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四边简支压电热弹性层合正交双曲壳的精确解

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摘　要：正交双曲坐标系下，对考虑温度梯度的压电热弹性材料，首先基于基本方程，结合对偶变量理论，将热传导关系并入到压电材料的本构关系中，得到压电热弹性材料包含温度的增维本构关系；由此增维本构关系，利用状态空间法消去曲面内的应力分量，便可直接推导出可独立求解的压电热弹性材料的齐次状态方程。齐次状态方程的导出，将大大简化层合正交双曲壳的求解过程，提高计算精度，且齐次状态方程可以直接进行有限元离散，为工程实践中复杂边界条件下层合双曲结构的分析求解提供一种新方法。

关键词：压电热弹性材料；层合正交双曲壳；精确解；状态空间法；齐次状态方程

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EXACT SOLUTION FOR LAMINATED PIEZOTHERMOELASTIC GENERIC SHELLS WITH FOUR SIMPLY SUPPORTED EDGES

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Abstract: In generic coordinates, firstly, by the basic equations and the theory of symplectic variable, an augmented dimension constitutive relationship was obtained by combining the heat exchange equations into the constitutive equations of piezoelectric materials. Utilizing the state space method and the augmented dimension constitutive relationships, the homogeneous state equation which can be solely solved is educed by eliminating the stress in the shell surface. The homogeneous state equation can simplify greatly the solution procedure of the laminated generic shell, improve numerical precision, and be straightly discreted in finite element format, as well as presents a new method for the semi-analytical solution of laminated structures with complicated boundary condition in practice.

Key words: piezothermoelastic materials; laminated generic shells; exact solution; state space method; homogenous state equation

压电材料因其具有的正压电效应、逆压电效应，作为一种重要的智能材料，越来越广泛的应用于各种智能结构中。板和壳是工程中最常用的结构形式。在求解层合板壳结构时，状态空间法[1]是一种十分简洁、有效的方法。Sosa 和 Castro[2]最先提出了压电材料的二维状态空间法；丁皓江等[3]对压电板壳的自由振动进行了三维精确分析；盛宏玉等[4－5]应用状态空间分析方法和传递矩阵法分别求解了一般边界条件下压电层合厚板和任意厚度压电层合闭口柱壳的精确解。

对压电热弹性材料，其机-电-热耦合效应是材料界研究的重点。Mindlin[6]已推出了机-电-热场耦合的压电热弹性材料的控制方程，Nowacki[7]和 Iesan[8]给出了压电热弹性材料的一般理论分析，

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Tauchert[9]和 Jonnalagadda 等[10]提出了压电热弹性材料层合板的理论。因机-电-热的耦合及材料固有的各向异性，很难得到压电热弹性材料三维控制方程的精确解。Xu 等[11]由压电热弹性材料控制方程推导出了其广义的状态向量方程，并研究了多层板的静态响应和敏度系数，Vel 和 Batra[12]分析了压电热弹性材料层合板的广义应变；以上文献中的控制方程都是非齐次的微分方程，不能独立求解，在稳态温度问题的求解过程中，还需结合关于热平衡方程和热传导方程的二阶微分方程[13]。刘艳红等[14]和田秀云[15]等，由状态空间法，结合对偶变量理论[16]，分别在直角坐标系和圆柱坐标系下，推导出了压电热弹性材料可以独立求解的齐次状态方程，大大简化压电热弹性材料的求解。

