ADVANCES ON THE LATE ARRIVALS PROBLEM

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We study a discrete time queueing system where deterministic arrivals have i.i.d. exponential delays ξ_i. The standard deviation σ of the delay is finite, but its value is much larger than the deterministic unit service time. We describe the model as a bivariate Markov chain and focus on the joint equilibrium distribution. We also prove that the latter decays super-exponentially fast in the quarter plane. Finally, we discuss the numerical computation of the stationary distribution, showing the effectiveness of a simple approximation scheme in a wide region of the parameters. The model, motivated by air and railway traffic, was proposed many decades ago by Kendall [45] with the name of "late arrivals problem", but no solution has been found so far.

1 INTRODUCTION

In this paper we consider a single-server queue with deterministic service time, which is assumed of unitary length for the sake of simplicity. The ith customer arrives to the system at time

$$t_i = i + \xi_i\,, \qquad i \in \mathbb{N}\,, \tag{1.1}$$

where $\{\xi_i\}_i$ are i.i.d. exponential random variables with parameter β.
In the limit $\beta \to 0$ the point process (1.1) weakly converges to a Poisson process of parameter 1, whereas for fixed β the arrivals are negatively autocorrelated, see [21, 37] and references therein.

Remark 1. Although the results in [37] are stated under the hypothesis that the probability density function of the delays $\{\xi_i\}_i$ has compact support, this assumption does not play any role in establishing the convergence to a Poisson process. As a consequence, the very same result applies here too.

We study the system described above for fixed β and we assume throughout the paper an independent *thinning* to the arrival process; in other words, each customer can be deleted independently with probability $1-\rho$ before joining the queue. Besides being a mathematical expedient that ensures the existence of a stationary state[1], the thinning is mainly a way to model empty intervals in a constant stream of customers. Similarly to the point process (1.1), also the thinned arrival process weakly converges to a Poisson process, but with

1 See Lemma 2 below.

parameter ρ, see again [37]. After Kendall (see below for his exact words) we name Exponentially Delayed Arrivals (EDA) the thinned version of the arrival process (1.1).

Service can be delivered by the unique server only at discrete times. The length of the queue at time t is n_t; it represents the number of customers waiting to be served, including the customer that will be served precisely at time t, if any. Due to the thinning procedure, it is immediate to see that the traffic intensity of the system is given by ρ; see [37] for details. Using Kendall's notation we hereafter refer to the queue model described so far as $EDA/D/1$.

The $EDA/D/1$ model is motivated by the description of public and private transportation systems, including buses, trains, aircraft [11, 37, 38, 41] and vessels [35, 42], appointment scheduling in outpatient services [10, 18, 50, 51] crane handling in dock operations [23, 28], and in general any system where scheduled arrivals are intrinsically subject to random variations. Preliminary results show that the model described above fits very well with actual data of inbound air traffic over a large hub, see [17].

The appearance of the stochastic point process (1.1) can be traced back to Winsten's seminal paper [63]. Winsten named such a queueing model *late process* and obtained results for the special case $\xi_i \in [0, 2]$ and service time exponentially distributed. At the end of [63] there is attached a discussion on Winsten's results by Lindley, Wishart, Foster, and Takács where they state that Winsten's paper can be considered as the first treatment of a queueing model with correlated arrivals [63, pages 22-28].

The same problem was investigated also by Kendall. In [45, page 11] he remarked the great importance of systems with arrivals like (1.1): *"[...] perhaps too much attention has been paid to rather uninteresting variations on the fundamental Poisson stream. As soon as one considers variations dictated by the exigencies of the real world, rather than by the pursuit of mathematical elegance, severe difficulties are encountered; this is particularly well illustrated by the notoriously difficult problem of* late arrivals." Kendall also provided the following elegant interpretation: if the random variables ξ_i are non-negative then the process defined by (1.1) is the output of the stationary $D/G/\infty$ queueing system. In particular, if the random variables ξ_i are exponentially distributed then $EDA/D/1$ can be viewed as a 2-stage tandem queueing network

$$D/M/\infty \to \cdot/D/1\,.$$

However, this is not the approach followed in this work.

Some years later, under the hypothesis that $\xi_i > 0$, Nelsen and Williams exactly characterised in [53] the distribution of the inter-arrival time intervals and the correlation coefficient between successive inter-arrival time intervals. They also gave an explicit expression of these quantities in the particular case of ξ_i's exponentially distributed.

After the '70s only approximations of the arrival process (1.1) [15, 61] or numerical studies of its output [5, 11, 57] seem to have appeared in the literature. In particular, in [37] the authors presented a self-contained study of an arrival process like (1.1), assuming for ξ_i a compact-support distribution. They also proposed an approximation scheme that keeps the correlation of the arrivals and is able to compute in a quite accurate way the quantitative features of the queue. To the best of our knowledge, a queueing system with arrivals described by (1.1) still remains an open problem and the best results obtained so far are due to Winstein in 1959.

$EDA/D/1$ is an example of a queueing system with correlated arrivals, a subject broadly studied in past years. There are many ways to impose a correlation to the arrival process. For instance, the parameters of the process may depend on their past realisation, as in [24], or on some on/off sources, as in [64]. Another relevant example of a queue model with correlated arrivals is the so-called Markov Modulated Queueing System. In Markov Modulated

Queueing Systems the parameters are driven by an independent external Markovian process, see [2, 9, 22, 49, 58] and references therein. Our model shares with Markov Modulated Queueing Systems the property that one can define an external and independent Markovian process that drives the arrival rates. However, we see in Section 2 that the output of this external drive also determines the evolution of the queue length. More precisely, $EDA/D/1$ can be interpreted as a single-server queue with deterministic service time and arrivals given by the reneging[2] from an auxiliary queue, which represents the customers that are *late* at time t, see (2.2) below. Due to the memoryless property of the exponential delays, each customer late at time t may be late in the unit time slot $(t, t+1]$ independently and with probability $q \equiv e^{-\beta}$. This means that the aforesaid reneging only happens at integer times, and clients perform synchronous independent abandonments leading to binomial transitions in the number of late customers.

In Section 2 we show that the $EDA/D/1$ model can be described as a bivariate Markov chain representing the queue length and the number of late customers. There exists an extensive literature about two-dimensional Markov models. Many methods for attacking the problem are available under two assumptions, namely, spatial homogeneity and finiteness of at least one marginal chain, see [12, 31, 36, 48, 52, 54, 56]. Unfortunately, the Markov chain defined in Section 2 does not satisfy any of the mentioned requirements.

When both components of the Markov chain are infinite but space homogeneity is still ensured, the problem is typically attacked by reduction to a Riemann-Hilbert Boundary Value Problem. Boundary Value Problems represent a broadly studied subject and several techniques have been developed in the last decades to solve them; among these, the uniformisation technique [46], conformal mappings [19, 20, 29], the compensation method [3], and the Power Series Approximation [13, 14, 47, 40]. The second issue with $EDA/D/1$ is the lack of spatial homogeneity, which is due to the aforesaid reneging with binomial transitions. This kind of transitions are often encountered in Mathematical Biology [8, 16, 25].

