A law of large numbers for random walks in random mixing environments

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Abstract

We prove a law of large numbers for a class of multidimensional random walks in random environments where the environment satisfies appropriate mixing conditions, which hold when the environment is a weak mixing field in the sense of Dobrushin and Shlosman. Our result holds if the mixing rate balances moments of some random times depending on the path. It applies in the non-nestling case, but we also provide examples of nestling walks that satisfy our assumptions. The derivation is based on an adaptation, using coupling, of the regeneration argument of Sznitman-Zerner.

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Short title: Random walk in mixing environment

1 Introduction and statement of results

Let S denote the 2d-dimensional simplex, and set $\Omega = S^{\mathbb{Z}^d}$. We consider Ω as an "environment" for the random walk that we define below in (1.1). We denote by $\omega(z, \cdot) = \{\omega(z, z+e)\}_{e\in\mathbb{Z}^d, |e|=1}$ the coordinate of $\omega \in \Omega$ corresponding to $z \in \mathbb{Z}^d$.

Conditional on a realization $\omega \in \Omega$, we define the Markov Chain $\{X_n\} = \{X_n; n \geq 0\}$ with state space $\mathbb{Z}^d$ started at $z \in \mathbb{Z}^d$ as the process satisfying $X_0 \equiv z$ and

$$P_\omega^z(X_{n+1} = x + e | X_n = x) = \omega(x, x+e), \quad e \in \mathbb{Z}^d, |e| = 1. \tag{1.1}$$

The law of the random walk in random environment (RWRE) $\{X_n\}$ under this transition kernel, denoted $P_\omega^z(\cdot)$, depends on the environment $\omega \in \Omega$ and is called the *quenched* law of $\{X_n\}$.

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Let P be a probability measure on Ω, stationary and ergodic with respect to the shifts in $\mathbb{Z}^d$. With a slight abuse of notations, we write $\mathbb{P}^z = P \otimes P^z_\omega$ for both the joint law on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$ of $\{X_n\}_n$ and ω, and for its marginal on $(\mathbb{Z}^d)^{\mathbb{N}}$; in the latter case, we refer to it as the *annealed* law of the process $\{X_n\}$. We will denote by $\mathbb{E}^z = E_{\mathbb{P}^z}, E^z_\omega = E_{P^z_\omega}$ the expectations corresponding to $\mathbb{P}^z, P^z_\omega$, respectively. Considering the annealed law rather than the quenched law, one takes advantage of the smoothing from the ω-average, but the Markov property is lost.

The RWRE with $d = 1$ is by now well studied, see [26] for a recent review. The multidimensional case is much less understood. A crucial simplification in the case $d = 1$ is that a nearest neighbor random walk tending to $+\infty$ has to visit all positive sites; then one can use ergodic theorem to smooth the environment out. In constrast, it is not clear how to take advantage of ergodicity of the medium in dimension $d > 1$. When P is a product measures, laws of large numbers and central limit theorems for $\{X_n\}$ were derived in an impressive sequence of papers [22], [19], [20], focusing on the ballistic regime. Our goal in this paper is to present a technique, based on an appropriate coupling, for extending some of the results of [22], [19] to the case where P is not a product measure.

A motivation for this question relates to an example in [25, Proposition 2], of a RWRE in dimension 2 with the non-standard asymptotics: $\mathbb{P}^o(\lim_n X_n/n = w) = 1/2$, $\mathbb{P}^o(\lim_n X_n/n = -w) = 1/2$ for some non zero vector w. There, the environment is ergodic (but not mixing), and the RWRE is strictly elliptic (but not uniformly). In view of this example, it seems important to clarify the specific role of the various assumptions used to get a standard law of large numbers, e.g. independent, identically distributed (i.i.d.) environment, uniform ellipticity and drift-condition in [22].

We work in the context of *ballistic* walks, i.e. walks X_n which tend to infinity in some direction $\ell \in \mathbb{R}^d \setminus \{0\}$, with a non-vanishing speed. Conditions for the first statement to occur have been explored by Kalikow [10] two decades ago, see below Assumption ($\mathcal{A}4$). With an i.i.d. environment, Sznitman and Zerner [22] introduced a sequence of *regeneration* times and showed, roughly, that the environments traversed by the walk between regeneration times, together with the path of the walk, form a sequence of i.i.d. random vectors under the *annealed* law $\mathbb{P}^o$. This allowed them to derive a law of large numbers under Kalikow's condition by studying the tail properties of these regeneration times.

In [26], a *coupling technique* was introduced that immediately allows one to adapt the construction of regeneration times to the case of measures P which are L-dependent, that is such that coordinates of the environment at distance larger than L, some fixed deterministic L, are independent. This coupling covers in particular the setup in [18], that deals with a particular 1-dependent environment (note however that we do not attempt to recover here all the results of [18]). Intuitively, the coupling idea is that, due to the uniform ellipticity property, the walk has positive probability for travelling the L next steps whithout looking at the environment. One then is reduced to the study of tails of these regeneration times.

Our purpose in this work is to further modify the construction of regeneration times and allow for more general type of mixing conditions on P. A complication arises from the destruction of the renewal structure due to the dependence in the environment: in fact, the environments between regeneration times need not even define a stationary sequence any more. Our approach is based on suitably approximating this sequence by an i.i.d. sequence, namely the so-called splitting representation [23].

Another approach to non product measures P, but with a rather mild dependence structure, has been proposed in [12], [13]. A comparison between their results and ours is presented at the end of the article. Our results here cover natural examples of environment distributions as Gibbs measures in a mixing regime.

We now turn to a description of our results. We deal with environments subject to various mixing conditions, and it is appropriate to consider closed positive *cones.* For $\ell \in \mathbb{R}^d \setminus \{0\}$, $x \in \mathbb{R}^d$ and $\zeta \in (0,1)$, define the cone of vertex x, direction ℓ and angle $\cos^{-1}(\zeta)$,

$$C(x,\ell,\zeta) = \{y \in \mathbb{R}^d \; ; \; (y-x)\cdot \ell \geq \; \zeta |y-x||\ell|\} \, . \tag{1.2}$$

(All through the paper, $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^d$, and $|\ell|_1 = \sum_i |\ell_i|$ the ℓ_1-norm.) Note that for $\zeta = 0$, this is just a usual half-space.

In the sequel, we fix an $\ell \in \mathbb{R}^d \setminus \{0\}$ such that ℓ has *integer coordinates.* With $\mathrm{sgn}(0) = 0$, let

$$\mathcal{E}_{\bar{\varepsilon}} = \{\mathrm{sgn}(\ell_i) e_i\}_{i=1}^d \setminus \{0\} \, . \tag{1.3}$$

Throughout, we make the following two assumptions on the environment:

Assumption 1.4

($\mathcal{A}1$) *P is stationary and ergodic, and satisfies the following mixing condition on ℓ-cones: for all positive ζ small enough there exists a function $\phi(r) \underset{r\to\infty}{\longrightarrow} 0$ such that any two events A, B with $P(A) > 0$, $A \in \sigma\{\omega_z; z\cdot\ell \leq 0\}$ and $B \in \sigma\{\omega_z; z \in C(r\ell,\ell,\zeta)\}$ it holds that*

$$\left| \frac{P(A\cap B)}{P(A)} - P(B) \right| \leq \phi(r|\ell|) \, .$$

($\mathcal{A}2$) *P is elliptic and uniformly elliptic with respect to ℓ: $P(\omega(0,e) > 0; |e| = 1) = 1$, and there exists a $\kappa > 0$ such that*

$$P(\min_{e\in\mathcal{E}_{\bar{\varepsilon}}} \omega(0,e) \geq \kappa) = 1 \, .$$

Note that ($\mathcal{A}1$) is equivalent to

$$|E(fg) - EfEg| \leq \phi(r|\ell|) \|f\|_1 \|g\|_\infty \tag{1.5}$$

for all bounded functions, f being $\sigma\{\omega_z; z\cdot\ell \leq 0\}$-measurable, and g being $\sigma\{\omega_z; z \in C(r\ell,\ell,\zeta)\}$-measurable. Properties of the type ($\mathcal{A}1$) are generically called ϕ-mixing or uniform mixing [4, Section 1.1]. When ($\mathcal{A}1$) holds for $\zeta = 0$, we will say, in this paper, that the field P is ϕ-mixing. But this condition is too restrictive, and we give in Section 4, examples of environments satisfying the mixing property ($\mathcal{A}1$), but not ϕ-mixing (with $\zeta = 0$). In most applications, one can find such a ϕ not depending on $\ell \in \mathbb{R}^d \setminus \{0\}$, for which the condition holds for all ℓ.

Next, we turn to conditions on the environment ensuring the ballistic nature of the walk. In order to do so, we introduce (a natural extension of) Kalikow's Markov chain [10] as follows. Let U be a finite, connected subset of $\mathbb{Z}^d$, with $0 \in U$, let

$$\mathcal{F}_{U^c} = \sigma\{\omega_z : z \notin U\} ,$$

and define on $U \cup \partial U$ an auxiliary Markov chain with transition probabilities

$$\hat{P}_U(x, x+e) = \begin{cases} \dfrac{\mathbb{E}^o\left[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}}\omega(x,x+e)|\mathcal{F}_{U^c}\right]}{\mathbb{E}^o\left[\sum_{n=0}^{T_{U^c}} \mathbf{1}_{\{X_n=x\}}|\mathcal{F}_{U^c}\right]}, & x \in U, |e| = 1 \\ 1 & x \in \partial U, e = 0 \end{cases} \tag{1.6}$$

where $T_{U^c} = \min\{n \geq 0 : X_n \in \partial U\}$ (note that the expectations in (1.6) are finite due the Markov property and ℓ-ellipticity). The transition kernel $\hat{P}_U$ weights the transitions $x \mapsto x+e$ according to the occupation time of the vertex x before exiting U. Define the Kalikow drift as $\hat{d}_U(x) = \sum_{|e|=1} e\hat{P}_U(x, x+e)$, with the RWRE's drift at x defined by $d(x,\omega) = \sum_{|e|=1} e\omega(x, x+e)$. Note that, unlike in the i.i.d. case, the Kalikow drift, as well as Kalikow's chain itself, here is random because it depends on the environment outside of U. This new Markov chain is useful because of the following property [10], which remains valid in our non-i.i.d. setup. Since U is finite and the walk is uniformly elliptic in the direction ℓ, under both $\hat{P}_U$ and $\mathbb{P}^o(\cdot|\mathcal{F}_{U^c})$, the exit time $T_{U^c} = \inf\{n \geq 0; X_n \in U^c\}$ is finite, and

$$X_{T_{U^c}} \text{ has the same law under } \hat{P}_U \text{ and } \mathbb{P}^o(\cdot|\mathcal{F}_{U^c}) . \tag{1.7}$$

In this paper, we will consider one of following *drift* conditions, which ensure a ballistic behavior for the walk:

Assumption 1.8

($\mathcal{A}3$) *Kalikow's condition: There exists a $\delta(\ell) > 0$ deterministic such that*

$$\inf_{U, x\in U} \hat{d}_U(x) \cdot \ell \geq \delta(\ell),\ P-a.s..$$

(The infimum is taken over all connected finite subsets of $\mathbb{Z}^d$ containing 0.)

($\mathcal{A}4$) *Non-nestling: There exists a $\delta(\ell) > 0$ such that*

$$d(x,\omega) \cdot \ell \geq \delta(\ell)\,, P-a.s.$$

We will always assume ($\mathcal{A}1$) and ($\mathcal{A}2$), and (except in the beginning of Section 3), also one of ($\mathcal{A}3$) or the stronger ($\mathcal{A}4$). Clearly, ($\mathcal{A}4$) implies that ($\mathcal{A}3$) holds with the same ℓ and $\delta(\ell)$. An inspection of the proof in [10] reveals that, under ($\mathcal{A}3$), the conclusion $\mathbb{P}^o(\liminf_{n\to\infty} X_n \cdot \ell = \infty) = 1$ remains valid in our non-i.i.d. setup (see e.g. the exposition in [26]). Note that the requirement from ℓ to possess integer coordinates is not a restrictive one for given an $\ell \in \mathbb{R}^d \setminus \{0\}$ satisfying either $(\mathcal{A}3)$ or $(\mathcal{A}4)$, one finds by continuity an ℓ with integer coordinates satisfying the same. We make this restriction for the convenience of defining the path $\bar{\varepsilon}$ in (2.2) below.

The statement of our fundamental result, Theorem 3.12, involves certain *modified regeneration times*, which are introduced in Section 2 below. The basic condition in this theorem is a trade-off between moments for these "regeneration times" and rate of mixing for the environment. A corollary can be readily stated here: it deals with the non-nestling case, and does not require any assumption on the mixing rate other than $\phi \to 0$.

Corollary 1.9 *Assume $(\mathcal{A}1, \mathcal{A}2)$ and $(\mathcal{A}4)$ hold for some $\ell \in \mathbb{Z}^d \setminus \{0\}$. Then there exists a deterministic $v \neq 0$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v\,, \quad \mathbb{P}^o - a.s..$$

Our fundamental Theorem 3.12 does not restrict attention to non-nestling cases, but applies also when only Kalikow's condition holds. A class of nestling examples based on Ising-like environments, satisfying the conditions of Theorem 3.12 and hence the law of large numbers, is provided in Theorem 5.15 below. Dealing with nestling walks is much more delicate than with non-nestling ones due to the existence of so-called *traps*, i.e., finite but large regions where the environment is atypical, confining the walk or creating an abnormal drift. In this case, a renormalization procedure ("coarse graining") is needed to control the size of the traps and their effect.

The structure of the article is as follows: in Section 2, we introduce the coupling representation (2.1) which allows us to deal with non-independent medium, and some *approximate regeneration times* $\tau_i^{(L)}$ parametrized by a parameter L: they are defined by (2.3) and (2.8), and they lead to an "approximate renewal" result, our Lemma 2.13 below. In Section 3, we prove the law of large numbers under $(\mathcal{A}1, 2, 3)$ and a suitable integrability condition $(\mathcal{A}5)$. We then give the proof of Corollary 1.9, by showing that Condition $(\mathcal{A}5)$ is trivially satisfied under the non-nestling assumption $(\mathcal{A}4)$. In Section 4, we make a short digression to show that the mixing assumption $(\mathcal{A}1)$ is satisfied in many cases of interest. In Section 5 we show how condition $(\mathcal{A}5)$ can be checked in the nestling case, and we construct a class of nestling, mixing environments satisfying our conditions for the law of large numbers, see Theorems 5.1 and 5.15 for precise statements. Finally, Section 6 is devoted to concluding remarks and extensions.