本文进一步在正交双曲坐标系下，结合状态空间法和对偶变量理论，推导四边简支的压电热弹性材料的齐次状态方程，并给出其对应的精确解。

1 基本方程

如图 1 所示，在正交双曲坐标系下，不计体积力，静态，各向异性压电热弹性体稳态温度问题的基本方程为：

本构关系：

$$\begin{Bmatrix}\sigma_\alpha\\ \sigma_\beta\\ \sigma_\gamma\\ \tau_{\beta\gamma}\\ \tau_{\alpha\gamma}\\ \tau_{\alpha\beta}\\ d_\alpha\\ d_\beta\\ d_\gamma\end{Bmatrix}=\begin{bmatrix}c_{11}&c_{12}&c_{13}&0&0&0\\ c_{12}&c_{22}&c_{23}&0&0&0\\ c_{13}&c_{23}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{55}&0\\ 0&0&0&0&0&c_{66}\\ 0&0&0&0&e_{15}&0\\ 0&0&0&e_{24}&0&0\\ e_{31}&e_{32}&e_{33}&0&0&0\end{bmatrix}\begin{Bmatrix}\varepsilon_\alpha\\ \varepsilon_\beta\\ \varepsilon_\gamma\\ \gamma_{\beta\gamma}\\ \gamma_{\alpha\gamma}\\ \gamma_{\alpha\beta}\end{Bmatrix}-$$

$$\begin{bmatrix}0&0&e_{31}\\ 0&0&e_{31}\\ 0&0&e_{31}\\ 0&e_{24}&0\\ e_{15}&0&0\\ 0&0&0\\ \kappa_\alpha&0&0\\ 0&\kappa_\beta&0\\ 0&0&\kappa_\gamma\end{bmatrix}\begin{Bmatrix}E_\alpha\\ E_\beta\\ E_\gamma\end{Bmatrix}-\begin{bmatrix}\lambda_{11}\\ \lambda_{22}\\ \lambda_{33}\\ 0\\ 0\\ 0\\ 0\\ 0\\ r_3\end{bmatrix}\begin{bmatrix}\lambda_{11}\\ \lambda_{22}\\ \lambda_{33}\\ 0\\ 0\\ 0\end{bmatrix}T \tag{1a}$$

$$p_\alpha=k_\alpha T_\alpha,\ p_\beta=k_\beta T_\beta,\ p_\gamma=k_\gamma T_\gamma \tag{1b}$$

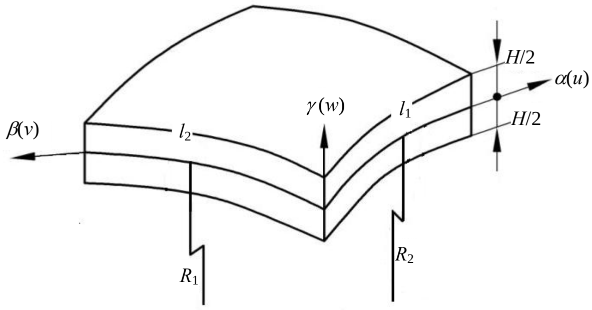

图 1 正交双曲坐标系

Fig.1 Generic coordinates

梯度关系：

$$\varepsilon_\alpha=\frac{1}{H_1}\frac{\partial u}{\partial\alpha}+\frac{K_1}{H_1}w,\quad \gamma_{\beta\gamma}=\frac{\partial v}{\partial\gamma}+\frac{1}{H_2}\frac{\partial w}{\partial\beta}-\frac{K_2}{H_2}v,$$

$$\varepsilon_\beta=\frac{1}{H_1}\frac{\partial v}{\partial\beta}+\frac{K_2}{H_2}w,\quad \gamma_{\alpha\gamma}=\frac{\partial u}{\partial\gamma}+\frac{1}{H_1}\frac{\partial w}{\partial\alpha}-\frac{K_1}{H_1}u,$$

$$\varepsilon_{rr}=\frac{\partial w}{\partial\gamma},\qquad \gamma_{\alpha\beta}=\frac{1}{H_2}\frac{\partial u}{\partial\beta}+\frac{1}{H_1}\frac{\partial v}{\partial\alpha}。 \tag{2a}$$

$$E_\alpha=-\frac{1}{H_1}\frac{\partial\phi}{\partial\alpha},\ E_\beta=-\frac{1}{H_2}\frac{\partial\phi}{\partial\beta},\ E_\gamma=-\frac{\partial\phi}{\partial\gamma} \tag{2b}$$

$$T_\alpha=-\frac{1}{H_1}\frac{\partial T}{\partial\alpha},\ T_\beta=-\frac{1}{H_2}\frac{\partial T}{\partial\beta},\ T_\gamma=-\frac{\partial T}{\partial\gamma} \tag{2c}$$