To the best of our knowledge, the functional equation (2.14) below has never previously appeared in the literature. Yet it is possible to mark some analogies with the functional equation in [26, 27, 44], the most important being that both equations the right hand side exhibits the generating function computed in a convex combination in the parameter $q = e^{-\beta}$, the probability of each independent abandonment. Other examples of chains with binomial transitions may be found in [1, 6, 55, 66].

In Section 3 we study the marginal distribution of the number of late customer, obtaining its exact analytical expression; this expression reveals the rich combinatorial structure of the problem. Then, we show that the stationary distribution of the $EDA/D/1$ queue has a super-exponential decay. Finally, in Section 4 we show that such a super-exponential behaviour enables a simple, yet very effective, numerical approximation scheme of the system balance equations. For a wide range of the system parameters, including typical values for real traffic applications of the model, we give a very good a priori estimate of the total-variation distance between the true and the approximate solution.

2 STATIONARY DISTRIBUTION: GENERATING FUNCTION AND BALANCE EQUATIONS

Let us consider the process n_t, which describes the length of the queue at time t. This process is governed by the stochastic recursion

$$n_{t+1} = n_t + m_{(t,t+1]} - (1 - \delta_{n_t,0}) , \tag{2.1}$$

2 A customer is said to perform a *reneging* when it abandons the queue before being served.

where $m_{(t,t+1]}$ is the number of arrivals in the interval $(t, t+1]$ according to the arrival process (1.1), and the term $1 - \delta_{n_t,0}$ represents the action of the service: if at time t the queue is non-empty then the first waiting customer is served.
The quantity $m_{(t,t+1]}$ depends in general on the whole previous history of the system. Indeed, if for some large value of T, $m_{(s,s+1]} = 0$ for any $s \in \{t-T, t-T+1, ..., t-1\}$ then $m_{(t,t+1]}$ is large with great probability. Conversely, if in the recent past the values of $m_{(s,s+1]}$ have been large then $m_{(t,t+1]}$ is expected to be small. This suggests that the arrival process is negatively autocorrelated, proof of this property can be found in [37]. Hence, the recursion (2.1) does not depend only on the present value of n_t, and the memory of the process is infinite since T can be arbitrarily large.
Let us now denote by l_t the number of customers that have not yet arrived at time t, that is to say,

$$l_t = \left|\{0 \le i \le t \quad \text{s.t.} \quad \xi_i > t - i\}\right| . \tag{2.2}$$

Let us next define $p = \int_0^1 f_\xi(t)dt = 1 - e^{-\beta}$, being f_ξ the probability density function of ξ, and $q = e^{-\beta}$. Given the value of l_t, the random variable $m_{(t,t+1]}$ is binomially distributed with parameters l_t and $1-q$. According to the memoryless property of the exponential delays ξ_i, each customer which is late at time t has probability q to continue being late in each of the following time slots; the process l_t is hence a discrete-time Markov chain.
For the sake of simplicity, let us use the notation $m_{(t,t+1]} = m_t$. If the customer expected to arrive in the slot $(t, t+1]$ has been deleted by the thinning procedure then

$$P\left(m_t = j \,|\, l_t = l\right) = \binom{l}{j} p^j q^{l-j} = b_{j,l} \,, \tag{2.3}$$

otherwise

$$P\left(m_t = j \,|\, l_t = l\right) = \binom{l+1}{j} p^j q^{l+1-j} = b_{j,l+1} \,.$$

All in all,

$$P\left(m_t = j \,|\, l_t = l\right) = b_{j,l}\,(1-\rho) + b_{j,l+1}\,\rho \,. \tag{2.4}$$

Since l_t is a Markov chain, if we assume that the state of the system is determined by the couple (n_t, l_t) then the evolution of the system is Markovian due to (2.1) and (2.4). Thus, the bivariate process (n_t, l_t), $t \in \mathbb{N}$, is a discrete-time Markov chain. The process (n_t, l_t) is usually called the *embedded chain* because we look at embedded points on the time axis, i. e. at departure instants.
The embedded Markov chain has the following transition probabilities:

For $n > 0$, (2.5)

$$\mathcal{P}\left((n,l),(n+a-1, l-a+1)\right) = \rho\, b_{a,l+1} \,, \qquad 0 \le a \le l{+}1 \,,$$
$$\mathcal{P}\left((n,l),(n+a-1, l-a)\right) = (1-\rho)\, b_{a,l} \,, \qquad 0 \le a \le l \,;$$

For $n = 0$, (2.6)

$$\mathcal{P}\left((0,l),(a, l-a+1)\right) = \rho\, b_{a,l+1} \,, \qquad 0 \le a \le l{+}1 \,,$$
$$\mathcal{P}\left((0,l),(a, l-a)\right) = (1-\rho)\, b_{a,l} \,, \qquad 0 \le a \le l \,.$$

Figure 1 displays those transitions having non-zero probability according to (2.5)–(2.6).
Then we have the following:

LEMMA 1 *The chain l_t is ergodic if and only if $q < 1$.*

Proof. If $q < 1$, the chain l_t is clearly irreducible. In order to prove its positive recurrence, we use Foster's criterion [60, Cor. 8.7] setting $f(l_t) = l_t + 1$ as the Lyapunov function. Thus, we need to show that there exist suitable positive constants K, γ such that

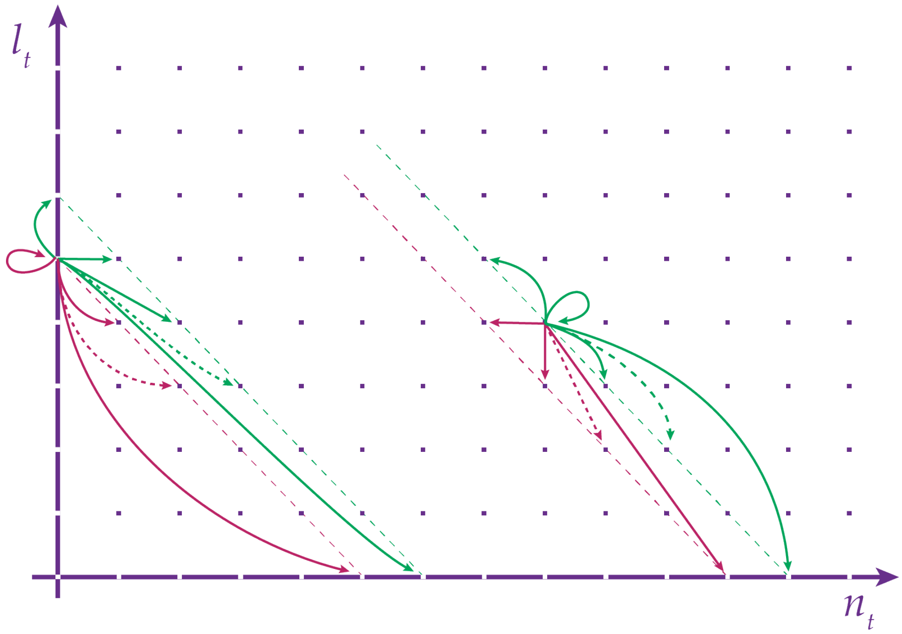

Figure 1: Transitions of the $EDA/D/1$ queueing system in the quarter plane. Transitions happen along the lines of Cartesian equation $x + y = n_t + l_t$ and $x + y = n_t + l_t - 1$ if $n_t \neq 0$, and along the lines of Cartesian equation $x + y = n_t + l_t$ and $x + y = n_t + l_t + 1$ if $n_t = 0$. Green transitions happen with probability ρ (no thinning), red transitions happen with probability $1 - \rho$ (thinning).

