

Symplectic Reduction of Sheaves of \mathcal{A} -modules

Anastasios Mallios, Patrice P. Ntumba*

Abstract

Given an arbitrary sheaf \mathcal{E} of \mathcal{A} -modules (or \mathcal{A} -module in short) on a topological space X , we define *annihilator sheaves* of sub- \mathcal{A} -modules of \mathcal{E} in a way similar to the classical case, and obtain thereafter the analog of the *main theorem*, regarding classical annihilators in module theory, see Curtis[[5], pp. 240-242]. The familiar classical properties, satisfied by annihilator sheaves, allow us to set clearly the *sheaf-theoretic version* of *symplectic reduction*, which is the main goal in this paper.

Subject Classification (2000): 55P05.

Key Words: \mathcal{A} -module, annihilator sheaves, ordered \mathbb{R} -algebraized space, , symplectic \mathcal{A} -module, affine Darboux theorem.

Introduction

This paper is part of our ongoing project of *algebraizing* classical symplectic geometry using the tools of abstract differential geometry (*à la* Mallios). Our main reference as far as abstract differential geometry is concerned is the first author's book [10]. For the sake of convenience, we recall here some of the objects of abstract differential geometry that recur all throughout.

*Is the corresponding author for the paper

Let X be a topological space. A *sheaf of \mathbb{C} -algebras* or a *\mathbb{C} -algebra sheaf*, on X , is a triple $\mathcal{A} \equiv (\mathcal{A}, \tau, X)$ satisfying the following conditions:

- (i) \mathcal{A} is a sheaf of rings.
- (ii) Fibers $\mathcal{A}_x \equiv \tau^{-1}(x)$, $x \in X$, are \mathbb{C} -algebras.
- (iii) The *scalar multiplication* in \mathcal{A} , viz. the map

$$\mathbb{C} \times \mathcal{A} \longrightarrow \mathcal{A} : (c, a) \longmapsto c \cdot a \in \mathcal{A}_x \subseteq \mathcal{A}$$

with $\tau(a) = x \in X$, is continuous; in this mapping, \mathbb{C} is assumed to carry the discrete topology.

The triple (\mathcal{A}, τ, X) is called a *unital \mathbb{C} -algebra sheaf* if the individual fibers of \mathcal{A} , \mathcal{A}_x , $x \in X$, are unital \mathbb{C} -algebras. A pair (X, \mathcal{A}) , with \mathcal{A} assumed to be unital and commutative, is called a *\mathcal{C} -algebraized space*. Next, suppose that $\mathcal{A} \equiv (\mathcal{A}, \tau, X)$ is a unital \mathbb{C} -algebra sheaf on X . A *sheaf of \mathcal{A} -modules* (or an *\mathcal{A} -module*), on X , is a sheaf, $\mathcal{E} \equiv (\mathcal{E}, \rho, X)$, on X such that the following properties hold:

- (iv) \mathcal{E} is a sheaf of abelian groups on X .
- (v) Fibers \mathcal{E}_x , $x \in X$, of \mathcal{E} are \mathcal{A}_x -modules.
- (vi) The *left action* $\mathcal{A} \circ \mathcal{E} \longrightarrow \mathcal{E}$, described by

$$(a, z) \longmapsto a \cdot z \in \mathcal{E}_x \subseteq \mathcal{E},$$

with $\tau(a) = \rho(z) = x \in X$, is continuous.

The sheaf-theoretic version of the classical notion of a *dual module* is defined in this manner: Given a \mathbb{C} -algebraized space (X, \mathcal{A}) and an \mathcal{A} -module \mathcal{E} on X , the \mathcal{A} -module (on X)

$$\mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$$

is called the *dual \mathcal{A} -module* of \mathcal{E} . For two given \mathcal{A} -modules on a topological space X , $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is the \mathcal{A} -module generated on X by the (complete)

presheaf, given by $U \mapsto \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U)$, where U runs over the open subsets of X ; the restriction maps of this presheaf are quite obvious. A most familiar consequence regarding dual \mathcal{A} -modules is that given a free \mathcal{A} -module \mathcal{E} of finite rank on X , one has $\mathcal{E} = \mathcal{E}^*$, within an \mathcal{A} -isomorphism.

Section 1 is concerned with *annihilator sheaves* of sub- \mathcal{A} -modules of arbitrary \mathcal{A} -modules on one hand, and φ -annihilator sheaves, i.e. annihilator sheaves (of sub- \mathcal{A} -modules) with respect to a non-degenerate bilinear \mathcal{A} -morphism $\varphi : \mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{A}$, where \mathcal{E} and \mathcal{F} are free \mathcal{A} -modules of finite rank. The sheaf-theoretic version of the main theorem on classical annihilators is examined. The section ends with the interesting result that given a sub- \mathcal{A} -module \mathcal{F} of an \mathcal{A} -module \mathcal{E} , the dual \mathcal{A} -module $(\mathcal{E}/\mathcal{F})^*$ is \mathcal{A} -isomorphic to the annihilator \mathcal{F}^\perp of \mathcal{F} .

Section 2 deals with properties of exterior *rankwise* \mathcal{A} -2-forms. We provide another proof for the *affine Darboux theorem*. The proof is derived from E. Cartan[4].

Section 3, which is the last section, outlines the *symplectic reduction* of an \mathcal{A} -module \mathcal{E} by a co-isotropic sub- \mathcal{A} -module \mathcal{F} of \mathcal{E} ; the \mathcal{A} -module \mathcal{E} carries a symplectic (\mathcal{A} -) structure, given by the \mathcal{A} -morphism $\omega : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A}$.

1 Annihilator Sheaves

Definition 1.1 Let (\mathcal{S}, π, X) be a sheaf. By a **subsheaf** of \mathcal{S} , we mean a sheaf \mathcal{E} on X , *generated by a presheaf* $(E(U), \sigma_V^U)$ which is such that, for all open $U \subseteq X$ and open $V \subseteq U$,

- $E(U) \subseteq \mathcal{S}(U)$,
- $\sigma_V^U = \rho_V^U|_{E(U)}$,

where $(\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S}), \rho_V^U) \equiv \Gamma(\mathcal{S})$ is the (complete) presheaf of sections of the sheaf \mathcal{S} , cf. Mallios[[10], Lemma 11.1, p. 48]. \square

We can now define the notion of sub- \mathcal{A} -module of a given \mathcal{A} -module, which will be of use in the sequel.

Definition 1.2 A subsheaf \mathcal{E} of an \mathcal{A} -module \mathcal{S} , defined on a topological space X , is called a **sub- \mathcal{A} -module** of \mathcal{S} if \mathcal{E} is an \mathcal{A} -module and the *inclusion* $i : \mathcal{E} \subseteq \mathcal{S}$ is an \mathcal{A} -morphism. \square

Lemma 1.1 *Subsheaves are open subsets, and conversely.*

Proof. Let \mathcal{S} be a sheaf on (X, \mathcal{T}) , \mathcal{E} a subsheaf, of \mathcal{S} , generated by the presheaf $(E(U), \sigma_V^U)$, and let us denote by \mathfrak{R} the set $\bigcup\{E(U) : U \in \mathcal{T}\}$. According to Mallios[[10], Theorem 3.1, p.14], the family

$$\mathcal{B} = \{s(U) : s \in \mathfrak{R} \text{ and } U \in \mathcal{T} \text{ with } U = \text{Dom}(s)\}$$

is a basis for the topology of \mathcal{E} , with respect to which \mathcal{E} is a sheaf on X . But $E(U) \subseteq \mathcal{S}(U)$ for every open $U \subseteq X$, therefore, for all $s \in \mathfrak{R}$, $s(U)$ is open in \mathcal{S} , and thus $\bigcup \mathcal{B} = \mathcal{E}$ is open in \mathcal{S} , as desired.

For the converse, see Mallios[[10], p. 5]. \blacksquare

It follows from Lemma 1.1 that Definition 1.1 and Mallios' definition of subsheaf, see Mallios[[10], p. 5], are equivalent.

Definition 1.3 Let \mathcal{E} be an \mathcal{A} -module on a topological space X , and \mathcal{F} a sub- \mathcal{A} -module of \mathcal{E} . Assume that $(\mathcal{E}(U), \sigma_V^U)$ is the (complete) presheaf of sections of \mathcal{E} . By the **\mathcal{A} -annihilator sheaf** (or **sheaf of \mathcal{A} -annihilators**, or just **\mathcal{A} -annihilator**) of \mathcal{F} , we mean the sheaf generated by the *presheaf*, given by the correspondence

$$U \longmapsto \mathcal{F}(U)^\perp,$$

where U is an open subset of X and

$$\mathcal{F}(U)^\perp = \{t \in \mathcal{E}^*(U) : t(s) = 0 \text{ for all } s \in \mathcal{E}(U)\},$$

along with restriction maps

$$(\rho^\perp)_V^U : \mathcal{F}(U)^\perp \longrightarrow \mathcal{F}(V)^\perp$$

such that

$$(\rho^\perp)_V^U := (\sigma^*)_V^U|_{\mathcal{F}(U)^\perp},$$

with the $(\sigma^*)_V^U : \mathcal{E}^*(U) \longrightarrow \mathcal{E}^*(V)$ being the restriction maps for the *dual presheaf* $(\mathcal{E}^*(U), (\sigma^*)_V^U)$. We denote by

$$\mathcal{F}^\perp$$

the annihilator sheaf of \mathcal{F} . □

It follows from Definition 1.3, that the annihilator \mathcal{F}^\perp of a sub- \mathcal{A} -module \mathcal{F} of an \mathcal{A} -module \mathcal{E} is a subsheaf of the dual \mathcal{A} -module \mathcal{E}^* .

Lemma 1.2 *Let \mathcal{E} be an \mathcal{A} -module on a topological space X , and \mathcal{F} a sub- \mathcal{A} -module of \mathcal{E} . Then, the correspondence*

$$U \longmapsto \mathcal{F}(U)^\perp$$

along with maps $(\rho^\perp)_V^U$, as defined above, yields a complete presheaf of \mathcal{A} -modules on X .

Proof. First, we notice that it is immediate that for every open $U \subseteq X$, $\mathcal{F}(U)^\perp$ is an $\mathcal{A}(U)$ -module and that $(\mathcal{F}(U)^\perp, (\rho^\perp)_V^U)$ is a presheaf of \mathcal{A} -modules on X . To see that the presheaf $(\mathcal{F}(U)^\perp, (\rho^\perp)_V^U)$ is complete, let us fix an open subset U of X and an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of U . Next, let s, t be two elements of $\mathcal{F}(U)^\perp$ such that

$$(\rho^\perp)_{U_\alpha}^U(s) \equiv s_\alpha = t_\alpha \equiv (\rho^\perp)_{U_\alpha}^U(t),$$

for every $\alpha \in I$. Since $\mathcal{F}(U)^\perp \subseteq \mathcal{E}^*(U)$, so $s, t \in \mathcal{E}^*(U)$; $(\mathcal{E}^*(U), (\rho^*)_V^U)$ being the complete presheaf of sections associated with the sheaf \mathcal{E}^* (for the completeness of $(\mathcal{E}^*(U), (\rho^*)_V^U)$, see Mallios[[10], (6.4), Definition 6.1, p.134, and (5.1), p.298]), and

$$(\rho^\perp)_{U_\alpha}^U(s) = (\rho^*)_{U_\alpha}^U(s) \quad \text{and} \quad (\rho^\perp)_{U_\alpha}^U(t) = (\rho^*)_{U_\alpha}^U(t) ,$$

we have that $s = t$. Therefore, axiom (S1) (cf. Mallios[[10], p.46]) is satisfied*.

