Duality in Equations of Motion from Spacetime Dependent Lagrangians

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Abstract

Starting from lagrangian field theory and the variational principle, we show that duality in equations of motion can also be obtained by introducing explicit spacetime dependence of the lagrangian. Poincare invariance is achieved precisely when the duality conditions are satisfied in a particular way. The same analysis and criteria are valid for both abelian and nonabelian dualities. We illustrate how (1) Dirac string solution (2) Dirac quantisation condition (3) t'Hooft-Polyakov monopole solutions and (4) a procedure emerges for obtaining *new* classical solutions of Yang-Mills (Y-M) theory. Moreover, these results occur in a way that is strongly reminiscent of the *holographic principle*.

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1. Introduction

Most theories with symmetries have their roots in lagrangian field theory as derived from the variational principle. Recently, a new symmetry has attracted attention,*viz.*,duality. Considerable literature exists where duality has been studied in the above framework. For free Maxwell theory, this was first done in a non-manifestly covariant approach by Zwanziger [1] and Deser and Teitelboim [1]. Zwanziger's [1] lagrangian depended on a fixed four vector and manifest isotropy was lost. This was regained when the electric and magnetic charges fulfilled a quantisation condition. Deser and Teitelboim [1] constructed duality transformations by a time-local generator and showed: free Maxwell action and stress tensor components were duality invariant and the generator conserved; in Y-M theory no transformation exists (at the level of fields A_μ) that gives the desired rotations and leaves the action invariant although stress tensor was duality invariant. Deser [1] clarified that duality transformations are not possible in nonabelian theories as these have minimal (self-) coupling.Action description of the so-called self-dual boson fields in any even spacetime dimension also uses this approach [2]. Manifestly Lorentz-covariant approaches for bosonic fields using an infinite set of auxiliary fields exist in [3]; in [4] actions with a finite set of auxiliary fields were used and the covariant version of refs. [1b-2] constructed. In [5] the covariant version of Zwanziger's action [1] was proposed and its connection to [4] shown in diverse dimensions with and without sources. Coupling of the actions [1b-2] to arbitrary external sources was done in [6]; in [7] this was solved for the covariant approach of [4]. Based on [4], the covariant worldvolume action for the M-theory super-5-brane coupled to the duality-symmetric D=11 supergravity was constructed as well as the Lorentz-covariant action for Type IIB supergravity [8]. Models with duality-symmetric or self-dual fields are in [9] while other approaches are in [10].

The motivation of the present work comes from the recent discoveries of the behaviour of field theories at the boundaries of spacetimes [13]. Specifically, gauge theories have dual description in gravity theories in one higher dimension. The theory in higher dimensions is encoded on the boundary (which has a lower dimension) through a *different* (i.e. the particle spectrum is different) field theory in a lower dimension. The operators in the lower dimensional theory are now *composite*. This encoding is through the phenomenon of *duality*, manifesting itself in relations between the coupling constants of the two theories. This discovery of Maldacena and others [13] is a concrete realisation of the *holographic principle* of t'Hooft [13] according to which the combination of quantum mechanics and gravity requires the three dimensional world to be an image of data that can be stored on a two dimensional projection much like a holographic image.

In this context, we give another approach to electromagnetic duality and show that some analogue of the holographic principle seems to exist even at length scales far larger than that of quantum gravity.*This is the formalism of spacetime dependent lagrangians coupled with Schwarz's view [11] that in situations with fields not defined everywhere there exist exotic solutions like monopoles.* (Note: x_ν dependence was already inherent in Zwanziger's work [1]).Such solutions are related to duality. In this work we will be confined to *classical solutions* of theories where *the fields do not couple to gravity*. The x_ν dependence of the lagrangian will be embodied through a function $\Lambda(x_\nu)$, whose *finite* behaviour at *spatial infinity* $\mathbf{x} = \pm\infty$ i.e. *boundary* (together with duality invariance of equations of motion), gives exotic solutions (Dirac string, Dirac monopole, t'Hooft-Polyakov monopole etc.).

