Marinescu and Yeganefar [22] states that any Sasakian manifold is embeddable. Sasakian manifolds are strictly pseudo-convex. The three dimensional ones -to which theorem 1 does not apply- are also covered by the aforementioned result of Marinescu and Yeganefar.

A similar result based on the solution of the $\bar{\partial}$-Neumann problem with L^2-estimates is the following:

Theorem 2. *(Ohsawa, Sibony* [30]*) Let (M, D, J) be a compact Levi-flat CR manifold and $L \to M$ a positive line bundle. Let h be any natural number. Then there exists another natural number $d(h)$ and a CR embedding $j\colon M \hookrightarrow \mathbb{CP}^{d(h)}$ of class C^h.*

Now let (M, D, J) a CR manifold (not necessarily projective). Let Y be either a complex or a CR manifold, and let us denote in either case by D' the maximal complex distribution.

Definition 6. *The bundle of CR 1-jets of maps from M to Y, denoted by $\mathcal{J}^1_D(M, Y) \to M$, is defined to be the bundle over M whose fiber over x is*

$$\mathcal{J}^1_D(M, Y)_x := \{(y, h) |\, y \in Y, h \in Hom_{\mathbb{C}}(D_x, D'_y)\}$$

Inside this bundle there is a distinguished submanifold

$$\Sigma = \{(x, y, h) \in \mathcal{J}^1_D(M, Y) | h = 0\} \tag{1}$$

The CR 1-jet of a CR map $\phi\colon M \to Y$ is by definition $j^1_D\phi := (\phi, \nabla_D\phi)$, where $\nabla_D\phi$ is the restriction of $\nabla\phi$ to D. It is a section of $\mathcal{J}^1_D(M, Y)$.

If (M, D, J) is a projective CR manifold, there is a second relevant complex distribution associated to M: the complex envelope of its tangent bundle, denoted by TM^J (at each point $x \in M$ we put the smallest complex subspace of $T_x\mathbb{CP}^N$

containing T_xM). Using the Kahler metric, both D and the convex envelope of TM can be extended by parallel transport along normal geodesics to complex distributions in $\mathcal{N}_\epsilon(M)$, a neighborhood of M. The extended bundles will be also denoted by D and TM^J whenever there is no risk of confusion.

We have the corresponding bundle $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), Y) \to \mathcal{N}_\epsilon(M)$ of 1-jets along TM^J of maps from $\mathcal{N}_\epsilon(M)$ to Y. We denote by Σ^J the submanifold corresponding to the 1-jets along TM^J whose degree 1 component is vanishing.

Similarly, the holomorphic 1-jet along TM^J of a holomorphic map $\Phi\colon \mathcal{N}_\epsilon(M) \to Y$ is by definition $j^1_{TM^J}\Phi := (\Phi, \nabla_{TM^J}\Phi)$. It is a section of $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), Y)$.

Definition 7. *Given $j\colon (M, D, J) \hookrightarrow \mathbb{CP}^N$ a projective CR manifold, a (CR) Lefschetz pencil structure (of degree k) on M is defined to be a pencil of (degree k) hypersurfaces of $\mathbb{CP}^N$, so that any two hypersurfaces intersect transversally in the base locus $\tilde{B}$, and*

1. *$\tilde{B}$ is transversal to M and therefore $B := \tilde{B} \cap M$ is a real codimension four CR submanifold of M.*
2. *If $\Phi\colon \mathbb{CP}^N \backslash \tilde{B} \to \mathbb{CP}^1$ denotes the holomorphic map associated to the pencil, and $\phi\colon M\backslash B \to \mathbb{CP}^1$ is its restriction, then $j^1_D\phi$ is transversal along D to Σ.*
3. *$j^1_{TM^J}\Phi\colon \mathcal{N}_\epsilon(M)\backslash\tilde{B} \to \mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M)\backslash\tilde{B}, \mathbb{CP}^1)$ is transversal along M to Σ^J.*

We will denote the Lefschetz pencil structure by the triple (ϕ, B, Δ), where Δ are the points of M where the derivative $\nabla_D\phi$ vanishes.

Using a different language, what we have defined is the notion of a generic rank one linear system of the CR line bundle $\mathcal{O}_M(k) := j^*\mathcal{O}(k)$.

Any point $x \in \Delta$ is called a singular point of the pencil. Points in B are base points, and points in $M\backslash(B \cup \Delta)$ are called regular. Similarly, $a \in \mathbb{CP}^1\backslash\phi(\Delta)$ is called a regular value, and singular otherwise. Notice that at singular point of the pencil, the map ϕ fails to be a submersion.

Recall that transversality along D to Σ of a section $\sigma\colon M \to \mathcal{J}^1_D(M, \mathbb{CP}^1)$ is defined as follows: the pullback of D by the projection $\mathcal{J}^1_D(M, \mathbb{CP}^1) \to M$ defines a distribution $\hat{D}$ in $\mathcal{J}^1_D(M, \mathbb{CP}^1)$. At every point x such that $\sigma(x) \in \Sigma$, $\hat{D}_{\sigma(x)}$ has to be spanned by $T_{\sigma(x)}\Sigma \cap \hat{D}_{\sigma(x)}$ and $T_x\sigma \cap \hat{D}_{\sigma(x)}$, where $T_x\sigma$ is the tangent space of the graph of σ at $\sigma(x)$.

The above definition extends to transversality along D to any submanifold S of a bundle $E \to M$.

Transversality along M is defined as transversality along D, where the role of D is played by TM. Since TM is integrable, transversality along M of $j^1_{TM^J}\Phi$ is equivalent to usual transversality for the restriction $j^1_{TM^J}\Phi\colon M\backslash B \to j^*\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M)\backslash\tilde{B}, \mathbb{CP}^1)$.

Transversality along distributions is stronger than usual transversality.

From the transversality assumption along D on $j^1_D\phi$, we conclude that Δ is a 1-dimensional submanifold transversal to D. A dimension count implies that transversality of $j^1_{TM^J}\Phi$ along M to Σ^J, is equivalent to the section not intersecting Σ^J. Therefore in the points of Δ the derivative $\nabla\phi\colon TM \to T\mathbb{CP}^1$ cannot vanish, because being Φ holomorphic that would imply the vanishing of $\nabla_{TM^J}\Phi$. Thus $\phi(\Delta) \subset \mathbb{CP}^1$ is an immersed curve, which is why condition 3 in definition 7 is required.

Remark 1. *If the projective CR manifold is Levi-flat and the inclusion $j\colon M \hookrightarrow \mathbb{CP}^N$ is of class C^h, $h \geq 2$, definition 7 also makes sense, for transversality (along any distribution) is a C^1-notion.*

Our first main result is the following:

Theorem 3. *Let* (M^{2n+1}, D, J) *be a (closed and oriented) projective CR manifold of class* C^h*,* $h \geq 3$*. Then for any* k *large enough* M *admits degree* k *Lefschetz pencil structures. Let* W *be the fiber of* ϕ *over any regular value, compactified by adding* B*. Then the morphisms at the level of homotopy (and homology groups)* $l_*\colon \pi_i(W) \to \pi_i(M)$ *induced by the inclusion are isomorphisms for* $0 \leq i \leq n-2$ *and a epimorphism for* $i = n-1$*.*

Theorem 3 is a result about the existence of generic rank one linear systems of the bundles $\mathcal{O}_M(k) \to M$, for k large.

It is not possible in general to give a normal form for ϕ in the vicinity of Δ and B, if we do not have CR coordinates. In the Levi-flat case we can answer that question and construct other generic maps.

For any projective Levi-flat CR manifold the behavior of the CR map ϕ around the base and singular points is the following (see [29], [18]):

Proposition 1.

(1) *For any* $x \in B$ *there exist CR coordinates* $z^1, \dots, z^n, s$ *centered at* x *and a holomorphic chart of* $\mathbb{CP}^1$*, such that*

$$B \equiv z^1 = z^2 = 0 \quad \text{and} \quad \phi(z^1, \dots, z^n, s) = z^2/z^1$$

(2) *For any* $x \in \Delta$ *there exists CR coordinates* $z^1, \dots, z^n, s$ *centered at* x*, a holomorphic chart of* $\mathbb{CP}^1$ *and a real map* t *of class* C^{h-1}*, such that*

$$\phi(z^1, \dots, z^n, s) = z_1^2 + \cdots + z_n^2 + t(s), \; t(0) = 0, \; t'(0) \neq 0$$

Corollary 1. *Let* $(M, \mathcal{F}, J)$ *be a projective Levi-flat CR manifold of dimension bigger than five, and let* (ϕ, B, Δ) *be one of the pencils provided by theorem 3. Then for any regular fiber* W *of* ϕ *(compactified adding the base points) and for every leaf* F *of* $\mathcal{F}$*, the intersection* $F \cap W$ *is connected.*

As a consequence, the inclusion $l\colon (W, \mathcal{F}_W) \hookrightarrow (M, \mathcal{F})$*, where* $\mathcal{F}_W$ *is the induced foliation, descends to a homeomorphism of leaf spaces.*

Corollary 1 is a leafwise Lefschetz hyperplane theorem for the π_0, and for sections (degree k hypersurfaces) which fit into a Lefschetz pencil for $(M, \mathcal{F}, J)$.

We would like to single out theorem 3 for 3-dimensional projective Levi-flat CR manifolds.