2 Some Random Times

To implement our coupling technique, we begin, following [26], by constructing an extension of the probability space, depending on the vector ℓ with integer coordinates: recall that the RWRE was defined by means of the law $\mathbb{P}^o = P \otimes P_\omega^o$ on the canonical space $(\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{F} \times \mathcal{G})$. Set $W = \{0\} \cup \mathcal{E}_{\bar{\varepsilon}}$ (recall the notation (1.3)), and let $\mathcal{W}$ be the cylinder σ-algebra on $W^{\mathbb{N}}$. We now define the measure

$$\overline{\mathbb{P}}^o = P \otimes Q \otimes \overline{P}^o_{\omega,\varepsilon}$$

on

$$\Big(\Omega \times W^{\mathbb{N}} \times (\mathbb{Z}^d)^{\mathbb{N}}, \quad \mathcal{F} \times \mathcal{W} \times \mathcal{G}\Big)$$

in the following way: Q is a product measure, such that with $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ denoting an element of $W^{\mathbb{N}}$, $Q(\varepsilon_1 = e) = \kappa$, for $e \in \mathcal{E}_{\bar{\varepsilon}}$, while $Q(\varepsilon_1 = 0) = 1 - \kappa|\mathcal{E}_{\bar{\varepsilon}}|$. For each fixed ω, ε, $\overline{P}^o_{\omega,\varepsilon}$ is the law of the Markov chain $\{X_n\}$ with state space $\mathbb{Z}^d$, such that $X_0 = 0$ and, for every $z, e \in \mathbb{Z}^d$, $|e| = 1$,

$$\overline{P}^o_{\omega,\varepsilon}(X_{n+1} = z+e \mid X_n = z) = \mathbf{1}_{\{\varepsilon_{n+1}=e\}} + \frac{\mathbf{1}_{\{\varepsilon_{n+1}=0\}}}{1-\kappa|\mathcal{E}_{\bar{\varepsilon}}|}[\omega(z, z+e) - \kappa \mathbf{1}_{\{e \in \mathcal{E}_{\bar{\varepsilon}}\}}] \,. \tag{2.1}$$

Clearly, the law of $\{X_n\}$ under $Q \otimes \overline{P}^o_{\omega,\varepsilon}$ coincides with its law under P^o_ω, while its law under $\overline{\mathbb{P}}^o$ coincides with its law under $\mathbb{P}^o$.

We fix now a particular sequence of ε in $\mathcal{E}_{\bar{\varepsilon}}$ of length $|\ell|_1$ with sum equal to ℓ: for definiteness, we take $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_{|\ell|_1})$ with

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = \dots \bar{\varepsilon}_{|\ell_1|} = \mathrm{sgn}(\ell_1)e_1 \,,\ \bar{\varepsilon}_{\ell_1+1} = \bar{\varepsilon}_{\ell_1+2} = \dots \bar{\varepsilon}_{|\ell_1|+|\ell_2|} = \mathrm{sgn}(\ell_2)e_2 \,,$$
$$\dots \ \bar{\varepsilon}_{|\ell|_1-|\ell_d|+1} = \dots \bar{\varepsilon}_{|\ell|_1} = \mathrm{sgn}(\ell_d)e_d \,.$$

We fix, from now on through the whole paper, $\zeta > 0$ small enough such that

$$\bar{\varepsilon}_1, \bar{\varepsilon}_1 + \bar{\varepsilon}_2, \dots, \bar{\varepsilon}_1 + \dots \bar{\varepsilon}_{|\ell|_1} = \ell \in C(0, \ell, \zeta) \quad , \tag{2.2}$$

and such that $(\mathcal{A}1)$ above is satisfied. Without mentioning it explicitely in the sequel, we always rotate the axes such that $\ell_1 \neq 0$.

For $L \in |\ell|_1\mathbb{N}^*$ we will denote by $\bar{\varepsilon}^{(L)}$ the vector

$$\bar{\varepsilon}^{(L)} = (\bar{\varepsilon}, \bar{\varepsilon}, \dots, \bar{\varepsilon})$$

of dimension L. In particular, $\bar{\varepsilon} = \bar{\varepsilon}^{(|\ell|_1)}$, and for ε with $(\varepsilon_{n+1}, \dots \varepsilon_{n+L}) = \bar{\varepsilon}^{(L)}$,

$$\overline{P}^o_{\omega,\varepsilon}(X_{n+L} = x + \frac{L}{|\ell|_1}\ell \mid X_n = x) = 1 \quad ,$$

and the path $X_n, X_{n+1}, \dots X_{n+L}$ remains in the cone $C(x, \ell, \zeta)$.

Define

$$D' = \inf\{n \geq 0 : \ X_n \notin C(X_0, \ell, \zeta)\} \quad . \tag{2.3}$$

(For $\zeta = 0$, this is D from [22].) Let us state a few direct consequences of $(\mathcal{A}3)$ and $(\mathcal{A}4)$.

Lemma 2.4 *Assume $(\mathcal{A}3)$. Let $f(y) = y \cdot \ell - \zeta|y||\ell|$ with $\zeta \leq \delta(\ell)/(3|\ell|)$.*

1) There exist a $\lambda_0 = \lambda_0(\delta(\ell)) > 0$ such that for all $\lambda \in (0, \lambda_0]$ and all connected, finite subset U of $\mathbb{Z}^d$ containing 0,

$$M^\lambda_n = \exp\{-3\lambda f(X_n) + \lambda\delta(\ell)(n \wedge T_{U^c})\}$$

is a supermartingale for Kalikow's Markov chain $\hat{P}_U$ from (1.6) (T_{U^c} is the exit time of U for X_n).

2) For $m > |\ell|$, consider the truncated cone $V_m = C(0,\ell,\zeta)\bigcap\{y \in \mathbb{R}^d; y\cdot\ell \le m\}$. We have $\hat{P}_{V_m}(X_{T_{V_m^c}}\cdot\ell > m) \ge 2\eta$ with some constant $\eta > 0$ depending on $\delta(\ell),\zeta$ but not on m,ω.

3) If also $(\mathcal{A}4)$ holds (and ℓ is general), choosing κ small enough such that $\delta(\ell) > 2\kappa$, it holds that M_n^λ is a supermartingale under the quenched measure $\overline{P}^o_{\omega,\varepsilon}$ for all $\omega,\varepsilon,\lambda \in (0,\lambda_0]$, and $\overline{P}^o_{\omega,\varepsilon}(X_{T_{V_m^c}}\cdot\ell > m) \ge 2\eta$. Moreover, with $W_r = \{x \in \mathbb{Z}^d; x\cdot\ell < r\}, r > 0$,

$$\overline{E}^o_{\omega,\varepsilon}(\exp\{\lambda\delta(\ell)T_{W_r^c}\}) \le \exp\{3\lambda r\}\ . \tag{2.5}$$

4) Assume $(\mathcal{A}3)$ holds with $\ell = e_1$, and $\delta = \delta(e_1)$. Then, there exists $\lambda_1 = \lambda_1(\delta) > 0$ with $\lambda_1 \to +\infty$ as $\delta \to 1^-$ such that $\exp\{-3\lambda f(X_n)\}$ is a supermartingale for Kalikow's Markov chain ($\lambda \in [0,\lambda_1)$). In particular, $\inf_{m,\omega}\hat{P}_{V_m}(X_{T_{V_m^c}}\cdot\ell > m) \to 1$ when $\delta \to 1^-$.

We stress that the above constants do not depend on ω outside U, in contrast to Kalikow's Markov chain $\hat{P}_U$ itself. Recall that, due to (1.7), estimates on the exit distribution for Kalikow's Markov chain yields the similar estimate for the RWRE, but on the other hand, exit time distribution for Kalikow's Markov chain and RWRE may be quite different.

Proof: 1) Since the chain has unit jumps, $(\mathcal{A}3)$ implies for $y \in U$

$$\hat{E}_U[(f(X_{n+1}) - f(X_n))|X_n = y] \ge (2/3)\delta(\ell)$$

and $f(X_{n+1}) - f(X_n)$ is uniformly bounded. Choosing λ_0 such that $\max\{u^{-2}(e^u - 1 - u); 0 < |u| \le \lambda_0(3|\ell|+2)\} \le \delta(\ell)/(\lambda_0[3|\ell|+2]^2)$, we have for $\lambda \in [0,\lambda_0]$ that $\hat{E}_U(\exp\{-3\lambda(f(X_{n+1}) - f(X_n)) + \lambda\delta(\ell)\}|X_n = y) \le 1$ uniformly in $y \in U$. Note that we can choose λ_0 increasing in $\delta(\ell)$.

2) Applying the stopping theorem for the exit time $T_{V_m^c}$ to the above supermartingale, we get for $y \in V_m$,

$$\exp\{-3\lambda f(y)\} \ge \hat{E}_{V_m}(\exp\{-3\lambda f(X_{T_{V_m^c}}) + \lambda\delta(\ell)T_{V_m^c}\}|X_0 = y) \ge \hat{P}_{V_m}(X_{T_{V_m^c}}\cdot\ell \le m|X_0 = y)\ ,$$

where the second inequality is due to $f < 0$ on the boundary of the cone $C(0,\ell,\zeta)$ and $\lambda\delta(\ell) > 0$. With $y = \bar{\varepsilon}_1 \in V_m$, we have, for the chain starting from 0,

$$\hat{P}_{V_m}(X_{T_{V_m^c}}\cdot\ell > m) \ge \hat{P}_{V_m}(X_1 = \bar{\varepsilon}_1)\hat{P}_{V_m}(X_{T_{V_m^c}}\cdot\ell \ge m|X_1 = \bar{\varepsilon}_1) \ge \kappa[1 - e^{-3\lambda f(\bar{\varepsilon}_1)}] \tag{2.6}$$

which is positive since f is positive in the interior of the cone.

3) It is straightforward to check that the above computations apply to $\overline{P}^o_{\omega,\varepsilon}$ under the assumption $(\mathcal{A}4)$ (the assumption $\delta(\ell) > 2\kappa$ is used to ensure that modified environment appearing in the definition (2.1) of $\overline{P}^o_{\omega,\varepsilon}$ also is uniformly elliptic in the direction ℓ). In addition, to prove (2.5), we apply the stopping theorem to the $\overline{P}^o_{\omega,\varepsilon}$-supermartingale M_n^λ at the exit time of the domain W_r intersected with large, finite boxes, and we get

$$\exp\{-3\lambda r\}\overline{E}^o_{\omega,\varepsilon}(\exp\{\lambda\delta(\ell)T_{W_r^c}\}) \le 1\ .$$

4) With $\delta=\delta(e_1)$, set $\mathcal{A}=\{(\alpha,\beta)\subset[0,1]^2:\alpha+\beta\le 1,\alpha-\beta\ge\delta\}$. Then, for $y\in V_m$, and $\lambda>0$, it follows from $(\mathcal{A}3)$ that

$$\begin{aligned}\hat{E}_U[\exp\{-3\lambda((X_{n+1}-X_n)\cdot e_1+3\lambda\zeta\}|X_n=y] &\le \exp\{3\lambda\zeta\}\sup_{(\alpha,\beta)\in\mathcal{A}}\Big[\alpha e^{-3\lambda}+\beta e^{3\lambda}+(1-\alpha-\beta)\Big]\\ &= \exp\{3\lambda\zeta\}\left[\cosh(3\lambda)-\delta\sinh(3\lambda)\right]=:A(\delta,\lambda,\zeta)\,.\end{aligned}$$

We see that $\lambda_1(\delta):=\sup\{\lambda:A(\delta,\lambda,\zeta)<1,\ \forall\zeta\le\delta/3\}\to\infty$ as $\delta\to 1$. In particular, the right-hand side of (2.6) can be arbitrary close to 1 as $\delta\to 1$. □

Assumption $(\mathcal{A}3)$ implies $\mathbb{P}^o(D'=\infty)>0$. Indeed, consider the truncated cone $V_m=C(0,\ell,\zeta)\bigcap\{y\in\mathbb{R}^d; y\cdot\ell\le m\}$, and Kalikow's Markov chain $\hat{P}_{V_m}$. From (1.7), the exit distribution for X out of V_m is the same under both $\hat{P}_{V_m}$ and $\mathbb{P}^o(\cdot|\mathcal{F}_{V_m^c})$. From part 2) of Lemma 2.4, it follows

$$\mathbb{P}^o(D'=\infty|\,\omega_x,x\cdot\ell\le 0)=\lim_{m\to\infty}\mathbb{P}^o(X_{T_{V_m^c}}\cdot\ell>m|\,\omega_x,x\cdot\ell\le 0)\ge 2\eta\,,\quad P-a.s..$$

By integration, we get

$$\mathbb{P}^o(D'=\infty\mid\omega_x,x\cdot\ell\le -r)\ge 2\eta,\quad P-a.s. \tag{2.7}$$

Recall that $(\mathcal{A}3)$ implies that $\mathbb{P}^o(X_n\cdot\ell\to_{n\to\infty}+\infty)=1$. Set

$$\mathcal{G}_n=\sigma((\varepsilon_i,X_i),\,i\le n)\,,$$

fix $L\in|\ell|_1\mathbb{N}$ and, setting $\overline{S}_0=0$, define, using θ_n to denote time shift and $\bar{\theta}_x$ to denote space shift,

$$\begin{aligned}&\overline{S}_1=\inf\Big\{n\ge L: X_{n-L}\cdot\ell>\max\{X_m\cdot\ell:\ m<n-L\},\,(\varepsilon_{n-1},\dots,\varepsilon_{n-L})=\bar{\varepsilon}^{(L)}\Big\}\le\infty\,,\\ &\overline{R}_1=D'\circ\theta_{\bar{S}_1}+\overline{S}_1\le\infty.\end{aligned} \tag{2.8}$$

Define further, by induction for $k\ge 1$,

$$\begin{aligned}&\overline{S}_{k+1}=\inf\Big\{n\ge R_k: X_{n-L}\cdot\ell>\max\{X_m\cdot\ell:\ m<n-L\},\,(\varepsilon_{n-1},\dots,\varepsilon_{n-L})=\bar{\varepsilon}^{(L)}\Big\}\le\infty\,,\\ &\overline{R}_{k+1}=D'\circ\theta_{\bar{S}_{k+1}}+\overline{S}_{k+1}\le\infty\,,\end{aligned}$$

Clearly, these are $\mathcal{G}_n$ stopping times (depending on L), and

$$0=\overline{S}_0\le\overline{S}_1\le\overline{R}_1\le\overline{S}_2\le\dots\le\infty$$

and the inequalities are strict if the left member is finite. On the set $A_\ell:=\{X_n\cdot\ell\to_{n\to\infty}\infty\}$, it is straightforward to check (using the product structure of Q), that the time $\overline{S}_1$ is $\overline{\mathbb{P}}^o$-a.s. finite, as is $\overline{S}_{k+1}$ on the set $A_\ell\bigcap\{\overline{R}_k<\infty\}$. Define:

$$\begin{aligned}K&=\inf\{k\ge 1:\overline{S}_k<\infty,\overline{R}_k=\infty\}\le\infty,\\ \tau_1^{(L)}&=\overline{S}_K\le\infty\,.\end{aligned}$$

This random time $\tau_1^{(L)}$ is the first time n when the walk performs as follow: at time $n-L$ it has reached a record value in the direction $+\ell$, then it travels using the ε-sequence only up to time n, and from time n on, it doesn't exit the positive cone $C(X_n,\ell,\zeta)$ with vertex X_n. In particular, $\tau_1^{(L)}$ is not a stopping time, and we emphasize its dependence on L. As in [26], the advantage in working with $\tau_i^{(L)}$ (as opposed to the more standard $\tau_i^{(0)}$) is that the $\{\varepsilon\}$ sequence creates a spacing where no information on the environment is gathered by the RWRE.