平衡方程：

$$H_2\frac{\partial\sigma_\alpha}{\partial\alpha}+H_1\frac{\partial\tau_{\alpha\beta}}{\partial\beta}+H_1H_2\frac{\partial\tau_{\alpha\gamma}}{\partial\gamma}+(2K_1+K_2+3K_1K_2\gamma)\tau_{\alpha\gamma}=0,$$

$$H_2\frac{\partial\tau_{\alpha\beta}}{\partial\alpha}+H_1\frac{\partial\sigma_\beta}{\partial\beta}+H_1H_2\frac{\partial\tau_{\beta\gamma}}{\partial\gamma}+(K_1+2K_2+3K_1K_2\gamma)\tau_{\beta\gamma}=0,$$

$$H_2\frac{\partial\tau_{\alpha\gamma}}{\partial\alpha}+H_1\frac{\partial\tau_{\beta\gamma}}{\partial\beta}+H_1H_2\frac{\partial\sigma_\gamma}{\partial\gamma}+(K_1+K_2+2K_1K_2\gamma)\sigma_\gamma-K_1H_2\sigma_\alpha-K_2H_1\sigma_\beta=0。 \tag{3a}$$

$$H_2\frac{\partial D_\alpha}{\partial\alpha}+H_1\frac{\partial D_\beta}{\partial\beta}+H_1H_2\frac{\partial D_\gamma}{\partial\gamma}+D_\gamma(H_1K_2+H_2K_1)=0 \tag{3b}$$

$$H_2\frac{\partial p_\alpha}{\partial\alpha}+H_1\frac{\partial p_\beta}{\partial\beta}+H_1H_2\frac{\partial p_\gamma}{\partial\gamma}+p_\gamma(H_1K_2+H_2K_1)=0 \tag{3c}$$

其中：σ_α、σ_β、σ_γ、$\tau_{\beta\gamma}$、$\tau_{\alpha\gamma}$、$\tau_{\alpha\beta}$ 为应力向量；$C_{ij}=C_{ji}$ ($i,j=1,2,3,4,5,6$)为刚度系数；ε_α、ε_β、ε_γ、$\gamma_{\beta\gamma}$、$\gamma_{\alpha\gamma}$、$\gamma_{\alpha\beta}$ 为应变分量；e_{15}、e_{24}、e_{31}、e_{32}、e_{33} 为压电系数；E_α、E_β、E_γ 为电场强度分量；λ_{11}、λ_{22}、λ_{33} 应力-温

度系数；D_α、D_β、D_γ 电位移分量；κ_α、κ_β、κ_γ 为介电系数；r_3 为热电系数；k_α、k_β、k_γ 为热传导系数；T 是温度增量；l_1、l_2 为中性面处弧长；R_1、R_2 为中性面曲线 $\overset{\frown}{o\alpha}$、曲率 $\overset{\frown}{o\beta}$ 的曲率半径；K_1、K_2 为相应的主曲率，即 $K_1 = 1/R_1, K_2 = 1/R_2$；H_1、H_2 分别为 α 方向、β 方向的拉梅(Lamè)系数，且：

$$H_1 = 1 + K_1\gamma, H_2 = 1 + K_2\gamma \tag{4}$$

从热梯度关系式(2c)和热平衡方程式(3c)可推出以下简明关系式：

$$\partial p_\gamma / \partial\gamma = \left(\frac{A^2}{H_1}k_\alpha + \frac{B^2}{H_2}k_\beta\right)T - p_r(H_1K_2 + H_2K_1) \tag{5}$$