Now, let us check that axiom (S2), see Mallios[[10], p.47], is also satisfied. So, let U and $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be as above. Furthermore, let $(t_\alpha) \in \prod_\alpha \mathcal{F}(U_\alpha)^\perp$ be such that for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ in \mathcal{U} , one has

$$(\rho^\perp)_{U_{\alpha\beta}}^{U_\alpha}(t_\alpha) \equiv t_\alpha|_{U_{\alpha\beta}} = t_\beta|_{U_{\alpha\beta}} \equiv (\rho^\perp)_{U_{\alpha\beta}}^{U_\beta}(t_\beta).$$

Since $(\rho^\perp)_{U_{\alpha\beta}}^{U_\alpha} = (\rho^*)_{U_{\alpha\beta}}^{U_\alpha}|_{\mathcal{F}(U_\alpha)^\perp}$, $\mathcal{F}(U_\alpha)^\perp \subseteq \mathcal{E}^*(U_\alpha)$ for all $\alpha, \beta \in I$, and the presheaf $(\mathcal{E}^*(U), (\rho^*)_{U_\alpha}^U)$ is *complete*, there exists an element $t \in \mathcal{E}^*(U)$ such that

$$(\rho^*)_{U_\alpha}^U(t) \equiv t|_{U_\alpha} = t_\alpha,$$

for every $\alpha \in I$. We should now show that t is indeed an element of $\mathcal{F}(U)^\perp$. To this end, suppose that there exists an $s \in \mathcal{E}(U)$ such that $t(s) \neq 0 \in \mathcal{A}(U)$; this implies that for some $\alpha \in I$,

$$(\rho^*)_{U_\alpha}^U(t)(\rho_{U_\alpha}^U(s)) \equiv t|_{U_\alpha}(s|_{U_\alpha}) = t_\alpha(s_\alpha) \neq 0,$$

which is impossible as $t_\alpha \in \mathcal{F}(U_\alpha)^\perp$ and $s_\alpha \in \mathcal{E}(U_\alpha)$. Thus, $t \in \mathcal{F}(U)^\perp$; hence (S2) is satisfied. ■

By virtue of Proposition 11.1, see Mallios[[10], p.51], if \mathcal{F} is a sub- \mathcal{A} -module of an \mathcal{A} -module \mathcal{E} , then

$$\mathcal{F}^\perp(U) = \mathcal{F}(U)^\perp \tag{1}$$

within an $\mathcal{A}(U)$ -isomorphism.

From the relation (1), we have the following corollary.

Corollary 1.1 *Let \mathcal{E} be an \mathcal{A} -module on a topological space, and \mathcal{F} a sub- \mathcal{A} -module of \mathcal{E} . Then*

$$\mathcal{F}^\perp = \{z \in \mathcal{E}^* : z(u) = 0 \text{ for all } u \in \mathcal{F}\}.$$

*We can however notice here that the presheaf of Lemma 1.2 can also be considered as a *functional* presheaf by its very definition, hence axiom (S1), see Mallios[[11]relation 1.12, p. 81]

Lemma 1.3 *Let \mathcal{E} be an \mathcal{A} -module on a topological space X , and U an open subset of X . Then,*

$$\mathcal{E}^*|_U = (\mathcal{E}|_U)^*$$

within an $\mathcal{A}|_U$ -isomorphism of the sheaves in question.

Proof. For any open subset $V \subseteq U$, we have

$$(\mathcal{E}^*|_U)(V) \equiv (\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})|_U)(V) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})(V) = \text{Hom}_{\mathcal{A}|_V}(\mathcal{E}|_V, \mathcal{A}|_V),$$

and

$$(\mathcal{E}|_U)^*(V) \equiv \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U)(V) = \text{Hom}_{\mathcal{A}|_V}(\mathcal{E}|_V, \mathcal{A}|_V).$$

We thus conclude that since these two sheaves have isomorphic (local) sections, they are $\mathcal{A}|_U$ -isomorphic. ■

Using the language of Category Theory, see MacLane[9] for Category Theory, Lemma 1.3 infers that the *dual- \mathcal{A} -module functor* (cf. Mallios[[10], (5.20), p.301]) commutes with the *restriction-(over U)-of- \mathcal{A} -modules functor*. Schematically, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{E}^* \\ \downarrow & & \downarrow \\ \mathcal{E}|_U & \longrightarrow & \mathcal{E}^*|_U = (\mathcal{E}|_U)^*. \end{array}$$

The following definition hinges on Lemma 1.3 and Mallios[[10], (5.2), p.298].

Definition 1.4 Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X , and let $\varphi \in \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(U)$ with U an open subset of X . The **adjoint** of φ is the sheaf $\mathcal{A}|_U$ -morphism

$$\begin{aligned} \varphi^* &\equiv (\varphi_V^*)_{U \supseteq V, \text{open}} \equiv ((\varphi_V)^*)_{U \supseteq V, \text{open}} \in \text{Hom}_{(\mathcal{A}|_U)^*}((\mathcal{F}|_U)^*, (\mathcal{E}|_U)^*) \\ &= \text{Hom}_{\mathcal{A}^*|_U}(\mathcal{F}^*|_U, \mathcal{E}^*|_U) = \text{Hom}_{\mathcal{A}|_U}(\mathcal{F}^*|_U, \mathcal{E}^*|_U) = \mathcal{H}om_{\mathcal{A}}(\mathcal{F}^*, \mathcal{E}^*)(U) \end{aligned}$$

such that for all $\omega \equiv (\omega_W)_{V \supseteq W, \text{open}} \in \text{Hom}_{\mathcal{A}|_V}(\mathcal{F}|_V, \mathcal{A}|_V) = \mathcal{F}(V)$, one has

$$\varphi_V^*(\omega) = (\omega \circ \varphi)|_{\mathcal{E}|_V},$$

that is

$$(\varphi_V^*(\omega))_W = \omega_W \circ \varphi_W$$

for all open $W \subseteq V$. □

Scholium 1.1 If the \mathcal{A} -modules \mathcal{E} and \mathcal{F} in Definition 1.4 are *vector sheaves* on X , then concerning the *adjoint* φ^* of an \mathcal{A} -morphism $\varphi \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, we have

$$\varphi^* \in \text{Hom}_{\mathcal{A}}(\mathcal{F}^*, \mathcal{E}^*) = \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})^* = \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{E}), \quad (2)$$

where as usual the displayed equalities of (2) are \mathcal{A} -isomorphisms of the \mathcal{A} -modules involved. For these foregone \mathcal{A} -isomorphisms, see Mallios[[10], Corollary 6.3, p.306].

Theorem 1.1 *Let \mathcal{E} and \mathcal{F} be an \mathcal{A} -module on a topological space X , and $\varphi \in \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U)$, where U is some open subset of X . Then,*

$$(\text{im}\varphi)^\perp = \ker \varphi^*$$

within an $\mathcal{A}|_U$ -isomorphism, that is for every open subset $V \subseteq U$, we have

$$(\text{im}\varphi_V)^\perp = \ker \varphi_V^*$$

within an $\mathcal{A}(V)$ -isomorphism of modules.

Proof. Let $\omega \in (\mathcal{F}|_U)^* = \mathcal{F}^*|_U$. Then,

$$\begin{aligned} \omega \in \ker \varphi_V^* &\Leftrightarrow \varphi_V^* \omega = 0 \\ &\Leftrightarrow \omega_W \circ \varphi_W = 0 \text{ for all open } W \subseteq V \\ &\Leftrightarrow (\omega_W \circ \varphi_W)(s) = 0 \text{ for all open } W \subseteq V \text{ and } s \in \mathcal{E}(W) \\ &\Leftrightarrow \omega_W(t) = 0 \text{ for all open } W \subseteq V \text{ and } t \in \varphi_W(\mathcal{E}(W)) \\ &\Leftrightarrow \omega_W \in \text{im}(\mathcal{E}(W))^\perp \text{ for all open } W \subseteq V \\ &\Leftrightarrow \omega \in (\text{im}\mathcal{E}(V))^\perp = (\text{im}\varphi_V)^\perp \end{aligned}$$

Thus, we have the sought $\mathcal{A}(V)$ -isomorphism for every open $V \subseteq U$; the proof is thus finished. ■

The notion of an annihilator sheaf may be generalized by considering any two \mathcal{A} -modules that are “dual” with respect to some non-degenerate bilinear \mathcal{A} -form; i.e. \mathcal{A} -isomorphic within an \mathcal{A} -isomorphism determined by the non-degenerate bilinear \mathcal{A} -form considered. Before we define the notion of non-degenerate bilinear \mathcal{A} -morphism, we would like to make the following convention: In fact, let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X . An \mathcal{A} -morphism $\varphi \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ will be denoted $\varphi \equiv (\varphi^U)_{U \in \mathcal{T}}$ or $\varphi \equiv (\varphi_U)_{U \in \mathcal{T}}$. These notations will depend on the situation at hand, but this will be done for the sole purpose of making indices a lot easier to handle.

Definition 1.5 Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on X . A bilinear \mathcal{A} -morphism $\varphi \equiv (\varphi^U)_{X \supseteq U, \text{ open}} : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ is said to be **non-degenerate** if for every open subset U of X , the following conditions hold.

$$\varphi^U(s, t) = 0 \quad \text{for all } t \in \mathcal{F}(U) \text{ implies that } s = 0$$

and

$$\varphi^U(s, t) = 0 \quad \text{for all } s \in \mathcal{E}(U) \text{ implies that } t = 0.$$

□

Proposition 1.1 Let \mathcal{E} and \mathcal{F} be each a free \mathcal{A} -module of finite rank, and $\varphi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ a non-degenerate bilinear \mathcal{A} -morphism. For every (local) section $t \in \mathcal{F}(U)$, let $\alpha_t^U : \mathcal{E}(U) \longrightarrow \mathcal{A}(U)$ be the mapping

$$\alpha_t^U(s) := \varphi^U(s, t) \tag{3}$$

for all $s \in \mathcal{E}(U)$. Then, the mapping α_t^U , as defined above, is an element of $\mathcal{E}(U)^* = \mathcal{E}^*(U) = \mathcal{E}(U)$, where the previous equalities are derived from Mallios[[10], (3.14), p.122, (5.2.1), p.298]. On the other hand, the mapping

$$\vartheta : \mathcal{F} \longrightarrow \mathcal{E}$$

defined by

$$\vartheta^U(t) = \alpha_t^U$$

yields an \mathcal{A} -isomorphism of \mathcal{F} onto \mathcal{E} .