Lemma 2.9 *Assume $(\mathcal{A}1,2,3)$, and $\zeta \le \delta(\ell)/(3|\ell|)$. Then, there exists a L_0 such that for $L \ge L_0$, $\tau_1^{(L)}$ is finite $\overline{\mathbb{P}}^o$-a.s..*

Proof: This amounts to proving $K<\infty$. Toward this end, write

$$\begin{aligned}\overline{\mathbb{P}}^o(\overline{R}_{k+1}<\infty) &= \overline{\mathbb{P}}^o(\overline{R}_k<\infty, D'\circ\theta_{\overline{S}_{k+1}}<\infty)\\ &= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}\overline{\mathbb{P}}^o(\overline{R}_k<\infty, D'\circ\theta_n<\infty, X_{\overline{S}_{k+1}}=z, \overline{S}_{k+1}=n) \qquad (2.10)\\ &= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\Big(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k<\infty, X_{\overline{S}_{k+1}}=z,\overline{S}_{k+1}=n, D'\circ\theta_n<\infty)\Big)\\ &= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\Big(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k<\infty, X_{\overline{S}_{k+1}}=z,\overline{S}_{k+1}=n)\cdot \overline{P}^o_{\theta_z\omega,\theta_n\varepsilon}(D'<\infty)\Big)\end{aligned}$$

using the strong Markov property under $\overline{P}^o_{\omega,\varepsilon}$. The point here is that $\overline{P}^o_{\theta_z\omega,\theta_n\varepsilon}(D'<\infty)$ is measurable on $\sigma(\omega_x:\ x\in C(z,\ell,\zeta))\otimes\sigma(\varepsilon_i, i\ge n)$, whereas $\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k<\infty, X_{\overline{S}_{k+1}}=z,\overline{S}_{k+1}=n)$ is measurable on $\sigma(\omega_x: x\cdot\ell\le z\cdot\ell - L|\ell|^2/|\ell|_1)\otimes\sigma(\varepsilon_i, i<n)$. Hence, by the ϕ-mixing property on cones of P, by the product structure of Q and by stationarity,

$$\begin{aligned}\overline{\mathbb{P}}^o(\overline{R}_{k+1}<\infty) &\le \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}\Big[E_{P\otimes Q}\Big(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k<\infty, X_{\overline{S}_{k+1}}=z,\overline{S}_{k+1}=n)\Big)\cdot E_{P\otimes Q}\Big(\overline{P}^o_{\omega,\varepsilon}(D'<\infty)\Big)\Big]\\ &\quad+\phi(L)\sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}}E_{P\otimes Q}\Big(\overline{P}^o_{\omega,\varepsilon}(\overline{R}_k<\infty, X_{\overline{S}_{k+1}}=z,\overline{S}_{k+1}=n)\Big)\\ &= \overline{\mathbb{P}}^o(\overline{R}_k<\infty)(\overline{\mathbb{P}}^o(D'<\infty)+\phi(L))\\ &\le (\overline{\mathbb{P}}^o(D'<\infty)+\phi(L))^{k+1} \qquad (2.11)\end{aligned}$$

by induction. Choosing L with $\phi(L)\le\eta$, and using (2.7), we see that $\overline{\mathbb{P}}^o(K\ge k)\le(1-\eta)^k$. $\square$

Consider now $\tau_1^{(L)}$ as a function of the path $(X_n)_{n\ge0}$ and set

$$\tau_{k+1}^{(L)} = \tau_k^{(L)}(X.) + \tau_1^{(L)}(X_{\tau_k^{(L)}+.} - X_{\tau_k^{(L)}})\,, \qquad (2.12)$$

with $\tau_{k+1}^{(L)}=\infty$ on $\{\tau_k^{(L)}=\infty\}$.

Under $(\mathcal{A}3)$, $\tau_k^{(L)}$ is $\overline{\mathbb{P}}^o$-a.s. finite for all k. Indeed, in view of the definition (2.12),

$$\begin{aligned}\overline{\mathbb{P}}^o(\tau_1^{(L)} < \infty, \tau_2^{(L)} = \infty) &= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\overline{P}^o_{\omega,\varepsilon}(\tau_1^{(L)} = n, X_n = z, \tau_2^{(L)} = \infty)\\ &\le \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} E_{P\otimes Q}\overline{P}^z_{\omega,\theta^n\varepsilon}(\tau_1^{(L)} = \infty)\\ &= \sum_{z\in\mathbb{Z}^d}\sum_{n\in\mathbb{N}} \overline{\mathbb{P}}^z(\tau_1^{(L)} = \infty) = 0\ ,\end{aligned}$$

since all summands are equal to $\overline{\mathbb{P}}^o(\tau_1^{(L)} = \infty) = 0$.

Define

$$\mathcal{H}_1 = \sigma\left(\tau_1^{(L)}, X_0, \varepsilon_0, X_1, \cdots, \varepsilon_{\tau_1^{(L)}-1}, X_{\tau_1^{(L)}}, \{\omega(y,\cdot); \ell\cdot y < \ell\cdot X_{\tau_1^{(L)}} - L|\ell|^2/|\ell|_1\}\right)\ ,$$

$$\mathcal{H}_k = \sigma\Big(\tau_1^{(L)}\ldots\tau_k^{(L)}, \quad X_0, \varepsilon_0, X_1, \cdots,\ \varepsilon_{\tau_k^{(L)}-1}, X_{\tau_k^{(L)}}, \quad \{\omega(y,\cdot); \ell\cdot y < \ell\cdot X_{\tau_k^{(L)}} - L|\ell|^2/|\ell|_1\}\Big)\ .$$

Note that since $\{D' = \infty\} = \{X_1, \ldots X_{\tau_1^{(L)}} \in C(0,\ell,\zeta)\}\bigcap\{\tau_1^{(L)} < \infty\}$, see (2.2), we have that

$$\{D' = \infty\} \in \mathcal{H}_1\ .$$

Then, we have the following crucial lemma. Recall the variational distance $\|\mu-\nu\|_{\mathrm{var}} = \sup\{\mu(A) - \nu(A); A \text{ measurable}\}$ between two probability measures on the same space.

Lemma 2.13 *Assume $(\mathcal{A}1,2,3)$, and $\zeta \le \delta(\ell)/(3|\ell|)$. Set $\phi'(L) = 2[\overline{\mathbb{P}}^o(D' = \infty) - \phi(L)]^{-1}\phi(L)$. (Here, and in the following, we consider L large enough so that $\phi(L) < \overline{\mathbb{P}}^o(D' = \infty)$.) Then, it holds a.s.,*

$$\|\overline{\mathbb{P}}^o\Big(\{X_{\tau_k^{(L)}+n} - X_{\tau_k^{(L)}}\}_{n\ge0} \in \cdot\,|\mathcal{H}_k\Big) - \overline{\mathbb{P}}^o\Big(\{X_n\}_{n\ge0} \in \cdot\,|D' = \infty\Big)\|_{\mathrm{var}} \le \phi'(L)\ .$$

Proof: We start with the case $k = 1$. Let A be a measurable subset of the path space, and write for short $\mathbf{1}_A = \mathbf{1}_{\{X_n - X_0\}_{n\ge0}\in A}$. Let $h \ge 0$ be a $\mathcal{H}_1$-measurable non-negative random variable. Then for all $l, n \ge 1, x \in \mathbb{Z}^d$, there exists a random variable $h_{x,l,n} \ge 0$, measurable with respect to $\sigma(\{\omega(y,\cdot); y\cdot\ell < x\cdot\ell - L|\ell|^2/|\ell|_1\}, \{X_i\})_{i\le n}$ such that, on the event $\{\tau_1^{(L)} = \overline{S}_l = n, X_{\overline{S}_l} = x\}$, it holds $h = h_{x,l,n}$. Recall that $\{K = l\} = \{\overline{S}_l < \infty, D'\circ\theta_{\overline{S}_l} = \infty\}$, and use the (weak) Markov property and shift invariance to write

$$
\begin{aligned}
\overline{\mathbb{E}}^o(h\,\mathbf{1}_A\circ\theta_{\tau_1^{(L)}}) &= \sum_{l\geq 1}\overline{\mathbb{E}}^o(h\,\mathbf{1}_A\circ\theta_{\tau_1^{(L)}}\mathbf{1}_{K=l}) \\
&= \sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1} E_{P\otimes Q}\overline{E}^o_{\omega,\mathcal{E}}(h_{x,l,n}\,\mathbf{1}_A\circ\theta_n\mathbf{1}_{\overline{S}_l=n,X_n=x,D'\circ\theta_n}=\infty) \\
&= \sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1} E_{P\otimes Q}\left[\overline{E}^o_{\omega,\mathcal{E}}(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x})\times\overline{P}^x_{\omega,\theta_n\mathcal{E}}(A\bigcap\{D'=\infty\})\right] \\
&= \sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x})\overline{\mathbb{P}}^x(A\bigcap\{D'=\infty\})+\rho_A \\
&= \overline{\mathbb{P}}^o(A\bigcap\{D'=\infty\})\sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x})+\rho_A\,. \qquad (2.14)
\end{aligned}
$$

The quantity ρ_A is defined by the above equalities, i.e., $\rho_A=\sum_{l,x,n}\mathrm{Cov}_{P\otimes Q}(f_{x,l,n},g_{x,n})$ with $f_{x,l,n}=\overline{E}^o_{\omega,\mathcal{E}}(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x}), g_{x,n}=\overline{P}^x_{\omega,\theta_n\mathcal{E}}(A\bigcap\{D'=\infty\})$. The point is that, from (1.5) it holds for a non-negative h

$$
|\rho_A|\leq\phi(L)\sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x}) \qquad (2.15)
$$

uniformly in A. In particular for A equal to the whole path space $(\mathbb{Z}^d)^{\mathbb{N}}$, one gets

$$
\sum_{l\geq 1, x\in\mathbb{Z}^d, n\geq 1}\overline{\mathbb{E}}^o(h_{x,l,n}\mathbf{1}_{\overline{S}_l=n,X_n=x})\leq[\overline{\mathbb{P}}^o(D'=\infty)-\phi(L)]^{-1}\overline{\mathbb{E}}^o(h) \qquad (2.16)
$$

as well as a formula for the sum in the left member above. Plugging this formula in (2.14), one obtains

$$
\begin{aligned}
|\overline{\mathbb{E}}^o(h\,\mathbf{1}_A\circ\theta_{\tau_1^{(L)}})-\overline{\mathbb{E}}^o(h)\overline{\mathbb{P}}^o(A|D'=\infty)| &= |\rho_A-\rho_{(\mathbb{Z}^d)^{\mathbb{N}}}\overline{\mathbb{P}}^o(A|D'=\infty)| \\
&\leq 2[\overline{\mathbb{P}}^o(D'=\infty)-\phi(L)]^{-1}\phi(L)\overline{\mathbb{E}}^o(h)
\end{aligned}
$$

where the second inequality follows from (2.15), (2.16). Since h is arbitrary, we have

$$
|\overline{\mathbb{P}}^o\Big(\{X_{\tau_1^{(L)}+n}-X_{\tau_1^{(L)}}\}_{n\geq 0}\in A|\mathcal{H}_1\Big)-\overline{\mathbb{P}}^o\Big(\{X_n\}_{n\geq 0}\in A|D'=\infty\Big)|\leq\phi'(L)\,.
$$

a.s., for all A's. But there are only countably many cylinders in the path space, so we can find a subset of Ω of P-measure one, where the previous inequality holds simultaneously for all measurable A. We have shown the lemma for $k=1$.

The case of a general $k\geq 1$ follows similarly from the above computations, and the definition (2.12).
□

3 Law of large numbers

Throughout this section we assume ($\mathcal{A}1,2$) for some ℓ with integer coordinates, and we assume also that the conclusions of Lemma 2.13 hold. For $L \in |\ell|_1\mathbb{N}^*$ we define $\tau_0^{(L)} = 0$, and for $k \geq 1$,

$$\overline{\tau}_k^{(L)} = \kappa^L \left(\tau_k^{(L)} - \tau_{k-1}^{(L)}\right) , \qquad \overline{X}_k^{(L)} = \kappa^L \left(X_{\tau_k^{(L)}} - X_{\tau_{k-1}^{(L)}}\right) . \tag{3.1}$$

The following uniform integrability condition is instrumental in our derivation:

Assumption 3.2

($\mathcal{A}5$) *There exist an $\alpha > 1$ and $M = M(L)$ such that $\phi'(L)^{1/\alpha'} M(L)^{1/\alpha} \underset{L\to\infty}{\longrightarrow} 0$ (with $1/\alpha' = 1 - 1/\alpha$), and*

$$P\Big(\overline{\mathbb{E}}^o\big((\overline{\tau}_1^{(L)})^\alpha \mid D' = \infty, \mathcal{F}_0^L\big) > M \Big) = 0\,, \tag{3.3}$$

where $\mathcal{F}_0^L = \sigma(\omega(y,\cdot) : \ell\cdot y < -L)$.