1. $\mathbb{E}\left[f(l_1) - f(l_0)\,|\,l_0 = L\right] \leq -\gamma$ for $f(L) > K$;
2. $\mathbb{E}\left[f(l_1)\,|\,l_0 = L\right] < \infty$ for $f(L) \leq K$;
3. the set $\{l \geq 0 \,:\, f(l) \leq K\}$ is finite.

First, by (2.4),

$$\begin{aligned}\mathbb{E}\left[f(l_1) - f(l_0)\,|\,l_0 = L\right] &= Lq(1-\rho) + (L+1)q\rho - L\,,\\ &= \rho q - L(1-q)\,. \end{aligned} \tag{2.7}$$

Therefore, point 1 is satisfied, for instance, by the choice $\gamma = 1$ and $K = 1 + \frac{1+\rho q}{1-q}$. Next, Figure 1 shows that at each iteration l_t increases at most by one unit and point 2 is satisfied:

$$\mathbb{E}\left[f(l_1)\,|\,l_0 = L\right] \leq L + 2\,. \tag{2.8}$$

By definition of $f(l_t)$ point 3 is also fulfilled. On the other hand, for $q = 1$ the chain l_t is not ergodic because it is no longer irreducible[3]. □

LEMMA 2 *The bivariate chain (n_t, l_t) is ergodic if and only if $q < 1$ and $\rho < 1$.*

Proof. If $q < 1$ and $\rho < 1$, the bivariate chain (n_t, l_t) is irreducible, see Figure 1. Let us consider the process $\alpha_t = n_t + l_t$, which represent the diagonal in the quarter plane where the point (n_t, l_t) lies on, see Figure 1; this process has the property that $|\alpha_{t+1} - \alpha_t| \leq 1$. Equations (2.5)-(2.6) yield

$$P(\alpha_{t+1} = \alpha_t + 1\,|\,n_t \neq 0) = 0\,, \tag{2.9}$$
$$P(\alpha_{t+1} = \alpha_t - 1\,|\,n_t \neq 0) = (1-\rho)\,, \tag{2.10}$$
$$P(\alpha_{t+1} = \alpha_t + 1\,|\,n_t = 0) = \rho\,, \tag{2.11}$$
$$P(\alpha_{t+1} = \alpha_t - 1\,|\,n_t = 0) = 0\,. \tag{2.12}$$

3 For $q = 1$, the chain satisfies in fact $l_{t+1} \geq l_t$.

In order to prove the positive recurrence of (n_t, l_t), we use again Foster's criterion setting $f(n_t, l_t) = M\alpha_t + l_t + 1$ with $M = 2/(1-\rho)$. From (2.7) and (2.9)-(2.12),

$$\begin{aligned}
&\mathbb{E}\left[f(n_1, l_1) - f(n_0, l_0) \,|\, n_0 = N, l_0 = L\right] \\
&\quad = M\mathbb{E}\left[\alpha_1 - \alpha_0 \,|\, n_0 = N, l_0 = L\right] + \mathbb{E}\left[l_1 - l_0 \,|\, l_0 = L\right] , \\
&\quad \le M\rho\delta_{N,0} - M(1-\rho)(1-\delta_{N,0}) + \rho q - L(1-q) .
\end{aligned}$$

Then, using a little algebra, it can be shown that the first point of Foster's criterion is satisfied by $\gamma = 1$ and $K > 1 + \frac{(M+1)(1+\rho q+M\rho)}{1-\rho}$ (for example, $K = \frac{7}{(1-\rho)^2(1-q)}$). Point 2 of Foster's criterion holds because

$$\mathbb{E}\left[f(n_1, l_1) \,|\, n_0 = N, l_0 = L\right] \le M(N + L + 1) + L + 2,$$

where we have used the property that α_t has only nearest-neighbour transitions and equation (2.8). Point 3 is fulfilled by simply considering the definition of $f(n_t, l_t)$.
For $\rho = 1$, the bivariate chain (n_t, l_t) is not ergodic because it is no longer irreducible[4]. □

Let us now consider the following bivariate generating function:

$$P(z, y) = \sum_{n,l\ge 0} z^n \, y^l \, P_{n,l} \, , \quad |z|, |y| \le 1 \, , \tag{2.13}$$

where $P_{n,l}$ is the stationary distribution of the ergodic chain (n_t, l_t), i. e.

$$P_{n,l} = \lim_{t\to\infty} P(n_t = n, l_t = l) \, .$$

The following result holds:
THEOREM 3 *The bivariate generating function* (2.13) *satisfies*

$$P(z, y) = \frac{1 + \rho\,(v-1)}{z} \left[(z-1)\, P(0, v) + P(z, v)\right] , \tag{2.14}$$

where

$$v = v(z, y) = z + q\,(y - z) \, .$$

Remark 2. The functional equation (2.14) and does not admit simple or immediate solutions. It is radically different from the functional equations typically studied in the literature[20, 30] and it is rather special in this respect. A simple solution can be found only in the particular case $z = 1$, see Section 3 below.

Proof of Theorem 3. For each $n, l \ge 0$, the balance equations of $EDA/D/1$ are the following:

$$\begin{aligned}
P_{n,l} = (1-\rho) &\left(\sum_{j=0}^{n} P_{j+1,l+n-j} \, b_{n-j,l+n-j} + P_{0,l+n} \, b_{n,l+n} \right) \\
&+ \rho \left(\sum_{j=0}^{n} P_{j+1,l+n-j-1} \, b_{n-j,l+n-j} + P_{0,l+n-1} \, b_{n,l+n} \right) ,
\end{aligned} \tag{2.15}$$

where $b_{j,l}$ are given by (2.3) and we agree that $P_{n,l} = 0$ whenever $n, l < 0$. The special cases $n = 0$ and $n = l = 0$ respectively lead to

$$P_{0,l} = (1-\rho)\,(P_{1,l} + P_{0,l}) \; b_{0,l} + \rho\,(P_{1,l-1} + P_{0,l-1}) \; b_{0,l} \, , \tag{2.16}$$