Proof. That α_t^U , as given in relation (3), is an element of $\mathcal{E}^*(U)$ is clear. It is also easy to see that the mapping

$$\vartheta^U(t) = \alpha_t^U$$

is a module morphism of $\mathcal{F}(U)$ into $\mathcal{E}^*(U)$, where, as above, U is an open subset of X . Now, let us show that every component ϑ^U of ϑ is an isomorphism of $\mathcal{A}(U)$ -modules $\mathcal{F}(U)$ and $\mathcal{E}(U)$. To this purpose, suppose first that $\vartheta^U(t) = \vartheta^U(t')$, for $t, t' \in \mathcal{F}(U)$; then

$$\varphi^U(s, t) = \varphi^U(s, t')$$

for all $s \in \mathcal{E}(U)$. By the bilinearity and non-degeneracy of φ , we have that $t = t'$. Therefore, ϑ^U is one-to-one.

Now, suppose that the ranks of the free \mathcal{A} -modules \mathcal{E} and \mathcal{F} are n and m , respectively, that is $\mathcal{E} = \mathcal{A}^n$ and $\mathcal{F} = \mathcal{A}^m$ within \mathcal{A} -isomorphisms. Since $\vartheta^U : \mathcal{F}(U) = \mathcal{A}^m(U) \rightarrow \mathcal{A}^n(U) = \mathcal{E}(U)$ is one-to-one and $\mathcal{A}^k(U)$ is a free module for every $k \in \mathbb{N}$, it follows that

$$\dim \mathcal{E}(U) \geq \dim \mathcal{F}(U).$$

By a similar argument, one shows that there exists a one-to-one $\mathcal{A}(U)$ -morphism of $\mathcal{E}(U)$ into $\mathcal{F}(U)$, and therefore

$$\dim \mathcal{F}(U) \geq \dim \mathcal{E}(U);$$

hence

$$\dim \mathcal{E}(U) = \dim \mathcal{F}(U);$$

which implies that ϑ^U is onto. Therefore, for every open $U \subseteq X$, $\mathcal{E}(U)$ is $\mathcal{A}(U)$ -isomorphic to $\mathcal{F}(U)$. The restriction maps of the associated complete presheaves of sections $(\Gamma(U, \mathcal{E}), \rho_V^U)$ and $(\Gamma(U, \mathcal{F}), \lambda_V^U)$ can be chosen in such a way that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\vartheta^U} & \mathcal{E}(U) \\ \lambda_V^U \downarrow & & \downarrow \rho_V^U \\ \mathcal{F}(V) & \xrightarrow{\vartheta^V} & \mathcal{E}(V) \end{array}$$

commutes. Hence, $\vartheta \equiv (\vartheta^U)$ is an \mathcal{A} -isomorphism between \mathcal{F} and \mathcal{E} . ■

From Proposition 1.1, we bring about the following definition.

Definition 1.6 A pair of free \mathcal{A} -modules \mathcal{E} and \mathcal{F} are said to be **\mathcal{A} -dual** or just **dual** with respect to a bilinear \mathcal{A} -morphism $\varphi : \mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{A}$ if φ is non-degenerate. If we want to stress the fact that φ is the bilinear map with respect to which the free \mathcal{A} -modules \mathcal{E} and \mathcal{F} are dual, we shall say that \mathcal{E} and \mathcal{F} are **φ -dual**. \square

We also need the following definition.

Definition 1.7 Let \mathcal{E} and \mathcal{F} be φ -dual free \mathcal{A} -modules, $S \equiv (S^U) \in \text{End}_{\mathcal{A}}\mathcal{E} := (\text{End}_{\mathcal{A}}\mathcal{E})(X)$, and $T \equiv (T^U) \in \text{End}_{\mathcal{A}}\mathcal{F} := (\text{End}_{\mathcal{A}}\mathcal{F})(X)$. Then, S and T are said to be **transposes** of each other provided that

$$\varphi^U(s, T^U(t)) = \varphi^U(S^U(s), t)$$

for all $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(U)$. \square

Theorem 1.2 Let \mathcal{E} and \mathcal{F} be free \mathcal{A} -modules which are dual with respect to a bilinear \mathcal{A} -morphism $\varphi : \mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{A}$. Moreover, let $S \equiv (S^U) \in \text{End}_{\mathcal{A}}\mathcal{E}$; then there exists a uniquely determined family

$$T \equiv (T^U) \in \prod_{X \supseteq U, \text{open}} \text{End}_{\mathcal{A}(U)}\mathcal{F}(U)$$

such that for all open $U \subseteq X$, S^U and T^U are transposes of each other. Furthermore, if $T \equiv (T^U)$ is an \mathcal{A} -endomorphism $\mathcal{F} \rightarrow \mathcal{F}$, then S and T are transposes of each other as sheaf morphisms.

Proof. We have to show that for all open $U \subseteq X$ and $t \in \mathcal{F}(U)$, there exists a unique element $t' \in \mathcal{F}(U)$ such that

$$\varphi^U(s, t') = \varphi^U(S^U(s), t), \quad s \in \mathcal{E}(U), \quad (4)$$

so that we can define $T^U(t)$ to be t' . For each open $U \subseteq X$, $S^U \in \text{End}_{\mathcal{A}(U)}\mathcal{E}(U)$; because of the latter the mapping

$$\psi^U : s \mapsto \varphi^U(S^U(s), t)$$

is an element of $\mathcal{E}^*(U) = \mathcal{E}(U)$. By Proposition 1.1, there exists a unique element $t' \in \mathcal{F}(U)$ such that

$$\vartheta^U(t') (= \alpha_{t'}^U) = \psi^U;$$

so for all $s \in \mathcal{E}(U)$, we have

$$\varphi^U(s, t') = \varphi^U(S^U(s), t).$$

Now, define $T^U : \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ as $T^U(r) = r'$, where r' is the solution of the equation obtained by substituting r for t in (4). Then, we have

$$\varphi^U(s, T^U(t)) = \varphi^U(S^U(s), t)$$

for all $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(U)$. The mapping T^U is an $\mathcal{A}(U)$ -endomorphism of $\mathcal{F}(U)$. (The details of checking that $T^U \in \text{End}_{\mathcal{A}(U)}\mathcal{E}(U)$ are presented in the proof of Theorem (27.7), [5], p.239.) ■

Lemma 1.4 *Let \mathcal{E} and \mathcal{F} be dual free \mathcal{A} -modules, with respect to a non-degenerate bilinear \mathcal{A} -morphism φ , and \mathcal{G} a sub- \mathcal{A} -module of \mathcal{E} . For all open $U \subseteq X$, let*

$$\mathcal{G}(U)^\perp := \{t \in \mathcal{F}(U) : \varphi^U(s, t) = 0, \text{ for all } s \in \mathcal{G}(U)\}.$$

Moreover, let $(\sigma^\perp)_V^U : \mathcal{G}(U)^\perp \longrightarrow \mathcal{G}(V)^\perp$, where $V \subseteq U$, with V and U open in X , be mappings

$$(\sigma^\perp)_V^U := \rho_V^U|_{\mathcal{G}(U)^\perp},$$

where the ρ_V^U are the restriction maps for the (complete) presheaf of sections $(\mathcal{F}(U), \rho_V^U)$. Then, the correspondence

$$U \longmapsto \mathcal{G}(U)^\perp, \tag{5}$$

along with the maps $(\sigma^\perp)_V^U$, yields a complete presheaf of \mathcal{A} -modules on X .

Proof. We first notice, by virtue of Proposition 1.1, that, for all open subset $U \subseteq X$, $\mathcal{F}(U)$ is $\mathcal{A}(U)$ -isomorphic to $\mathcal{E}(U)$, so that restricting a map such as $\rho_V^U : \mathcal{E}(U) \longrightarrow \mathcal{E}(V)$ to $\mathcal{G}(U)^\perp$ makes sense.

The rest of the proof is similar to the proof of Lemma 1.2. ■

Lemma 1.4 makes the following definition rather natural.

Definition 1.8 Let \mathcal{E} , \mathcal{G} and \mathcal{F} be as in Lemma 1.4. We denote by

$$\mathcal{G}^\perp$$

the sheaf on X generated by the (complete) presheaf defined by (5). We call it the **\mathcal{A} -annihilator sheaf** of \mathcal{G} with respect to the non-degenerate bilinear \mathcal{A} -morphism φ (or **\mathcal{A}_φ -annihilator**) of \mathcal{G} . \square

Corollary 1.2 For any φ -dual free \mathcal{A} -modules \mathcal{E} and \mathcal{F} on X , the annihilator sheaf \mathcal{G}^\perp of a subsheaf \mathcal{G} of \mathcal{E} , as defined above, is an \mathcal{A} -module on X .

We are now set for the *main theorem* on \mathcal{A} -annihilator sheaves. The results of the analog theorem in classical module theory can be found in Curtis[[5], pp. 240-242], Adkins and Weintraub[[2], pp. 345-349]. But before we state the theorem, we open a breach for the analog of a submodule of a module M , generated by the set $\cup_{i \in I} M_i$, where every M_i is a submodule of M .

Lemma 1.5 Let \mathcal{E} be an \mathcal{A} -module on X , and $(\mathcal{F}_i)_{i \in I}$ a family of sub- \mathcal{A} -modules of \mathcal{E} . For every open $U \subseteq X$, let

$$F(U) := \langle \cup_{i \in I} \mathcal{F}_i(U) \rangle,$$

that is $F(U)$ is the $\mathcal{A}(U)$ -submodule of $\mathcal{E}(U)$, generated by $\cup_{i \in I} \mathcal{F}_i(U)$, i.e. $F(U)$ is the sum of the family $(\mathcal{F}_i(U))_{i \in I}$. The presheaf, given by

$$U \mapsto F(U) := \langle \cup_{i \in I} \mathcal{F}_i(U) \rangle, \tag{6}$$

where U runs over the open subsets of X , along with restriction maps $\sigma_V^U = \rho_V^U|_{\mathcal{F}(U)}$ ($(\mathcal{E}(U), \rho_V^U)$ is the presheaf of sections of \mathcal{E}), is complete.