We define

$$\beta_L := \overline{\mathbb{E}}^o(\overline{\tau}_1^{(L)} \mid D' = \infty) < \infty\,, \tag{3.4}$$

and

$$\gamma_L := \overline{\mathbb{E}}^o(\overline{X}_1^{(L)} \mid D' = \infty) \in \mathbb{R}^d\,, \tag{3.5}$$

where the moments β_L and γ_L are finite due to (3.3). Further, we note that $\beta_L \geq 1$ for all L.

From Lemma 2.13, we have a.s., that for $k \geq 2$,

$$\|\overline{\mathbb{P}}^o\left((\overline{\tau}_k^{(L)}, \overline{X}_k^{(L)}) \in \cdot \mid \mathcal{H}_{k-1}\right) - \mu^{(L)}(\cdot)\|_{\mathrm{var}} \leq \phi'(L)\,, \tag{3.6}$$

where $\mu^{(L)}$ is defined by

$$\mu^{(L)}(A\times B) = \overline{\mathbb{P}}^o\left(\overline{\tau}_1^{(L)} \in A, \overline{X}_1^{(L)} \in B \,|\, D' = \infty\right) ,$$

for any sets $A \subset \kappa^L\mathbb{N}^*, B \subset \kappa^L\mathbb{Z}^d$. This will allow us to implement a coupling procedure. We recall the following splitting representation: if $\overline{X}, \tilde{X}$ are random variables of laws $\overline{P}, \tilde{P}$ such that $\|\overline{P} - \tilde{P}\|_{\mathrm{var}} \leq a$ then one may find, on an enlarged probability space, independent random variables $Y, \Delta, Z, \tilde{Z}$ where Δ is Bernoulli distributed on $\{0,1\}$ with parameter a, and

$$\overline{X} = (1-\Delta)Y + \Delta Z\ , \quad \tilde{X} = (1-\Delta)Y + \Delta\tilde{Z}\,,$$

(see e.g. [1, Appendix A.1] for the proof); in particular,

$$\overline{X} = (1-\Delta)\tilde{X} + \Delta Z\ , \quad |\Delta Z| \leq |\overline{X}|,\ |\Delta\tilde{Z}| \leq |\tilde{X}|\,.$$

Thus, due to (3.6) –see for similar constructions [2] or [23, Chapter 3]–, we can enlarge our probability space where is defined the sequence $\{(\overline{\tau}_i^{(L)}, \overline{X}_i^{(L)})\}_{i\geq 1}$ in order to support also:

- a sequence $\{(\tilde\tau_i^{(L}, \tilde X_i^{(L)}, \Delta_i^{(L)})\}_{i\geq 1}$ of i.i.d. random vectors (with values in $\kappa^L\mathbb{N}^* \times \kappa^L\mathbb{Z}^d \times \{0,1\}$) such that $\{(\tilde\tau_1^{(L)}, \tilde X_1^{(L)})\}$ is distributed according to $\mu^{(L)}$ while $\Delta_1^{(L)} \in \{0,1\}$ is such that $P(\Delta_1^{(L)} = 1) = \phi'(L)$,
- and another sequence $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\geq 1}$ such that

$$(\overline\tau_i^{(L)}, \overline X_i^{(L)}) = (1-\Delta_i^{(L)})(\tilde\tau_i^{(L)}, \tilde X_i^{(L)}) + \Delta_i^{(L)}(Z_i^{(L)}, Y_i^{(L)}),$$

  and such that, with

$$\mathcal{G}_i = \sigma(\{\tilde\tau_j^{(L}\}_{j\leq i-1}, \{\tilde X_j^{(L)})\}_{j\leq i-1}, \{\Delta_j^{(L)}\}_{j\leq i-1}),$$

  it holds that $\Delta_i^{(L)}$ is independent of $\mathcal{G}_i$ and of $(Z_i^{(L)}, Y_i^{(L)})$.

The joint law of the variables $\{(Z_i^{(L)}, Y_i^{(L)})\}_{i\geq 1}$ is complicated, but it holds that $|Y_i^{(L)}| \leq Z_i^{(L)}$ while, due to (3.3) and since $|\Delta_i^{(L)} Z_i^{(L)}| \leq \tau_i^{(L}$, it holds almost surely that

$$\overline{\mathbb{E}}^o[(\Delta_i^{(L)} Z_i^{(L)})^\alpha \,|\, \mathcal{G}_i] = \phi'(L)\overline{\mathbb{E}}^o[(Z_i^{(L)})^\alpha \,|\, \mathcal{G}_i] \leq M(L). \tag{3.7}$$

We next have the

Lemma 3.8 *Assume the integrability condition (3.3). Then, there exists a sequence* $\eta_L \underset{L\to\infty}{\longrightarrow} 0$ *such that*

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n \overline\tau_i^{(L)} - \beta_L\right| < \eta_L, \quad \overline{\mathbb{P}}^o - a.s., \tag{3.9}$$

and

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^n \overline X_i^{(L)} - \gamma_L\right| < \eta_L, \quad \overline{\mathbb{P}}^o - a.s., \tag{3.10}$$

Proof of Lemma 3.8 We prove (3.9), the proof of (3.10) being similar. Simply write

$$\frac{1}{n}\sum_{i=1}^n \overline\tau_i^{(L)} = \frac{1}{n}\sum_{i=1}^n \tilde\tau_i^{(L)} - \frac{1}{n}\sum_{i=1}^n \Delta_i^{(L)}\tilde\tau_i^{(L)} + \frac{1}{n}\sum_{i=1}^n \Delta_i^{(L)} Z_i^{(L)}.$$

Note first that by independence,

$$\frac{1}{n}\sum_{i=1}^n \tilde\tau_i^{(L)} \underset{n\to\infty}{\longrightarrow} \beta_L, \quad \overline{\mathbb{P}}^o - a.s.,$$

while

$$|\frac{1}{n}\sum_{i=1}^n \Delta_i^{(L)}\tilde\tau_i^{(L)}| \leq \left(\frac{1}{n}\sum_{i=1}^n (\Delta_i^{(L)})^{\alpha'}\right)^{1/\alpha'} \left(\frac{1}{n}\sum_{i=1}^n (\tilde\tau_i^{(L)})^{\alpha}\right)^{1/\alpha} \tag{3.11}$$

and hence

$$\limsup_{n\to\infty} |\frac{1}{n}\sum_{i=1}^{n} \Delta_i^{(L)}\tilde{\tau}_i^{(L)}| \le \phi'(L)^{1/\alpha'} M(L)^{1/\alpha}, \quad \overline{\mathbb{P}}^o - a.s.$$

We next consider the term involving $Z_i^{(L)}$. Set $\overline{Z}_i^{(L)} := \overline{\mathbb{E}}^o(Z_i^{(L)}|\mathcal{G}_i)$, and note that $M_n := \sum_{i=1}^n \Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)})/i$ is a zero mean martingale with respect to the filtration $\mathcal{G}_i$. By the Burkholder-Gundy maximal inequality [24, 14.18], for $\gamma = \alpha \wedge 2$,

$$E|\sup_n M_n|^\gamma \le C_\gamma E\left(\sum_i \frac{(\Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}))^2}{i^2}\right)^{\gamma/2} \le C_\gamma \sum_i E\left(\frac{(\Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}))^\gamma}{i^\gamma}\right) \le C'_\gamma\,,$$

for some constants C_γ, C'_γ. Hence, M_n converges P-a.s. to an integrable random variable, and by the Kronecker lemma [24, 12.7], it holds that $n^{-1}\sum_i \Delta_i^{(L)}(Z_i^{(L)} - \overline{Z}_i^{(L)}) \to 0$, almost surely. On the other hand, there is nothing to prove if $\phi'(L) = 0$ while, if $\phi'(L) > 0$ then

$$|\overline{Z}_i^{(L)}| \le \left(\overline{\mathbb{E}}^o(|Z_i^{(L)}|^\alpha \,|\, \mathcal{G}_i)\right)^{1/\alpha} \le \left(\frac{M(L)}{\phi'(L)}\right)^{1/\alpha}$$

by (3.3), and hence

$$|\frac{1}{n}\sum_{i=1}^{n} \overline{Z}_i^{(L)}\Delta_i^{(L)}| \le \left(\frac{M(L)}{\phi'(L)}\right)^{1/\alpha} \frac{1}{n}\sum_{i=1}^{n}\Delta_i^{(L)} \underset{n\to\infty}{\longrightarrow} M(L)^{1/\alpha}\phi'(L)^{1/\alpha'}\,,\ \mathbb{P}^o - a.s.\,,$$

yielding (3.9) by choosing $\eta_L = 2M(L)^{1/\alpha}\phi'(L)^{1/\alpha'}$. □

Using that $\tilde{\tau}_i^{(L)} \ge \kappa^L$ and that $\beta_L \ge 1$, we conclude from Lemma 3.8 that for all L large enough,

$$\limsup_{n\to\infty}\left|\frac{\frac{1}{n}\sum_{i=1}^n \overline{X}_i^{(L)}}{\frac{1}{n}\sum_{i=1}^n \overline{\tau}_i^{(L)}} - \frac{\gamma_L}{\beta_L}\right| \le 3\eta_L\,, \quad \overline{\mathbb{P}}^o - a.s.,$$

from which one deduces by standard arguments that

$$\limsup_{n\to\infty}\left|\frac{X_n}{n} - \frac{\gamma_L}{\beta_L}\right| \le 4\eta_L\,, \quad \overline{\mathbb{P}}^o - a.s..$$

Thus, we conclude both the existence of the limit $v := \lim_{L\to\infty}\gamma_L/\beta_L$ and the $\overline{\mathbb{P}}^o$ convergence of X_n/n to it. Summarizing, we have proved the following

Theorem 3.12 *Assume the conclusion of Lemma 2.13 and the integrability condition (3.3). Then, there exists a deterministic vector v with $v\cdot\ell > 0$ such that*

$$\lim_{n\to\infty}\frac{X_n}{n} = v\,, \quad \mathbb{P}^o - a.s..$$

Proof of Theorem 3.12 The only statement left to be shown is $v \cdot \ell > 0$. Actually we show that for all $L \in |\ell|_1 \mathbb{N}^*$, we have $\gamma_L = v\beta_L$, from which the desired claim follows, since $\gamma_L \cdot \ell \geq \kappa^L$ and $\beta_L < \infty$. Let us fix L. We already know that $X_n/n \to v$, $\mathbb{P}^o$- a.s., and since $\tau_n^{(L)} \to \infty$, we have

$$\frac{X_{\tau_n^{(L)}}}{\tau_n^{(L)}} - v \;=\; \left(\frac{X_{\tau_n^{(L)}}}{n} - v \frac{\tau_n^{(L)}}{n} \right) \frac{n}{\tau_n^{(L)}} \underset{n\to\infty}{\longrightarrow} 0\,, \quad \mathbb{P}^o - a.s.$$

By (3.3) $\overline{\mathbb{E}}^o \tau_n^{(L)}/n \leq M(L)\kappa^{-L}$, so that $\tau_n^{(L)}/n$ is bounded in probability, yielding

$$\lim_{n\to\infty} \left(\frac{X_{\tau_n^{(L)}}}{n} - v \frac{\tau_n^{(L)}}{n} \right) = 0\,, \quad \mathbb{P}^o - a.s. \tag{3.13}$$

Again from (3.3) and Jensen's inequality, we have

$$\overline{\mathbb{E}}^o \left(\frac{1}{n} [\tau_{n+1}^{(L)} - \tau_1^{(L)}] \right)^\alpha \leq \overline{\mathbb{E}}^o \frac{1}{n} \sum_{i=1}^n \left(\kappa^{-L} \bar{\tau}_{i+1}^{(L)} \right)^\alpha \leq M(L)\kappa^{-\alpha L}\ .$$

So the sequence of random variables $\{[\tau_n^{(L)} - \tau_1^{(L)}]/n\}_{n\geq 1}$ is uniformly integrable, and the similar conclusion holds for $\{[X_{\tau_n^{(L)}} - X_{\tau_1^{(L)}}]/n\}_{n\geq 1}$. Therefore, taking the expectation in (3.13), we obtain

$$\kappa^{-L}(\gamma_L - v\beta_L) = 0$$

for arbitrary L. □

Proof of Corollary 1.9. Throughout, we write $\overline{P}_\omega^z$ for $E_Q \overline{P}_{\omega,\varepsilon}^z$, and assume w.l.o.g. that κ is taken small enough such that $\delta(\ell) > 2\kappa$. (Note that $\overline{P}_\omega^z$, in contrast to P_ω^z, allows considering the ε random sequence.) Recall that from Lemma 2.4,

$$q = \inf_\omega \overline{P}_\omega^o(D' = \infty) = \inf_\omega P_\omega^o(D' = \infty) \in (0,1]\,.$$

For $a > 0$, set

$$\phi_0^L(a) = \sup_\omega \overline{E}_\omega^o(\exp a\kappa^L \overline{S}_1)\,, \quad \phi_1^L(a) = \sup_\omega \overline{E}_\omega^o(\exp a\kappa^L \overline{S}_1 \,|\, D' < \infty)\,.$$

We derive below estimates on $\phi_0^L(a), \phi_1^L(a)$ and in particular show that under a non-nestling assumption, these functions are bounded (uniformly in L) for a small enough, and hence can be made arbitrarily close (again, uniformly in L) to 1 by reducing $a > 0$. Assuming that, we get from the

Markov property and independence of the ε sequence,

$$\begin{aligned}
\overline{E}^o_\omega(\exp\{a\overline{\tau}^{(L)}_1)\} &= \sum_{k=1}^\infty \overline{E}^o_\omega(\exp\{a\overline{\tau}^{(L)}_1\}\,\mathbf{1}_{K=k}) \\
&= \sum_{z\in\mathbb{Z}^d} \overline{E}^o_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z})\cdot\overline{P}^z_\omega(D'=\infty) \\
+\sum_{z,z_1\in\mathbb{Z}^d} & \overline{E}^o_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z})\cdot\overline{P}^z_\omega(D'<\infty)\overline{E}^z_\omega(\exp\{a\kappa^L\overline{S}_1\}\mathbf{1}_{X_{\overline{S}_1}=z_1}|D'<\infty)\cdot\overline{P}^{z_1}_\omega(D'=\infty)+\dots \\
&\le \sup_\omega \overline{P}^o_\omega(D'=\infty)\left[\phi^L_0(a)+\phi^L_0(a)\phi^L_1(a)(1-\inf_\omega\overline{P}^o_\omega(D'=\infty))+\dots\right] \\
&\le \phi^L_0(a)\left[\sum_{k=0}^\infty \phi^L_1(a)^k(1-q)^k\right] \\
&= \frac{\phi^L_0(a)}{(1-(1-q)\phi^L_1(a))} := g(a,L) \qquad (3.14)
\end{aligned}$$

where $\sup_L g(a,L)<\infty$ for small enough, positive a. Thus,

$$\mathbb{E}^o((\overline{\tau}^{(L)}_1)^\alpha|D'=\infty,\mathcal{F}^L_0)\le\frac{\sup_\omega\overline{E}^o_\omega[(\overline{\tau}^{(L)}_1)^\alpha]}{\inf_\omega\overline{P}^o_\omega(D'=\infty)}\le\frac{\text{Const.}}{qa^\alpha}\sup_{\omega,L}\overline{E}^o_\omega e^{a\overline{\tau}^{(L)}_1}\ ,$$

which, together with (3.14) and choosing $a>0$ small enough, yields (3.3) with any $\alpha>1$ and some M not depending on L. Hence, Theorem 3.12 applies for any rate $\phi\to 0$.