$$\partial T / \partial\gamma = -1/k_\gamma p_\gamma \tag{6}$$

依据对偶变量理论可以证明上述两式中的热流密度 $p_\gamma(\alpha,\beta,\gamma)$ 和温度增量 $T(\alpha,\beta,\gamma)$ 是对偶变量，同理，可以证明 $p_\alpha(\alpha,\beta,\gamma)$、$p_\beta(\alpha,\beta,\gamma)$ 与 $T(\alpha,\beta,\gamma)$ 也是对偶变量。即可将热流密度分量 $p_i(i=\alpha,\beta,\gamma)$ 类比为应力分量，$T_{,i}(i=\alpha,\beta,\gamma)$ 类比为应变分量，则温度增量 T 是位移。那么，热梯度关系式(2c)也是本构关系方程，可将其直接并入到压电材料的本构关系中，从而得到热弹性压电材料正交双曲坐标系下，包含温度梯度的增维本构关系为：

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \\ \tau_{\alpha\gamma} \\ \tau_{\beta\gamma} \\ \tau_{\alpha\beta} \\ D_\alpha \\ D_\beta \\ D_\gamma \\ p_\alpha \\ p_\beta \\ p_\gamma \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} & 0 & 0 & 0 & -\lambda_{11} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 & 0 & 0 & -e_{32} & 0 & 0 & 0 & -\lambda_{22} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} & 0 & 0 & 0 & -\lambda_{33} \\ 0 & 0 & 0 & c_{44} & 0 & 0 & 0 & -e_{24} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 & -e_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{15} & 0 & \kappa_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 & 0 & \kappa_{22} & 0 & 0 & 0 & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 & 0 & 0 & \kappa_{33} & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{33} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \\ \gamma_{\beta\gamma} \\ \gamma_{\alpha\gamma} \\ \gamma_{\alpha\beta} \\ E_{,\alpha} \\ E_{,\beta} \\ E_{,\gamma} \\ T_{,\alpha} \\ T_{,\beta} \\ T_{,\gamma} \\ T \end{Bmatrix} \tag{7}$$

由此增维本构关系，以状态空间法消去平面内的应力分量 σ_α、σ_β、$\tau_{\alpha\beta}$、D_α、D_β、p_α、p_β，便可直接得到正交双曲坐标系下的齐次状态方程为：

$$\partial \boldsymbol{R} / \partial\gamma = [D_A \quad D_B]\boldsymbol{R} \tag{8}$$

其中：

$$\boldsymbol{R} = [\tau_{\alpha\gamma} \quad \tau_{\beta\gamma} \quad \sigma_\gamma \quad D_\gamma \quad p_\gamma \quad u \quad v \quad w \quad \phi \quad T]'$$

曲面内的应力分量 σ_α、σ_β、$\tau_{\alpha\beta}$、D_α、D_β、p_α、p_β 可由状态向量 $\sigma_{\alpha\gamma}$、$\sigma_{\beta\gamma}$、σ_γ、D_γ、p_γ、u、v、w、ϕ、T 求出，即：

$$\boldsymbol{P} = I\ \boldsymbol{R} \tag{9}$$

其中：

$$\boldsymbol{P} = [\sigma_\alpha \quad \sigma_\beta \quad \tau_{\alpha\beta} \quad D_\alpha \quad D_\beta \quad p_\alpha \quad p_\beta]'$$

2　状态方程的求解

如图 1 所示的四边简支压电热弹性材料层合正交双曲壳的边界条件为：

$$\begin{cases} \sigma_\alpha = w = v = \phi = T = 0, & \alpha = 0, l_1 \\ \sigma_\beta = w = u = \phi = T = 0, & \beta = 0, l_2 \end{cases} \tag{10}$$