Proof. That $(F(U), \sigma_V^U)$ is a presheaf of \mathcal{A} -modules on X is easy to see. To see that the presheaf $(F(U), \sigma_V^U)$ is complete, we need check axioms (S1) and (S2) in Mallios[[10], pp. 46-47]. It is easy to see that axiom (S1) is satisfied. To verify that axiom (S2) is satisfied, let U be an open subset of X and

$\mathcal{U} = (U_\alpha)_{\alpha \in I}$ an open covering of \mathcal{U} . Furthermore, let $(t_\alpha) = \prod_\alpha F(U_\alpha)$ be such that for any $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta \neq \emptyset$ in \mathcal{U} , one has

$$\sigma_{U_{\alpha\beta}}^{U_\alpha}(t_\alpha) \equiv t_\alpha|_{U_{\alpha\beta}} = t_\beta|_{U_{\alpha\beta}} \equiv \sigma_{U_{\alpha\beta}}^{U_\beta}(t_\beta).$$

Since $\sigma_{U_{\alpha\beta}}^{U_\alpha} = \rho_{U_{\alpha\beta}}^{U_\alpha}|_{F(U_\alpha)}$, $F(U_\alpha) \subseteq \mathcal{E}(U_\alpha)$ for all $\alpha, \beta \in I$, and the presheaf $(\mathcal{E}(U), \rho_V^U)$ is complete, there exists an element $t \in \mathcal{E}(U)$ such that

$$\rho_{U_\alpha}^U(t) \equiv t|_{U_\alpha} = t_\alpha,$$

for every $\alpha \in I$. It remains to show that t is indeed an element of $F(U)$. Suppose that t is not an element of $F(U)$, so it follows that t cannot be written as $t = \sum_{k \in J \subseteq I}^m a_{i_k} t^{i_k}$, with J finite, $a_{i_k} \in \mathcal{A}(U)$, and $t^{i_k} \in \mathcal{F}_{i_k}(U)$. This means that for some $x \in U$, $t_x \equiv t(x)$ cannot be written as a linear combination of finitely many $t_x^{i_k} \equiv t^{i_k}(x)$, where $t^{i_k} \in \mathcal{F}_{i_k}(U)$ and $k \in J$ with J a finite subset of I . But this is a contradiction as $x \in U_\alpha$ for some $\alpha \in I$, and $t|_{U_\alpha} = \sum_{k=1}^m a_{i_k} t^{i_k}$, where $m \in \mathbb{N}$, $a_{i_k} \in \mathcal{A}(U_\alpha)$, and $t^{i_k} \in \mathcal{F}_{i_k}(U_\alpha)$. Thus, $t \in F(U)$, and the proof is finished. ■

Definition 1.9 Keeping with the notations of Lemma 1.5, we denote by

$$\mathcal{F} \equiv \sum_{i \in I} \mathcal{F}_i$$

the sub- \mathcal{A} -module, on X , of \mathcal{E} , generated by the presheaf defined by (6). We call the sub- \mathcal{A} -module $\sum_{i \in I} \mathcal{F}_i$ the **sum of the family** $(\mathcal{F}_i)_{i \in I}$. In the case where the index set I is finite, say $I = \{1, \dots, m\}$, we shall often write $\sum_{i \in I} \mathcal{F}_i$ as $\sum_{i=1}^m \mathcal{F}_i$, or $\mathcal{F}_1 + \dots + \mathcal{F}_m$. □

On another side, it is readily verified that :

Given an \mathcal{A} -module \mathcal{E} and a family $(\mathcal{F}_i)_{i \in I}$ of sub- \mathcal{A} -modules, the correspondence

$$U \longmapsto (\cap_{i \in I} \mathcal{F}_i)(U) \equiv \cap_{i \in I} (\mathcal{F}_i(U)), \quad (7)$$

where U is any open set in X , along with the obvious restriction maps yield a *complete presheaf of \mathcal{A} -modules* on X .

The sheaf generated by the presheaf given by (7) is called the **intersection sub- \mathcal{A} -module** of the family $(\mathcal{F}_i)_{i \in I}$ and is denoted

$$\bigcap_{i \in I} \mathcal{F}_i.$$

Thus, based on Mallios[[10], Proposition 11.1, p. 51], one has

$$(\bigcap_{i \in I} \mathcal{F}_i)(U) = \bigcap_{i \in I} (\mathcal{F}_i(U))$$

for every open set $U \subseteq X$.

Theorem 1.3 *Let \mathcal{E} and \mathcal{F} be free \mathcal{A} -modules of finite rank on X , dual with respect to a non-degenerate bilinear \mathcal{A} -form φ , and let \mathcal{G} and \mathcal{H} be sub- \mathcal{A} -modules of \mathcal{E} . Then*

- (a) $\dim \mathcal{G}(U) + \dim \mathcal{G}^\perp(U) = \dim \mathcal{E}(U)$, for all open $U \subseteq X$.
- (b) $(\mathcal{G}^\perp)^\perp = \mathcal{G}$.
- (c) $\mathcal{G}^\perp \cap \mathcal{H}^\perp = (\mathcal{G} + \mathcal{H})^\perp$.
- (d) $(\mathcal{G} \cap \mathcal{H})^\perp = \mathcal{G}^\perp + \mathcal{H}^\perp$.
- (e) \dagger *The mapping $\mathcal{G} \mapsto \mathcal{G}^\perp$ is a one-to-one mapping of the set of sub- \mathcal{A} -modules of \mathcal{E} onto the set of sub- \mathcal{A} -modules of \mathcal{F} , such that $\mathcal{G} \subseteq \mathcal{H}$ implies that $\mathcal{G}^\perp \supseteq \mathcal{H}^\perp$.*
- (f) *If $\mathcal{E} = \mathcal{G} \oplus \mathcal{H}$, then $\mathcal{F} = \mathcal{G}^\perp \oplus \mathcal{H}^\perp$.*
- (g) *The \mathcal{A} -modules \mathcal{G} and $\mathcal{F}/\mathcal{G}^\perp$ are dual \mathcal{A} -modules with respect to the non-degenerate bilinear \mathcal{A} -form $\tilde{\varphi}$, defined by*

$$\tilde{\varphi}^U(s, [t]) = \varphi^U(s, t)$$

for all $s \in \mathcal{G}(U)$ and $t \in (\mathcal{F}/\mathcal{G}^\perp)(U) = \mathcal{F}(U)/\mathcal{G}^\perp(U)$, with U running over the open subsets of X .

\dagger Assertion (e) is otherwise stated as \mathcal{E} and \mathcal{F} having isomorphic “projective geometries”, that is $p(\mathcal{E})$ is isomorphic to $p(\mathcal{F})$. See Gruenberg and Weir[[6], p.29].

- (h) Suppose that $S \in \text{End}_{\mathcal{A}}\mathcal{E}$ and $T \in \text{End}_{\mathcal{A}}\mathcal{F}$ are transposes of each other with respect to φ , and suppose that \mathcal{G} is an S -invariant sub- \mathcal{A} -module of \mathcal{E} , i.e.

$$S^U(s) \in \mathcal{G}(U)$$

for all $s \in \mathcal{G}(U)$ and U open in X . Then, \mathcal{G}^\perp is T -invariant, and the restriction $S|_{\mathcal{G}} \equiv (S^U|_{\mathcal{G}(U)})_{X \supseteq U, \text{open}}$ and the induced \mathcal{A} -morphism $T^* \equiv T_{\mathcal{F}/\mathcal{G}^\perp}$ are transposes of each other with respect to the bilinear \mathcal{A} -form $\tilde{\varphi}$, defined in part (g).

Proof. (a) Suppose that the rank of \mathcal{E} is n ($n \in \mathbb{N}$), i.e. $\mathcal{E} = \mathcal{A}^n$ within an \mathcal{A} -isomorphism, so that $\mathcal{E}(U)$ is $\mathcal{A}(U)$ -isomorphic to $\mathcal{A}^n(U)$ for every open $U \subseteq X$. Now, let us fix an open subset U of X ; then we have

$$\mathcal{G}(U) = \mathcal{A}^{k(U)}(U) \equiv \mathcal{A}^k(U)$$

within an $\mathcal{A}(U)$ -isomorphism and such that $1 \leq k \leq n$. Next, let $\{e_i^U\}_{1 \leq i \leq n}$ be the canonical basis of $\mathcal{E}(U)$, obtained from the Kronecker gauge $\{\varepsilon_i^U\}_{1 \leq i \leq n}$ (cf. Mallios[[10], p.123]) through the \mathcal{A} -isomorphism $\mathcal{E} = \mathcal{A}^n$. Since $\mathcal{F}(U)$ is $\mathcal{A}(U)$ -isomorphic to $\mathcal{E}(U) = \mathcal{E}(U)^* = \mathcal{E}^*(U)$, we can find, see Blyth[[3], Theorem 9.1, p.116], a basis $\{f_j^U\}_{1 \leq j \leq n}$ such that, using Proposition 1.1, we have

$$\varphi^U(e_i^U, f_j^U) = \begin{cases} 0^U, & i \neq j \\ 1^U, & i = j. \end{cases}$$

We assert that $\{f_{k+1}^U, \dots, f_n^U\}$ is a basis of $\mathcal{G}(U)^\perp$. This is clearly established as $\{e_i^U\}_{1 \leq i \leq n}$ is a basis of $\mathcal{G}(U)$, $\varphi^U(e_i^U, f_j^U) = 0^U$, for all $1 \leq i \leq k$, $k+1 \leq j \leq n$, and f_{k+1}^U, \dots, f_n^U are linearly independent and generate $\mathcal{G}(U)^\perp$. To see that f_{k+1}^U, \dots, f_n^U generate $\mathcal{G}(U)^\perp$, let $s = \sum_{i=1}^n \alpha_i f_i^U \in \mathcal{G}(U)^\perp$. Since $\varphi^U(e_i^U, s) = 0^U$, $1 \leq i \leq k$, we have that $\alpha_i = 0$ for $1 \leq i \leq k$. Thus, assertion (a) is proved.

- (b) We have, for every open $U \subseteq X$,

$$\mathcal{G}(U) \subseteq (\mathcal{G}^\perp(U))^\perp = (\mathcal{G}(U)^\perp)^\perp \equiv \mathcal{G}(U)^{\perp\perp}.$$

From part (a), we have, for all open $U \subseteq X$,

$$\begin{aligned} \dim(\mathcal{G}^\perp(U))^\perp &= \dim \mathcal{F}(U) - \dim \mathcal{G}^\perp(U) \\ &= \dim \mathcal{F}(U) - (\dim \mathcal{E}(U) - \dim \mathcal{G}(U)) \\ &= \dim \mathcal{G}(U), \end{aligned}$$

form which we deduce that for all open $U \subseteq X$,

$$\mathcal{G}(U) = (\mathcal{G}^\perp(U))^\perp$$

within an $\mathcal{A}(U)$ -isomorphism. Hence, the $\mathcal{A}(U)$ -isomorphisms $\mathcal{G}(U) = (\mathcal{G}^\perp(U))^\perp$, U running over the open subsets of X , along with the restriction maps σ_V^U yield a complete presheaf, defined by

$$U \longmapsto \mathcal{G}^{\perp\perp} \equiv (\mathcal{G}(U))^{\perp\perp} := (\mathcal{G}^\perp(U))^\perp.$$

It follows that if $(\mathcal{G}^\perp)^\perp \equiv \mathcal{G}^{\perp\perp}$ is the sheaf corresponding to the preceding (complete) presheaf, then we have

$$(\mathcal{G}^\perp)^\perp = \mathcal{G}$$

within an \mathcal{A} -isomorphism.