We thus turn to the proof of the claimed (uniform in L) finiteness of $\phi^L_0(a)$ and $\phi^L_1(a)$ for a small enough. Since $\phi^L_1(a)\le\frac{\phi^L_0(a)}{1-q}$ it clearly suffices to consider $\phi^L_0(a)$. Let us denote here by $T_m (m=1,2\dots)$ the hitting time of the half-space $\{x\cdot\ell\ge mL|\ell|^2/|\ell|_1\}$, limited by the hyperplanes through points $m(L/|\ell|_1)\ell$ and orthogonal to ℓ. Time T_m is called *L-successful* if

$$(\varepsilon_{T_m+1},\varepsilon_{T_m+2},\cdots\varepsilon_{T_m+L})=\bar{\varepsilon}^{(L)}\ .$$

We denote by $I=\inf\{m\ge1; T_m \text{ is L}-\text{successful}\}$, and we note that, by definition, $\overline{S}_1\le T_I+L$, and that I is geometrically distributed on $\mathbb{N}^*$ with failure probability κ^L. Let

$$\psi^L_0(a)=\sup_\omega\overline{E}^o_\omega(\exp a\kappa^L(T_1+L))\,,\quad \psi^L_1(a)=\sup_\omega\overline{E}^o_\omega(\{\exp a\kappa^L T_1\}\,\mathbf{1}_{\{(\varepsilon_1,\cdots\varepsilon_L)\neq\bar{\varepsilon}^{(L)}\}})\,.$$

Similar to (3.14),

$$\begin{aligned}
\overline{E}^o_\omega(\exp\{a\kappa^L\overline{S}_1\} &\le \sum_{m=1}^\infty \overline{E}^o_\omega(\exp\{a\kappa^L(T_m+L)\}\,\mathbf{1}_{I=m}) \\
&= \overline{E}^o_\omega(\exp\{a\kappa^L(T_1+L)\})\kappa^L+ \\
&\quad +\sum_{z\in\mathbb{Z}^d}\overline{E}^o_\omega(\exp\{a\kappa^L(T_1+L)\}\mathbf{1}_{X_{T_1}=z})\overline{E}^o_\omega(\exp\{a\kappa^L(T_2-T_1)\}\mathbf{1}_{I>1})\kappa^L+\dots \\
&\le \kappa^L\psi^L_0(a)\left[\sum_{k=0}^\infty\psi^L_1(a)^k\right]=\frac{\kappa^L\psi^L_0(a)}{(1-\psi^L_1(a))_+}\,,
\end{aligned}$$

with $(r)_+ = \max\{r, 0\}$. But, if $a \le \lambda_0 \delta(\ell)\kappa^{-|\ell|_1}$,

$$\begin{aligned} \psi_1^L(a) &\le (1-\kappa^L) \sup_{\omega,\mathcal{E}} \overline{E}^o_{\omega,\mathcal{E}}(\{\exp a\kappa^L T_1\} \\ &\le (1-\kappa^L)\exp\{3a\kappa^L/\delta(\ell)\}\,, \end{aligned}$$

from (2.5). Hence, we can choose $a > 0$ small enough so that $\psi_1^L(a) < \infty$ for all L and such that

$$\sup_L \frac{\kappa^L}{(1-\psi_1^L(a))_+} < \infty\,, \quad \sup_L \psi_0^L(a) < \infty.$$

This implies that $\sup_L \phi_0^L(a) < \infty$ for $a > 0$ small. □

4 Mixing

Here are the main examples of distributions P of the environment field which are ϕ-mixing on cones.

Definition 4.1 *1. A random field P is ϕ-mixing if there exists a function $\phi(r) \underset{r\to\infty}{\to} 0$ such that any two r-separated events A, B with $P(A) > 0$,*

$$\left| \frac{P(A\cap B)}{P(A)} - P(B) \right| \le \phi(r)\,.$$

*2. Let $k \ge 1$, and let $\partial\Lambda^k = \{z \in \lambda^c; \mathrm{dist(z,\Lambda)} \le \mathrm{k}\}$ be the k-boundary of $\Lambda \subset \mathbb{Z}^d$. (*dist *and* $|.|$ *both denote the Euclidean distance). A random field P is* k-**Markov** *if there exists a family π of transition kernels — called specification — $\pi_\Lambda = \pi_\Lambda(\prod_{y\in\Lambda} d\omega_y |\, \mathcal{F}_{\partial\Lambda})$ for finite $\Lambda \subset \mathbb{Z}^d$ such that*

$$P\big((\omega_x)_{x\in\Lambda} = \cdot \mid \mathcal{F}_{\Lambda^c}\big) = \pi_\Lambda\big(\ \cdot\ \mid \mathcal{F}_{\partial\Lambda}\big), \quad P--a.s. \tag{4.2}$$

In addition, a k-Markov field P is called **weak-mixing** *if there exist constants $\gamma > 0$, $C < \infty$ such that for all finite subsets $V \subset \Lambda \subset \mathbb{Z}^d$,*

$$\sup\left\{\|\pi_\Lambda(\cdot\mid\omega) - \pi_\Lambda(\cdot\mid\omega')\|_V;\quad \omega,\omega' \in \Sigma^{\Lambda^c}\right\} \le C \sum_{y\in V,\, z\in\partial\Lambda^k} \exp(-\gamma|z-y|)\,, \tag{4.3}$$

with $\|.\|_V = \|.\|_{\mathrm{var},V}$ the variational norm on V, $\|\mu-\nu\|_V = \sup\{\mu(A)-\nu(A); A \subset \sigma((\omega_x)_{x\in V})\}$.

These notions of mixing are different and both of practical interest. Refer to [4] for the first one. The second one describes environments produced by a Gibbsian particle system at equilibrium in the uniqueness regime [5, 15].

Proposition 4.4 *Assume P is stationary and ergodic. If P is ϕ-mixing, then P is ϕ-mixing on cones, i.e. Assumption $(\mathcal{A}1)$ is satisfied. When P is weak-mixing k-Markov of constant γ, then Assumption $(\mathcal{A}1)$ is satisfied with the function $\phi(r) = \mathrm{Const.}(\zeta)e^{-\gamma' r}$, and $\gamma' = \gamma/\sqrt{2}$.*

Proof:

1. In the ϕ-mixing case, the statement directly follows from the definition. We can even take $\zeta = 0$, i.e. we can replace cones by hyperplanes as in [22], all through the paper.

2. We assume now that P is a weak-mixing k-Markov field. Fix some $\zeta > 0$ and $\ell \in \mathbb{R}^d \setminus \{0\}$. For $m, M, N > 0$, define the truncated cone V

$$V_m = V = C(r\ell, \ell, \zeta) \bigcap \{y \in \mathbb{R}^d ; (y - r\ell) \cdot \ell \leq m|\ell|^2\} ,$$

and the cylinder Λ

$$\Lambda = \{y \in \mathbb{R}^d ; 0 \leq y \cdot \ell \leq (r+m+M)|\ell|^2, |y - (y \cdot \ell)\ell/|\ell|^2| \leq N|\ell| + m|\ell| \tan(\cos^{-1}(\zeta))\} ,$$

depicted in Figure 1.

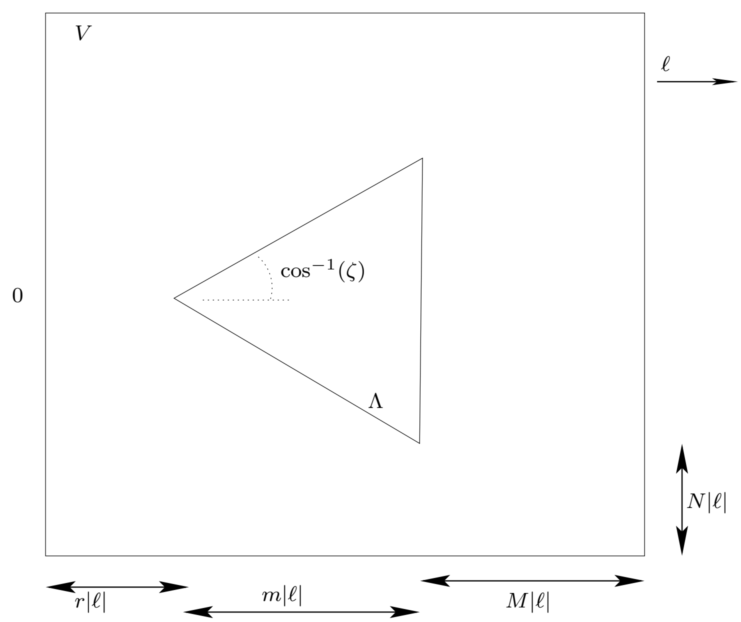

Figure 1: Cone definition

Then, $V \subset \Lambda$, and we split the sum over z in (4.3) into three terms, according to $\partial\Lambda^k = (\partial_-\Lambda) \bigcup (\partial_+\Lambda) \bigcup (\partial_O\Lambda)$ with

$$\partial_-\Lambda = \partial\Lambda^k \bigcap \{z \cdot \ell \leq 0\} , \ \partial_+\Lambda = \partial\Lambda^k \bigcap \{z \cdot \ell \geq (r + m + M)|\ell|^2\}$$

$$\partial_O\Lambda = \partial\Lambda^k \bigcap \{|z - (z \cdot \ell)\ell/|\ell|^2| \geq N|\ell| + m|\ell| \tan(\cos^{-1}(\zeta)\} .$$

Since $\zeta > 0$, the number of points in the cone $C(r\ell, \ell, \zeta)$ at distance l from the hyperplane $z \cdot \ell = 0$ grows linearly in l. Using $\sqrt{a^2 + b^2} \geq (a + b)/\sqrt{2}$ and the previous remark, we find that

$$\begin{aligned} \sum_{y \in V, z \in \partial_-\Lambda} \exp(-\gamma|z - y|) &\leq \sum_{y \in V} \text{Const.} \exp(-\gamma' y \cdot \ell) \\ &\leq \text{Const.} \exp(-\gamma' r|\ell|) \end{aligned}$$

with Const. some constant (depending on k, ζ) which may change from line to line. Similarly,

$$\sum_{y \in V, z \in \partial_O \Lambda} \exp(-\gamma|z-y|) \le \text{Const.}(N+m)^{d-2} \exp(-\gamma' N|\ell|) ,$$

and

$$\sum_{y \in V, z \in \partial_+ \Lambda} \exp(-\gamma|z-y|) \le \text{Const.} m^{d-1} \exp(-\gamma' M|\ell|) .$$

Letting now $M \to \infty$ and then $N \to \infty$ in (4.3), we get for all $A \in \sigma((\omega)_x, x \cdot \ell \le 0)$, and all $B \in \sigma((\omega)_x, x \in V_m)$, from (4.3)

$$|P(B|A) - P(B)| \le \text{Const.} \exp(-\gamma' r|\ell|) ,$$

with arbitrary m. But for any $B \in \sigma((\omega)_x, x \in C(r\ell, \ell, \zeta))$, there exists a sequence $B_m \in \sigma((\omega)_x, x \in V_m)$ with $\lim_m P(B \Delta B_m) = 0$. Therefore such a B satisfies also the previous estimate, and P satisfies $(\mathcal{A}1)$ with exponentially vanishing ϕ. □

5 A nestling example

For simplicity, we work here with $\ell = e_1$. The example below can be modified to work for any ℓ, at the expense of more cumbersome notations. Note that in this case, $(\mathcal{A}2)$ can be rephrased as the directional ellipticity condition:

$(\mathcal{A}2')$: There exists a $\kappa > 0$ such that $P(\omega(0, e_1) > \kappa) = 1$.

Our goal is to provide a family of *examples* of environments which are nestling and to which the results of this paper apply. The examples can be considered as a perturbation of the environment with $\omega(x, x+e_1) = 1$. Alternatively, they may also be considered as a perturbation of i.i.d. environments satisfying Kalikow's condition, perturbed by a slight dependence. (Indeed, it will appear from the proof that $\delta_0 \searrow 0$ as $\gamma' \to +\infty$ in the statement below.)

We claim the following:

Theorem 5.1 *Assume P satisfies $(\mathcal{A}1)$ with $\phi(L) \le \text{Const.}(\zeta) e^{-\gamma' L}$ and $(\mathcal{A}2')$. Then there exists a $\delta_0 = \delta_0(\kappa, \gamma', d) < 1$ such that*

if P satisfies $(\mathcal{A}3)$ with $\delta(e_1) > \delta_0$, then P satisfies $(\mathcal{A}5)$.

A class of explicit examples satisfying the conditions of Theorem 5.1 is provided at the end of this section, following the:

Proof of Theorem 5.1: It is useful to consider the marked point process of fresh times and fresh points. Formally, fresh times are those times when the random walk achieves a new record value in the ℓ direction.