$$P_{0,0} = (1-\rho)(P_{1,0} + P_{0,0}) \, . \tag{2.17}$$

4 For $\rho = 1$, the process α_t satisfies in fact $\alpha_{t+1} \ge \alpha_t$.

To show that (2.15)–(2.17) hold, it suffices to write $P_{n_{t+1},l_{t+1}}$ in terms of P_{n_t,l_t} and then neglect the time dependency. Take for example (2.17): the system is found at time $t+1$ in state $(0,0)$, i.e. with empty queue and no late customers, only if at time t it was either in state $(0,0)$ or in state $(1,0)$, and the $(t+1)$th scheduled customer[5] is deleted by thinning. Indeed, if at time t the system was in state $(0,0)$ then nothing happens and the state remains unchanged, whereas if it was in state $(1,0)$ then the customer in queue is served and at time $t+1$ the system is in state $(0,0)$. Similarly, there are four cases such that the system is found at time $t+1$ in state $(0,l)$, i.e. with an empty queue and l customers late. In the first two cases the system is in state $(1,l)$ or in state $(0,l)$ at time t, the $(t+1)$th customer is deleted, and no one of the l late customers arrives in the interval $[t,t+1)$ (this event has in fact probability $q^l = b_{0,l}$). In the remaining cases the system is in state $(1,l-1)$ or in state $(0,l-1)$ at time t , the $(t+1)$th customer is not deleted[6], and no one of the $(l-1)+1$ late customers arrives in the interval $(t,t+1]$. The latter argument gives (2.16) while an easy generalisation to the case $n \geq 1$ leads to (2.15).

Let us take (2.15), multiply both sides by $z^n\, y^l$, and then sum over n and l. The summation of all terms multiplied by $(1-\rho)$ yields

$$(1-\rho)\left\{ \sum_{n,l\geq 0} \left[\sum_{j=0}^{n} P_{j+1,l+n-j} \binom{n+l-j}{n-j} z^j (zp)^{n-j} (yq)^l + P_{0,l+n} \binom{l+n}{n} (zp)^n (yq)^l \right] \right\},$$

or equivalently,

$$(1-\rho)\left\{ \sum_{j\geq 0} \sum_{n\geq j} \left[\sum_{l\geq 0} P_{j+1,l+n-j} \binom{n+l-j}{n-j} z^j (zp)^{n-j} (yq)^l + P_{0,l+n} \binom{l+n}{n} (zp)^n (yq)^l \right] \right\}.$$

The change of indices $k = n-j$ and $m = l+n-j = l+k$ yields

$$\begin{aligned}
&(1-\rho)\left\{ \sum_{j\geq 0} \sum_{n\geq j} \left[\sum_{l\geq 0} P_{j+1,l+n-j} \binom{n+l-j}{n-j} z^j (zp)^{n-j} (yq)^l \right\} \\
&= (1-\rho)\left\{ \sum_{j\geq 0} z^j \sum_{m\geq 0} \left[\sum_{k=0}^{n} P_{j+1,m} \binom{m}{k} (zp)^k (yq)^m \right] \right\}, \\
&= \frac{1-\rho}{z} \sum_{j\geq 1} \sum_{m\geq 0} P_{j,m} z^j (zp+yq)^m, \\
&= \frac{1-\rho}{z} \left[P(z, zp+yq) - P(0, zp+yq) \right],
\end{aligned}$$

to which we still have to sum the contribution

$$(1-\rho) \sum_{n,l\geq 0} P_{0,l+n} \binom{l+n}{n} (zp)^n (yq)^l = (1-\rho) P(0, zp+yq).$$

All in all, the sum of all terms multiplied by $(1-\rho)$ is

$$(1-\rho)\left[P(0, zp+yq) + \frac{1}{z}\left(P(z, zp+yq) - P(0, zp+yq) \right) \right].$$

In a completely analogous way we can compute the sum of the terms multiplied by ρ, which turns out to be

$$\rho(zp+yq)\Big[P(0, zp+yq) + \frac{1}{z}\left(P(z, zp+yq) - P(0, zp+yq) \right) \Big].$$

Summing up the two contributions, we get (2.14). □

5 Cf. formula (1.1).

6 Therefore, the $(t+1)$th customer is added to the set of the $l-1$ customers that are already late.

Remark 3. We end this section with a discussion of the special case $q = 0$. In this regime, the right-hand side of equation (2.14) does not depend on y anymore, and $P(z, y) \equiv Q(z)$. The number of late customers is in fact $l_t \equiv 0$ as the ith customer can not have a delay $\xi_i \geq 1$. Then, equation (2.14) yields directly

$$Q(z) = \frac{1 + \rho(z-1)}{z}[(z-1)Q(0) + Q(z)], \tag{2.18}$$

where $Q(0) = 1 - \rho$ is the stationary probability of a void queue. Therefore, equation (2.18) is equivalent to

$$Q(z) = 1 + \rho(z-1),$$

which is the classical result of a $D/D/1$ queue with balking[7].

3 THE MARGINAL DISTRIBUTION OF LATE CUSTOMERS

In this section we focus on P_l, the marginal distribution of late customers. First, we iterate the functional equation (2.14) to obtain the generating function of P_l in the form of an infinite product. Then, we invert the generating function and find the exact analytical expression of P_l. Finally, we derive the asymptotic behaviour of P_l and use it to infer asymptotics for $P_{n,l}$.
The marginal distribution of late customers is

$$P_l = \sum_{n \geq 0} P_{n,l}.$$

Thus, the generating function of this distribution is

$$\sum_{l \geq 0} P_l y^l = \sum_{n,l \geq 0} P_{n,l}\, y^l = P(1, y).$$

Setting $z = 1$ into equation (2.14) yields

$$P(1, y) = [1 + \rho q(y-1)]P(1, 1 + q(y-1)). \tag{3.1}$$

Using (3.1),

$$P(1, 1 + q(y-1)) = [1 + \rho q^2(y-1)]P(1, 1 + q^2(y-1)),$$

iterating N times hence yields

$$P(1, y) = \left[\prod_{k=0}^{N-1} 1 + \rho q^{k+1}(y-1)\right] P(1, 1 + q^N(y-1)).$$

The limit of $\prod_{k=0}^{N-1}[1 + \rho q^{k+1}(y-1)]$ for $N \to \infty$ exists for each $q < 1$ and $y \in \mathbb{C}$, see [4]. Therefore, we have proven the following

COROLLARY 4 *For $q < 1$ and $|y| \leq 1$,*

$$P(1, y) = \prod_{k \geq 0} \left[1 + \rho q^{k+1}(y-1)\right]. \tag{3.2}$$

Remark 4. The infinite product (3.2) has an interesting combinatorial interpretation:

$$P(1, y) = \prod_{k \geq 0} \left[1 + \rho q^{k+1}(y-1)\right] = \frac{(\rho(1-y); q)_\infty}{1 + \rho(y-1)}, \tag{3.3}$$

where $(a; q)_\infty = \prod_{k \geq 0}[1 - aq^k]$ is the *infinite q-Pochhammer symbol*, also known as *infinite q-ascending factorial in a*. For $y = 1 - q/\rho$, $P(1, y) = \phi(q)\,(1-q)^{-1}$, where $\phi(q)$ is the well-known *Euler function*.