(c) For every open $U \subseteq X$, one has

$$\begin{aligned} (\mathcal{G}^\perp \cap \mathcal{H}^\perp)(U) &= \mathcal{G}^\perp(U) \cap \mathcal{H}^\perp(U) \\ &= \mathcal{G}(U)^\perp \cap \mathcal{H}(U)^\perp \\ &= (\mathcal{G}(U) + \mathcal{H}(U))^\perp \\ &= ((\mathcal{G} + \mathcal{H})(U))^\perp \\ &= (\mathcal{G} + \mathcal{H})^\perp(U); \end{aligned}$$

it follows that $\mathcal{G}^\perp \cap \mathcal{H}^\perp = (\mathcal{G} + \mathcal{H})^\perp$ within an \mathcal{A} -isomorphism.

(d) is shown by combining (b) and (c).

(e) Clearly for all open $U \subseteq X$, $\mathcal{G}(U) \subseteq \mathcal{H}(U)$ implies that

$$\mathcal{G}^\perp(U) = \mathcal{G}(U)^\perp \supseteq \mathcal{H}(U)^\perp = \mathcal{H}^\perp(U).$$

So, if

$$\{((\sigma^\perp)_V^U) : \mathcal{G}^\perp(U) \longrightarrow \mathcal{G}^\perp(V) \mid V, U \text{ are open in } X \text{ and } V \subseteq U\}$$

is the set of restriction maps for the (complete) presheaf of sections of the annihilator sheaf \mathcal{G}^\perp , then by taking

$$(\lambda^\perp)_V^U := (\sigma^\perp)_V^U |_{\mathcal{H}^\perp(U)} = \rho_V^U |_{\mathcal{H}^\perp(U)}$$

we obtain the (complete) presheaf of sections of the sheaf \mathcal{H}^\perp . Therefore, we have $\mathcal{H}^\perp \subseteq \mathcal{G}^\perp$. For the one-to-one property, suppose that $\mathcal{G}^\perp = \mathcal{H}^\perp$. Applying (b), we have

$$\mathcal{G} = (\mathcal{G}^\perp)^\perp = (\mathcal{H}^\perp)^\perp = \mathcal{H},$$

where the previous equalities are actually \mathcal{A} -isomorphisms. The proof that every sub- \mathcal{A} -module \mathcal{N} of the \mathcal{A} -module \mathcal{F} has the form \mathcal{G}^\perp for some sub- \mathcal{A} -module \mathcal{G} of \mathcal{E} is immediate. In effect, applying (b), we have

$$\mathcal{N} = (\mathcal{N}^\perp)^\perp$$

within an \mathcal{A} -isomorphism. Taking $\mathcal{G} = \mathcal{N}^\perp$ corroborates the assertion.

(f) It suffices to show that if U is an open subset of X , then $\mathcal{E}(U) = \mathcal{G}(U) \oplus \mathcal{H}(U)$ implies that $\mathcal{F}(U) = \mathcal{G}^\perp(U) \oplus (\mathcal{H})^\perp(U)$. But this is shown in Curtis[[5], p.242, part (d) of proof of Theorem (27.12)].

(g) That $\tilde{\varphi}$ is well defined is immediate. In fact, fix an open subset U of X ; then for every $s \in \mathcal{G}(U)$ and $t = t' \in (\mathcal{F}/(\mathcal{E})^\perp)(U) = \mathcal{F}(U)/(\mathcal{G})^\perp(U)$, we have

$$\varphi^U(s, t') = \varphi^U(s, t + z) = \varphi^U(s, t)$$

since $s \in \mathcal{G}(U)$, and $t' = t + z$ with $z \in \mathcal{G}(U)^\perp = (\mathcal{G})^\perp(U)$. It is obvious that $\tilde{\varphi}$ is bilinear. The proof that $\tilde{\varphi}^U$ is non-degenerate for all open $U \subseteq X$ can be found in Curtis[[5], p242, part (e) of proof of Theorem (27.12)].

(h) We show first that $T(\mathcal{G}^\perp) \subseteq \mathcal{G}^\perp$, that is

$$T(\mathcal{G}^\perp)(U) := T^U(\mathcal{G}^\perp(U)) = T^U(\mathcal{G}(U)^\perp) \subseteq \mathcal{G}^\perp(U) = \mathcal{G}(U)^\perp,$$

for all open $U \subseteq X$. Let us consider arbitrarily any open subset $U \subseteq X$, and let $s \in \mathcal{G}(U)$ and $t \in \mathcal{G}(U)^\perp$. Then,

$$\varphi^U(s, T^U(t)) = \varphi^U(S^U(s), t) = 0^U,$$

because $S^U(\mathcal{G}(U)) \subseteq \mathcal{G}(U)$; therefore $T^U(\mathcal{G}(U)^\perp) \subseteq \mathcal{G}(U)^\perp$. For the remaining part of (f), we start by noticing that for every open $U \subseteq X$,

$$(T^*)^U \circ q^U = q^U \circ T^U$$

where q is the quotient \mathcal{A} -morphism $\mathcal{F} \longrightarrow \mathcal{F}/\mathcal{G}^\perp$. It is sufficient to prove that for $s \in \mathcal{G}(U)$, $t \in \mathcal{F}(U)$,

$$\tilde{\varphi}^U(S^U|_{\mathcal{G}(U)}(s), t) = \tilde{\varphi}^U(s, (T^*)^U(t));$$

this statement is equivalent to showing that

$$\varphi^U(S^U(s), t) = \varphi^U(s, T^U(t)),$$

which is exactly the condition that S and T are transposes of each other. ■

The last part of this section concerns with the \mathcal{A} -isomorphism of the \mathcal{A} -annihilator of a sub- \mathcal{A} -module \mathcal{F} of an \mathcal{A} -module \mathcal{E} and the dual $(\mathcal{E}/\mathcal{F})^*$ of the quotient \mathcal{A} -module \mathcal{E}/\mathcal{F} . This question requires some preparation.

Definition 1.10 Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X , U an open subset of X , and $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(U) = \mathcal{H}om_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{F}|_U)$. For any \mathcal{A} -module \mathcal{G} on X , we define an $\mathcal{A}(U)$ -morphism

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{E})(U) \xrightarrow{\varphi_*} \mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{F})(U)$$

by setting

$$\varphi_*(f) = \varphi \circ f \equiv (\varphi_V \circ f_V)_{U \supseteq V, \text{open}} \equiv ((\varphi_V)_*(f_V))_{U \supseteq V, \text{open}}$$

for all $f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{G}, \mathcal{E})(U)$. Likewise, we can define an $\mathcal{A}(U)$ -morphism

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})(U) \xrightarrow{\varphi^*} \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(U)$$

by the assignment

$$\varphi^*(f) = f \circ \varphi \equiv (f_V \circ \varphi_V)_{U \supseteq V, \text{open}} \equiv ((\varphi_V)^*(f_V))_{U \supseteq V, \text{open}}$$

for all $f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})(U)$.

Proposition 1.2 Let \mathcal{E} , \mathcal{F} , and \mathcal{G} be \mathcal{A} -modules on X , $\varphi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(U)$ and $\psi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(U)$, where U is an open subset of X . Then, we have

- (1) $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$
 (2) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proof. Immediate. ■

Our main interest in the above induced $\mathcal{A}(U)$ -morphisms φ_* and φ^* transpires in the following theorem.

Theorem 1.4 *Consider a short exact sequence*

$$0 \longrightarrow \mathcal{E}' \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\psi} \mathcal{E}'' \longrightarrow 0$$

of \mathcal{A} -modules (on X) and \mathcal{A} -morphisms. For an arbitrary \mathcal{A} -module \mathcal{F} , the induced sequences of $\mathcal{A}(X)$ -modules and $\mathcal{A}(X)$ -morphisms

- (1) $0 \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E}')(X) \xrightarrow{\varphi_*} \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E})(X) \xrightarrow{\psi_*} \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E}'')(X)$
 (2) $0 \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{E}'', \mathcal{F})(X) \xrightarrow{\psi^*} \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})(X) \xrightarrow{\varphi^*} \mathcal{H}om_{\mathcal{A}}(\mathcal{E}', \mathcal{F})(X)$

are exact. The diagrams above are \mathcal{A} -isomorphic to the diagrams

- (1') $0 \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E}') \xrightarrow{\varphi_*} \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E}) \xrightarrow{\psi_*} \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E}'')$
 (2') $0 \longrightarrow \mathcal{H}om_{\mathcal{A}}(\mathcal{E}'', \mathcal{F}) \xrightarrow{\psi^*} \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F}) \xrightarrow{\varphi^*} \mathcal{H}om_{\mathcal{A}}(\mathcal{E}', \mathcal{F})$

Proof. We shall show that (1) is exact. The sequence (2) is established in a similar way.

First, let $f \in \ker \varphi_*$. We have $0 = \varphi_*(f) = \varphi \circ f \in \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{E})(X)$, whence $f = (0_U)_{X \supseteq U, \text{open}}$ with $0_U : \mathcal{F}(U) \longrightarrow \mathcal{E}(U)$, $0_U(s) = 0$ for all $s \in \mathcal{F}(U)$ and all open subset $U \subseteq X$, which means that φ_* is one-to-one.