Definition 5.2 *A point $x \in \mathbb{Z}^d$ is called a fresh point for the RWRE (X_n), and a time s is called a fresh time for (X_n), if*

$$\{X_n \cdot \ell < x \cdot \ell, \quad n < s\} \cap \{X_s = x\}\,.$$

We label these random couples (s, x) according to increasing times, $0 = s_0 < s_1 < \ldots, x_n = X_{s_n} (n \geq 0)$. For transient walks in the direction ℓ, there are infinitely many fresh times $\{s_i\}_{i \geq 0}$ and fresh points $\{x_i\}_{i \geq 0}$, and, in the present case $\ell = e_1$, it holds $x_{i+1} \cdot \ell = x_i \cdot \ell + 1$.

Like in the proof of Corollary 1.9, call a fresh time s *L-successful* if $\varepsilon_{s+1} = \varepsilon_{s+2} = \cdots = \varepsilon_{s+L} = e_1$. Note that all $\overline{S}_k$ are L-successful fresh times, and that an L-successful fresh time s leads to an L-regeneration time (more accurately, an "approximate L-regeneration time") $s + L$ if $\theta_{s+L} D' = \infty$.

Define $F \geq 0$ by $L + s_F = \tau_1^{(L)}$, so that it holds that

$$X_{\tau_1^{(L)}} \cdot \ell = F + L\,.$$

As a general feature, fresh points have much nicer tail properties than fresh times. The following summarizes some properties of fresh points and regeneration positions, and does not require any additional assumptions. It is slightly stronger than what we need in the sequel.

Lemma 5.3 *Assume $(\mathcal{A}1, 2, 3)$. Then there exist deterministic constants $\zeta_0 > 0$ and $\lambda_2 = \lambda_2(\delta(\ell), \kappa, d)$ and a function $Q(\lambda)$ (depending on $\delta(\ell), \kappa$) such that for any $\zeta < \zeta_0$, $\lambda < \lambda_2$ and all $L > L_0(\lambda)$,*

$$\overline{\mathbb{E}}^o\left(e^{\lambda \kappa^L X_{\tau_1^{(L)}} \cdot \ell} | \mathcal{F}_0^0\right) = e^{\lambda L \kappa^L} \overline{\mathbb{E}}^o\left(e^{\lambda \kappa^L F} | \mathcal{F}_0^0\right) < Q(\lambda) < \infty\,,\ P - a.s.,$$

with $\mathcal{F}_0^0$ defined in (3.3). Further, $\lambda_2 \underset{\delta(\ell) \to 1}{\longrightarrow} 1$.

Proof of Lemma 5.3

Set $W := \min\{i : s_i \text{ is L-successful}\}$. By definition, $s_W + L = \overline{S}_1$ and $X_{\overline{S}_1} \cdot \ell = W + L$. We first evaluate the exponential moments of W. To every i, attach a random variable $\chi_i = \mathbf{1}_{\{\overline{\varepsilon}_{s_i+1} = e_1\}}$. Note that the χ_i are i.i.d., Bernoulli distributed with parameter κ and

$$W = \min\{j : \ \chi_j = \chi_{j+1} = \cdots = \chi_{j+L-1} = 1\}\,.$$

Consider the inter-failure times $\{\mu_i\}_{i \geq 1}$, i.e. the sequence in $\mathbb{N}^*$ defined by

$$\{j > 0; \chi_j = 0\} = \{\mu_1, \mu_1 + \mu_2, \ldots\}\,,$$

which is i.i.d., geometrically distributed with failure probability $(1 - \kappa)$, and note that $W = \mu_1 + \ldots + \mu_i$ when $\mu_1 \leq L, \ldots, \mu_i \leq L, \mu_{i+1} > L$.

With the notation $\overline{E}_\omega^o = E_Q \overline{E}_{\omega,\varepsilon}^o$ as in the proof of Corollary 1.9, we have for all ω such that the walk is $\overline{P}^o_\omega$-a.s. transient in the direction ℓ,

$$\begin{aligned}\overline{E}^o_\omega \exp\{\lambda\kappa^L W\} &= \overline{E}^o_\omega \sum_{i\geq 0} \exp\{\lambda\kappa^L(\mu_1+\ldots+\mu_i)\}\, \mathbf{1}_{\mu_1<L,\ldots\mu_i<L,\mu_{i+1}\geq L} \\ &= \sum_{i\geq 0}\kappa^L\left(\overline{E}^o_\omega[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}]\right)^i \\ &= \frac{\kappa^L}{\left(1-\overline{E}^o_\omega[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}]\right)_+}\ ,\end{aligned}$$

though

$$\overline{E}^o_\omega[\mathbf{1}_{\mu_1<L}\exp\{\lambda\kappa^L\mu_1\}] = (1-\kappa)e^{\lambda\kappa^L}\frac{1-(\kappa e^{\lambda\kappa^L})^L}{1-(\kappa e^{\lambda\kappa^L})} = 1+(\lambda-1)\kappa^L+o(\kappa^L)\ .$$

Hence, for all $\lambda<1$, there exists a finite $L_1(\lambda)$ with

$$\sup_{L\geq L_1(\lambda)} \text{ess sup}_{\omega\in\text{supp}(P)}\overline{E}^o_\omega\exp\{\lambda\kappa^L W\}<\infty\ . \tag{5.4}$$

Note that the estimate in (5.4) is *quenched*, i.e. for P-almost all environments.

Set

$$M_0 = \max\Big\{X_n\cdot\ell - X_0\cdot\ell\,,\ 0\leq n<D'\Big\}\in(0,\infty]\ ,$$

which is a.s. finite on the set $\{D'<\infty\}$ and infinite otherwise. Next, let $\overline{s}$ denote an L-successful fresh time (i.e., $\overline{s}=s_k$ for some k), and consider

$$M = M(\overline{s}) := M_0\circ\theta_{\overline{s}+L} = \max\Big\{X_n\cdot\ell - X_{\overline{s}+L}\cdot\ell\,,\ \overline{s}+L\leq n<\overline{s}+L+\theta_{\overline{s}+L}D'\Big\}\ .$$

Define

$$\overline{\mathcal{F}_{\overline{s}}} = \sigma(\omega_z : z\cdot\ell\leq X_{\overline{s}}\cdot\ell)\vee\sigma(X_t, t\leq\overline{s})\ ,$$

recall the definition from (3.3) $\mathcal{F}^L_0=\sigma(\omega(y,\cdot):\ell\cdot y<-L)$ and the notation $\bar\theta_x$ for space shift. It is useful to note that the annealed law of paths after fresh points has the following property,

$$\overline{\mathbb{P}}^o(M>r|\overline{\mathcal{F}_{\overline{s}}}) = \overline{\mathbb{P}}^o(M_0>r|\mathcal{F}^L_0)\circ\bar\theta_{X_{\overline{s}+L}}\ ,\quad r>0\ , \tag{5.5}$$

which implies also that $\overline{\mathbb{P}}^o(M>r, D'\circ\theta_{\overline{s}+L}<\infty|\overline{\mathcal{F}_{\overline{s}}}) = \overline{\mathbb{P}}^o(M_0>r, D'<\infty|\mathcal{F}^L_0)\circ\bar\theta_{X_{\overline{s}+L}}$, since $\{M_0>r, D'<\infty\}=\{M_0\in(r,\infty)\}$ almost surely. Indeed, fix $n\geq L, \{x^*_i\}_{0\leq i<n}$ a path for walk with $x^*_{i+1}=x^*_i+e_1$ for $n-L\leq i\leq n$, fix some $\omega^*\in S^n$ and some measurable set $A\subset S^{\{y:y\cdot\ell\leq x^*_{n-L}\cdot\ell-L\}}$ such that $\{\omega(x^*_i)\}_{0\leq i\leq n-L}=\omega^*$ for all $\omega\in A$, and $P(A)>0$. Then, by the

Markov property, and for arbitrary $\omega^{**} \in A$,

$$
\begin{aligned}
&\overline{\mathbb{P}}^o(M > r \mid \overline{s} = n - L, \{X_i\}_{0\le i\le n} = \{x_i^*\}_{0\le i\le n}, \omega \in A) \ = \\
&= \frac{E_{P\otimes Q}[\overline{P}^o_{\omega,\mathcal{E}}(M > r, \overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\mathbf{1}_{\omega\in A}]}{\overline{\mathbb{P}}^o(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n}, \omega\in A)} \\
&= \frac{E_{P\otimes Q}[\overline{P}^{x_n^*}_{\omega,\theta_n\mathcal{E}}(M_0 > r)\overline{P}^o_{\omega,\mathcal{E}}(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\mathbf{1}_{\omega\in A}]}{\overline{\mathbb{P}}^o(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n}, \omega\in A)} \\
&= \frac{E_{P\otimes Q}[\overline{P}^{x_n^*}_{\omega,\theta_n\mathcal{E}}(M_0 > r)\overline{P}^o_{\omega^{**},\mathcal{E}}(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\mathbf{1}_{\omega\in A}]}{\overline{\mathbb{P}}^o(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n}, \omega\in A)} \\
&= \frac{E_Q[E_P\left[\overline{P}^{x_n^*}_{\omega,\theta_n\mathcal{E}}(M_0 > r)\mathbf{1}_{\omega\in A}\right] \times \overline{P}^o_{\omega^{**},\mathcal{E}}(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})]}{\overline{\mathbb{P}}^o(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n}, \omega\in A)} \\
&= \frac{E_{P\otimes Q}\left[\overline{P}^{x_n^*}_{\omega,\theta_n\mathcal{E}}(M_0 > r)\mathbf{1}_{\omega\in A}\right] \times E_Q\left[\overline{P}^o_{\omega^{**},\mathcal{E}}(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\right]}{E_{P\otimes Q}\left[\mathbf{1}_{\omega\in A}\right] \times E_Q\left[\overline{P}^o_{\omega^{**},\mathcal{E}}(\overline{s}=n-L, \{X_i\}_{0\le i\le n}=\{x_i^*\}_{0\le i\le n})\right]} \\
&= \frac{E_{P\otimes Q}\overline{P}^{x_n^*}_{\omega,\theta_n\mathcal{E}}(M_0 > r, \omega\in A)}{P(\omega\in A)} \ = \ \overline{\mathbb{P}}^o(M_0 > r \mid \omega \in \overline{\theta}_{x_n^*}A)
\end{aligned}
$$

performing the same computations on the denominator, yielding (5.5).

We next derive tail estimates on M, considering only the case of M_0 in view of (5.5). For $x \in \mathbb{N}^*$, define

$$U_x^{(L)} = \{z \in \mathbb{Z}^d : -L \le z\cdot\ell \le x\}$$

and set $\tau_{x,L} = \min\{n > 0 : \ X_n \notin U_x^{(L)}\}$. Finally, let Π denote the orthogonal projection with respect to $\ell = e_1$, i.e. $\Pi(z) = z - z_1 e_1$, and K_0 a positive constant. Then, for any $\zeta > 0$, using the Markov property and a union bound,

$$
\begin{aligned}
\overline{\mathbb{P}}^o(M_0 \in (x,\infty) \mid \mathcal{F}_0^L) &\le \overline{\mathbb{P}}^o\left(X_{\tau_{x,L}}\cdot\ell \ge x, \left|\Pi(X_{\tau_{x,L}})\right| > K_0 x \mid \mathcal{F}_0^L\right) \\
&\quad + (2K_0x)^{d-1} \max_{z: z\cdot\ell = x, |\Pi(z)|\le K_0 x} \overline{\mathbb{P}}^z\left(X_\cdot \text{ hits } C(0,\ell,\zeta)^c \mid \mathcal{F}_0^L\right) =: \mathrm{I}+\mathrm{II}\,,
\end{aligned}
$$

as indicated in Figure 2.

Note that the term I does not depend on ζ. We argue below that, by Kalikow's condition,

$$\mathrm{I} \le e^{-g(K_0,\delta(\ell))x} \tag{5.6}$$

where $g(K_0,\delta(\ell)) \underset{K_0\to\infty}{\longrightarrow} \infty$ is monotone non decreasing in $\delta(\ell)$ and does not depend on ζ.

Fix now K_0 such that $g(K_0,\delta(\ell)) > 0$ and set ζ_0 such that

$$\{z : z\cdot\ell = x, |\Pi(z)| \le 2K_0 x\} \subset C(0,\ell,\zeta_0)\}\,. \tag{5.7}$$

We will also argue that

$$\mathrm{II} \le e^{-\beta'_{K_0}(\delta(\ell))x} \tag{5.8}$$

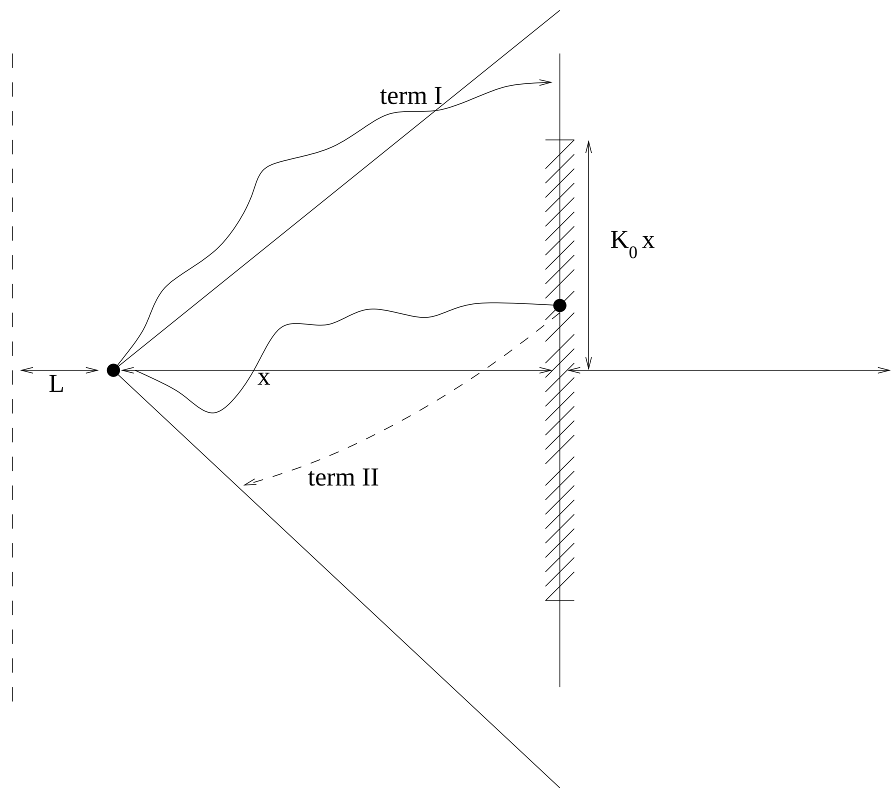

Figure 2: Escape events

where β'_{K_0} comes from Kalikow's condition and $\beta'_{K_0}(\delta) \underset{\delta \to 1}{\longrightarrow} \infty$. Therefore, we obtain for all x large, and all $\zeta < \zeta_0$,

$$\sup_{\omega} \overline{\mathbb{P}}^o(M \in (x, \infty) | \overline{\mathcal{F}}_{\overline{s}}) \leq e^{-\beta(\delta(\ell))x} \tag{5.9}$$

for some function $\beta = \beta(\ell)$ such that $\beta \to \infty$ as $\delta \to 1$.