设层合双曲壳任意一层满足边界条件式(10)的级数解为：

$$(\tau_{\alpha\gamma}, u) = \sum_m^\infty \sum_n^\infty (\tau_{\alpha\gamma}^{mn}(\gamma), u^{mn}(\gamma))\cos(\zeta\alpha)\sin(\eta\beta),$$

$$(\tau_{\beta\gamma}, v) = \sum_m^\infty \sum_n^\infty (\tau_{\beta\gamma}^{mn}(\gamma), v^{mn}(\gamma))\sin(\zeta\alpha)\cos(\eta\beta),$$

$$(\sigma_\gamma, D_\gamma) = \sum_m^\infty \sum_n^\infty (\sigma_\gamma^{mn}(\gamma), D_\gamma^{mn}(\gamma))\sin(\zeta\alpha)\sin(\eta\beta),$$

$$(p_\gamma, w) = \sum_m^\infty \sum_n^\infty (p_\gamma^{mn}(\gamma), w^{mn}(\gamma))\sin(\zeta\alpha)\sin(\eta\beta),$$

$$(\phi, T) = \sum_m^\infty \sum_n^\infty (\phi^{mn}(\gamma), T^{mn}(\gamma))\sin(\zeta\alpha)\sin(\eta\beta)。 \tag{11}$$

式中，$\zeta = m\pi / l_1, \eta = n\pi / l_2$。

将式(11)代入式(8)和式(9)对于任意一对

$m-n$，得到：

$$\partial \boldsymbol{R}/\partial \gamma = [\,D_{AS} \quad D_{BS}\,]\boldsymbol{R} \tag{12}$$

$$\boldsymbol{P} = I_S \boldsymbol{R} \tag{13}$$

由微分方程理论，式(12)的解为：

$$\boldsymbol{R}(\gamma) = \boldsymbol{D}(\gamma)\boldsymbol{R}(\gamma_0) \tag{14}$$

其中，$\boldsymbol{D}(\gamma) = \mathrm{e}^{[D_{AS}(\gamma) \quad D_{BS}(\gamma)]\cdot d}$，$d$ 为沿 γ 向的厚度；$\boldsymbol{R}(\gamma_0)$ 为初始值，$\boldsymbol{D}(\gamma)$ 可用精细积分法[17]计算。

对于 n 层正交双曲壳，可令式(14)中 $\gamma = \gamma_j$，$d = d_j$，γ_j 为第 j 子层的外径，$\gamma_{j\mathrm{m}}$ 为第 j 子层的中径，d_j 为第 j 子层的厚度，有：

$$\boldsymbol{R}(\gamma_j) = \boldsymbol{D}(\gamma_{jm})\,\boldsymbol{R}(\gamma_{j-1}) \tag{15}$$

根据层间应力和位移的连续条件，得到：

$$\boldsymbol{R}(\gamma_i) = \boldsymbol{T}\boldsymbol{R}(\gamma_0),\quad i = 1,\cdots,n \tag{16}$$

其中：

$$\boldsymbol{T} = D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1)$$

这样，由初始值 $\boldsymbol{R}(\gamma_0)$，便可求出层合双曲壳任意空间位置的应力和位移及曲面内的应力。

上述各公式中的变量及矩阵如下所示：

$$\boldsymbol{D}_A = \begin{bmatrix} s_a & 0 & \dfrac{As_9}{H_1} & \dfrac{As_{11}}{H_1} & 0 \\ 0 & s_b & \dfrac{Bs_{10}}{H_2} & \dfrac{Bs_{12}}{H_2} & 0 \\ -\dfrac{A}{H_1} & -\dfrac{B}{H_2} & -\dfrac{K_1 s_9}{H_1}-\dfrac{K_2 s_{10}}{H_2}-s_c & -\dfrac{K_1 s_{11}}{H_1}-\dfrac{K_2 s_{12}}{H_2} & 0 \\ \dfrac{As_7}{H_1} & \dfrac{Bs_8}{H_2} & 0 & -\dfrac{K_1}{H_1}-\dfrac{K_2}{H_2} & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{K_1}{H_1}-\dfrac{K_2}{H_2} \\ s_1 & 0 & 0 & 0 & 0 \\ 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & s_3 & s_4 & 0 \\ 0 & 0 & s_4 & s_5 & 0 \\ 0 & 0 & 0 & 0 & s_6 \end{bmatrix},$$