Let M^3 be an orientable closed 3-manifold endowed with a smooth co-orientable foliation $\mathcal{F}$ by surfaces. We recall the following elementary result.

Lemma 1. $(M^3, \mathcal{F})$ *admits the structure of a Levi-flat CR manifold (with CR foliation* $\mathcal{F}$*) with a positive CR line bundle, if and only* $\mathcal{F}$ *is a taut foliation.*

Proof. The existence of a Levi-flat CR structure with a positive CR line bundle L clearly implies tautness, since following Sullivan [31] this is equivalent to the existence of a closed 2-form restricting to a leafwise area form. In our case i times the curvature of L has this property.

Conversely, if the foliation is taut we do have an integral closed 2-form ω which is non-degenerate when restricted to each leaf. Let J be a leafwise compatible almost complex structure. Then J is integrable, for the leaves are 2-dimensional, and ω is of type (1,1) w.r.t J. Therefore we can construct (L, ∇), the corresponding pre-quantum line bundle. Being ω of type (1,1), the bundle admits a leafwise holomorphic structure, and hence a CR one. Positivity is also clear. □

Corollary 2. *Let* $(M^3, \mathcal{F})$ *be a closed orientable 3-manifold endowed with a co-orientable taut foliation. Then there exists a CR structure in* $(M^3, \mathcal{F})$ *and maps* $\phi\colon M^3 \to \mathbb{CP}^1$ *with the following properties:*

(1) *For any fixed* $h \geq 3$*,* ϕ *is of class* C^h *and leafwise holomorphic.*

(2) *The restriction of ϕ to each leaf is a branched cover with index 2 singular points.*
(3) *The leafwise singular sets fit into a C^{h-1}-transversal link Δ.*
(4) *Around each point $a \in \Delta$ there exist local CR coordinates (z, s) and a complex coordinate in $\mathbb{CP}^1$, so that $\phi(z, s) = z^2 + t(s)$, where t is C^{h-1} and $t(0) = 0$, $t'(0) \neq 0$.*

Corollary 3. *Let $(M^3, \mathcal{F})$ be a closed orientable 3-manifold endowed with a co-orientable foliation. Then $\mathcal{F}$ is taut is and only if there exist a leafwise complex structure for which $(M^3, \mathcal{F}, J)$ admits a (CR) Lefschetz pencil structure.*

Recall that in the holomorphic context the concept of an r-generic holomorphic map $\Phi\colon X \to Y$ can be defined as follows: the map is 1-generic if the partition of X according to the rank of the differential (as a complex linear map) defines an stratification $\Sigma(\Phi)_r$, $r = 0, ..., |\dim X - \dim Y|$ by smooth strata (r denotes how much the rank drops from the expected generic dimension). 1-generic maps are 2-generic when their restriction to any of the strata $\Sigma(\Phi)_r$ are 1-generic; r-genericity is defined by induction.

Definition 8. *Let $(M, \mathcal{F}, J)$ be a Levi-flat CR manifold and Y a complex manifold. A CR map $\phi\colon M \to Y$ is said to be r-generic if its restriction to each leaf is an r-generic holomorphic function.*

Definition 8 works for any manifold $(M, \mathcal{F})$ foliated by complex leaves. An equivalent formulation is the following: consider the bundles of foliated holomorphic r-jets, and there the foliated Thom-Boardman stratification $\mathcal{T}^{\mathcal{F}}$ (over each leaf it amounts to considering the usual Thom-Boardman stratification [7]). A CR map ϕ is r-generic if its foliated holomorphic r-jet is *leafwise* transversal to $\mathcal{T}^{\mathcal{F}}$. For such a map the pullback of each strata $\Sigma_I^{\mathcal{F}}(\phi)$ would be transversal to the leaves; its intersection to each leaf F would be the Thom-Boardman stratum $\Sigma_I(\phi_{|F})$.

Let us for simplicity forget about holomorphic functions and consider the foliated genericity problem in the smooth setting. A strategy to solve transversality problems for foliated smooth jets, is to use the canonical submersion from the bundle of jets to the bundle of foliated jets [6], to pull back the leafwise Thom-Boardman stratification to the bundle of r-jets. Foliated r-genericity is equivalent to strong transversality *along the leaves of* $\mathcal{F}$ to the pulled back stratification.

The existence of foliated r-generic maps for $(M, \mathcal{F})$ is obstructed. Obstructions are deduced from the strata being transversal to the foliation. For example, there cannot be leafwise Morse functions in foliated 3-manifolds with Reeb components.

To develop a similar strategy to solve the leafwise holomorphic genericity problem, we would need to embed M inside a complex manifold X, and transfer the problems for foliated holomorphic jets to problems for full holomorphic jets. The next step would be to solve the foliated (strong) transversality problem in the bundle of full holomorphic r-jets. This is always possible locally, but there is no reason why a global solution should exist.

If $(M, \mathcal{F}, J)$ is Levi-flat and posses a positive CR line bundle, theorem 2 implies that for any natural number h we obtain a leafwise holomorphic embedding in $\mathbb{CP}^{d(h)}$ of class C^h. This projective space plays the role of the ambient complex space X.

The target space of our generic maps cannot be an arbitrary complex manifold, but complex projective space, for our maps will come from linear systems.

Theorem 4. *Let $(M^{2n+1}, \mathcal{F}, J)$ be a closed Levi-flat CR manifold endowed with a positive CR line bundle $L \to M$. Fix $h, r \in \mathbb{N}$, $r \leq h - 2$. Then for any integer m there exists $\phi\colon M \backslash B \to \mathbb{CP}^m$ an r-generic map. More precisely, we obtain the following:*

(1) *A Levi-flat CR submanifold B of real codimension 2m+2 and class C^h.*
(2) *A CR map $\phi\colon M\backslash B \to \mathbb{CP}^m$ -restriction of a holomorphic map $\Phi\colon \mathbb{CP}^N\backslash\mathcal{B} \to \mathbb{CP}^m$- such that for each leaf of $\mathcal{F}$, the holomorphic r-jet of its restriction is transversal to the corresponding Thom-Boardman stratification of the bundle of holomorphic r-jets. This bundle fits into a bundle of class C^{h-r} –the bundle of CR r-jets– and the same holds for the strata of the Thom-Boardman stratifications. The CR r-jet is leafwise transversal to this stratification by Levi-flat manifolds of class C^{r-h}. Therefore, the pullback of each strata is a Levi-flat submanifold of the expected codimension and of class C^{r-h}.*
(3) *The 1-jet of Φ along TM^J is transversal along M to the stratification of $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$, whose strata are defined according to the rank of the degree 1 component of $\sigma \in \mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$.*

If $L^{\otimes d(h)} \to M$ is the line bundle providing embedding of class C^h in projective space, theorem 4 asserts the existence of generic linear systems of arbitrary rank for large enough powers of $L^{\otimes d(h)}$. Observe that genericity is defined to be leafwise genericity (properties 1 and 2), together with a requirement on the full derivative of the induced map analogous to the third one in the definition of a Lefschetz pencil structure. This condition gives supplementary information about the differential of ϕ in the direction transversal to D, in those points where the rank along D decreases. For example, when $m = 1, n-1, n$, it implies that $\nabla\phi$ has always maximal rank and therefore that $\nabla_{D^\perp}\phi(x) \neq 0$, $\forall x \in \Sigma_1^{\mathcal{F}}(\phi)$.

2. Dual geometry of projective CR manifolds

For any projective manifold $X \hookrightarrow \mathbb{CP}^N$ generic linear systems of $\mathcal{O}(1)$ can be constructed by applying elementary transversality results to its dual variety $X^* \to \mathbb{CP}^{N*}$. It might happen that the dual variety is not a divisor, but it is indeed a divisor if we twist the embedding (compose with the k-th Veronese embedding, for any $k \geq 2$). So in the worst case there exist always degree 2 Lefschetz pencils [11].

Let $(M, D, J) \hookrightarrow \mathbb{CP}^N$ be a projective CR variety. We define its dual set

$$M^* = \{H \in \mathbb{CP}^{N*} | D_x \subset H_x \text{ for some } x \in M\}$$

By pulling back the usual diagram of flag varieties, it can be shown that there exists $\pi\colon S_D \to M$ a smooth (resp. C^{h-1} for C^h-embeddings of Levi-flat manifolds) fiber bundle of real dimension 2N-1, and a smooth (resp. C^{h-1}) map $\nu\colon S_D \to \mathbb{CP}^{N*}$ -the dual map- such that $M^* = \nu(S_D)$.

The difference w.r.t. the complex setting -where X^* is known to be a (singular) complex variety because it is the image of a complex manifold by a complex map- stems from the fact that S_D has only the structure of smooth manifold, and ν is just a smooth map.