Indeed, to see (5.6) and thus control I, we use (1.7) for the truncated strip $U_x^{(L)} \bigcap \{|\Pi(z)| \leq \tilde{L}\}$ for large $\tilde{L} < \infty$, and apply the stopping theorem to the supermartingale $\exp\{-3\lambda_0 f(X_n)\}$ as in Lemma 2.4-1) (with the function $f(y) = y \cdot e_1 - \zeta'|y|, \zeta' = \delta(\ell)/2$), and to Kalikow's chain at the exit time from this strip. Letting then $\tilde{L} \to \infty$, one readily gets (5.6) with $g(K_0) = 1.5\lambda_0(\delta(\ell)\sqrt{1 + K_0^2} - 2)_+$. To prove (5.8), we proceed similarly with truncated cones, using (1.7) and, this time, the supermartingale $\exp\{-3\lambda_1 f(X_n)\}$ as in Lemma 2.4-4) (with the function $f(y) = y \cdot e_1 - \zeta_0|y|$ and assuming $\zeta_0 \leq \delta(\ell)/3$ w.l.o.g.), to get

$$\overline{\mathbb{P}}^z(X_. \text{ hits } C(0, \ell, \zeta)^c | \mathcal{F}_0^L) \times \exp\{0\} \leq \exp\{-3\lambda_1 x(1 - \zeta_0\sqrt{K_0^2 + 1})\}$$

which yields (5.8) with $\beta'_{K_0}(\delta) = 3\lambda_1(\delta)(1 - \zeta_0\sqrt{K_0^2 + 1})$, where the factor $1 - \zeta_0\sqrt{K_0^2 + 1} \geq 1 - (\sqrt{K_0^2 + 1}/\sqrt{4K_0^2 + 1}) > 0$ from (5.7).

Next, from Lemma 2.4-4), it follows also that, for any L-successful fresh time $\overline{s}$,

$$\overline{\mathbb{P}}^o(D' \circ \theta_{\overline{s}+L} = \infty | \overline{\mathcal{F}}_{\overline{s}}) \overset{(5.5)}{\geq} q_1 := \inf_{\omega} \overline{\mathbb{P}}^o(D' = \infty | \overline{\mathcal{F}}_0^L) > 0 \tag{5.10}$$

where $q_1 \underset{\delta \to 1}{\longrightarrow} 1$ uniformly in L and $\zeta < \zeta_0$.

By definition, $X_{\overline{S}_1} \cdot e_1 = W + L$, and on $\{\overline{S}_k < \infty\}$, $(X_{\overline{S}_{k+1}} - X_{\overline{S}_k}) \cdot e_1 = M_k + W_{k+1} + L$, with the notations $M_k = M(\overline{S}_k)$, $W_{k+1} = W \circ \theta_{\overline{S}_k + M_k}$ if $M_k < \infty$, and $W_{k+1} = \infty$ on $\{M_k = \infty\} = \{D' \circ \theta_{\overline{S}_k} = \infty\}$. Put also $W_1 = W$, write

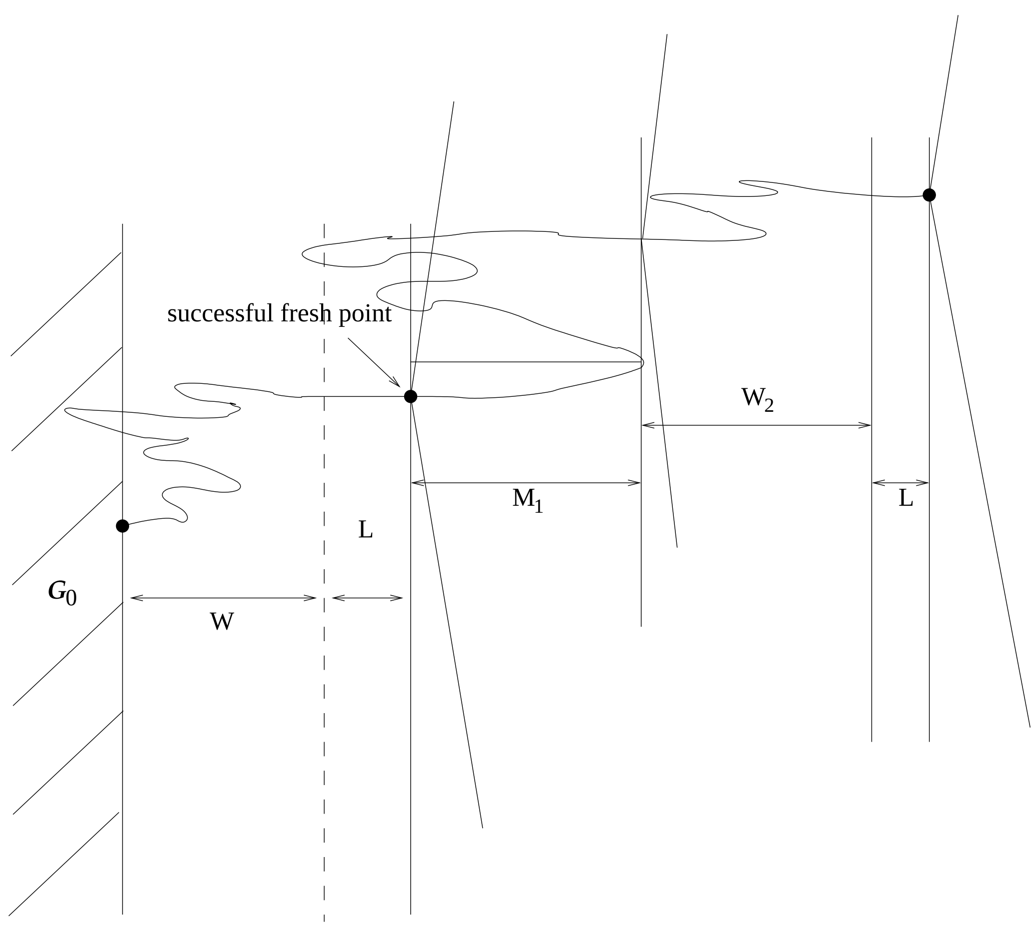

Figure 3: Successful fresh times leading to regeneration.

$$F + L = \sum_{k \geq 0} (M_k \mathbf{1}_{k \neq 0} + W_{k+1} + L) \mathbf{1}_{k < K} \ ,$$

(see Figure 3) and, with $\lambda > 0$,

$$\overline{\mathbb{E}}^o(e^{\lambda\kappa^L(F+L)} \mid \mathcal{F}_0^0) \le \overline{\mathbb{E}}^o(\sum_{k\ge 0} e^{\lambda\kappa^L \sum_{l=0}^{k}(M_l \mathbf{1}_{l\neq 0}+W_{l+1}+L)}\mathbf{1}_{M_1,\dots M_k<\infty} \mid \mathcal{F}_0^0)$$

$$= \sum_{k\ge 0}\overline{\mathbb{E}}^o\left(e^{\lambda\kappa^L \sum_{l=0}^{k-1}(M_l\mathbf{1}_{l\neq 0}+W_{l+1}+L)}\mathbf{1}_{M_1,\dots M_{k-1}<\infty}\times\overline{\mathbb{E}}^o\left[e^{\lambda\kappa^L(M_k\mathbf{1}_{k\neq 0}+W_{k+1}+L)}\mathbf{1}_{M_k<\infty}|\overline{\mathcal{F}}_{\overline{S}_k}\vee\mathcal{F}_0^0\right] \mid \mathcal{F}_0^0\right)$$

$$\le \sum_{k\ge 0}\overline{\mathbb{E}}^o\left(e^{\lambda\kappa^L \sum_{l=0}^{k-1}(M_l\mathbf{1}_{l\neq 0}+W_{l+1}+L)}\mathbf{1}_{M_1,\dots M_{k-1}<\infty}|\mathcal{F}_0^0\right)\times\overline{\mathbb{E}}^o[e^{\lambda\kappa^L(W+L)}]\sup_\omega\overline{\mathbb{E}}^o[e^{\lambda\kappa^L M_0}\mathbf{1}_{D'<\infty}|\mathcal{F}_0^0]$$

$$\le \sum_{k\ge 0}\left(\overline{\mathbb{E}}^o[e^{\lambda\kappa^L(W+L)}]\right)^{k+1}\left(\sup_\omega\overline{\mathbb{E}}^o[e^{\lambda\kappa^L M_0}\mathbf{1}_{D'<\infty}|\mathcal{F}_0^0]\right)^k$$

$$= \frac{e^{\lambda L\kappa^L}\overline{\mathbb{E}}^o[e^{\lambda\kappa^L W})]}{\left(1-e^{\lambda L\kappa^L}\overline{\mathbb{E}}^o[e^{\lambda\kappa^L W}]\sup_\omega\overline{\mathbb{E}}^o[e^{\lambda\kappa^L M_0}\mathbf{1}_{D'<\infty}|\mathcal{F}_0^0]\right)_+} =: Q(\lambda)\ ,$$

where the third line comes from independence and from (5.5), and the fourth one from a recursion. From (5.4), $\sup_{L>L_1(\lambda)}\{e^{\lambda L\kappa^L}\overline{\mathbb{E}}^o[e^{\lambda\kappa^L W}]\}<\infty$ for $\lambda<1$, while by Schwarz's inequality,

$$\sup_\omega\overline{\mathbb{E}}^o[e^{\lambda\kappa^L M_0}\mathbf{1}_{D'<\infty}|\mathcal{F}_0^0] \le \left[(1-q_1)\sup_\omega\overline{\mathbb{E}}^o[e^{2\lambda\kappa^L M_0}\mathbf{1}_{M_0<\infty}|\mathcal{F}_0^0]\right]^{1/2}\ .$$

From (5.9), we can choose L large enough so that the supremum in the right-hand side is finite, and therefore there exists some $\lambda_2>0$ such that $Q(\lambda)<\infty$ for $\lambda<\lambda_2$. This completes the proof of the first claim in the Lemma. To obtain the second claim, we make $1-q_1$ arbitrary small by taking $\delta(\ell)$ close to 1, thus keeping Q finite for λ arbitrarily close to 1.

This completes the proof of Lemma 5.3. □

We now complete the proof of Theorem 5.1, by deriving probability estimates for tails of $\tau_1^{(L)}$. Fix $n>0$, $T=3\kappa^{-L}\ln n$, and a large constant $K_1>0$. Set

$$B=\{z\in\mathbb{Z}^d:\ -L\le z\cdot\ell\le T\}\,,$$
$$\tilde B_{K_1}=\{z\in B:\ |\,\Pi(z)\,|\ \le K_1 T\},\ \partial_+\tilde B_{K_1}=\{z\in\partial\tilde B_{K_1}:\ z\cdot\ell\ge T\}\,.$$

(Recall that Π is the projection on the hyperplane orthogonal to ℓ.) With T_{B^c} [resp. $T_{\tilde B^c_{K_1}}$,] the exit time from B [resp. $\tilde B_{K_1}$], we decompose the set $\{\tau_1^{(L)}>\kappa^{-L}n\}\cap\{D'=\infty\}$ according to $X_\cdot$ exiting $\tilde B_{K_1}$ after time $\kappa^{-L}n$, and then decompose the latter case according to $X_\cdot$ exiting $\tilde B_{K_1}$ from $\partial_+\tilde B_{K_1}$ or not:

$$\begin{aligned}\overline{\mathbb{P}}^o\left(\tau_1^{(L)}>\kappa^{-L}n\ ,\ D'=\infty\mid\mathcal{F}_0^L\right) &\le \overline{\mathbb{P}}^o\left(T_{\tilde B^c_{K_1}}>\kappa^{-L}n\ ,\ D'=\infty\mid\mathcal{F}_0^L\right)+\overline{\mathbb{P}}^o\left(X_{\tau_1^{(L)}}\cdot\ell>T\mid\mathcal{F}_0^L\right)+\\ &\quad+\overline{\mathbb{P}}^o\left(X_{T_{\tilde B^c_{K_1}}}\notin\partial_+\tilde B_{K_1}\ ,\ D'=\infty\mid\mathcal{F}_0^L\right)\\ &\le \overline{\mathbb{P}}^o\left(T_{\tilde B^c_{K_1}}>\kappa^{-L}n\ ,\ D'=\infty\mid\mathcal{F}_0^L\right)+e^{-3\lambda\ln n}+0\ ,\end{aligned}\tag{5.11}$$

where $\lambda<\lambda_2$ from Lemma 5.3, and where the last term is made equal to zero by fixing $K_1>\zeta^{-1}(1-\zeta^2)^{1/2}$, in which case all path contained in $C(0,\ell,\zeta)$ have to exit $\tilde B_{K_1}$ from $\partial_+\tilde B_{K_1}$.

In $\tilde{B}_{K_1}$, consider strips of width $\Delta = \delta_1 \ln n$, with $-\delta_1 \ln \kappa < 1$ and T/Δ integer,

$$B_i = \{z \in \tilde{B}_{K_1} : \ (i-1)\Delta < z \cdot \ell < (i+1)\Delta\} , \quad i = 0, 1, \cdots , T/\Delta$$

consider the truncated hyperplanes (slices) $A_i = \{z \in \tilde{B}_{K_1} : \ z \cdot \ell = [i\Delta]\}$, and define the random variables ($T_{B_i^c}$ = exit time of B_i)

$$Y_i := \sup_{z \in A_i} P_\omega^z(X_{T_{B_i^c}} \cdot \ell < (i-1)\Delta) ,$$

i.e., the smallest quenched probability starting from the middle slice A_i to exit the strip B_i from the left.