A field, by definition, is a quantity defined at all spacetime points. Fields in a lagrangian must be defined everywhere. However, there are theories e.g. the Dirac theory of monopoles [11] and unified theories of strong,weak and electromagnetic interactions where fields are defined only in a region. These theories have monopole solutions and also duality invariance. We show that these solutions can also be understood by demanding the finite behaviour of Λ on the boundary at spatial infinity. Within the boundary, Λ is like a *constant background external field and is non-dynamical.* Hence it is ignorable. On the boundary, finiteness of Λ encodes the exotic solutions, restores Poincare invariance for the full theory and also implies existence of *a new vector field* as a *classical* solution of Yang-Mills theory. Both abelian and nonabelian cases are treated similarly. *We stress that although our results occur at length scales very much larger than those of string theory (quantum gravity), some analogue of the holographic principle still seems to exist.* The dynamics of Λ on the boundary and our results for the full quantum theories will be reported in subsequent communications.

We first develop this formalism. Section 2 clarifies the role of Λ. In Section 3 we show how equations of motion themselves lead to Dirac-string like configurations and how Dirac quantisation condition is obtained by introducing a complex interaction between electric and magnetic charges. In Section 4, t'Hooft-Polyakov monopole solutions [11] are obtained and a procedure for obtaining *new* solutions to classical Y-M theory outlined. We also show that the solutions can accommodate a new vector boson. Section 5 lists the advantages of our method over those currently available.

Let the lagrangian L' be a function of fields η_ρ, their derivatives $\eta_{\rho,\nu}$ *and the spacetime coordinates* x_ν, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$. Variational principle

[12] yields :

$$\int dV \left(\partial_\eta L' - \partial_\mu \partial_{\partial_\mu \eta} L'\right) = 0 \tag{1}$$

Assuming a separation of variables : $L'(\eta_\sigma, \eta_{\sigma,\nu}, ..x_\nu) = L(\eta_\sigma, \eta_{\sigma,\nu})\Lambda(x_\nu)$ ($\Lambda(x_\nu)$ is the x_ν dependent part and is a finite non-vanishing function) gives

$$\int dV \left(\partial_\eta (L\Lambda) - \partial_\mu \partial_{\partial_\mu \eta} (L\Lambda)\right) = 0 \tag{2}$$

2. The Role of Lambda

In this work we will be confined to *classical solutions* of theories where *the fields do not couple to gravity.* Under these circumstances,

(1)Λ is *not* dynamical and is a finite, non-vanishing function given once and for all at all x_ν multiplying the primitive lagrangian L. *It is like an external field*, any allusion to the dilaton is unfounded and equations of motion for Λ meaningless.

(2)Duality invariance is related to finiteness of Λ. When fields are not defined everywhere and equations of motion are duality invariant, finiteness of Λ on the spatial boundary at infinity leads to new solutions for the fields. Requiring $\Lambda = 1$ at $\mathbf{x} = \pm\infty$ gives back usual L together with exotic solutions. Within the boundary Λ is an ignorable costant.

(3)Poincare invariance and duality invariance is achieved through same behaviour of Λ. Λ finite,but not a constant, gives theories with duality invariance but not Poincare invariance.

(4)The finite behaviour of Λ on the boundary *encodes the exotic solutions of the theory within the boundary.* In this way we are reminded of the holographic principle.

3. The Dirac string-like field configuration and Dirac quantisation condition

For a modified electrodynamics L' may be written as

$$L' = [-(1/4)F^{\mu\nu}F_{\mu\nu} + j^{\mu}A_{\mu}]\Lambda(x_{\nu}) \tag{3}$$

with $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ and j^{μ} a current. Dual of $F^{\mu\nu}$ is $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. This corresponds to $\mathbf{E} \Rightarrow \mathbf{B}$ and $\mathbf{B} \Rightarrow -\mathbf{E}$ in the field strength $F_{\mu\nu}$. Equations of motion obtained from (2) are

$$\Lambda(\partial^{\mu}F_{\mu\nu}) + (\partial^{\mu}\Lambda)F_{\mu\nu} - \Lambda j_{\nu} = 0 \tag{4a}$$

while the dual $\tilde{F}_{\mu\nu}$ satisfies

$$\partial^{\mu}\tilde{F}_{\mu\nu} = 0 \tag{4b}$$

Duality invariance means identical equations of motion for F and $\tilde{F}$,i.e. $\partial^{\mu}F_{\mu\nu} = 0$. This implies

$$(\partial^{\mu}\Lambda)F_{\mu\nu} = \Lambda j_{\nu} \tag{5}$$

There are two possibilities: (1)Finiteness of Λ is assumed to be independent of the behaviour of the fields and Λ can be put equal to the constant unity *a priori*. One then has *usual* electrodynamics and in the absence of sources duality is present.(2)Λ satisfies (5) : L' is a lagrangian whose equations of motion and Bianchi identity are invariant under duality rotations *even in presence of sources.* Λ is finite only if the fields behave in a certain way and this precisely corresponds to the solutions mentioned above.