When $(M, \mathcal{F}, J)$ is Levi-flat the situation is slightly better, but still not good enough to do geometry with the dual variety. Indeed, the fiber bundle S_D is a Levi-flat manifold and the dual map $\nu\colon S_D \to \mathbb{CP}^{N*}$ is a CR map. If $y \in S_D$ is a regular point for the leafwise dual map, then it is easy to check that is is a regular point for the full dual map. Hence, ν fails to be regular where the leafwise dual map is degenerate. That might lead to some control on the singular points S^* of M^*, which in turn might imply for example the existence of pencils of hyperplanes (complex projective lines in $\mathbb{CP}^{N*}$) avoiding S^*. Being $(M, \mathcal{F}, J)$ a Levi-flat manifold of class at least C^3, it cannot have codimension one in $\mathbb{CP}^{N*}$ [9, 10]. Inside of S_D there is a real codimension 2 sub-bundle S_0 corresponding to those hyperplanes, that as well as containing some D_x, also contain T_xM^J. Let $S_0^* := \nu(S_0)$. In order to construct degree 1 Lefschetz pencil structures as defined in 7 we would need to find complex projective lines L which avoid $S^* \cup S_0^*$ (and transversal to its complement in M^*). The difficulty comes from the fact that it is not clear that S_0 is a Levi-flat manifold (it is not clear that the leaves of the obvious codimension one foliation are complex). Thus we cannot argue that the set on pencils not intersecting S_0^* is non-empty.

As we mention, at most Sard's theorem can be used to argue that M^* has measure zero and hence that hyperplane sections do exist. We recall that the topology of the hyperplane section can only be related to that of M when the Levi form has some degeneracy, in which case the results of Ni and Wolfson apply [28].

One might think heuristically of theorems 3 and 4, as a manifestation of the existence some sort of dual geometry for the re-embeddings provided by the k-th Veronese maps for k very large. That is, the corresponding dual sets M_k^* are in "most of its points" close to be stratified varieties; similarly whenever M has real codimension bigger than one, and for $k \gg 1$, the image of $S_{0,k}$ should be thought of being close to be a stratified variety with complex strata of complex codimension at least two. Therefore, a generic hyperplane should be able to avoid both the closed strata of M_k^* and the image of $S_{0,k}$. However, we were not able to come up with a proof of this fact and hence with a geometric proof of theorems 3 and 4.

In a similar vein the results about the topology of the smooth fiber of a Lefschetz pencil should not come as a surprise. They coincide with the results of [28] for Levi-flats manifolds. Approximately holomorphic theory is based on the study of the CR manifold at a very small scale, where it looks like a Levi-flat one.

2.1. **Strictly $\mathbb{C}$-convex hypersurfaces.** In this subsection we construct degree 1 Lefschetz pencils structures, for a class of hypersurfaces in $\mathbb{CP}^N$ using dual geometry.

Definition 9. *(definition 2.5.10 in* [1]*) A bounded domain $\Omega \subset \mathbb{C}^N$ with boundary of class C^h, $h \geq 2$, is strictly $\mathbb{C}$-convex if for any point $x \in \partial\Omega$ the restriction of the shape operator to D_x is strictly positive.*

Our main result is the following:

Theorem 5. *Let $\Omega \subset \mathbb{C}^N$ be a bounded, connected, strictly $\mathbb{C}$-convex domain of class C^h, $h \geq 2$. For every pencil of complex hyperplanes L whose base locus $\mathcal{B}$ intersects $\partial\Omega$ transversely, one of the following two possibilities hold:*

(i) *If the base of the pencil does not intersect $\partial\Omega$, then the associated function $\phi_L \colon \partial\Omega \to \mathbb{CP}^1$ defines a Lefschetz pencil structure and its critical set Δ is a circle. Moreover ϕ_L embeds Δ in $\mathbb{CP}^1$ splitting the 2-sphere into 2 open disks. One of them is the set of regular values of ϕ_L and the other is missed by ϕ_L. The inverse image of $\phi_L(\Delta)$ is exactly Δ.*

   The above CR function gives rise to a diffeomorphism of class C^{h-1} from $\partial\Omega$ to the sphere.

(ii) *If the base $\mathcal{B}$ intersects $\partial\Omega$ transversely in B, another strictly $\mathbb{C}$-convex sphere inside $\mathcal{B}$, then $\phi_L \colon \partial\Omega \backslash B \to \mathbb{CP}^1$ is a submersion. The fibers of ϕ_L compactified by adding the base B, are strictly $\mathbb{C}$-convex spheres inside the corresponding hyperplanes.*

Bounded and connected strictly $\mathbb{C}$-convex domains containing the origin have dual domains $\Omega^* \subset \mathbb{CP}^{N*}$ with the same properties (proposition 2.5.12 in [1], where the boundaries are assumed to be smooth). They are also known to be homeomorphic to balls (theorem 2.4.2 in [1] or theorem 4.6.12 in [17]).

Theorem 5 recovers the known results about the topology of strictly $\mathbb{C}$-convex domains which follow from the fact of being $\mathbb{C}$-convex domains. The main novelty is the use of tools coming from differential topology, rather than the more topological ones used in [1, 17], which in particular allows extra regularity in the identification of $\partial\Omega$ with S^{2N-1}.

The notion of $\mathbb{C}$-convexity (and thus also the strict $\mathbb{C}$-convexity) is relevant from the point of view of analytic function theory (see [23] where it was introduced under the name of strong linear convexity, and also [1, 17]). Strictly $\mathbb{C}$-convex domains have also interesting features as complex hyperbolic spaces [21, 19]. Perhaps this new description of strictly $\mathbb{C}$-convex domains may reveal interesting features of their analytic function theory and hyperbolic metric properties.

Let M be a hypersurface inside $\mathbb{CP}^N$. The complex dual map is

$$\begin{aligned} \nu\colon M &\longrightarrow \mathbb{CP}^{N*} \\ x &\longmapsto D_x M\ , \end{aligned} \tag{2}$$

where D_xM is understood as the tangent projective hyperplane. Its image is M^* the set of complex hyperplanes which are not transversal to M.

Let $\mathbb{CP}^N$ with fixed homogeneous coordinates $Z_0, \dots, Z_N$. Let $\mathbb{C}^N$ be for example the canonical affine chart with affine coordinates $z_1, \dots, z_N$, $z_j = Z_j/Z_0$, $z_j = x_j + iy_j$. For any $x \in M$, we can find a unitary transformation so that x is sent to z_0 (the origin), D_x to the complex hyperplane D_0 with equation $z_N = 0$, and T_xM to the real hyperplane with equation $y_N = 0$.

The parametrization

$$\begin{aligned} \psi\colon \mathbb{C}^{N-1} \times \mathbb{R} &\longrightarrow M \cap \mathbb{C}^N \\ (w_1, \dots, w_{N-1}, t) &\longmapsto (w_1, \dots, w_{N-1}, t + i\varphi(w,t)), \end{aligned} \tag{3}$$

is obtained by inverting the orthogonal projection $\pi\colon M \to T_xM$.

Since D_x is $T_xM \cap JT_xM$, around z_0 we have

$$D_{\psi(w,t)}M \equiv \sum_{j=1}^{N-1} \left(\frac{\partial\varphi}{\partial v_j} + i\frac{\partial\varphi}{\partial u_j} \right) z_j + \left(-1 + i\frac{\partial\varphi}{\partial t} \right) z_N = 0, \tag{4}$$

where $w_j = u_j + iv_j$. Equivalently,

$$D_{\psi(w,t)}M \equiv \sum_{j=1}^{N-1} 2i\frac{\partial\varphi}{\partial w_j} z_j + \left(-1 + i\frac{\partial\varphi}{\partial t} \right) z_N = 0$$

The complex Gauss map $G\colon M \cap \mathbb{C}^N \to \mathbb{CP}^{N-1*}$ has components

$$G(w,t)_j = \frac{\left(-1 - i\frac{\partial\varphi}{\partial t}\right)}{1 + \left(\frac{\partial\varphi}{\partial t}\right)^2} \left(\frac{\partial\varphi}{\partial v_j} + i\frac{\partial\varphi}{\partial u_j} \right) = \frac{2\left(\frac{\partial\varphi}{\partial t}\right) - 2i}{1 + \left(\frac{\partial\varphi}{\partial t}\right)^2} \frac{\partial\varphi}{\partial w_j},\ j = 1, \dots, N-1$$

The dual map $\nu\colon \mathbb{C}^{N-1} \times \mathbb{R} \to \mathbb{C}^N$ is obtained by adjoining the component

$$\nu_N(w,t) = \sum_{j=1}^{N-1} \frac{\left(-2\frac{\partial\varphi}{\partial t} + 2i\right)}{1 + \left(\frac{\partial\varphi}{\partial t}\right)^2} \frac{\partial\varphi}{\partial w_j} w_j - (t + i\varphi)$$

One easily computes $\nabla\nu_N(0) = \frac{\partial}{\partial t}\nu_N(0) = -1$. Hence the injectivity of the differential of ν at 0, is equivalent to $\nabla G(0)$ being an isomorphism when restricted to the hyperplane $t = 0$.

Definition 10. *A hypersurface in $\mathbb{CP}^N$ is said to have immersed dual set if the (complex) dual map (eq. 2) is a local embedding.*

Lemma 2. *If $M \subset \mathbb{C}^N$ is a strictly $\mathbb{C}$-convex hypersurface then the (complex) dual map is a local embedding.*

Proof. Using the parametrizations introduced above and the description of the dual map , the non-degeneracy of the differential of the dual map, is equivalent to the non-degeneracy of the of the hessian of $\varphi(u,v,0)$ at 0 in the coordinates u,v. By the construction of the charts, the corresponding matrix is the matrix of the shape operator along D_0 in the basis $\partial/\partial x_1, \partial/\partial y_1, \ldots, \partial/\partial x_{N-1}, \partial/\partial y_{N-1}$. □

If M has immersed dual set, compactness implies the existence of a bound for the number of points in the fiber of ν. Therefore, for each $x^* \in M^*$ there is a finite number of branches through it.