7 That is, the thinning of intensity ρ, cf. Section 1.

Remark 5. For $q < 1$, the right-hand side of (3.3) is analytic for each $y \in \mathbb{C}$. Therefore, the power series $P(1,y) = \sum_{l \geq 0} P_l \, y^l$, convergent for each $|y| \leq 1$, can be analytically continued in the whole complex plane. As such, we expect the marginal distribution P_l to decrease super-exponentially fast in l.

Following the insight given by Remark 5, we shift now the focus to the asymptotic behaviour of P_l and $P_{n,l}$. Expanding the product and rearranging it in powers of $\rho(y-1)$ yields

$$\begin{aligned} P(1,y) &= \prod_{k \geq 0} \left(1 + \rho\, q^{k+1}(y-1)\right) , \\ &= 1 + \sum_{k \geq 1} \rho^k (y-1)^k \left[\sum_{m \geq \binom{k+1}{2}} d(m;k) q^m \right] , \end{aligned} \tag{3.4}$$

where $d(m;k)$ is the number of partitions of m in k distinct parts.
The following theorem holds:

THEOREM 5 *Let P_l be the equilibrium marginal distribution of the number of late customers and $P(1,y)$ its generating function. Then,*

$$P(1,y) = \sum_{k \geq 0} \frac{\rho^k q^{\binom{k+1}{2}} (y-1)^k}{\prod_{i=1}^k [1-q^i]} , \tag{3.5}$$

$$P_l = \sum_{k \geq l} \frac{(-1)^{k-l} \rho^k q^{\binom{k+1}{2}} \binom{k}{l}}{\prod_{i=1}^k [1-q^i]} . \tag{3.6}$$

Theorem 5 is a direct consequence of two results from number theory. The first result can be found in [65] and links the number of partitions in k distinct parts with the number of partitions into at most k parts.

LEMMA 6 *If $m > \binom{k+1}{2}$ then the number of partitions of m in k distinct parts equals the number of partitions of $m - \binom{k+1}{2}$ into at most k parts (not necessarily distinct).*

The second lemma states that the number of partitions into at most k parts equals the number of partitions in parts less or equal than k.

LEMMA 7 *Let $p_{\leq k}(m)$ be the number of partitions of m in parts that do not exceed k. Then $p_{\leq k}(m)$ equals the number of partitions of m into at most k parts. The generating function of the number of partitions of m into at most k parts is*

$$P_{\leq k}(x) = \sum_{m \geq 0} p_{\leq k}(m)\, x^m = \prod_{i=1}^{k} \frac{1}{1-x^i} .$$

Lemma 7 is relatively easy to prove. Using Ferrers diagrams, it is readily seen that a partition in parts that do not exceed k and a partition into at most k parts are conjugate to one another; see [7, 39] for more details.

Proof of Theorem 5. Using Lemma 6 and 7 we can recast (3.4) as

$$\begin{aligned} P(1,y) &= 1 + \sum_{k \geq 1} \rho^k (y-1)^k \left[\sum_{m \geq \binom{k+1}{2}} d(m;k) q^m \right] , \\ &= 1 + \sum_{k \geq 1} \rho^k (y-1)^k q^{\binom{k+1}{2}} \left[1 + \sum_{m > 0} d\left(\binom{k+1}{2} ; k \right) q^m \right] , \\ &= 1 + \sum_{k \geq 1} \rho^k (y-1)^k q^{\binom{k+1}{2}} \left[1 + \sum_{m > 0} p_{\leq k}(m) q^m \right] , \\ &= 1 + \sum_{k \geq 1} \rho^k (y-1)^k q^{\binom{k+1}{2}} \prod_{i=1}^{k} \frac{1}{1-x^i} , \\ &= \sum_{k \geq 0} \frac{\rho^k q^{\binom{k+1}{2}} (y-1)^k}{\prod_{i=1}^k [1-q^i]} , \end{aligned}$$

where we have used the usual convention according to which $p_{\leq k}(0) = 1$ and $\prod_{i=a}^{b} f_k = 1$ when $b < a$.
The second part of the Theorem is proved from (3.5) as follows:

$$
\begin{aligned}
P_l &= \frac{1}{l!}\frac{d^l}{dy^l}P(1,y)\Big|_{y=0}, \\
&= \sum_{k\geq l} \frac{\rho^k q^{\binom{k+1}{2}}}{\prod_{i=1}^{k}[1-q^i]}\frac{1}{l!}\frac{d^l}{dy^l}(y-1)^k\Big|_{y=0}, \\
&= \sum_{k\geq l} \frac{(-1)^{k-l}\rho^k q^{\binom{k+1}{2}}\binom{k}{l}}{\prod_{i=1}^{k}[1-q^i]}.
\end{aligned}
$$

□

Equation (3.6) can be used to obtain an upper bound on P_l. Let

$$
(q;q)_l = \prod_{i=1}^{l}\left[1-q^i\right]
$$

be the q-Pochammer symbol of the pair (q,q). The following inequalities hold:

$$
\begin{aligned}
\binom{m+l}{l} &= \prod_{k=1}^{m}\left(1+\frac{l}{k}\right) \leq (1+l)^m, \\
\prod_{i=l+1}^{l+m}\left[1-q^i\right] &\geq \prod_{i=1}^{m}\left[1-q^i\right] = (q;q)_m, \\
(q;q)_l &\geq (q;q)_\infty.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_l &= \frac{\rho^l\, q^{\binom{l+1}{2}}}{(q;q)_l}\sum_{m\geq 0}\frac{(-\rho)^m q^{(l+1)m} q^{\binom{m}{2}}\binom{m+l}{l}}{\prod_{i=l+1}^{l+m}[1-q^i]}, \\
&\leq \frac{\rho^l\, q^{\binom{l+1}{2}}}{(q;q)_\infty}\sum_{m\geq 0}\frac{q^{\binom{m}{2}}[\rho\, q^{l+1}\,(l+1)]^m}{(q;q)_m}, && (3.7)\\
&= \rho^l\, q^{\binom{l+1}{2}}\frac{\prod_{k\geq 0}\left[1+q^{k+l+1}\rho(l+1)\right]}{(q;q)_\infty}, && (3.8)
\end{aligned}
$$

where from (3.7) to (3.8) we have used the properties of q-ascending factorials and q-binomial coefficients [32]. If l is sufficiently large then $q^l\rho(l+1) \leq 1$, and (3.8) yields

$$
P_l \leq \rho^l\, q^{\binom{l+1}{2}}\frac{(-q;q)_\infty}{(q;q)_\infty}. \tag{3.9}
$$