$$\boldsymbol{D}_{AS} = \begin{bmatrix} s_a & 0 & \dfrac{\zeta s_9}{H_1} & \dfrac{\zeta s_{11}}{H_1} & 0 \\ 0 & s_b & \dfrac{\eta s_{10}}{H_2} & \dfrac{\eta s_{12}}{H_2} & 0 \\ \dfrac{\zeta}{H_1} & -\dfrac{\eta}{H_2} & -\dfrac{K_1 s_9}{H_1}-\dfrac{K_2 s_{10}}{H_2}-s_c & -\dfrac{K_1 s_{11}}{H_1}-\dfrac{K_2 s_{12}}{H_2} & 0 \\ -\dfrac{\zeta s_7}{H_1} & -\dfrac{\eta s_8}{H_2} & 0 & -\dfrac{K_1}{H_1}-\dfrac{K_2}{H_2} & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{K_1}{H_1}-\dfrac{K_2}{H_2} \\ s_1 & 0 & 0 & 0 & 0 \\ 0 & s_3 & 0 & 0 & 0 \\ 0 & 0 & s_3 & s_4 & 0 \\ 0 & 0 & s_4 & s_5 & 0 \\ 0 & 0 & 0 & 0 & s_6 \end{bmatrix},$$

$$s_a = -\frac{2K_1}{H_1H_2}-\frac{K_2}{H_1H_2}-\frac{3\gamma K_1K_2}{H_1H_2},\quad s_b = -\frac{K_1}{H_1H_2}-\frac{2K_2}{H_1H_2}-\frac{3\gamma K_1K_2}{H_1H_2},\quad s_c = \frac{K_1}{H_1H_2}+\frac{K_2}{H_1H_2}+\frac{2\gamma K_1K_2}{H_1H_2},$$

$$\boldsymbol{D}_B=\begin{bmatrix} -\frac{A^2 s_{13}}{H_1^2}-\frac{B^2 s_{16}}{H_2^2} & -\frac{AB}{H_1H_2}(s_{14}+s_{16}) & -\frac{AK_1 s_{13}}{H_1^2}-\frac{AK_2 s_{14}}{H_1H_2} & 0 & -\frac{As_{21}}{H_1} \\ -\frac{AB}{H_1H_2}(s_{14}+s_{16}) & -\frac{A^2 s_{16}}{H_1^2}-\frac{B^2 s_{15}}{H_2^2} & -\frac{BK_2 s_{15}}{H_2^2}-\frac{BK_1 s_{14}}{H_1H_2} & 0 & -\frac{Bs_{22}}{H_2} \\ \frac{AK_1 s_{13}}{H_1^2}+\frac{AK_2 s_{14}}{H_1H_2} & \frac{BK_2 s_{15}}{H_2^2}+\frac{BK_1 s_{14}}{H_1H_2} & \frac{K_1^2 s_{13}}{H_1^2}-\frac{K_2^2 s_{15}}{H_2^2}+\frac{2K_1K_2 s_{14}}{H_1H_2} & 0 & \frac{K_1 s_{21}}{H_1}+\frac{K_2 s_{22}}{H_2} \\ 0 & 0 & 0 & -\frac{A^2 s_{17}}{H_1^2}-\frac{B^2 s_{18}}{H_2^2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{A^2 s_{19}}{H_1^2}-\frac{B^2 s_{20}}{H_2^2} \\ \frac{K_1}{H_1} & 0 & -\frac{A}{H_1} & \frac{As_7}{H_1} & 0 \\ 0 & \frac{K_2}{H_2} & -\frac{B}{H_2} & \frac{Bs_8}{H_2} & 0 \\ \frac{As_9}{H_1} & \frac{Bs_{10}}{H_2} & \frac{K_1 s_9}{H_1}+\frac{K_2 s_{10}}{H_2} & 0 & s_{23} \\ \frac{As_{11}}{H_1} & \frac{Bs_{12}}{H_2} & \frac{K_1 s_{11}}{H_1}+\frac{K_2 s_{12}}{H_2} & 0 & s_{24} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{R}(\gamma)=[\tau_{\alpha\gamma}^{mn}(\gamma)\quad \tau_{\beta\gamma}^{mn}(\gamma)\quad \sigma_{\gamma}^{mn}(\gamma)\quad D_{\gamma}^{mn}(\gamma)\quad p_{\gamma}^{mn}(\gamma)\quad u^{mn}(\gamma)\quad v^{mn}(\gamma)\quad w^{mn}(\gamma)\quad \phi^{mn}(\gamma)\quad T^{mn}(\gamma)]',$$