Definition 11. *A (projective) line $L \subset \mathbb{CP}^{N*}$ intersects M^* (immersed) transversely if for any $x^* \in L \cap M^*$, L has transverse intersection with all the branches of M^* through x^*.*

We give the following definition for hypersurfaces with immersed dual set (see also [11] for complex projective hypersurfaces).

Definition 12. *A projective line $L \subset \mathbb{CP}^{N*}$ is a Lefschetz pencil for M if it intersects M^* transversely.*

Sard's theorem implies that the set of pencils in $\mathbb{CP}^{N*}$ which are not Lefschetz pencils for M (with immersed dual set), has measure zero.

For a pencil of hyperplanes $L \equiv \lambda H_0 + \mu H_1$ in $\mathbb{CP}^N$, let

$$\begin{array}{rcl} \phi_L\colon M\backslash B & \longrightarrow & \mathbb{CP}^1 \\ x & \longmapsto & [H_0(x) : H_1(x)] \end{array} \tag{5}$$

be its associated map.

Proposition 2. *Let $M \subset \mathbb{CP}^N$ be a hypersurface with immersed dual set. Let L be a pencil of hyperplanes. Then the associated map ϕ_L of equation 5 defines a Lefschetz pencil structure for M (as in definition 7), if and only if L is a Lefschetz pencil for M.*

Proof. Let $L \equiv \lambda H_0 + \mu H_1$ be a Lefschetz pencil for M.

It is an elementary duality result that $\mathcal{B} := H_0 \cap H_1$ is transversal to M if and only if L is transversal to M^*. We notice that the base locus $B = \mathcal{B} \cap M$ is transversal to the CR distribution D. Therefore, it is a real codimension four CR submanifold of M.

Now we want to check the assertion about the CR 1-jet of $\phi_L\colon M\backslash B \to \mathbb{CP}^1$ being transversal to Σ.

We use affine coordinates and the parametrization of equation 3. By using an affine chart of $\mathbb{CP}^1$, we can assume that the pencil $L \equiv [H_0 : H_1]$ induces a map $\phi_L\colon M\backslash B \to \mathbb{C}$.

Recall that the parametrization was defined by the assignment

$$(w,t) \mapsto (w, t + i\varphi(t,w))$$

The composition

$$\begin{array}{rcl} g := \phi_L \circ \psi\colon \mathbb{C}^n \times \mathbb{R} & \longrightarrow & \mathbb{C} \\ (w,t) & \longmapsto & \dfrac{t + i\varphi(w,t)}{H_1(\psi(w,t))}, \end{array}$$

has vanishing 1-jet along D at the origin (the parametrization is J-complex at the origin).

Let
$$\pi\colon \mathbb{C}^N = \mathbb{C}^{N-1}\times\mathbb{R}\times\mathbb{R}\to\mathbb{C}^{N-1}\times\mathbb{R}$$
be the projection onto the first 2N-1 real coordinates, and let
$$\pi_{\mathbb{C}}\colon \mathbb{C}^N = \mathbb{C}^{N-1}\times\mathbb{C}\to\mathbb{C}^{N-1}$$
be the projection onto the first N-1 complex coordinates.

We have two local bundles of 1-jets (actually of degree 1 components):
$$D^{*1,0}\to M\cap\mathbb{C}^N$$
and
$$D_h^{*1,0} := \mathbb{C}^{N-1*}\times(\mathbb{C}^{N-1}\times\mathbb{R})\to\mathbb{C}^{N-1}\times\mathbb{R}$$
There is a bundle map $\pi_*\colon D\to D_h$ which sends the complex hyperplane over z to its projection by $\pi_{\mathbb{C}}$ (over $\pi(z)$). It is a map of complex vector bundles, and hence there is an induced dual map $\pi^*\colon D_h^{*1,0}\to D^{*1,0}$ of complex vector bundles.

We have the following commutative diagram:

$$\begin{array}{ccc} D_h^{*1,0} & \xrightarrow{\pi^*} & D^{*1,0} \\ \downarrow & & \downarrow \\ \mathbb{C}^{N-1}\times\mathbb{R} & \xrightarrow{\psi} & M \end{array} \tag{6}$$

Usual transversality of a submanifold to the zero section of a vector bundle is preserved by an isomorphism of vector bundles. It is also true that transversality along D at z_0 is equivalent to transversality along D_h at 0, because those subspaces are preserved by the map $\pi=\psi^{-1}$ between the base spaces of the bundles at z_0 (which is the identity).

Therefore, $\nabla_D\phi_L\in\Gamma(D^{*1,0})$ is transversal along D to the zero section, if and only if $\pi_*\nabla_D\phi_L := (\pi^*)^{-1}\circ\nabla_D\phi_L\circ\psi$ is transversal along D_h to the zero section of $D_h^{*1,0}$.

Observe that in general the bundle map does not commute with the covariant derivatives along D (because the coordinates are never CR). In any case, we are not stating that transversality of $\nabla_D\phi_L$ along D at z_0 is equivalent to transversality of $\nabla_{D_h}g$ along D_h at 0. We do not need to compute the latter, but $\pi_*\nabla_D\phi_L$.

Having into account equation 4,
$$\pi_*^{-1}\left(\frac{\partial}{\partial u_j}\right) = \frac{\partial}{\partial x_j} + \frac{-\frac{\partial\varphi}{\partial u_j}\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial v_j}}{1+\left(\frac{\partial\varphi}{\partial t}\right)^2}\frac{\partial}{\partial x_N} + \frac{\frac{\partial\varphi}{\partial v_j}\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial u_j}}{1+\left(\frac{\partial\varphi}{\partial t}\right)^2}\frac{\partial}{\partial y_N} \tag{7}$$

The pencil L induces a map $\Phi\colon \mathbb{CP}^N\backslash\mathcal{B}\to\mathbb{C}$ which in our charts has the formula
$$(z_1,\ldots,z_N)\mapsto\frac{z_N}{H_1(z_1,\ldots,z_N)}$$
Its differential is
$$\frac{dz_N}{H_1(z)} - \frac{z_N\nabla H_1(z)}{H_1^2(z)}$$
Thus, up to terms of order 2, and only along the hyperplane $t=0$, equation 7 implies

$$\pi_*\nabla_D\phi_L(u_1,v_1,\ldots,u_{N-1},v_{N-1}) = \sum_{j=1}^{N-1}\left(\sum_{p=1}^{N-1}\frac{\partial^2\varphi}{\partial v_j\partial u_p}u_p+\frac{\partial^2\varphi}{\partial v_j\partial v_p}v_p\right)du_j \tag{8}$$
$$+i\left(\sum_{p=1}^{N-1}\frac{\partial^2\varphi}{\partial u_j\partial u_p}u_p+\frac{\partial^2\varphi}{\partial u_j\partial v_p}v_p\right)du_j \tag{9}$$

Transversality of the above section to the zero section at 0, is equivalent to the linear independence *over the reals* of the following vectors in $\mathbb{C}^N = \mathbb{R}^{2N}$:

$$\frac{\partial}{\partial u_1}\pi_* \nabla_D \phi(0), \frac{\partial}{\partial v_1}\pi_* \nabla_D \phi(0), \ldots, \frac{\partial}{\partial u_{N-1}}\pi_* \nabla_D \phi(0), \frac{\partial}{\partial v_{N-1}}\pi_* \nabla_D \phi(0) \tag{10}$$

According to equation 10 the corresponding matrix is

$$\begin{pmatrix} \frac{\partial^2\varphi}{\partial v_1 \partial u_1} & \frac{\partial^2\varphi}{\partial u_1 \partial u_1} & \frac{\partial^2\varphi}{\partial v^2 \partial u_1} & \frac{\partial^2\varphi}{\partial u^2 \partial u_1} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1} \partial u_1} & \frac{\partial^2\varphi}{\partial u_{N-1} \partial u_1} \\ \frac{\partial^2\varphi}{\partial v_1 \partial v_1} & \frac{\partial^2\varphi}{\partial u_1 \partial v_1} & \frac{\partial^2\varphi}{\partial v^2 \partial v_1} & \frac{\partial^2\varphi}{\partial u^2 \partial v_1} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1} \partial v_1} & \frac{\partial^2\varphi}{\partial u_{N-1} \partial v_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2\varphi}{\partial v_1 \partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u_1 \partial u_{N-1}} & \frac{\partial^2\varphi}{\partial v^2 \partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u^2 \partial u_{N-1}} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1} \partial u_{N-1}} & \frac{\partial^2\varphi}{\partial u_{N-1} \partial u_{N-1}} \\ \frac{\partial^2\varphi}{\partial v_1 \partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u_1 \partial v_{N-1}} & \frac{\partial^2\varphi}{\partial v^2 \partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u^2 \partial v_{N-1}} & \cdots & \frac{\partial^2\varphi}{\partial v_{N-1} \partial v_{N-1}} & \frac{\partial^2\varphi}{\partial u_{N-1} \partial v_{N-1}} \end{pmatrix}$$

It coincides up to permutation of columns with the Hessian of $\varphi(w, 0)$ at 0.