Next, let us show that $\text{im}\varphi_*$ is an $\mathcal{A}(X)$ -submodule of the $\mathcal{A}(X)$ -module $\ker\psi_*$. (See Mallios[[10], pp 108, 109] for a proof of the statement: Given \mathcal{A} -modules \mathcal{E} , \mathcal{F} , and an \mathcal{A} -morphism $\phi : \mathcal{E} \rightarrow \mathcal{F}$. Then, $\ker\phi := \{z \in \mathcal{E} : \phi(z) = 0\}$ and $\text{im}\phi := \phi(\mathcal{E}) \subseteq \mathcal{F}$ are \mathcal{A} -modules; consequently $\ker\phi(X)$ and $\text{im}\phi(X)$ are $\mathcal{A}(X)$ -modules.) If $f \in \text{im}\varphi_*$, there exists $f' \in \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{E}')(X)$ such that $f = \varphi_*(f') = \varphi \circ f'$. Consequently

$$\psi_*(f) = \psi_*(\varphi_*(f')) = (\psi \circ \varphi)_*(f') = 0$$

because $\psi \circ \varphi = 0$. Thus, $f \in \ker\psi_*$, and we have established that

$$\text{im}\varphi_* \subseteq \ker\psi_*.$$

Finally, let us show that $\ker\psi_*$ is an $\mathcal{A}(X)$ -submodule of $\text{im}\varphi_*$. To this end, let $f \in \ker\psi_*$; then for every $s \in \mathcal{F}(U)$ where U is an open subset of X ,

$$\psi_U(f_U(s)) = (\psi_U \circ f_U)(s) = [(\psi_U)_*(f_U)](s) = 0 \in \mathcal{E}''(U)$$

and so $f_U(s) \in \ker\psi_U = \text{im}\varphi_U$. Thus, there exists $s' \in \mathcal{E}'(U)$ such that $f_U(s) = \varphi_U(s')$; and since φ is one-to-one, such an element s' is unique. We can therefore define a mapping $f'_U : \mathcal{F}(U) \rightarrow \mathcal{E}'(U)$ by setting $f'_U(s) = s'$. Clearly, f'_U yields an $\mathcal{A}(U)$ -morphism of $\mathcal{A}(U)$ -modules $\mathcal{F}(U)$ and $\mathcal{E}'(U)$, which by abuse of language we also call f'_U . But

$$f_U = \varphi_U \circ f'_U = (\varphi_U)_*(f'_U) \in \text{im}(\varphi_U)_*,$$

where U is an arbitrary subset of X . Thus, $\ker(\psi_U)_* \subseteq \text{im}(\varphi_U)_*$, for every open $U \subseteq X$. Hence, $\ker\psi_* \subseteq \text{im}\varphi_*$, which ends the proof. ■

For the notion in the following definition, we refer to Mallios[[10], pp. 301, 302] for specific details.

Definition 1.11 Let \mathcal{E} and \mathcal{F} be \mathcal{A} -modules on a topological space X . By the **transpose** of an \mathcal{A} -morphism $\varphi : \mathcal{E} \rightarrow \mathcal{F}$, we mean the \mathcal{A} -morphism

$${}^t\varphi \equiv ({}^t\varphi_U)_{X \supseteq U, \text{open}} : \mathcal{F}^* \rightarrow \mathcal{E}^*,$$

given by the assignment

$${}^t\varphi_U(u) := (u_V \circ \varphi_V)_{U \supseteq V, \text{open}}$$

for every $u \in \mathcal{F}^*(U)$, with U open in X . □

The principal properties of *transposition* in classical module theory apply in the setting as well, and are easily verified.

Proposition 1.3 *Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be \mathcal{A} -modules on a topological space X . Then*

- (1) ${}^t(id_{\mathcal{E}}) = id_{\mathcal{E}^*}$.
- (2) If $\varphi, \psi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, then ${}^t(\varphi + \psi) = {}^t\varphi + {}^t\psi$.
- (3) If $\varphi \in Hom_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ and $\psi \in Hom_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$, then ${}^t(\psi \circ \varphi) = {}^t\psi \circ {}^t\varphi$.

Corollary 1.3 *If $\varphi : \mathcal{E} \rightarrow \mathcal{F}$ is an \mathcal{A} -isomorphism of the \mathcal{A} -modules \mathcal{E} and \mathcal{F} , then so is ${}^t\varphi : \mathcal{F}^* \rightarrow \mathcal{E}^*$; and we also have in this case that $({}^t\varphi)^{-1} = {}^t(\varphi^{-1})$.*

Proof. By hypothesis and items (1), (3) of Proposition 1.3, we have

$${}^t\varphi \circ {}^t\varphi^{-1} = {}^t(\varphi^{-1} \circ \varphi) = {}^t(id_{\mathcal{E}}) = id_{\mathcal{E}^*}$$

and

$${}^t\varphi^{-1} \circ {}^t\varphi = {}^t(\varphi \circ \varphi^{-1}) = {}^t(id_{\mathcal{F}}) = id_{\mathcal{F}^*}.$$

The proof is finished. ■

Corollary 1.4 *If \mathcal{F} is a sub- \mathcal{A} -module of \mathcal{E} , then*

$$(\mathcal{E}/\mathcal{F})^* = \mathcal{F}^{\perp},$$

within an \mathcal{A} -isomorphism.

Proof. Let $q : \mathcal{E} \rightarrow \mathcal{E}/\mathcal{F}$ be the quotient \mathcal{A} -morphism and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{E} \xrightarrow{q} \mathcal{E}/\mathcal{F} \longrightarrow 0$$

the natural short exact sequence (see Mallios[[10], Lemma 2.1, p.116]). By Theorem 1.4, we have the induced short exact sequence

$$\mathcal{F}^* \xleftarrow{{}^t\iota} \mathcal{E}^* \xleftarrow{{}^tq} (\mathcal{E}/\mathcal{F})^* \longleftarrow 0.$$

Then, since

$$\ker({}^t\iota) = (\text{im } \iota)^\perp,$$

it follows from the exactness of the foregoing sequence that

$$(\mathcal{E}/\mathcal{F})^* = \text{im}({}^tq) = \ker({}^t\iota) = (\text{im } \iota)^\perp = \mathcal{F}^\perp.$$

■

2 Properties of exterior \mathcal{A} -2-forms

In this section, we examine some properties of exterior \mathcal{A} -2-forms. The most useful property is the *normal* (or *Darboux*) *form* for exterior \mathcal{A} -2-forms, see [[12], Theorem 3.3], which we prove this time by following Libermann and Marle[[8], Theorem 2.3, pp. 4,5] and Sternberg[[13], Theorem 5.1, p. 24].

Throughout this section, \mathcal{E} stands for an \mathcal{A} -module on a topological space $X \equiv (X, \tau)$, and $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ for an *exterior bilinear \mathcal{A} -form*, unless otherwise specified.

Definition 2.1 Let $\eta : \mathcal{E} \oplus \dots \oplus \mathcal{E} \longrightarrow \mathcal{A}$ be a non-zero exterior \mathcal{A} - k -form. For every $s \equiv (s_U)_{U \in \mathcal{T}} \in \prod_{U \in \mathcal{T}} \mathcal{E}(U)$, let

$$i : \text{Hom}_{\mathcal{A}}(\mathcal{E}, \text{Hom}_{\mathcal{A}}(\bigwedge^k \mathcal{E}^*, \bigwedge^{(k-1)} \mathcal{E}^*))$$

be an \mathcal{A} -morphism, whose U -component, for an arbitrary open subset U of X , is the $\mathcal{A}(U)$ -morphism

$$\begin{aligned} i_U &\in \text{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \text{Hom}_{\mathcal{A}(U)}((\bigwedge^k \mathcal{E}^*)(U), (\bigwedge^{(k-1)} \mathcal{E}^*)(U))) \\ &\equiv \text{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \text{Hom}_{\mathcal{A}(U)}((\bigwedge^k \mathcal{E})^*(U), (\bigwedge^{(k-1)} \mathcal{E})^*(U))), \end{aligned}$$

which is given by

$$i(s_U)\eta_U(s_{1U}, \dots, s_{k-1U}) = \eta_U(s_U, s_{1U}, \dots, s_{k-1U})$$

for all $s_{1U}, \dots, s_{k-1U} \in \mathcal{E}(U)$. We call

$$i(s)\eta \equiv (i_U(s_U)\eta_U)_{U \in \mathcal{T}} : \mathcal{E} \oplus \dots \oplus \mathcal{E} \longrightarrow \mathcal{A}$$

the **inner \mathcal{A} -product** of η and s . □

Now, let us move our attention to \mathcal{A} -2-forms. Suppose $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ is an \mathcal{A} -2-form on a free \mathcal{A} -module \mathcal{E} ; for a family $s \equiv (s_U)_{U \in \mathcal{T}} \in \prod_{U \in \mathcal{T}} \mathcal{E}(U)$, the following mapping in $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}^{**} = \mathcal{E} = \mathcal{E}^*)$, see Mallios[[10], relation 5.4, p. 298], given by

$$s \longmapsto -i(s)\omega \equiv -(i(s_U)\omega_U)_{U \in \mathcal{T}},$$

will be denoted, keeping with Libermann and Marle[[8], p. 3], by ω^{\flat} . Next, for every open $U \subseteq X$, we consider the canonical basis $(e_i^U)_{1 \leq i \leq n} \subseteq \mathcal{E}(U)$. Suppose that $\omega \equiv (\omega^U)_{U \in \mathcal{T}}$ is such that

$$\omega_{ij}^U \equiv \omega^U(e_i^U, e_j^U)$$

for any open subset $U \subseteq X$, and such that

$$\text{rank}(\omega_{ij}^U) = \text{rank}(\omega_{ij}^V),$$

then the **rank** of ω is by definition the rank of the matrix (ω_{ij}^U) for any open $U \subseteq X$. Throughout this paper, *all exterior \mathcal{A} -2-forms $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ on the free \mathcal{A} -module \mathcal{E} of rank n are assumed to have a rank*. We shall call such \mathcal{A} -2-forms **rankwise \mathcal{A} -2-forms**.

As in Libermann and Marle[[8], pp. 3, 4], given an exterior \mathcal{A} -2-form $\omega : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ on a free \mathcal{A} -module \mathcal{E} , we denote by ${}^{\flat}\mathcal{E}$ the sub- \mathcal{A} -module $\text{im } \omega^{\flat} \subseteq \mathcal{E}^* = \mathcal{E}$; see Mallios[[10], p. 109] for a proof of the following statement:

If $\varphi \equiv (\varphi_U) \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$, then $\text{im } \varphi := \varphi(\mathcal{E})$ is a subsheaf of the sheaf \mathcal{F} .

Since $\omega_U^b(\mathcal{E}(U))$ is an $\mathcal{A}(U)$ -submodule of $\mathcal{E}^*(U) = \mathcal{E}(U)^* = \mathcal{E}(U)$ for every open $U \subseteq X$, it follows that $\text{im } \omega^b$ is a sub- \mathcal{A} -module of $\mathcal{E}^* = \mathcal{E}$.

It is worth noting too that ${}^b\mathcal{E} = (\ker \omega^b)^\perp$ within an \mathcal{A} -isomorphism. If $\ker \omega^b \neq 0$, we have

$$\mathcal{E} / \ker \omega^b = {}^b\mathcal{E}$$

within an \mathcal{A} -isomorphism, see Mallios[[10], Lemma 2.1, p. 116, relation (2.19), p. 110].

The first part of the following theorem was proved in our previous paper, see [12], however, here, we are presenting another proof for the same first part of the theorem; the relevance of this approach consists in the fact it provides hints, which are necessary for the proof of the second part of the theorem. This theorem in its classical form is proved in Libermann and Marle[[8], Theorem 2.3, p. 4] and Sternberg[[13], Theorem 5.1, p. 24].