$$\boldsymbol{R}(\gamma_0)=[\tau_{\alpha\gamma}^{mn}(0)\quad \tau_{\beta\gamma}^{mn}(0)\quad \sigma_{\gamma}^{mn}(0)\quad D_{\gamma}^{mn}(0)\quad p_{\gamma}^{mn}(0)\quad u^{mn}(0)\quad v^{mn}(0)\quad w^{mn}(0)\quad \phi^{mn}(0)\quad T^{mn}(0)]',$$

$$I=\begin{bmatrix} 0 & 0 & -s_9 & -s_{11} & 0 & \frac{As_{13}}{H_1} & \frac{Bs_{14}}{H_2} & \frac{K_1 s_{13}}{H_1}+\frac{K_2 s_{14}}{H_2} & 0 & s_{21} \\ 0 & 0 & -s_{10} & -s_{12} & 0 & \frac{As_{14}}{H_1} & \frac{Bs_{15}}{H_2} & \frac{K_1 s_{14}}{H_1}+\frac{K_2 s_{15}}{H_2} & 0 & s_{22} \\ 0 & 0 & 0 & 0 & 0 & \frac{Bs_{16}}{H_2} & \frac{As_{16}}{H_1} & 0 & 0 & 0 \\ -s_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{As_{17}}{H_1} & 0 \\ 0 & -s_8 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Bs_{18}}{H_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{As_{19}}{H_1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{Bs_{20}}{H_B} \end{bmatrix},$$

$$
I_S=\begin{bmatrix}
0 & 0 & -s_9 & -s_{11} & 0 & -\dfrac{\zeta s_{13}}{H_1} & -\dfrac{\eta s_{14}}{H_2} & \dfrac{K_1 s_{13}}{H_1}+\dfrac{K_2 s_{14}}{H_2} & 0 & s_{21}\\
0 & 0 & -s_{10} & -s_{12} & 0 & -\dfrac{\zeta s_{14}}{H_1} & -\dfrac{\eta s_{15}}{H_2} & \dfrac{K_1 s_{14}}{H_1}+\dfrac{K_2 s_{15}}{H_2} & 0 & s_{22}\\
0 & 0 & 0 & 0 & 0 & \dfrac{\eta s_{16}}{H_2} & \dfrac{\zeta s_{16}}{H_1} & 0 & 0 & 0\\
-s_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\zeta s_{17}}{H_1} & 0\\
0 & -s_8 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\eta s_{18}}{H_2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\zeta s_{19}}{H_1}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{\eta s_{20}}{H_B}
\end{bmatrix},
$$