Therefore, the CR 1-jet of ϕ_L at z_0 is transversal along D to Σ if and only if the dual map is an immersion at z_0.

Since the hyperplane H_0 cannot contain the full tangent space $T_{z_0}M$, it follows that $\nabla\phi_L(z_0)$ is non-vanishing. In particular $\phi_L(\Delta)$ is automatically an immersed curve, and this finishes the proof of one of the implications of the proposition.

Conversely, if the map ϕ_L induced by a pencil L induces Lefschetz pencil structure for M, in particular $\mathcal{B}$ is transversal to M. As we saw, under the hypothesis of M^* being immersed L this is equivalent to L intersecting M^* transversely. □

Corollary 4. *Let $M \subset \mathbb{CP}^N$ a hypersurface with immersed dual set. Let L be a pencil of hyperplanes. Then L is a Lefschetz pencil (and therefore defines a Lefschetz pencil structure) if and only if its base locus $\mathcal{B}$ intersects M transversely.*

Sketch of the proof of theorem 5. We will proof only part (i) and left the proof of part (ii) for the interested reader.

If $M \subset \mathbb{CP}^N$ is a hypersurface with immersed dual set, there is an integer λ attached to M which is the index of the shape operator restricted to D. If M misses a hyperplane, i.e it is a strictly $\mathbb{C}$-convex hypersurface, then its index is 2N-2.

Let L be a Lefschetz pencil with associated map $\phi_L \colon M \backslash B \to \mathbb{CP}^1$. The set $K := \phi_L(\Delta)$ is a collection of immersed curves of class C^{h-1}. Therefore, any two regular values $a, b \in \mathbb{CP}^1 \backslash K$ can be joined by a smooth curve transversal to K (for self-intersection points that means being transversal to all the branches). Let W_a the compactification of the fiber over a (that by abusing notation also call the fiber over a). If we had the normal forms for ϕ_L around the points of Δ in the statement of proposition 1, then proposition 6.2 in [29] will imply that the fibers W_a and W_b are cobordant by a cobordism which amounts to adding λ-handles. The same result holds in our setting, though the proof is slightly different.

We can assume -maybe after a projective transformation- that the base of the pencil lies in the hyperplane at infinity of $\mathbb{C}^N \subset \mathbb{CP}^N$.

Let us fix coordinates around $x \in \Delta$ as in equation 3

$$\begin{aligned} \psi \colon \mathbb{C}^{N-1} \times \mathbb{R} &\longrightarrow M \cap \mathbb{C}^N \\ (w_1, \ldots, w_{N-1}, t) &\longmapsto (w_1, \ldots, w_{N-1}, t + i\varphi(w, t)) \end{aligned}$$

In the above affine coordinates of $\mathbb{C}^N$, the map ϕ_L is nothing but

$$(z_1, \ldots, z_N) \longmapsto z_N,$$

and hence the composition $g = \phi_L \circ \psi$ is

$$(w_1, \ldots, w_{N-1}, t) \longmapsto t + i\varphi(w, t)$$

Consider a small segment $[-ia, ia] \subset i\mathbb{R} \subset \mathbb{C}$, in the target space ϕ_L. For a small enough the segment is transversal to ϕ_L (it is clearly transversal near x, but since we have a finite number of critical points for the same critical value, after a suitable rotation in $\mathbb{C}$ we get transversality in all M). Therefore $\phi_L^{-1}([-ia, ia]) \subset M$ is a cobordism form W_{-a} to W_a. Near x we can parametrize it as follows

$$\begin{array}{ccc} \mathbb{C}^{N-1} & \longrightarrow & \phi_L^{-1}([-ia, ia]) \\ (w_1, \ldots, w_{N-1}) & \longmapsto & (w_1, \ldots, w_{N-1}, i\varphi(w, t)) \end{array}$$

If we compose the parametrization with the imaginary part of ϕ_L, we obtain

$$(w_1, \ldots, w_{N-1}) \longmapsto \varphi(w, t)$$

the imaginary part of g. It is routine to check that near $0 \in \mathbb{C}^{N-1}$ this is a Morse function, and 0 is a critical point of index λ.

Let Ω a strictly $\mathbb{C}$-convex domain. We claim that for any CR function associated to a Lefschetz pencil whose base set misses $\partial\Omega$, which always exist, the set of singular points Δ has a unique connected component which is embedded by ϕ_L into $\mathbb{CP}^1$. Its image separates the sphere into two disks, one containing all the regular points of ϕ_L and the other missed by ϕ_L. This CR function gives a homeomorphism from $\partial\Omega$ to the sphere.

Let L be a Lefschetz pencil for $\partial\Omega$ with empty base locus. This implies that the fibers of ϕ_L are already compact. Let c be a point in the complement of $\phi_L(\partial\Omega)$. It follows that any other regular fiber is built by adding a finite number of (2N-2)-handles to the empty set (the fiber over c). Thus, any fiber is a finite collection of spheres $\coprod_{j\in J} S_j^{2N-3}$.

It also follows from the previous local analysis, that around the points of Δ any fiber is a sphere which collapses into a point of Δ. Hence the restriction $\phi_L \colon \partial\Omega\backslash\Delta \to \mathbb{CP}^1$ is proper. Therefore, it is a fiber bundle whose fiber is a union of spheres.

The image of $\phi_L \colon \partial\Omega\backslash\Delta \to \mathbb{CP}^1$ is an open, connected subset of $\mathbb{CP}^1$ in the complement of a small disk. Let us denote it by $\mathcal{V}$.

We want to prove that (i) $\bar{\mathcal{V}}$ -which is $\phi_L(\partial\Omega)$- is C^{h-1}-diffeomorphic to the closed unit disk, (ii) Δ has a unique connected component, and (iii) ϕ_L sends Δ diffeomorphically to the boundary of the unit disk. We also want to show that the fiber is a unique sphere.

Each connected component Δ_α of Δ has to be the in the boundary of $\mathcal{V}$: if $p \in K \cap \mathcal{V}$, we can choose a small arc γ transversal to K whose unique singular value is p. By hypothesis the fiber over p is $\coprod_{j\in J} S_j^{2N-3}$ plus the points in Δ that map into p. When we approach p by points in the arc γ the corresponding pre-images are spheres. For each point in Δ over p, a one parameter family of spheres should contract into the point. But those spheres should also have as limit one of the spheres $S_{j_0}^{2N-3}$. Thus, there are no points approaching the points of Δ over p.

Therefore, Δ is sent to $\bar{\mathcal{V}}\backslash\mathcal{V}$. Let Δ_α be one of the components parametrized by the interval $[0, 1]$. Since ϕ_L immerses Δ_α, if $\phi_L(\Delta_\alpha)$ is not embedded there has to be a first value t_0, $0 < t_0 < 1$ where the first self-intersection occurs. The interval $[0, t_0]$ is sent to a curve S homeomorphic to S^1. By the Jordan curve theorem it splits the sphere $\mathbb{CP}^1$ into two open disks. The open set $\mathcal{V}$ is connected, so it lies inside one of the two disks, and it accumulates into S (it fills one of the sides of the set of the tubular neighborhood of some radius ϵ). Since the image of the whole interval $[0, 1]$ has to be contained in $\partial\mathcal{V}$, it has to be sent to S. Therefore, $[t_0, t_0 + \delta]$ has to be a (continuous) reparametrization of the image of $[0, \delta']$. This, together with ϕ_L being a local immersion when restricted to Δ_α, implies that S is a differentiable curve of class C^{h-1} and $\phi_{L|\Delta_\alpha}$ a covering map.

Hence we deduce that $\bar{\mathcal{V}}$ is in principle diffeomorphic to D^2_Λ, a closed disk with Λ small open disks removed. Also Δ is sent to the boundary of $\partial\mathcal{V}$ (by covering maps).

Assume that $\Lambda = 0$, i.e. the image is the closed disk. It follows that $\partial\Omega$ is homeomorphic to the sphere S^{2N-1}; it is just a generalization of the construction of S^3 obtained by starting with a solid torus $D^2 \times S^1$ (in higher dimensions $D^2 \times S^{2N-3}$), and then collapsing each circle $\{\cdot\} \times S^1$ (resp. $\{\cdot\} \times S^{2N-3}$) with a disk (resp. a ball B^{2N-2}).

If $\Lambda = 1$ then $\partial\Omega$ is diffeomorphic to a fiber bundle over S^1 whose fiber is homeomorphic to S^{2N-2}. In principle the fiber of ϕ_L is a collection of S^{2N-3}. By parametrizing the annulus as $S^1 \times [1,2]$, the inverse image of each segment is a collection of S^{2N-2}. The degree of the covering of the restriction of ϕ_L to the two components of Δ is the number of spheres in the fiber of ϕ_L.

On the other hand we know that $\partial\Omega$ is has the structure of a bundle over $\mathbb{CP}^{N-1}$ with one dimensional fibers, because the complex Gauss map $G\colon \partial\Omega \to \mathbb{CP}^{N-1*}$ is by lemma 2 a submersion along D; the structural maps are of class C^{h-1}. Since $\mathbb{CP}^{N-1}$ is simply connected and M is connected, the fiber has to be also connected). In particular $\pi_1(M)$ has to be abelian.