Theorem 2.1 *Let $(X, \mathcal{A}, \mathcal{P}, |\cdot|)$ be an ordered \mathbb{R} -algebraized space, endowed with an absolute value morphism (see [12]), such that every strictly positive section of \mathcal{A} is invertible. Moreover, let ω be a rankwise \mathcal{A} -2-form on the free \mathcal{A} -module \mathcal{E} of rank n . Then, for every $x \in X$, there exist an open neighborhood $U \subseteq X$ of x and a basis*

$$s_U^1, \dots, s_U^{2m} \in {}^b\mathcal{E}(U), \quad 2 \leq 2m \leq n$$

such that

$$\omega_U = \sum_{k=1}^m s_U^{2k-1} \wedge s_U^{2k};$$

furthermore, s_U^2 may be chosen arbitrarily in ${}^b\mathcal{E}(U)$.

Proof. Let $(e_{1X}, \dots, e_{nX}) \equiv (e_1, \dots, e_n)$ be a basis of $\mathcal{E}(X)$, whose corresponding dual basis is $(e_X^1, \dots, e_X^n) \equiv (e^1, \dots, e^n)$. The X -component of the \mathcal{A} -2-form ω may be expressed as

$$\omega_X = \frac{1}{2} \sum_{(i,j)} a_{ij} e^i \wedge e^j,$$

where the coefficients a_{ij} are sections of \mathcal{A} over X , i.e. $a_{ij} \in \mathcal{A}(X) \equiv \Gamma(X, \mathcal{A})$, and satisfy the condition $a_{ji} = -a_{ij}$, $1 \leq i, j \leq n$. By hypothesis, at every $x \in X$, the coefficients $a_{ij}(x)$ are not all zero. Let us fix a point $x \in X$; we can rearrange the basis (e^1, \dots, e^n) as to obtain $a_{12}(x) \neq 0$. Therefore, for some open neighborhood U of x , we have

$$|\rho_U^X(a_{12})| \in \mathcal{P}^*(U),$$

where $\mathcal{P}^* := \mathcal{P} - \{0\} \subseteq \mathcal{A}^\bullet$, cf. Mallios[[10], relation (10.1), p. 335], and the $(\rho_W^V)_{V \supseteq W, \text{open}}$, with V running over the open subsets of X , are the restriction maps for the (complete) presheaf of sections of the coefficient sheaf \mathcal{A} . Let us assume likewise that the restriction maps for the (complete) presheaf of sections of \mathcal{E} are maps $(\sigma_W^V)_{V \supseteq W, \text{open}}$, with V being any open set in X ; we set

$$s_U^1 = \frac{1}{|\rho_U^X(a_{12})|} i(\sigma_U^X(e_1)) \omega_U,$$

and

$$s_U^2 = i(\sigma_U^X(e_2)) \omega_U$$

i.e.

$$s_U^1 = \sigma_U^X(e^2) + \frac{1}{|\rho_U^X(a_{12})|} \sum_{k=3}^n \rho_U^X(a_{1k}) \sigma_U^X(e^k),$$

and

$$s_U^2 = -|\rho_U^X(a_{12})| \sigma_U^X(e^1) + \sum_{k=3}^n \rho_U^X(a_{2k}) \sigma_U^X(e^k).$$

It is clear that $(s_U^1, s_U^2, \sigma_U^X(e^3), \dots, \sigma_U^X(e^n))$ is a basis of $\mathcal{E}(U)$. Next, we set

$$\omega_{1U} = \omega_U - s_U^1 \wedge s_U^2;$$

ω_{1U} does not contain any expression involving $\sigma_U^X(e^1)$ or $\sigma_U^X(e^2)$. If $\omega_{1U} = 0$, then $\omega_U = s_U^1 \wedge s_U^2$, and we are done. Otherwise, we continue the same process until we achieve the desired form, that is if

$$\omega_{1U} = \frac{1}{2} \sum_{i,j=3}^n b_{ij} \rho_U^X(e^i) \wedge \rho_U^X(e^j)$$

with $b_{ji} = -b_{ij} \in \mathcal{A}(U)$, $3 \leq i, j \leq n$, then there exists a $b_{ij} \in \mathcal{A}(U)$ such that $b_{ij}(x) \neq 0$. As above, there exists an open neighborhood $V \subseteq U$ such

that $\rho_V^U(b_{ij}) \neq 0$. Through a convenient rearrangement of the basis vectors $\rho_U^X(e^3), \dots, \rho_U^X(e^n)$, we may assume that $\rho_U^X(b_{34}) \neq 0$. So, as before, we shall get an \mathcal{A} -2-form

$$\omega_{2V} = \omega_{1V} - s_V^3 \wedge s_V^4,$$

where

$$s_V^3 = \frac{1}{|\rho_V^U(b_{34})|} i(\sigma_V^U(e_3))\omega_{1V}, \quad s_V^4 = i(\sigma_V^U(e_4))\omega_{1V},$$

and so on \dots

Let t be a non-zero element in ${}^b\mathcal{E}(U)$. There exists a non-zero vector $s_2 \in \mathcal{E}(U)$ such that $t = i(s_2)\omega_U$. Since $t \neq 0$, there exists a section $s_1 \in \mathcal{E}(U)$ such that $t(s_1) \neq 0$, hence $\omega_U(s_1, s_2) \neq 0$. We choose the basis (e_1, \dots, e_n) of $\mathcal{E}(X)$ such that $\sigma_U^X(e_1) = s_1$, and $\sigma_U^X(e_2) = s_2$ so that $s_U^2 = t$. ■

3 Symplectic Reduction

We start by observing that Definition 1.5 hints that if \mathcal{E} is a free \mathcal{A} -module of rank n , then an \mathcal{A} -bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A}$ is non-degenerate if and only if ω is rankwise, and of rank n . A pair (\mathcal{E}, ω) , where \mathcal{E} is an arbitrary \mathcal{A} -module and $\omega : \mathcal{E} \oplus \mathcal{E} \rightarrow \mathcal{A}$ a non-degenerate \mathcal{A} -bilinear morphism, is called a **symplectic \mathcal{A} -module**.

Throughout this section, we will be particularly interested in symplectic free \mathcal{A} -modules of finite rank.

The most important examples of sub- \mathcal{A} -modules of a symplectic \mathcal{A} -module (\mathcal{E}, ω) (\mathcal{E} is not assumed necessarily free) are the following

Definition 3.1 Let (\mathcal{E}, ω) be a symplectic \mathcal{A} -module and $\mathcal{F} \subseteq \mathcal{E}$ a sub- \mathcal{A} -module. We say that

- (i) \mathcal{F} is **isotropic** if $\mathcal{F} \subseteq \mathcal{F}^\perp$, that is $\omega|_{\mathcal{F} \oplus \mathcal{F}} \equiv \omega|_{\mathcal{F}} = 0$.
- (ii) \mathcal{F} is **co-isotropic** if $\mathcal{F}^\perp \subseteq \mathcal{F}$, that is $\omega|_{\mathcal{F}^\perp} = 0$.

- (iii) \mathcal{F} is a **symplectic sub- \mathcal{A} -module** if $\omega|_{\mathcal{F}} : \mathcal{F} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ is non-degenerate.
- (iv) \mathcal{F} is **Lagrangian** if it is isotropic and has an isotropic complement, that is $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$, where \mathcal{G} is isotropic. \square

The next result will often be used to define *Lagrangian sub- \mathcal{A} -modules*.

Proposition 3.1 *Let (\mathcal{E}, ω) be a symplectic \mathcal{A} -module of finite rank on X , and $\mathcal{F} \subseteq \mathcal{E}$ a sub- \mathcal{A} -module. Then, the following assertions are equivalent:*

- (i) \mathcal{F} is Lagrangian.
- (ii) $\mathcal{F} = \mathcal{F}^\perp$, within an \mathcal{A} -isomorphism.
- (iii) \mathcal{F} is isotropic and $\text{rank } \mathcal{F} = \frac{1}{2} \text{rank } \mathcal{E}$.

Proof. The following proof is derived from the proof of Proposition 5.3.3, in Abraham and Marsden[[1], p. 404].

First, we prove that (i) implies (ii). We have $\mathcal{F} \subseteq \mathcal{F}^\perp$ by hypothesis. Next, we have to show the converse, i.e. $\mathcal{F}^\perp \subseteq \mathcal{F}$. To this end, for every open subset U of X , let $s^U \in \mathcal{F}^\perp(U) = \mathcal{F}(U)^\perp$; since $\mathcal{E}(U) = \mathcal{F}(U) + \mathcal{G}(U)$, where \mathcal{G} , according to Definition 3.1(iv), is an isotropic complement of \mathcal{F} , write $s^U = s_0^U + s_1^U$ for some $s_0^U \in \mathcal{F}(U)$ and $s_1^U \in \mathcal{G}(U)$. We shall show that $s_1^U = 0$. Indeed, let $s_1^U \in \mathcal{G}(U) \subseteq \mathcal{G}^\perp(U) = \mathcal{G}(U)^\perp$, and $s_1^U = s^U - s_0^U \in \mathcal{F}^\perp(U)$. Thus,

$$\begin{aligned} s_1^U \in \mathcal{G}^\perp(U) \cap \mathcal{F}^\perp(U) &= (\mathcal{G}(U) + \mathcal{F}(U))^\perp, \text{ by virtue of Theorem 1.3(c)} \\ &= \mathcal{E}^\perp(U) \\ &= \{0\}, \text{ by the non-degeneracy of } \omega_U. \end{aligned}$$

Thus, $s_1^U = 0$, so $\mathcal{F}^\perp(U) \subseteq \mathcal{F}(U)$; since U is arbitrary, $\mathcal{F}^\perp \subseteq \mathcal{F}$, that is (ii) holds.

The implication (ii) \implies (iii) is immediate.