Now, at the theoretical level, neither of the equations $\partial^{\mu}\tilde{F}_{\mu\nu} = 0$ and $\partial^{\mu}F_{\mu\nu} = 0$ are more fundamental than the other.Remembering this let us discuss the second possibility (i.e. Λ satisfies (5)) by considering some specific forms for j_{μ}.

Case (a): Consider electrodynamics with only electric charge e. Let $j_0(x) = e\delta(x_1)\delta(x_2)\delta(x_3)$; $j_i(x) = 0$; $\Lambda(x_\nu) = \Lambda(x_3)$. i runs over spatial coordinates. Eq.(5) then splits into two sets of equations : one for the temporal index and another set of three for the spatial indices. As $F_{3i} \neq 0$, the second set gives a solution of Λ as a constant function of x_3. This solution when put in the first set gives $F_{30}(0,0,0,x_0) = E_3(0,0,0,t) \Rightarrow \infty$. E_3 is the electric field along the third direction. As this is valid for all times, the solution is effectively time independent.

Case (b):Let $j_0(x) = e\sum_{n=0}^{\infty}\delta(x_1)\delta(x_2)\delta(an + x_3)$; $j_i(x) = 0$; $\Lambda(x_\nu) = \Lambda(x_3)$. i runs over spatial coordinates and a is small and always positive. Again, Λ is constant and we get $F_{30}(0,0,an,x_0) = E_3(0,0,an,t) \Rightarrow \infty$ as one possible configuration for the electric field. E_3 is again time independent for reasons already mentioned.

As stated earlier, $\partial^\mu F_{\mu\nu} = 0$ and $\partial^\mu \tilde{F}_{\mu\nu} = 0$ must be placed on equal footing. So the just concluded analysis is also valid for $\partial^\mu \tilde{F}_{\mu\nu} = 0$. The only differences will be (1) coupling e (electric charge) replaced by coupling m (magnetic charge) (2) Maxwell's equations now have a corresponding magnetic vector potential (3) Maxell's equations modified with $div\mathbf{B} \neq \mathbf{0}$ and $div\mathbf{E} = \mathbf{0}$.The other Maxwell equations will be accordingly modified. Consider case (a). Suppose we remove the singular field right upto the origin.Then situation is similar to Dirac construction of "infinite solenoid minus the string ".In the same spirit case(b) reminds us of the Dirac-string configuration. If now quantum considerations of single-valuedness of magnetic wave function are imposed *a la* Dirac, then Dirac quantisation conditions follow . This is how the Dirac monopole solutions can be understood in our formalism.

Now consider a $U(1) \otimes U(1)$ gauge invariant theory. A_μ, B_μ are four-vector potentials corresponding to electric and magnetic charges respectively; $F_{\mu\nu}$, $G_{\mu\nu}$ the respective field strengths; j_μ, k_μ the electric and magnetic (current) sources with interactions between respective currents and potentials introduced in usual way:

$$L_1 = -(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} + j^\mu A_\mu + k^\mu B_\mu \qquad (6a)$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$; $G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$; $\tilde{G}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$.
$\partial^\mu j_\mu = \partial^\mu k_\mu = 0$ (current conservation); $\partial^\mu A_\mu = \partial^\mu B_\mu = 0$ (transversality)
Therefore $\partial^\mu F_{\mu\nu} = j_\nu$; $\partial^\mu \tilde{F}_{\mu\nu} = 0$; $\partial^\mu G_{\mu\nu} = k_\nu$; $\partial^\mu \tilde{G}_{\mu\nu} = 0$.
Defining

$$\xi^{\mu\nu} = F^{\mu\nu} + \tilde{G}^{\mu\nu}; \;\; \beta^{\mu\nu} = \tilde{F}^{\mu\nu} - G^{\mu\nu} = \tilde{\xi}^{\mu\nu} \qquad (6b)$$

gives

$$\partial^\mu \xi_{\mu\nu} = j_\nu \;\; ; \;\; \partial^\mu \tilde{\xi}_{\mu\nu} = -k_\nu \qquad (6c)$$