Assume that Λ is bigger that one. Let us fix a base point x_0 in D_Λ and two loops α, β, which are each the boundary of one puncture plus a segment that goes to the base point (and with empty intersection). Then one can fix any lift of x_0 and loops $\hat{\alpha}, \hat{\beta}$ such that:

- Both loops are a component of Δ plus a lift of the corresponding segment (once on each direction).
- They project onto nontrivial multiples of α and β respectively.

The contradiction comes from the fact that $\hat{\alpha}$ and $\hat{\beta}$ have to commute in $\partial\Omega$, but their images cannot.

Hence D_Λ is either the disk or the annulus.

If $N \geq 3$, then D_Λ has to be the disk. The reason is that from the previous considerations the fundamental group has to be the integers. Hence, $\partial\Omega$ is $S^1 \times \mathbb{CP}^{N-1}$. On the other hand, from the description of $\partial\Omega$ as a bundle over S^1 with fiber S^{2N-2} and the exact sequence of homotopy groups, it follows that $\pi_2(\partial\Omega)$ has to be trivial, but that contradicts $\partial\Omega \approx S^1 \times \mathbb{CP}^{N-1}$.

The conclusion is that for $N > 2$, $\partial\Omega$ is the sphere and for $N = 2$ is either the 3-sphere or $S^1 \times S^2$. The proof that rules out the case $S^1 \times S^2$ is left for the interested reader.

We just saw that the functions ϕ_L for Lefschetz pencils with empty base locus, give rise to homeomorphism form $\partial\Omega$ to the sphere. On the other hand using the Gauss map we obtain diffeomorphisms of class C^{h-1} (because we know $\partial\Omega$ is C^{h-1}-diffeomorphic to a principal S^1-bundle, the map a fiber bundle morphism lifting the identity). Notice that these are fiber bundle maps to the Hopf fibration of S^{2N-1}; for any base point (hyperplane through the origin), the fiber of $\partial\Omega$ are the points whose complex tangent space is the base point. □

3. Approximately holomorphic geometry for projective CR manifolds

Let us fix some notation. Given $s_k\colon \mathbb{CP}^N \to \underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$, its projectivization is denoted by Φ_k. The restrictions to M of s_k and Φ_k will be denoted by τ_k and ϕ_k respectively. The holomorphic vector bundles $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$ -which will be also denoted by E_k- carry a natural connection ∇_k coming from the flat one in $\underline{\mathbb{C}}^{m+1}$

and the connection in $\mathcal{O}(k)$ associated to the Fubini-Study form. We will use the same notation for the restriction of ∇_k to $E_{k|M}$ if there is no risk of confusion.

For a projective Levi-flat CR manifold of class C^h, let $\mathcal{J}^r_{\mathcal{F}}(M, \mathbb{CP}^m)$ denote the bundle of leafwise holomorphic (CR) r-jets of maps to $\mathbb{CP}^m$. This is a bundle of class C^{h-r} and it inherits an obvious CR-structure. There is a leafwise Thom-Boardman stratification $\mathbb{P}\mathcal{T}^{\mathcal{F}}$, whose strata are Levi-flat CR submanifolds of class C^{h-r}. For any CR map ϕ to $\mathbb{CP}^m$ of class C^h, its r-jet prolongation $j^r_{\mathcal{F}}\phi$ is a CR section of class C^{h-r}.

In order to prove points 1 and 2 of theorem 4, we need to find suitable sequences of sections s_k of E_k such that (i) ϕ_k has a base locus which is a CR submanifold of the expected dimension, and (ii) $j^r_{\mathcal{F}}\phi$ is transversal along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}^{\mathcal{F}}$.

What we will do is showing that

(1) The aforementioned transversality problem can be linearized, i.e. both the bundle $\mathcal{J}^r_{\mathcal{F}}(M, \mathbb{CP}^m)$ and the notion of CR jet.
(2) The linearization has a *CR solution* ϕ_k.
(3) The solution of the linearized problem also solves the original problem.

We need to recall a number of notions and results from approximately holomorphic geometry.

If $s_k \in \Gamma(E_k)$, using J the complex structure of $\mathbb{CP}^N$ we can write

$$\nabla s_k = \partial s_k + \bar{\partial} s_k,\ \partial s_k \in \Gamma(T^{*1,0}\mathbb{CP}^N \otimes E_k),\ \bar{\partial} s_k \in \Gamma(T^{*0,1}\mathbb{CP}^N \otimes E_k)$$

Similarly, if τ_k is a section of $E_{k|M}$, the restriction of $\nabla \tau_k$ to D can be written

$$\nabla_D \tau_k = \partial \tau_k + \bar{\partial} \tau_k,\ \partial \tau_k \in \Gamma(D^{*1,0} \otimes E_k),\ \bar{\partial} \tau_k \in \Gamma(D^{*0,1} \otimes E_k)$$

Let g denote the Fubini-Study metric and let g_k denote the rescaled metric kg. We use the same notation for the restriction of these metrics to M.

Definition 13. *A sequence of sections s_k of E_k is approximately J-holomorphic (or approximately holomorphic or simply A.H.), if positive constants $(C_j)_{j\geq 0}$ exist such that:*

$$|\nabla^j s_k|_{g_k} \leq C_j,\ \ |\nabla^{j-1}\bar{\partial} s_k|_{g_k} \leq C_j k^{-1/2}$$

If in an A.H. sequence the sections s_k turns out to be holomorphic, we speak of a uniformly bounded holomorphic sequence.

Similarly, a sequence of sections τ_k of $E_{k|M}$ is approximately J-holomorphic (or approximately holomorphic or simply A.H.), if positive constants $(C_j)_{j\geq 0}$ exist such that:

$$|\nabla^j \tau_k|_{g_k} \leq C_j,\ \ |\nabla^{j-1}\bar{\partial} \tau_k|_{g_k} \leq C_j k^{-1/2},$$

If the sections in the sequence are CR, we say it is a uniformly bounded CR sequence.

The previous definition can also be given requiring control on a finite number of covariant derivatives, so we have C^h-A.H.(C) sequences of sections, where $C \geq C_j$ in the above inequalities.

3.1. **Linearization of the bundles of CR jets and the notion of CR r-jet.** Over the CR manifold M we consider the sequence of bundles of pseudo-holomorphic jets

$$\mathcal{J}^r_D E_{k|M} := (\sum_{j=0}^{r} D^{*1,0} \odot \cdots \overset{j}{\odot} \cdots \odot D^{*1,0}) \otimes E_{k|M}, \tag{11}$$

which carry a natural connection $\nabla_{k,r}$ and a metric (see sections 5 and 6 in [24] for more details).

Let $j_D^{r-1}\tau_k \in \mathcal{J}_D^{r-1}E_{k|M}$ be the (r-1)-jet of τ_k. It has homogeneous components of degrees $0, 1, \dots, r-1$. We will denote the homogeneous component of degree $j \in \{0, \dots, r-1\}$ by $\partial^j_{\text{sym}}\tau_k \in \Gamma((D^{*1,0})^{\odot j} \otimes E_{k|M})$.

The connection $\nabla_{k,r-1}$ is actually a direct sum of connections defined on the direct summands $(D^{*1,0})^{\odot j} \otimes E_{k|M}$, $j = 0, \dots, r-1$. For simplicity and if there is no risk of confusion, we will use the same notation for the restriction of $\nabla_{k,r-1}$ to each of the summands.

The restriction of $\nabla_{k,r-1}\partial^{r-1}_{\text{sym}}\tau_k$ to D defines a section $\nabla_{k,r-1,D}\partial^{r-1}_{\text{sym}}\tau_k \in \Gamma(D^* \otimes (D^{*1,0})^{\odot r-1} \otimes E_{k|M})$. For each $x \in M$, it is a form on D with values in the complex vector space $(D^{*1,0})^{\odot r-1} \otimes E_{k|M}$. Therefore, we can consider its (1,0)-component $\partial\partial^{r-1}_{\text{sym}}\tau_k \in \Gamma(D^{*1,0} \otimes (D^{*1,0})^{\odot r-1} \otimes E_{k|M})$. By applying the symmetrization map

$$\text{sym}_j \colon (D^{*1,0})^{\otimes j} \to (D^{*1,0})^{\odot j}$$

we obtain $\partial^r_{\text{sym}}\tau_k \in \Gamma((D^{*1,0})^{\odot r} \otimes E_{k|M})$.

Definition 14. *Let τ_k be a section of $(E_{k|M}, \nabla_k)$. The pseudo-holomorphic r-jet $j^r_D\tau_k$ is a section of the bundle $\mathcal{J}^r_D E_{k|M} = (\sum_r^{j=0}(D^{*1,0})^{\odot j}) \otimes E_{k|M}$, defined out of the (r-1)-jet by the formula $j^r_D\tau_k := (j^{r-1}_D\tau_k, \partial^r_{\text{sym}}\tau_k)$.*

Let $Z^0, \dots, Z^m$ be the complex coordinates associated to the trivialization of $\underline{\mathbb{C}}^{m+1}$ and let $\pi \colon \mathbb{C}^{m+1}\backslash\{0\} \to \mathbb{CP}^m$ be the canonical projection. Consider the canonical affine charts

$$\begin{array}{rcl} \varphi_i^{-1} \colon U_i & \longrightarrow & \mathbb{C}^m \\ [Z_0 : \cdots : Z_m] & \longmapsto & \left(\dfrac{Z^1}{Z^0}, \dots, \dfrac{Z^{i-1}}{Z^0}, \dfrac{Z^{i+1}}{Z^0}, \dots, \dfrac{Z^m}{Z^0}\right) \end{array}$$

For each chart φ_i, $i = 0, \dots, m$, we consider the bundle

$$\mathcal{J}^r_D(M, \mathbb{C}^m)_i := (\sum_{j=0}^{r} (D^{*1,0})^{\odot j}) \otimes \underline{\mathbb{C}}^m \tag{12}$$

On each bundle $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ we have a notion of pseudo-holomorphic r-jet as given in definition 14, where we use instead of $E_{k|M}$ the trivial bundle $\underline{\mathbb{C}}^m$ with trivial connection associated to the frame $\xi_{i,1}, \dots, \xi_{i,m}$ given by the above affine coordinates.