By Kalikow's condition, for $z \in A_i$,

$$E\Big(P_\omega^z(X_{T_{B_i^c}} \cdot \ell < (i-1)\Delta) \mid \omega(x, \cdot), x \notin B_i\Big) \le e^{-c\Delta}$$

with $c = c(\delta) \underset{\delta \to 1}{\longrightarrow} \infty$ (using the supermartingale $\exp\{-2\lambda_1 f(X_n)\}$ from Lemma 2.4, see proof of (5.9)). Hence,

$$P\Big(Y_i > e^{-c\Delta/2} \mid \omega(x, \cdot), x \notin B_i\Big) \le (2K_1 T)^{d-1} \, e^{-c\Delta/2}$$

In particular, the set

$$\mathcal{A} := \{\exists i \le T/\Delta : \ Y_i > e^{-c\Delta/2}\} \ \text{ is such that } \ P\Big(\mathcal{A} \mid \mathcal{F}_0^L\Big) \le \frac{T}{\Delta}(2K_1 T)^{d-1} \, e^{-c\Delta/2} .$$

Let now $\mathcal{B}$ denote the event that the walk, sampled at hitting times of neighboring slices, successively visits $A_1, A_2, \ldots A_{T/\Delta}$ without backtracking to the neighboring slice on the right. Then, for $\omega \in \mathcal{A}^c$,

$$P_\omega^0 \left(\mathcal{B}^c\right) \le 1 - \Big(1 - e^{-c\Delta/2}\Big)^{\left(\frac{T}{\Delta}\right)} \le \ \frac{T}{\Delta} e^{-c\Delta/2} ,$$

by convexity ($T/\Delta > 1$). On the other hand, the times spent inside each block are stochastically dominated above by $2(\Delta + 1)G_i$, where G_i are independent random variables, geometrically distributed with parameter $\kappa^{2(\Delta+1)}$; indeed, by ellipticity, the walk starting from any point in B_i has probability larger than $\kappa^{2(\Delta+1)}$ to exit B_i by traveling only with steps to the right, with at most $2(\Delta + 1)$ steps. Now, since

$$\overline{\mathbb{P}}^o\Big(T_{\tilde{B}_{K_1}^c} > \kappa^{-L} n \ , \ D' = \infty | \mathcal{F}_0^L\Big) \le \overline{\mathbb{P}}^o\Big(\mathcal{A} | \mathcal{F}_0^L\Big) + \overline{\mathbb{P}}^o\Big(\mathcal{A}^c \bigcap \mathcal{B}^c | \mathcal{F}_0^L\Big) + E\Big(P_\omega^o(T_{\tilde{B}_{K_1}^c} > \kappa^{-L} n \ , \ \mathcal{B}) \mathbf{1}_{\mathcal{A}^c} | \mathcal{F}_0^L\Big) ,$$

we have

$$\begin{aligned} &\overline{\mathbb{P}}^o\Big(\{\tau_1^{(L)} > \kappa^{-L} n\} \cap \{D' = \infty\} \mid \mathcal{F}_0^L\Big) \\ &\quad \le \left[n^{-3\lambda} + \frac{T}{\Delta}(2K_1 T)^{d-1} \, e^{-c\Delta/2} + \frac{T}{\Delta} e^{-c\Delta/2} + P\left(\sum_{i \le T/\Delta} G_i > \frac{\kappa^{-L} n}{2(\Delta + 1)} \right) \right] \wedge 1 . \end{aligned}$$

But, on the last event, at least one of the G_i's is larger than $\Delta\kappa^{-L}n/[2(\Delta+1)T]$, so

$$P\left(\sum_{i\leq T/\Delta} G_i > \frac{\kappa^{-L}n}{2(\Delta+1)}\right) \leq \frac{T}{\Delta}P\left(G_1 > \frac{n}{12\ln n}\right)$$
$$= \frac{T}{\Delta}\left(1-\kappa^{2(\Delta+1)}\right)^{\frac{n}{12\ln n}} \leq \frac{3\kappa^{-L}}{\delta_1}\exp\{-\frac{n}{12n^{-2\delta_1\ln\kappa}\ln n}\},$$

and finally

$$\overline{\mathbb{P}}^o\left(\{\tau_1^{(L)} > \kappa^{-L}n\}\cap\{D'=\infty\}\mid \mathcal{F}_0^L\right)$$
$$\leq \left[n^{-3\lambda} + \frac{3\kappa^{-L}}{\delta_1}\left(n^{-\delta_1 c/2}[(6K_1\kappa^{-L}\ln n)^{d-1}+1] + e^{-\frac{n}{12n^{-2\delta_1\ln\kappa}\ln n}}\right)\right]\wedge 1$$
$$\leq \begin{cases} 1, & \text{for } K_2\kappa^{-dL}(\ln n)^{d-1} > n^{+\delta_1 c/4}, \\ 2n^{-[(\delta_1 c/4)\vee(2\lambda)]}, & \text{else}, \end{cases}$$

for some constant K_2, all n large enough and all $\lambda < \lambda_2$. Hence, with $\alpha = 2$,

$$M(L) = \mathbb{E}^o\left((\tau_1^{(L)}\kappa^L)^2 1_{\{D=\infty\}}|\mathcal{F}_0^L\right) \leq K_3\kappa^{-8dL/c\delta_1},$$

for some constant K_3 independent of L, as soon as $c = c(\delta)$ is large enough, which happens if κ is kept fixed and $\delta \to 1$. Thus, as soon as $\phi'(\cdot)$ decreases exponentially one may find a $\delta(\ell)$ close enough to 1 such that $M(L)\phi'(L) \to_{L\to\infty} 0$. For such $\delta(\ell)$, we thus conclude that $(\mathcal{A}5)$ is satisfied with $\alpha = 2$. □

We are now ready to describe the class of examples satisfying the assumptions of Theorem 5.1. Let $(\sigma(x), x\in\mathbb{Z}^d)$ be a d-dimensional nearest neighbor Ising model with $\beta \geq 0, h > 0$ and π the corresponding probability measure, i.e.

$$\pi(\sigma(x) = \pm 1|\sigma(y), y\neq x) = \exp\pm\{\beta\mathtt{S}+h\}/(\exp\{\beta\mathtt{S}+h\} + \exp-\{\beta\mathtt{S}+h\}),$$

with $\mathtt{S} = \sum_{e:|e|=1}\sigma(x+e)$. Fix now two probability vectors $\omega^\pm = (\omega^\pm(e); e\in\mathbb{Z}^d, |e|=1)$, and assume that

$$\omega^\pm(e) > 0\,,\quad |e| = 1\,, \tag{5.12}$$

$$d^+ := \left(\sum_{|e|=1}\omega^+(e)e\right)\cdot e_1 > 0\,,\quad -d^- := \left(\sum_{|e|=1}\omega^-(e)e\right)\cdot e_1 < 0\,. \tag{5.13}$$

Consider the random environment given by

$$\omega(x, x+e) = \omega^\pm(e) \quad \text{according to } \sigma(x) = \pm 1\,. \tag{5.14}$$

Note that the RWRE is nestling in the case where the local drifts points in opposite directions, e.g. $\sum_{|e|=1}\omega^+(e)e \in (0,1)\cdot e_1$ and $\sum_{|e|=1}\omega^-(e)e \in (-1,0)\cdot e_1$.

Theorem 5.15 *For all choice of $\omega^\pm$ with (5.12) and (5.13), there exist a finite number h_0 and a positive function $\beta_0(h)$ with $\lim_{h\to+\infty}\beta_0(h)=\infty$ such that for $h>h_0$ and $\beta<\beta_0(h)$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}X_n=v\,,\quad \overline{\mathbb{P}}^o-a.s..$$

for some deterministic vector v ($v\cdot e_1>0$).

Note that, since the above function β_0 is unbounded, the result applies to arbitrary low temperature - but with large external field.

Proof of Theorem 5.15 We start by giving a sufficient condition for Kalikow's condition ($\mathcal{A}3$).

Lemma 5.16 *If for a deterministic $\delta>0$ we have P-a.s.*

$$\inf_{f:\{\pm e_i\}_1^d\to(0,1]}\left[E\left(\frac{1}{\sum_{|e|=1}f(e)\omega(x,x+e)}|\mathcal{F}_{\{x\}^c}\right)^{-1}E\left(\frac{d(x,\omega)\cdot\ell}{\sum_{|e|=1}f(e)\omega(x,x+e)}|\mathcal{F}_{\{x\}^c}\right)\right]\geq\delta\,, \tag{5.17}$$

then ($\mathcal{A}3$) holds with ℓ and $\delta(\ell)=\delta$.

The proof is similar to [10], p.759-760, replacing μ therein with our Gibbs measure $\pi(.|\sigma(y),y\neq x)$. We will omit it here, and we will use Lemma 5.16 with $\ell=e_1$.

Lemma 5.18 *For condition (5.17) to hold with $\delta>0$ and $\ell=e_1$ in the example (5.14), it is sufficient that*

$$\delta<d^+\,,\quad 2h-4\beta d\geq\ln\frac{(d^-+\delta)}{d^+-\delta}+\ln\max_{|e|=1}\frac{\omega^+(e)}{\omega^-(e)}\,.$$

Proof of Lemma 5.18 To simplify notations, we set $\tilde\pi(\pm|\mathtt{S})=\pi(\sigma(x)=\pm1|\sigma(y),y\neq x)$ on the set $\{\sum_{|e|=1}\sigma(x+e)=s\}$, and $\mathcal{S}=\{-2d,-2d+2,\dots,2d\}$ the set of possible values for s. For $\delta>0$,

$$\begin{aligned}(5.17)\qquad &\Longleftrightarrow\ \forall f,s,\quad \frac{\tilde\pi(+|s)\frac{d^+}{\sum_{|e|=1}f(e)\omega^+(e)}+\tilde\pi(-|s)\frac{-d^-}{\sum_{|e|=1}f(e)\omega^-(e)}}{\tilde\pi(+|s)\frac{1}{\sum_{|e|=1}f(e)\omega^+(e)}+\tilde\pi(-|s)\frac{1}{\sum_{|e|=1}f(e)\omega^-(e)}}\geq\delta\\ &\Longleftrightarrow\ \forall f,s,\quad \frac{\tilde\pi(+|s)}{\tilde\pi(-|s)}\geq\frac{\sum_{|e|=1}f(e)\omega^+(e)}{\sum_{|e|=1}f(e)\omega^-(e)}\times\frac{d^-+\delta}{d^+-\delta}\,,\end{aligned} \tag{5.19}$$

since $\delta<d^+$. Note that for the Ising measure π,

$$\inf_{s\in\mathcal{S}}\frac{\tilde\pi(+|s)}{\tilde\pi(-|s)}=\inf_{s\in\mathcal{S}}\exp(2h+2\beta s)\geq\exp(2h-4\beta d)\,,$$

while on the other hand,

$$\sup_{f:\{\pm e_i\}_1^d \to (0,1]} \frac{\sum_{|e|=1} f(e)\omega^+(e)}{\sum_{|e|=1} f(e)\omega^-(e)} = \max_{|e|=1} \frac{\omega^+(e)}{\omega^-(e)} .$$

Lemma 5.18 is proved. □

Let $c = c(\beta, h)$ be Dobrushin's contraction coefficient (for example, definition (2.7) in [6]). If $c = c(h, \beta) < 1$ then π is weak-mixing, with a constant γ depending only on c - as can be checked from (2.8) in [6]. (See also [4, Theorem 3, Section 2.2.1.3].) According to Proposition 4.4, P satisfies ($\mathcal{A}1$) with $\phi(L) \le e^{-\gamma' L}$, where $\gamma' = \gamma/\sqrt{2}$ depends only on c. From Theorem 5.1 and Lemma 5.18, Condition ($\mathcal{A}5$) is implied by

$$c(h,\beta) < 1 \ , \ \ \delta < d^+ \ , \ \ 2h - 4\beta d \ge \ln \frac{(d^- + \delta)}{d^+ - \delta} + \ln \max_{|e|=1} \frac{\omega^+(e)}{\omega^-(e)} \ , \ \ \delta > \delta_0(\omega^-(e_1), \gamma'(c(h,\beta)), d) \ .$$

Note that $c(h, \beta) \le c_0 < 1$ contains a region $0 \le \beta < \beta_0'(h)$ with $\beta_0'(h) \to \infty$ as $h \to \infty$. The proof of Theorem 5.15 is complete. □

6 Concluding remarks

1. While working on the first draft of this work, we learnt of [12], where the authors consider law of large numbers for L-dependent non-nestling environments. The approach of [12] is quite different from ours, as it relies on constructing an invariant measure, absolutely continuous with respect to the law P, which makes the regeneration sequence $\{\bar{\tau}_i^{(0)}, \bar{X}_i^{(0)}\}$ (with $L = 0$ and $\zeta = 0$ as introduced in [22]) a stationary sequence. These results are covered by our approach.

   In another preprint [13], the same authors obtain the law of large numbers for L-dependent environments with Kalikow's condition, under the crucial assumption that with positive probability the walker jumps at a distance larger than L. An appropriate modification of the regeneration times introduced in [22] leads in that case to a renewal structure as in the independent environment.

   We also mention that [17] has announced results related to ours, obtained by the method of the environment viewed from the point of view of the article.

2. It is reasonable to expect that under ($\mathcal{A}1, 2, 3$), the integrability condition ($\mathcal{A}5$), with $\alpha > 2$, implies that the CLT for $X_n/\sqrt{n}$ holds true. However, the proof of such a statement does present some challenges. We hope to return to this question in future work.

3. It is worthwhile to note that in the case of i.i.d. environments, under Kalikow's condition it holds that ($\mathcal{A}5$) with any $\alpha > 1$ is satisfied as soon as $d \ge 2$, by the results of [19]. It is not clear whether, under reasonable mixing conditions which are not of the L-dependent type, the law of large numbers holds for the whole range of Kalikow's condition, or more generally what should the analogue of Sznitman's T'-condition (see [20]) be in the mixing setup.

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