Note that for $\xi_{\mu\nu} \to \tilde{\xi}_{\mu\nu}$ one has $j_\nu \to -k_\nu$, and for $\tilde{\xi}_{\mu\nu} \to -\xi_{\mu\nu}$ one has $k_\nu \to j_\nu$. Contrast this to the usual case: for $F^{\mu\nu} \to \tilde{F}^{\mu\nu}$, one has $j_\nu \to k_\nu$ while for $\tilde{F}^{\mu\nu} \to -F^{\mu\nu}$, one has $k_\nu \to -j_\nu$; with $\partial^\mu F_{\mu\nu} = j_\nu$; $\partial^\mu \tilde{F}_{\mu\nu} = k_\nu$. In the absence of j_ν and k_ν one gets back the usual case.

Here both the $U(1)$ gauge fields (A_μ, B_μ) are independent. To obtain a single independent field one can start with the fields as defined in equation ($6b, c$) and proceed like Zwanziger [1]. In fact, $6(c)$ is identical to Zwanziger's starting equation ([1], eq.(2.1)) modulo a sign. The so-called one independent field can be obtained as a *superposition* of two independent fields— each of which separately describes an *electric-charge-only* world or a *magnetic-*

charge-only world. This is our view. From here we take a different route. We introduce a new interaction between the electric and magnetic charges and invoke the finiteness of Λ. The result is Dirac quantisation condition.

Consider a complex interaction L_2' between the electric and magnetic charges via their respective four-vector potentials and (current) sources: $L_2' = i\alpha A^\mu B_\mu j^\nu k_\nu$. α is a constant. In the classical theory the action has dimension of angular momentum. It is then straightforward to verify that α has the dimension of *inverse* angular momentum i.e. $(\hbar)^{-1}$. Dirac in his derivation of the charge quantisation using Maxwell's theory assumed that the electron obeyed quantum mechanics. In our approach we are therefore motivated to employ this semi-classical approach in the sense that we will be using a complex interaction. If we take the lagrangian as

$$L' = L_1 + L_2' \tag{7}$$

one gets

$$\partial^\mu \xi_{\mu\nu} = (j_\nu + ic_\nu') \ ; \ \partial^\mu \tilde{\xi}_{\mu\nu} = (-k_\nu - id_\nu') \tag{8a}$$

with

$$c_\nu' = \alpha j^\mu k_\mu B_\nu \ ; \ d_\nu' = \alpha j^\mu k_\mu A_\nu \tag{8b}$$

Here $j^\mu \rightarrow -k^\mu$ and $k^\mu \rightarrow j^\mu$ so that $j^\mu k_\mu \rightarrow -j^\mu k_\mu$. Hence the duality transformation gives

$$\partial^\mu \tilde{\xi}_{\mu\nu} = (-k_\nu - ic_\nu') \ ; \partial^\mu \xi_{\mu\nu} = (j_\nu - id_\nu') \tag{8c}$$

Comparing (8c) with (8a) we see that duality invariance cannot be obtained starting from the lagrangian (7).

Introducing x_ν dependence through $\Lambda(x)$,and putting $L_2 = f(\Lambda)L'_2,\ \ c_\nu = f(\Lambda)c'_\nu,\ \ d_\nu = f(\Lambda)d'_\nu$ the lagrangian is

$$L = L'\Lambda(x) = [L_1 + L_2]\Lambda(x)$$

$$= [-(1/4)F^{\mu\nu}F_{\mu\nu} - (1/4)G^{\mu\nu}G_{\mu\nu} + j^\mu A_\mu + k^\mu B_\mu + if(\Lambda)\alpha A^\mu B_\mu j^\nu k_\nu]\Lambda(x) \quad (9)$$

$f(\Lambda)$ a dimensionless function of Λ such that

$$f(\Lambda) = 0 \ \ or\ \ a\ \ finite\ \ constant$$

when $\Lambda = 1$.