We have the following results:

- Proposition 5 in [25] states that the vector bundles $\mathcal{J}^r_D(M, \mathbb{C}^m)_i$ can be glued to define the almost complex fiber bundles $\mathcal{J}^r_D(M, \mathbb{CP}^m)$ of pseudo-holomorphic r-jets of maps from M to $\mathbb{CP}^m$ (Proposition 5 in [24]).
- Given $\phi_k \colon M \to \mathbb{CP}^m$, proposition 5 in [24] proves the existence of a notion of pseudo-holomorphic r-jet extension $j^r_D\phi_k \colon M \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$, which is compatible with the notion of pseudo-holomorphic r-jet of definition 14 for the sections $\varphi_i^{-1} \circ \phi_k \colon M \to \mathbb{C}^m$.
- Define $\mathcal{J}^r_D E^*_k := \mathcal{J}^r_D E_k \backslash Z_k$, where Z_k denotes the sequence of strata of $\mathcal{J}^r_D E_k$ of r-jets whose degree 0-component vanishes. Then by proposition 6 in [24] there exists a bundle map $j^r\pi \colon \mathcal{J}^r_D E^*_k \to \mathcal{J}^r_D(M, \mathbb{CP}^m)$ such that for any section τ_k of E_k, in the points where it does not vanish and its projectivization ϕ_k is defined the following relation holds:

$$j^r\pi(j^r_D\tau_k) = j^r_D\phi_k \tag{13}$$

The sequence of bundles $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ and the notion of pseudo-holomorphic r-jet for $\phi_k\colon M\to\mathbb{CP}^m$, are the right linearization of the bundles $\mathcal{J}^r_{\mathcal{F}}(M,\mathbb{CP}^m)$ and the notion of CR r-jet for a CR map to $\mathbb{CP}^m$. To see that, we fix a family of *approximately holomorphic charts* $\varphi_{k,x}\colon(\mathbb{C}^n\times\mathbb{R},0)\to(M,x)$ (see definitions 8 and 13 in [24]) which are CR charts as well (so in particular D_h -the canonical foliation of $\mathbb{C}^n\times\mathbb{R}$ by complex hyperplanes- is sent to $\mathcal{F}$).

For each point $x\in M$ and associated to the CR coordinates $z^1_k,\dots,z^n_k,s_k$, over each of the balls $B_{g_k}(x,O(1))$ we have the local bundles $\mathcal{J}^r_{D_h,n,m}$ of CR r-jets with a canonical bundle map

$$\Psi^{\mathrm{lin}}_{k,x,i}\colon\mathcal{J}^r_D(M,\mathbb{C}^m)_i\to\mathcal{J}^r_{D_h,n,m} \tag{14}$$

The basis $dz^1_k,\dots,dz^n_k\in\Gamma(D^{*1,0})$ identifies $D^{*1,0}$ with $T^{*1,0}\mathbb{C}^n$; let I be an (N+2)-tuple $I=(i_0,i_1,\dots,i_n,i)$, $1\le i_0\le m$, $0\le i_j\le r$, $i=0,\dots,m$, $i_1+\dots+i_n=r$. The frame

$$\mu_{k,x,I}:=dz_k^{1\odot i_1}\odot\dots\odot dz_k^{n\odot i_n}\otimes\xi_{i,i_0} \tag{15}$$

defines the bundle map of equation 14.

The local bundles $\mathcal{J}^r_{D_h,n,m}$ glue into the non-linear bundle $\mathcal{J}^r_{\mathcal{F}}(M,\mathbb{C}^m)_i$: let $y\in M$ be a point belonging to two different charts centered at x_0 and x_1 respectively. If we send y in both charts to the origin via a translation, then the change of coordinates restricts to the leaf through the origin to a holomorphic map fixing the origin. The fibers over y are related by the action of the holomorphic r-jet of the bi-holomorphism $\Psi_{k,x_0,x_1,i}$. If we only take the linear part of the action we have a vector bundle map

$$\Psi^{\mathrm{lin}}_{k,x_0,x_1,i}\colon\mathcal{J}^r_{D_h,n,m}\to\mathcal{J}^r_{D_h,n,m} \tag{16}$$

which defines a vector bundle, for the cocycle condition still holds. Besides, since we only use the linear part we do not need either D or J to be integrable. This vector bundle is $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ as defined in equation 12 (it is rather a sequence of bundles in which the metric in the $D^{*1,0}$ factors is induced from g_k). Thus for Levi-flat CR manifolds the vector bundles $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ are "linear approximations" of the non-linear bundles $\mathcal{J}^r_{\mathcal{F}}(M,\mathbb{C}^m)_i$.

What is more, each local bundle $\mathcal{J}^r_{D_h,n,m}$ carries a corresponding CR Thom-Boardman stratification $\mathcal{T}^{\mathcal{F}}_{n,m}$ (or rather a refinement which is a Whitney (A) stratification [26]). The CR Thom-Boardman stratification $\mathbb{P}\mathcal{T}^{\mathcal{F}}_i$ of $\mathcal{J}^r_{\mathcal{F}}(M,\mathbb{C}^m)_i$ is the result of gluing the local stratifications $\mathcal{T}^{\mathcal{F}}_{n,m}$, for these are preserved by the maps $\Psi_{k,x_0,x_1,i}$; these stratifications in turn are used to build $\mathbb{P}\mathcal{T}^{\mathcal{F}}$ the Thom-Boardman stratification of $\mathcal{J}^r_{\mathcal{F}}(M,\mathbb{CP}^m)$. The relevant observation is that $\mathcal{T}^{\mathcal{F}}_{n,m}$ is also preserved by $\Psi^{\mathrm{lin}}_{k,x_0,x_1,i}$, thus giving rise to the Thom-Boardmann-Auroux stratification $\mathbb{P}\mathcal{T}_{k,i}$ of $\mathcal{J}^r_D(M,\mathbb{C}^m)_i$ as defined in [24], definition 26; these stratifications are also compatible with the gluing that defines $\mathcal{J}^r_D(M,\mathbb{CP}^m)$ giving rise to stratifications $\mathbb{P}\mathcal{T}_k$, and this completes step 1.

3.2. **Existence of CR solutions to the linearized problem.** Solving step 2 amounts to finding a uniformly bounded sequence of holomorphic sections s_k of E_k whose restriction $\tau_k\colon M\to E_{k|M}$ is such that:

(1) τ_k is uniformly transversal along $\mathcal{F}$ to Z_k, and therefore its zero set B_k is a CR submanifold of the expected codimension (see definition 17 in [24] for the precise notion of uniform transversality along distributions to stratifications).
(2) The CR r-jet of the projectivization $\phi_k\colon M\backslash B_k\to\mathbb{CP}^m$ is uniformly transversal to along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}^{\mathcal{F}}$.

We have the following result:

Theorem 6. *[Theorem 3 in* [25]*] Fix any $\delta > 0$ and $r, h \in \mathbb{N}$, $h - r \geq 2$. Then a constant $\eta > 0$ and a natural number k_0 exist such that for any C^h-A.H.(C) sequence s_k of E_k it is possible to find a C^h-A.H. sequence σ_k of E_k so that for any k bigger than k_0,*

1. *$|\nabla^j(s_k - \sigma_k)|_{g_k} < \delta, j = 0, \dots, h$ (i.e. $s_k - \sigma_k$ is C^h-A.H.(δ))*
2. *Let τ_k denote the restriction of σ_k to M, and ϕ_k its projectivization. Then τ_k is η-transversal along $\mathcal{F}$ to Z_k and $j^r\phi_k$ is η-transversal along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$*

Theorem 6 is proven by pulling back $\mathbb{P}\mathcal{T}_k$ to a stratification $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ of $\mathcal{J}^r_D E_{k|M}$, and finding σ_k so that $j^r_D\tau_k$ is uniformly transversality to it; this giving a projectivization ϕ_k with the required properties is mostly a consequence of equation 13. Again, the solution to this uniform transversality problem in $\mathcal{J}^r_D E_k \to M$ is obtained by "thickening" $j^r\pi^*\mathbb{P}\mathcal{T}_k \cup Z_k$ to an appropriate stratification $\mathcal{T}_k$ of the bundle of pseudo-holomorphic r-jets

$$\mathcal{J}^r E_k := (\sum_{j=0}^{r} T^{*1,0}\mathbb{CP}^N \odot \cdots \overset{j}{\odot} \cdots \odot T^{*1,0}\mathbb{CP}^N) \otimes E_k,$$

and making sure that $j^r\sigma_k$ (defined as in 14) is uniformly transversal along M to $\mathcal{T}_k$ (which is seen to be equivalent to uniform transversality along $\mathcal{F}$).