$$
\boldsymbol{D}_{BS}=\begin{bmatrix}
\dfrac{\zeta^2 s_{13}}{H_1^2}+\dfrac{\eta^2 s_{16}}{H_2^2} & \dfrac{\zeta\eta}{H_1H_2}(s_{14}+s_{16}) & -\dfrac{\zeta K_1 s_{13}}{H_1^2}-\dfrac{\zeta K_2 s_{14}}{H_1H_2} & 0 & -\dfrac{\zeta s_{21}}{H_1}\\
\dfrac{\zeta\eta}{H_1H_2}(s_{14}+s_{16}) & -\dfrac{\zeta^2 s_{16}}{H_1^2}-\dfrac{\eta^2 s_{15}}{H_2^2} & -\dfrac{\eta K_2 s_{15}}{H_2^2}-\dfrac{\eta K_1 s_{14}}{H_1H_2} & 0 & -\dfrac{\eta s_{22}}{H_2}\\
-\dfrac{\zeta K_1 s_{13}}{H_1^2}-\dfrac{\zeta K_2 s_{14}}{H_1H_2} & -\dfrac{\eta K_2 s_{15}}{H_2^2}-\dfrac{\eta K_1 s_{14}}{H_1H_2} & \dfrac{K_1^2 s_{13}}{H_1^2}-\dfrac{K_2^2 s_{15}}{H_2^2}+\dfrac{2K_1K_2 s_{14}}{H_1H_2} & 0 & \dfrac{K_1 s_{21}}{H_1}+\dfrac{K_2 s_{22}}{H_2}\\
0 & 0 & 0 & \dfrac{\zeta^2 s_{17}}{H_1^2}+\dfrac{\eta^2 s_{18}}{H_2^2} & 0\\
0 & 0 & 0 & 0 & \dfrac{\zeta^2 s_{19}}{H_1^2}+\dfrac{\eta^2 s_{20}}{H_2^2}\\
\dfrac{K_1}{H_1} & 0 & -\dfrac{\zeta}{H_1} & \dfrac{\zeta s_7}{H_1} & 0\\
0 & \dfrac{K_2}{H_2} & -\dfrac{\eta}{H_2} & \dfrac{\eta s_8}{H_2} & 0\\
-\dfrac{\zeta s_9}{H_1} & -\dfrac{\eta s_{10}}{H_2} & \dfrac{K_1 s_9}{H_1}+\dfrac{K_2 s_{10}}{H_2} & 0 & s_{23}\\
-\dfrac{\zeta s_{11}}{H_1} & -\dfrac{\eta s_{12}}{H_2} & \dfrac{K_1 s_{11}}{H_1}+\dfrac{K_2 s_{12}}{H_2} & 0 & s_{24}\\
0 & 0 & 0 & 0 & 0
\end{bmatrix}。
$$

3 数值算例

例. 取文献[14]中的例子来验证本文方法的正确性，分析 5 层的混合层板[$90_p/0/90/0/0_p$]，上下两层为压电材料 PZT-5A，中间三层均为弹性材料 Gr/Ep。层合板厚度 H=10mm，长宽 l_1=l_2=50H，压电层厚度均为 $h_p=0.05H$，各弹性层厚度均为 $h_e=0.9H/3$。工况为：$\sigma_\gamma=\sin(\zeta\alpha)\sin(\eta\beta)$，$T=\sin(\zeta\alpha)\sin(\eta\beta)$。选取 $m=n=1$，取不同半径值，其计算结果如表 1 所示。

从表 1 可看出，随着双曲壳半径的增大，其内表面、外表面上的广义应力和位移逐步逼近文献[14]，从而间接验证在正交双曲坐标系下，本文方法的正确性。

表 1 分析结果与比较

Table 1 Result and comparison

半径值	u_t (×10^{-7})	w_t (×10^{-5})	$D_{\gamma t}$ (×10^{-4})	$p_{\gamma t}$	u_b (×10^{-7})	w_b (×10^{-5})	$D_{\gamma b}$ (×10^{-4})	$p_{\gamma b}$
1×10^5H	−5.38995	−1.27216	1.22208	70.7228	2.56029	−1.26439	1.21864	70.2005
1×10^7H	−5.37841	−1.27383	1.22176	70.7221	2.58229	−1.26606	1.21835	70.2012
$1\times10^{10}H$	−5.37829	−1.27385	1.22176	70.7221	2.58251	−1.26608	1.21835	70.2012
文献[14]	−5.37829	−1.27385	1.22176	70.7221	2.58251	−1.26608	1.21835	70.2012

4 结论

本文在正交双曲坐标系下，根据状态空间法和对偶变量理论，推导了考虑温度梯度，压电材料稳态温度问题的齐次状态方程。齐次状态方程的导出，不仅大大简化了层合壳结构的求解，且其计算步骤简单，数据累积误差小，提高了计算精度。本文的方法可进一步扩展到有限元方法中，分析考虑温度梯度的复合压电材料有限元形式的齐次状态方程，为复杂边界条件下的半解析解提供一种新方法。

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