The conditions of duality invariance now become

$$\Lambda\partial^\mu\xi_{\mu\nu} + [(\partial^\mu\Lambda F_{\mu\nu} - \Lambda(j_\nu + ic_\nu)] = 0 \quad (10a)$$

$$\Lambda\partial^\mu\beta_{\mu\nu} - [(\partial^\mu\Lambda G_{\mu\nu} - \Lambda(k_\nu + id_\nu)] = 0 \quad (10b)$$

(Note that $\beta_{\mu\nu} = \tilde{\xi}_{\mu\nu}$). Duality invariance is obtained if

$$[(\partial^\mu\Lambda F_{\mu\nu} - \Lambda(j_\nu + ic_\nu)] = 0 \quad (11a)$$

and

$$[(\partial^\mu\Lambda G_{\mu\nu} - \Lambda(k_\nu + id_\nu)] = 0 \quad (11b)$$

Let $\Lambda(x_\nu)$ be a function of x_3 only and the sources have only time components. So we have an electric charge at the origin and a magnetic charge at $x_3 = a$; $\Lambda = \Lambda(x_3)\ ;\ j^i = k^i = 0\ ;\ j^0 = e\delta(x_1)\delta(x_2)\delta(x_3); k^0 = g\delta(x_1)\delta(x_2)\delta(x_3 - a)$. Then one has for $\nu = 0, 1, 2$

$$(\partial^3\Lambda)F_{3\nu} = \Lambda(j_\nu + ic_\nu) \quad (12a)$$

$$(\partial^3\Lambda)G_{3\nu} = \Lambda(k_\nu + id_\nu) \quad (12b)$$

For $\nu = 3$, $G_{33} = F_{33} = 0$. So $c_3 = d_3 = 0$. For $\nu = 0$,solutions to (12) are

$$\Lambda_\infty = \Lambda_{-\infty} exp[e\delta(x_1)\delta(x_2)/F_{30}(x_1, x_2, 0, x_0)] exp[if(\Lambda)\alpha eg P_0(x_1, x_2, x_3, x_0)] \tag{13a}$$

$$\Lambda_\infty = \Lambda_{-\infty} exp[g\delta(x_1)\delta(x_2)/G_{30}(x_1, x_2, 0, x_0)] exp[if(\Lambda)\alpha eg Q_0(x_1, x_2, x_3, x_0)] \tag{13b}$$

$$P_0(x_1, x_2, a, x_0) = (\delta(x_1))^2(\delta(x_2))^2\delta(a) B_0(x_1, x_2, a, x_0)/F_{30}(x_1, x_2, a, x_0) \tag{14a}$$

$$Q_0(x_1, x_2, a, x_0) = (\delta(x_1))^2(\delta(x_2))^2\delta(a) A_0(x_1, x_2, a, x_0)/G_{30}(x_1, x_2, a, x_0) \tag{14b}$$

Λ must be finite. Let $\Lambda_\infty = \Lambda_{-\infty} = unity$. Consider the set $(13a)$ and $(14a)$. Obviously the two exponentials must reduce to unity. For the first exponential (as seen in last section) *this corresponds to the Dirac string configuration where* $F_{30} \to \infty$ *so that the exponential essentially becomes unity.*For the second exponential, we see that in $(14a)$ the numerator has singular $\delta-$functions and together with $B_0 \to \infty$ would lead to a finite P_0 since $F_{30} \to \infty$. So second exponential is 1 if $exp[if(\lambda)\alpha eg P_0] = 1$ i.e.$(exp[if(\Lambda)\alpha eg])^{P_0} = 1$ (as P_0 is finite). Therefore

$$f(\Lambda)\alpha eg = 2\pi n \tag{15}$$

There are two possibilities:

(a)$f(\Lambda) = 0$. Then the $U(1) \otimes U(1)$ invariance of L_1 is not broken (from which Zwanziger's lagrangian can be retrieved via eq.$(6c)$) and we just have the Dirac string configuration (from the first exponential, $F_{30} \to \infty$).

(b)$f(\Lambda) =$ *a finite constant. Then the* $U(1) \otimes U(1)$ *invariance of* L_1 *is broken; and putting* $\alpha = (\hbar)^{-1}$ *we see that* (15) *is just the Dirac quantisation rule.*

So the Dirac quantisation rule is obtained by breaking the $U(1) \otimes U(1)$ invariance of the *unphysical* (i.e. two independent fields) lagrangian L_1. For $\nu = 1, 2$ a similar analysis will again lead to (15) and similarly for (13b) and (14b).