In order to find a CR solutions, we just need to make sure that whenever we start with s_k a uniformly bounded sequence of holomorphic sections of E_k, then the perturbations σ_k of theorem 6 can be chosen to be holomorphic; but this is essentially proven in section 7 of [12] (a result which is already present in [32]): if $s^{\mathrm{ref}}_{k,x,i_0}$, $0 \leq i_0 \leq m$, $x \in M$, $k \gg 1$, is an appropriate family of *reference frames* of $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$ (see definiton 2.3. in [4]), then their L^2-projection into the holomorphic sections defines a family of holomorphic reference frames. Exactly the same ideas show that for $I = (i_1, \dots, i_N, i)$, $0 \leq i \leq m$, $0 \leq i_j \leq r$, $i_1 + \cdots + i_N = r$, the L^2-projection of

$$\nu^{\mathrm{ref}}_{k,x,I} := z_k^{1\,i_1} \cdots z_k^{N\,i_N} s^{\mathrm{ref}}_{k,x,i}$$

are holomorphic sections whose pseudo-holomorphic r-jets define reference frames of $\mathcal{J}^r E_k$.

3.3. **Comparison between pseudo-holomorphic jets and foliated holomorphic jets.** Let s_k a uniformly bounded holomorphic sequence of sections of $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k) \to \mathbb{CP}^N$ provided by theorem 6 (for example a perturbation of the sequence of zero sections). Its restriction to M is a sequence τ_k of CR sections, whose zero set B_k is a CR submanifold of the expected dimension. Let $\phi_k \colon M \backslash B_k \to \mathbb{CP}^m$ the corresponding sequence of CR maps. By hypothesis $j^r_D\phi_k \in \Gamma(\mathcal{J}^r_D(M\backslash B_k, \mathbb{CP}^m))$ is uniformly transversal along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$ for $k \gg 1$.

The CR r-jet $j^r_{\mathcal{F}}\phi_k$ is a section of $\mathcal{J}^r_{\mathcal{F}}(M\backslash B_k, \mathbb{CP}^m)$ (the same bundle for all k, apart from the submanifold of base points). For each $k \gg 1$ the pairs $(\mathcal{J}^r_D(M, \mathbb{CP}^m), \mathbb{P}\mathcal{T}_k)$ and $(\mathcal{J}^r_{\mathcal{F}}(M, \mathbb{CP}^m), \mathbb{P}\mathcal{T}^{\mathcal{F}})$ are constructed using the same building blocks $(\mathcal{J}^r_{D_h,n,m}, \mathcal{T}^{\mathcal{F}}_{n,m})$, but with different transition functions.

For each $x \in M\backslash B_k$, fix i such that $j^r_D\phi_k \colon B_{g_k}(x, O(1))\backslash B_k \to \mathcal{J}^r_D(M\backslash B_k, \mathbb{C}^m)_i$. Over $B_{g_k}(x, O(1))$ we use the bundle map of equation 14 to see $j^r_D\phi_k$ as a section of $\mathcal{J}^r_{D_h,n,m}$. Similarly, we can see $j^r_{\mathcal{F}}\phi_k$ over $B_{g_k}(x, O(1))\backslash B_k$ as a section of $\mathcal{J}^r_{D_h,n,m}$.

Let g_0 be the natural metric induced on $\mathcal{J}^r_{D_h,n,m}$, which is comparable to the one induced by g_k (with comparison constant independent of k).

On $B_{g_k}(x, O(1))$ one easily checks

$$|j^r_D\phi_k - j^r_{\mathcal{F}}\phi_k|_{C^1, g_0} \leq O(k^{-1/2}), \tag{17}$$

for all $k \gg 1$.

Uniform transversality along $\mathcal{F}$ is an open condition (see [24]), meaning that if $j^r_D\phi_k$ is uniformly transversal along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$ and $|j^r_D\phi_k - \xi_k|_{C^1} \ll \delta$, then ξ_k is uniformly transversal along $\mathcal{F}$ to $\mathbb{P}\mathcal{T}_k$. This, together with equation 17, has the following consequences:

- $j^r_{\mathcal{F}}\phi_k$ over $B_{g_k}(x, O(1))\backslash B_k$ is also inside $\mathcal{J}^r_{\mathcal{F}}(M\backslash B_k, \mathbb{C}^m)_i$.
- By hypothesis in $U_{k,\epsilon}$ a small tubular neighborhood of B_k of some g_k-radius ϵ independent of k, $j^r_D\phi_k$ stays within bounded distance $\epsilon > 0$ of the closed strata of $\mathbb{P}\mathcal{T}_k$. We deduce that for $k \gg 1$ on $B_{g_k}(x, O(1)) \cap U_{k,\epsilon/2}$, the g_0-distance of the section $j^r_{\mathcal{F}}\phi_k$ to the closed strata of $\mathcal{T}^{\mathcal{F}}_{n,m}$ is bigger than $\epsilon/2$, with $\epsilon/2$ independent of the point x in which the chart is centered, of k and on the chart φ_i of $\mathbb{CP}^m$.
- On the points of $B_{g_k}(x, O(1))\backslash B_k$ the CR r-jet $j^r_{\mathcal{F}}\phi_k$ is uniformly transversal along $\mathcal{F}$ to $\mathcal{T}^{\mathcal{F}}_{n,m}$ (using again the metric g_0 to measure).

Notice that we might introduce suitable metrics g'_k in the sequence on fiber bundles $\mathcal{J}^r_{\mathcal{F}}(M\backslash B_k, \mathbb{CP}^m)$, and conclude uniform transversality along $\mathcal{F}$ of $j^r_{\mathcal{F}}\phi_k$ to $\mathbb{P}\mathcal{T}^{\mathcal{F}}$.

3.4. **Proof of theorem 4.** The previous three subsections imply that we can always construct s_k, uniformly bounded sequences of holomorphic sections of $\underline{\mathbb{C}}^{m+1} \otimes \mathcal{O}(k)$, such that

(1) the set of base points B_k of the restriction to M is a CR submanifold of codimension 2m+2 and class C^h,
(2) the projectivization of the restriction $\phi_k \colon M\backslash B_k \to \mathbb{CP}^m$ is a foliated r-generic map.

To prove point (iii) in the statement of the theorem we have to solve another uniform transversality problem:

One considers the bundles $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{CP}^m)$ of pseudo-holomorphic 1-jets along TM^J for maps to $\mathbb{CP}^m$, defined in section 1. Actually we can define them using affine coordinates of $\mathbb{CP}^m$ and then gluing the corresponding vector bundles $\mathcal{J}^1_{TM^J}(\mathcal{N}_\epsilon(M), \mathbb{C}^m)_i$. These are sequences of vector bundles with metric depending on k. Inside we have the submanifolds Σ_k of 1-jets along TM^J with vanishing degree 1-component. It is possible to pull them back to strata Σ^J_k of the bundles $\mathcal{J}^1E_k \to \mathcal{N}_\epsilon(M)$ of pseudo-holomorphic 1-jets of E_k.

The stratification $Z_k \cup \Sigma^J_k$ of $\mathcal{J}^1E_k$ is such that there is a version of theorem 6 (theorem 3 in [24]), asserting that for any $\delta' > 0$ there exist $\eta' > 0$ and s'_k a uniformly bounded sequence of holomorphic section of E_k, such that for all $k \gg 1$

(1) $|\nabla^j(s_k - s'_k)|_{g_k} < \delta', j = 0, \dots, h$
(2) $j^1_{TM^J}\Phi'_k$ is η'-transversal along M to Σ_k, where Φ'_k is the projectivization of s'_k.

By choosing δ'' small enough, the sequence s'_k will still have properties (1) and (2) at the beginning of this subsection, and this proves theorem 4.

3.5. **Existence of Lefschetz pencil structures.** The proof of theorem 3 is contained in the proof of theorem 4, because we are looking for sections whose projectivization is 1-generic. Notice that 1-genericity is well defined for general CR projective manifolds; integrability of D is not required at all.

The part concerning the "Lefschetz hyperplane theorem" for homotopy (homology) groups of a hyperplane section, is a standard result valid for Lefschetz pencil structures for 2-calibrated structures [18].

Proposition 1 is also elementary: coordinates around B_k are again constructed as for Lefschetz pencil structures for 2-calibrated structures (see theorem 1.2 in

[29]), but we obtain CR coordinates for the two components of $\tau_k \in \Gamma(\underline{\mathbb{C}}^2 \otimes \mathcal{O}(k))$ are already CR; CR Morse coordinates around Δ_k are obtained by applying the complex Morse lemma with parameters.

We observe that unlike the general case of 2-calibrated structures, CR Morse coordinates around a point $x \in \Delta_k$ are defined in a neighborhood of the form $B_{g_k,\mathcal{F}}(x, O(1)) \times [-\epsilon_k, \epsilon_k] \subset \mathbb{C}^n \times \mathbb{R}$, because we cannot apply generic perturbations to obtain a domain of the form $B_{g_k}(x, O(1))$. This is the same reason why the curve $\phi_k(\Delta)$ is only immersed, and might have non-generic self-intersections. In any case, the properties we obtain are enough to conclude the leafwise Lefschetz pencil hyperplane for π_0 (theorem 8 in [25]), and this proves corollary 1.

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