Remark 6. Theorem 5 and (3.9) show that, asymptotically in l, the leading order of P_l is $\rho^l q^{\binom{l+1}{2}}$. This fact can be directly implied from arrival process (1.1). In fact, the most likely way to have l late customers (l large) is that each of the customers originally scheduled in the interval $[t-l,t)$ are late at time t, an event of probability $q^l\, q^{l-1}\cdots q = q^{\binom{l+1}{2}}$.
Since $P_{n,l} \leq P_l$, we have just obtained the following asymptotic result:
THEOREM 8 *Uniformly in n, the equilibrium distribution $P_{n,l}$ decays super-exponentially fast in l. More precisely,*

$$
P_{n,l} = O\left(\rho^l q^{\binom{l+1}{2}}\right) \quad \textit{for } l \to \infty. \tag{3.10}
$$

As a matter of fact the super-exponential decay of $P_{n,l}$ may be proved also for $n \to \infty$. Let us consider the auxiliary process $\alpha_t = n_t + l_t$, which we have already encountered in the proof of Lemma 2. There we have interpreted α_t as the diagonal in the quarter plane where the point (n_t, l_t) lies on. Under equilibrium conditions, the probability of finding the system on the ath diagonal is just

$$p_a \equiv P(\alpha_t = a) = \sum_{\substack{n,l \geq 0 \\ n+l=a}} P_{n,l}\,, \quad a \geq 0\,.$$

Then, we have the following:

LEMMA 9 *The generating function of p_a is $P(z,z)$.*

Proof. Let $A(z) = \sum_{a\geq 0} p_a\, z^a$ be the generating function of p_α. Then,

$$A(z) = \sum_{a\geq 0} p_a\, z^a = \sum_{a\geq 0} \sum_{\substack{n,l \geq 0 \\ n+l=a}} P_{n,l}\, z^{n+l} = \sum_{n,l\geq 0} P_{n,l}\, z^n\, z^l = P(z,z)\,.$$

□

Substituting $y = z$ into (2.14) yields

$$P(z,z) = \frac{1+\rho(z-1)}{1-\rho} P(0,z)\,. \tag{3.11}$$

Remark 7. Equation (3.11) gives an interesting connection between the equilibrium distribution of the quantity $\alpha_t = n_t + l_t$ and the stationary probability of having l late customers given that the queue is void. Figure 1 shows that the latter event drives the dynamic of α_t through an independent Bernoulli random variable with parameter ρ, which explains the factor $1 + \rho(z-1)$.

From (3.11) we can compute as follows p_a in terms of $P_{0,a}$:

$$\begin{aligned} p_a &= \frac{1}{a!} \frac{d^a}{dz^a} P(z,z)\Big|_{z=0} \\ &= \frac{1}{a!} \Big[\frac{1+\rho(z-1)}{1-\rho} \frac{d^a}{dz^a} P(0,z) + \frac{\rho\, a}{1-\rho} \frac{d^{a-1}}{dz^{a-1}} P(0,z)\Big]_{z=0}, \\ &= P_{0,a} + \frac{\rho}{1-\rho} P_{0,a-1}\,. \end{aligned} \tag{3.12}$$

For $a = n + l$, formulas (3.10) and (3.12) then yield

$$P_{n,l} \leq p_a = P_{0,a} + \frac{\rho}{1-\rho} P_{0,a-1} = O(\rho^a q^{\binom{a}{2}})\,. \tag{3.13}$$

Therefore, the following asymptotic result holds:

THEOREM 10 *The equilibrium distribution $P_{n,l}$ decays super-exponentially fast as either $n \to \infty$ or $l \to \infty$. More precisely,*

$$P_{n,l} = O\left(\rho^{n+l} q^{\binom{n+l}{2}}\right) \quad \textit{for } n, l \to \infty\,. \tag{3.14}$$

4 NUMERICAL APPROXIMATION OF THE JOINT STATIONARY MEASURE

In this final section we examine the possibility to approximately compute the joint stationary distribution $P_{n,l}$. Due to the very broad range of applications of the queueing model $EDA/D/1$, an efficient approximate computation of the solution may prove itself crucial in contexts where practical solutions are needed.

In Section 3 we have shown that the joint stationary probability $P_{n,l}$ decreases super-exponentially fast in the limit of either $n, l \to \infty$. The natural question arising is then whether a bare truncation of the infinite linear system (2.15)-(2.17) is sufficient to obtain a satisfactory numerical expression of $P_{n,l}$. As we will see, in this case the answer is definitely yes due to (3.14).

The idea we present is not new and has been already discussed, for instance, in [62] for stationary distributions with geometric tail. To help the reader, we introduce it first in a general setting. Assume we have a numerable set of linear equations

$$\pi_j = \sum_{l \geq 0} \pi_i P_{i,j}\,, \qquad j \geq 0\,,$$
$$\sum_{i \geq 0} \pi_i = 1\,,$$

where π_j and $P_{i,j}$ denote the (unique) stationary distribution and the transition matrix of a Markov chain, respectively. Assume moreover that, for a fixed integer k, we are able to provide the following estimates:

$$\sum_{i > k} \pi_i\, P_{i,j} \leq \varepsilon_j\,, \qquad j = 0, 1, \ldots, k-1\,,$$
$$\sum_{i > k} \pi_i \leq \varepsilon_k\,.$$

Define the following $k \times k$ matrix

$$A_{ij} = \begin{cases} \delta_{i,j} - P_{i,j}\,, & i = 0, 1, \ldots, k-1\,, \\ 1 & i = k\,, \end{cases} \tag{4.1}$$

where $\delta_{i,j}$ is the usual Kronecker's delta. Finally, consider the norm-1 condition number of the matrix A

$$\kappa(A) = \|A\|_1 \|A^{-1}\|_1\,,$$

where $\|A\|_1 = \max_{0 \leq j \leq k} \sum_{i=0}^{k} |A_{ij}|$. Then, we have the following:

PROPOSITION 11 *With the notation introduced above, let b be a vector of dimension k whose ith entry satisfies $b_i = \delta_{i,k}$. The solution of the linear system $\tilde{\pi}\, A = b$ satisfies*

$$\|\pi - \tilde{\pi}\|_1 \leq \kappa(A) \sum_{i=0}^{k} \varepsilon_i\,. \tag{4.2}$$

Proof. Let us define

$$\sum_{i > k} \pi_i\, P_{i,j} \equiv \delta b_j \leq \varepsilon_j\,, \qquad j = 0, 1, \ldots, k-1\,,$$
$$\sum_{i > k} \pi_i \equiv -\delta b_k \leq \varepsilon_k\,.$$

Due to the unicity of π, the values of π_j, $j = 0, 1, \ldots, k$, are solution of

$$\pi_j = \sum_{i=0}^{k} \pi_i\, P_{i,j} + \delta b_j\,, \qquad j = 0, 1, \ldots, k-1\,,$$
$$\sum_{i=0}^{k} \pi_i = 1 + \delta b_k\,.$$

The last linear system can be recast in a compact form as

$$\pi A = b + \delta b\,.$$

The proposition follows from a well known result of *perturbation theory*, see [34, §2.6.2]. □

To fit the general idea illustrated above in the context of $EDA/D/1$, we fix a positive integer $\alpha_{\max}$ and consider the truncated linear system for the