Finally, we prove that (iii) implies (i). First, observe that (iii) implies $\dim \mathcal{F}(U) = \dim \mathcal{F}^\perp(U)$ for any open subset U of X . Since $\mathcal{F} \subseteq \mathcal{F}^\perp$, we have that $\mathcal{F} = \mathcal{F}^\perp$. Now, we construct the isotropic complement \mathcal{G} of \mathcal{F} as follows. For every open $U \subseteq X$, choose arbitrarily $s_1^U \notin \mathcal{F}(U)$, and let

$$\mathcal{F}_1(U) := \{as_1^U \mid a \in \mathcal{A}(U)\} \equiv \mathcal{A}s_1^U.$$

It is easy to see that the correspondence

$$U \longmapsto \mathcal{F}_1(U) \tag{8}$$

along with the obvious restrictions yield a complete presheaf of \mathcal{A} -modules. (If $\rho_V^U : \mathcal{E}(U) \rightarrow \mathcal{E}(V)$ is a restriction map, $\rho_V^U|_{\mathcal{F}_1(U)} : \mathcal{F}_1(U) \rightarrow \mathcal{F}_1(V)$ is the corresponding restriction map for the presheaf defined in (8).) The sheaf \mathcal{F}_1 generated by the presheaf defined in (8) is clearly a free \mathcal{A} -module of rank 1. For every open $U \subseteq X$, $\mathcal{F}(U) \cap \mathcal{F}_1(U) = \{0\}$; consequently

$$\begin{aligned} \mathcal{F}(U)^{\perp\perp} + \mathcal{F}(U)^\perp &= (\mathcal{F}(U)^\perp \cap \mathcal{F}_1(U))^\perp, \text{ by virtue of Theorem 1.3} \\ &= (\mathcal{F}(U) \cap \mathcal{F}_1(U))^\perp, \text{ since } \mathcal{F} = \mathcal{F}^\perp \\ &= \{0\}^\perp \\ &= \mathcal{E}(U). \end{aligned}$$

Now, choose, for every open $U \subseteq X$, an element $s_2^U \in \mathcal{F}_1(U)^\perp = \mathcal{F}_1^\perp(U)$ such that $s_2^U \notin \mathcal{F}(U) + \mathcal{F}_1(U)$. Next, let

$$\mathcal{F}_2(U) := \mathcal{F}_1(U) + \mathcal{A}s_2^U;$$

proceed inductively as before until one gets

$$\mathcal{F}(U) + \mathcal{F}_k(U) = \mathcal{E}(U) \tag{9}$$

for every open $U \subseteq X$. The correspondence

$$U \longmapsto \mathcal{F}_2(U)$$

defines a complete presheaf of \mathcal{A} -modules. The sheaf \mathcal{F}_2 generated by the foregoing presheaf is a free \mathcal{A} -module of rank 2. Similarly, the sheaf \mathcal{F}_k obtained by sheaffying the complete presheaf, given by

$$U \longmapsto \mathcal{F}_k(U)$$

is a free \mathcal{A} -module of rank k . Equation (9) yields the following \mathcal{A} -isomorphism

$$\mathcal{F} + \mathcal{F}_k = \mathcal{E}.$$

By construction, $\mathcal{F}(U) \cap \mathcal{F}_k(U) = \{0\}$ for every open $U \subseteq X$, so $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}_k$. Also, by construction,

$$\begin{aligned} \mathcal{F}_2^\perp(U) = \mathcal{F}_2(U)^\perp &= (\mathcal{F}_1(U) + \mathcal{A}s_2^U)^\perp \\ &= \mathcal{F}_1(U)^\perp + (\mathcal{A}s_2^U)^\perp \\ &\supseteq \mathcal{A}s_1^U + \mathcal{A}s_2^U \\ &= \mathcal{F}_2(U). \end{aligned}$$

It follows that $\mathcal{F}_2 \subseteq \mathcal{F}_2^\perp$. In the same way, one shows that \mathcal{F}_k is isotropic as well. Thus, $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}_k$, with $\mathcal{F}_k \subseteq \mathcal{F}_k^\perp$ as desired. ■

Lemma 3.1 *Let (\mathcal{E}, ω) be a symplectic \mathcal{A} -module, and $\mathcal{F} \subseteq \mathcal{E}$ a sub- \mathcal{A} -module. Then, $\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp$ has a natural symplectic structure.*

Proof. Indeed, let

$$\widehat{\omega}_U(s + (\mathcal{F} \cap \mathcal{F}^\perp)(U), s' + (\mathcal{F} \cap \mathcal{F}^\perp)(U)) := \omega_U(s, s') \quad (10)$$

for all $s, s' \in \mathcal{F}(U)$, where U is an open set in X . Equation (10) can equivalently be written as

$$\widehat{\omega}_U(s + (\mathcal{F}(U) \cap \mathcal{F}^\perp(U)), s' + (\mathcal{F}(U) \cap \mathcal{F}^\perp(U))) = \omega_U(s, s'),$$

because $\mathcal{F} \cap \mathcal{F}^\perp$ is a sub- \mathcal{A} -module of \mathcal{E} and $(\mathcal{F} \cap \mathcal{F}^\perp)(U) = \mathcal{F}(U) \cap \mathcal{F}^\perp(U) = \mathcal{F}(U) \cap \mathcal{F}(U)^\perp$ for all open subset $U \subseteq X$. Now, denote by

$$((\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(U) := \mathcal{F}(U)/\mathcal{F}(U) \cap \mathcal{F}^\perp(U), \sigma_V^U)$$

the (complete) presheaf of sections associated with the sheaf $\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp$, and by

$$(\mathcal{A}(U), \lambda_V^U)$$

the corresponding presheaf of sections for the coefficient sheaf \mathcal{A} . It is clearly easy to see that

$$\begin{array}{ccc} (\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(U) \oplus (\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(U) & \xrightarrow{\widehat{\omega}_U} & \mathcal{A}(U) \\ \sigma_V^U \oplus \sigma_V^U \downarrow & & \downarrow \lambda_V^U \\ (\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(V) \oplus (\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(V) & \xrightarrow{\widehat{\omega}_V} & \mathcal{A}(V) \end{array}$$

commutes for all open subsets $U, V \subseteq X$ such that $V \subseteq U$. Thus,

$$\widehat{\omega} : \mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp \oplus \mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp \longrightarrow \mathcal{A}$$

is an \mathcal{A} -morphism.

We need now show that $\widehat{\omega}$ is well defined and is a symplectic \mathcal{A} -form. Indeed, let $t, t' \in \mathcal{F}(U) \cap \mathcal{F}^\perp(U)$, where U is open in X ; then

$$\begin{aligned} \omega_U(s+t, s'+t') &= \omega_U(s, s') + \omega_U(s+t, t') + \omega_U(t, s') \\ &= \omega_U(s, s'), \end{aligned}$$

since $\omega_U(s+t, t') = 0 = \omega_U(t, s')$. Thus, $\widehat{\omega}$ is well defined. It is easy to see that $\widehat{\omega}$ is \mathcal{A} -bilinear. Let us now show that $\widehat{\omega}$ is non-degenerate. Suppose $s \in \mathcal{F}(U)$ such that

$$\widehat{\omega}_U(s + \mathcal{F}(U)\mathcal{F}^\perp(U), s' + \mathcal{F}(U) \cap \mathcal{F}^\perp(U)) = 0 \quad (11)$$

for all $s' \in \mathcal{F}(U)$. By virtue of the definition of $\widehat{\omega}$, see (10), Equation (11) becomes

$$\omega_U(s, s') = 0$$

for all $s' \in \mathcal{F}(U)$. Therefore, $s \in \mathcal{F}^\perp(U)$, so in $(\mathcal{F}/\mathcal{F} \cap \mathcal{F}^\perp)(U)$, is zero. ■

We now introduce some terminology in connection with the preceding lemma.

Definition 3.2 Let (\mathcal{E}, ω) be a symplectic \mathcal{A} -module, and $\mathcal{F} \subseteq \mathcal{E}$ a co-isotropic sub- \mathcal{A} -module of \mathcal{E} . The symplectic \mathcal{A} -module $(\mathcal{F}/\mathcal{F}^\perp, \widehat{\omega})$, where $\widehat{\omega}$ is given by (10), is called a **reduced symplectic \mathcal{A} -module** or the **\mathcal{A} -module \mathcal{E} reduced by \mathcal{F}** . The notation \mathcal{E}/\mathcal{F} will also be used to denote the underlying \mathcal{A} -module $\mathcal{F}/\mathcal{F}^\perp$ of the reduced symplectic \mathcal{A} -module $(\mathcal{F}/\mathcal{F}^\perp, \widehat{\omega})$. □

Proposition 3.2 *Let (\mathcal{E}, ω) be a symplectic free \mathcal{A} -module of finite rank, $\mathcal{G} \subseteq \mathcal{E}$ a Lagrangian sub- \mathcal{A} -module and $\mathcal{F} \subseteq \mathcal{E}$ a co-isotropic sub- \mathcal{A} -module of \mathcal{E} . Then,*

$$(\mathcal{G} \cap \mathcal{F})/\mathcal{F}^\perp \subseteq \mathcal{E}_{\mathcal{F}}$$

is Lagrangian in the reduced symplectic \mathcal{A} -module.

Proof. Take an open subset $U \subseteq X$ and $s, s' \in (\mathcal{G} \cap \mathcal{F})(U) = \mathcal{G}(U) \cap \mathcal{F}(U)$. One has

$$\widehat{\omega}_U(s + \mathcal{F}^\perp(U), s' + \mathcal{F}^\perp(U)) = \omega_U(s, s') = 0;$$

therefore $(\mathcal{G} \cap \mathcal{F})/\mathcal{F}^\perp$ is isotropic.

Next, we need show that

$$\dim((\mathcal{G} \cap \mathcal{F})/\mathcal{F}^\perp)(U) = \frac{1}{2} \dim(\mathcal{F}/\mathcal{F}^\perp)(U),$$

for every open $U \subseteq X$, to complete the proof of the proposition. The proof of this fact can be found in Abraham and Marsden[[1], Proposition 5.3.10, pp 407-408]. ■

References

- [1] R. Abraham, J.E. Marsden: *Foundations of Mechanics*. The Benjamin/Cummings Publishing Company, Inc, London, 1978.
- [2] W.A. Adkins, S.H. Weintraub: *Algebra. An approach via Module Theory*. Springer-Verlag New York, Inc. 1992.
- [3] T.S. Blyth: *Module Theory. An approach to linear algebra. Second edition*. Oxford Science Publications. Clarendon Press. Oxford 1990.
- [4] E. Cartan: *Leçons sur les invariants intégraux*. Hermann, Paris, 1968 (première édition: 1922).
- [5] CW. Curtis: *Linear Algebra. An introductory approach. Third edition*. Allyn and Bacon, Inc. Boston, 1974.

-
- [6] K.W. Gruenberg, A.K. Weir: *Linear Geometry. Second Edition*. Springer, 1977.
- [7] S. Lang: *Algebra. Revised Third Edition*. Springer, 2002.
- [8] P. Libermann, C-M. Marle: *Symplectic Geometry and Analytical Mechanics* D. Reidel Publishing Company, Dordrecht, 1987.
- [9] S. Mac Lane: *Categories for the Working Mathematician. Second Edition* Springer Verlag New York, Inc. 1998.
- [10] A. Mallios: *Geometry of Vector Sheaves. An Axiomatic Approach to Differential Geometry. Volume I: Vector Sheaves. General Theory*. Kluwer Academic Publishers. Netherlands, 1998.
- [11] A. Mallios: *On localizing topological algebras*. Topological algebras and their applications, 79- 95, Contemp. Math., **341**, Amer. Math. Soc., Providence, RI, 2004.
- [12] A. Mallios, P.P. Ntumba: *Fundamentals for Symplectic \mathcal{A} -modules. Affine Darboux Theorem* Under refereeing.
- [13] S. Sternberg: *Lectures on Differential Geometry* AMS Chelsea Publishing, Providence, Rhode Island, 1999.

Anastasios Mallios
Department of Mathematics
University of Athens
Athens, Greece
Email: amallios@math.uoa.gr

Patrice P. Ntumba
Department of Mathematics and Applied Mathematics
University of Pretoria
Hatfield 0002, Republic of South Africa
Email: patrice.ntumba@up.ac.za