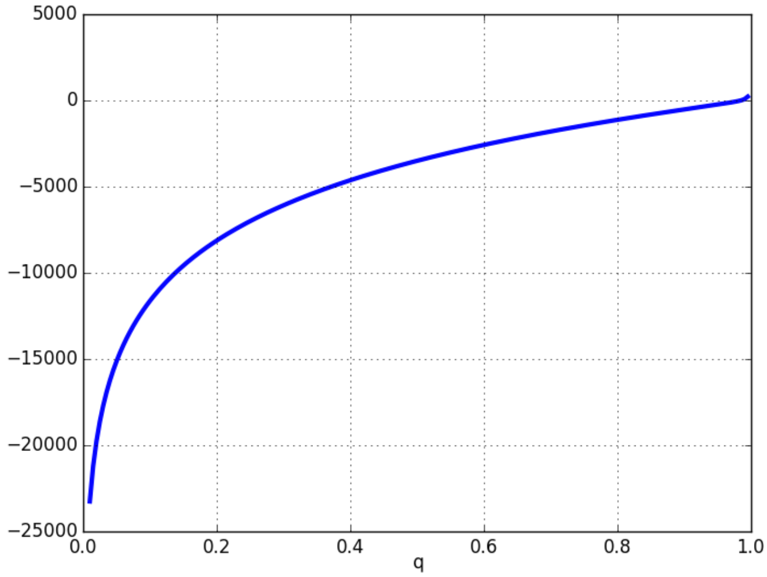

Figure 2: Behaviour of $\log\left(\frac{(-q;q)_\infty}{(q;q)_\infty} q^{\binom{\alpha_{\max}}{2}}\right)$ as a function of q for $\alpha_{\max} = 100$.

unknowns $P_{n,l}$ such that $0 \le n + l \le \alpha_{\max}$. By product of (3.14) and direct inspection of (2.5)–(2.6), for $i + j = \alpha_{\max}$,

$$\sum_{\substack{n,l \ge 0 \\ n+l > \alpha_{\max}}} P_{n,l} \mathcal{P}((n,l),(i,j)) \le (1-\rho) \frac{(-q;q)_\infty}{(q;q)_\infty} \rho^{\alpha_{\max}+1} q^{\binom{\alpha_{\max}+1}{2}}, \tag{4.3}$$

while for $i + j < \alpha_{\max}$,

$$\sum_{\substack{n,l \ge 0 \\ n+l > \alpha_{\max}}} P_{n,l} \mathcal{P}((n,l),(i,j)) = 0. \tag{4.4}$$

Also,

$$\sum_{\substack{n,l \ge 0 \\ n+l > \alpha_{\max}}} P_{n,l} \le 2 \frac{(-q;q)_\infty}{(q;q)_\infty} \rho^{\alpha_{\max}+1} q^{\binom{\alpha_{\max}+1}{2}}. \tag{4.5}$$

Figure 2 shows the behaviour of $\log\left(\frac{(-q;q)_\infty}{(q;q)_\infty} q^{\binom{\alpha_{\max}}{2}}\right)$ as a function of q for $\alpha_{\max} = 100$. We will see in a while that $\frac{(-q;q)_\infty}{(q;q)_\infty} q^{\binom{\alpha_{\max}}{2}}$ is the factor that actually scales down the right-hand side of (4.2). Therefore, unless the conditioning of the system is very large, we expect a truncation at the level $\alpha_{\max} = 100$ to give a very good approximation of the stationary probabilities of $EDA/D/1$.

Let us then focus on the system conditioning, which is the remaining ingredient needed to apply Proposition 11. Estimating the condition number of a matrix is a notably difficult problem and a vast literature exists on this topic, see e. g. [33, 59, 43]. Since the aim of the present section is showing that a bare truncation of the balance equations (2.5)–(2.6) may be sufficient for an approximate computation of $P_{n,l}$, we fall back on numerical computations. To this purpose, let us introduce the following map of the quarter plane onto the set of the non-negative integers and its inverse:

$$F(n,l) \mapsto m = \binom{n+l}{2} + l,$$
$$G(m) = (g(m), m - g(m)),$$
$$g(m) = \max\left\{ j \ge 0 \text{ such that } \binom{j+1}{2} \le m \right\} = \left\lfloor \frac{-1+\sqrt{1+8m}}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part, i. e. the *floor* operation.
Fixed a positive integer $\alpha_{\max}$, let $k_{\max} = \binom{\alpha_{\max}+1}{2}$. Define the $k_{\max} \times k_{\max}$ matrix

$$A_{ij} = \begin{cases} \delta_{i,j} - \mathcal{P}(G(i), G(j)), & i = 0, 1, \ldots, k_{\max} - 1, \\ 1 & i = k_{\max}, \end{cases} \tag{4.6}$$

where $\mathcal{P}(\cdot, \cdot)$ is defined by (2.5)–(2.6). Figure 3 displays the value of $\kappa(A)$ in the ρq-plane when $\alpha_{\max} = 100$. We see that the condition number is not larger than 6×10^5 for $\rho, q \leq 0.99$.
Consider also the new set of unknowns

$$\pi_j = P_{g(j), j-g(j)}, \qquad j = 0, 1, \ldots, k_{\max}, \tag{4.7}$$

corresponding to the set of all $P_{n,l}$ for which $0 \leq n + l \leq \alpha_{\max}$. With respect to Proposition 11, equations (4.3)–(4.5) yield

$$\sum_{j=0}^{k_{\max}} \varepsilon_j \leq 2\alpha_{\max} \frac{(-q;q)_\infty}{(q;q)_\infty} \rho^{\alpha_{\max}+1} q^{\binom{\alpha_{\max}+1}{2}} \leq 2\alpha_{\max} \frac{(-q;q)_\infty}{(q;q)_\infty} q^{\binom{\alpha_{\max}+1}{2}}. \tag{4.8}$$

Table 1 gives the numerical value of the right-hand side of (4.8) for q between 0.9 and 0.99, and $\alpha_{\max} = 100$. Comparison of Table 1 with Figure 3 shows that, uniformly in $\rho \leq 0.99$, the a priori norm-1 approximation error is less than 10^{-12} for q up to 0.98.

Remark 8. For air traffic applications, $q = 0.98$ corresponds to typical delays of the order of one hour. The same value of q for other transport systems, e. g. trains or buses, correspond to even higher delays. Consider also that typical values of ρ in extremely congested systems, e. g. London Heathrow Airport, do not exceed 0.98, see [17]. Therefore, the approximation scheme presented in this section is very fit for real life applications.

0.90	0.91	0.92	0.93	0.94
1.4×10^{-224}	9.8×10^{-200}	5.1×10^{-175}	2.4×10^{-150}	1.3×10^{-125}
0.95	0.96	0.97	0.98	0.99
1.3×10^{-100}	5.6×10^{-75}	8.7×10^{-48}	2.3×10^{-16}	2.6×10^{33}

Table 1: Value of $2\alpha_{\max} \frac{(-q;q)_\infty}{(q;q)_\infty} q^{\binom{\alpha_{\max}}{2}}$ for $0.90 \leq q \leq 0.99$ and $\alpha_{\max} = 100$.

Figure 3 suggests that the system conditioning is decreasing in q for fixed ρ. Figure 4 validates this insight by showing a log-log plot of the condition number for $\alpha_{\max}$ ranging between 10 and 100 for a high fixed value of ρ. The very same figure also suggests that the condition of the system may grow polynomially with $\alpha_{\max}$. In particular, $\kappa(A)$ as a function of $\alpha_{\max}$ seems to grow as $\alpha_{\max}$ to the power 1.038 for $\rho = 0.95$ and $q = 0.0$.

Remark 9. The matrix (4.6) is rather sparse, as shown by Figure 5. We recommend to exploit this property by using sparse storage formats and dedicated libraries when q is set larger than 0.98. In this regime $\alpha_{\max} = 100$ could be no longer sufficient to achieve a good approximation of $P_{n,l}$, but enlarging $\alpha_{\max}$ while using dense formats could quickly lead to memory shortage and a severe computational slowdown.

Remark 10. Figures 2–5 were obtained using Python v. 2.7.9, numpy v. 1.9.1, matplotlib v. 1.4.2, and mpmath v. 0.19. The code to generate them is freely available on GitHub at the following address: `https://github.com/clancia/EDA`.

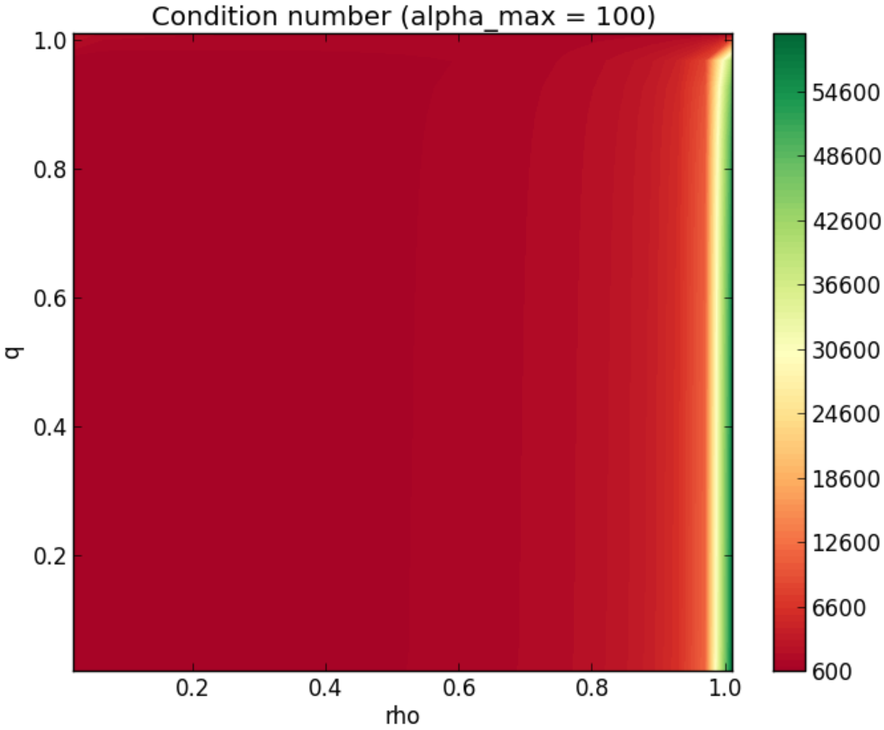

Figure 3: Condition number $\kappa(A)$ as a function of ρ and q for $\alpha_{\max} = 100$. The values of ρ and q vary between 0.0 and 0.99 by steps of 0.05.

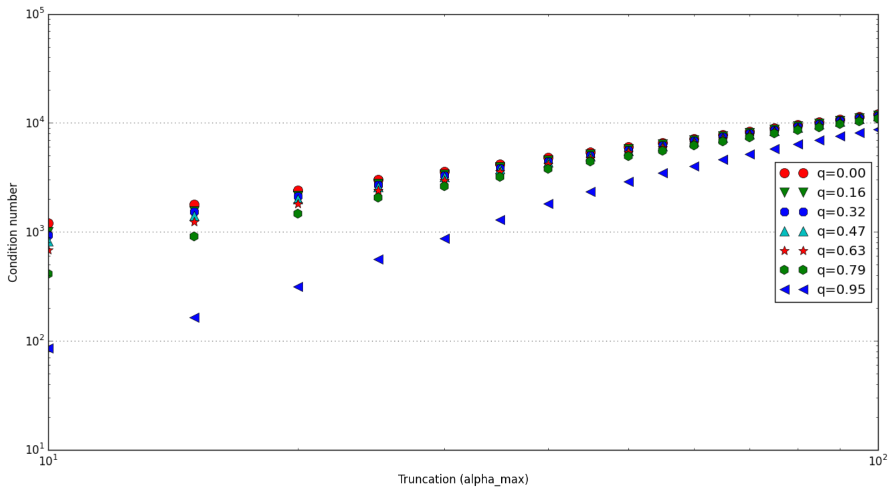

Figure 4: Log-Log plot for the growth of the condition number $\kappa(A)$ when $\alpha_{\max}$ varies between 10 and 100, with $\rho = 0.95$ and different values of q. The slope of the curve for $q = 0.0$ is approximately 1.038.

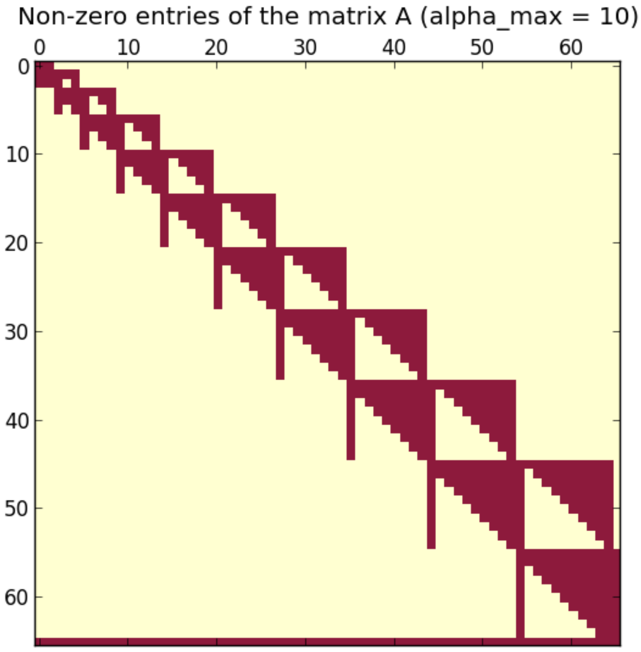

Figure 5: Sparsity structure of the matrix A with $\alpha_{\max} = 10$.

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