4. The t'Hooft-Polyakov Monopole Solutions

Consider now a simple nonabelian generalisation of (3):

$$L' = [-(1/4)G_a^{\mu\nu} G_{a\ \mu\nu} + j_a^\mu W_{a\ \mu}]\Lambda(x_\nu) \tag{16}$$

a, b, c are $SO(3)$ indices and $G_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - e\epsilon_{abc} W_b^\mu W_c^\nu$. j_a^μ is an external current and $\tilde{G}_a^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} G_{a\ \rho\sigma}$. Analogues of (4$a$), (4$b$) are:

$$\Lambda(D^\mu G_{a\ \mu\nu}) + (\partial^\mu \Lambda) G_{a\ \mu\nu} - \Lambda j_{a\ \nu} = 0 \tag{17a}$$

$$D^\mu \tilde{G}_{a\ \mu\nu} = 0 \tag{17b}$$

Again duality invariance of the equations (17) imply

$$(\partial^\mu \Lambda) G_{a\ \mu\nu} = \Lambda j_{a\ \nu} \tag{18}$$

If we take $\Lambda = \Lambda(r)$, $j_{a\ i} = 0$ for all i and a and $j_{a\ 0} = 0$ then one solution is a radial magnetic field $\mathbf{B_a} = \mathbf{H_a}(\mathbf{r})$ with $\mathbf{H_a}(\mathbf{r})$ satisfying $D^\mu \tilde{G}_{a\ \mu\nu} = 0$ and $D^\mu G_{a\ \mu\nu} = 0$.

Now consider the Georgi-Glashow model with L' defined as

$$L' = [-(1/4)G_a^{\mu\nu} G_{a\ \mu\nu} + (1/2)(D^\mu \phi)_a (D_\mu \phi)_a - V(\phi)]\Lambda(x_\nu) \tag{19}$$

The gauge group is $SO(3)$, $G_a^{\mu\nu}$ is as defined before, and the matter fields ϕ are in the adjoint representation of $SO(3)$. Equations of motion are :

$$\Lambda(D^\mu G_{a\ \mu\nu}) + (\partial^\mu \Lambda) G_{a\ \mu\nu} + \Lambda e\epsilon_{abc}(\partial_\nu \phi)_b (\phi)_c - \Lambda e^2 \epsilon_{abc}\epsilon_{bc'd'} W_{\nu\ c'} \phi_c \phi_{d'} = 0 \tag{20a}$$

$$(D^\mu D_\mu \phi)_a \Lambda + (D_\mu \phi)_a \partial_\mu \Lambda = -(\partial_{\phi^a} V)\Lambda \tag{20b}$$

and the Bianchi identities are:

$$D^\mu \tilde{G}_{a\ \ \mu\nu} = 0 \tag{20c}$$

Duality invariance then leads to

$$(\partial^\mu \Lambda) G_{a\ \ \mu\nu} = -\Lambda e \epsilon_{abc} (D_\nu \phi)_b (\phi)_c \tag{21}$$

For $\Lambda = \Lambda(r)$ we have

$$\Lambda_\infty = \Lambda_0 exp[-e \int_0^\infty dr \left((\epsilon_{abc} (D_\nu \phi)_b \phi_c)(\partial^i r G_{a\ \ i\nu})^{-1} \right)] \tag{22}$$

where Λ_p is the value of Λ at $r = p$; a, ν are fixed; and there is a sum over indices i, b and c. Λ_∞ must be finite. Choose this to be the constant unity.This may be realised in the following ways :

(I) $(D_\nu(\phi)_b \Rightarrow 0, \quad (\phi)_c \Rightarrow finite,$ and the product $(D_\nu \phi)_b (\phi)_c$ falls off faster than $G_{a\ \ i\nu}$ for large r. Then a constant value for Λ is perfectly consistent with (20b) and the conditions become analogous to the Higgs' vacuum condition for the t'Hooft-Polyakov monopole solutions where the duality invariance of the equations of motion and Bianchi identities are attained at large r by demanding $(D_\mu \phi)_a \Rightarrow 0$ and $\phi_a \Rightarrow a\delta_{a3}$ at large r. *Note that our results are perfectly consistent with the usual choice for the Higgs' potential $V(\phi)$ even though nothing has been assumed regarding this.* Thus, the t'Hooft-Polyakov monopole solutions follow naturally in our formalism.

(II)

$$(\epsilon_{abc} (D_\nu \phi)_b \phi_c) \Rightarrow 0 \tag{23}$$

and falls off faster than $G_{a\ \ i\nu}$ for large r (a and ν are fixed). A solution is when

$$D_\nu \phi = \alpha_\nu \phi$$

where α_ν can be (a) any Lorentz four vector field that is consistent with all the relvant equations of motion and the finiteness of energy constraint. (b) any Lorentz four vector field as in (a) but which may also carry $SO(3)$ indices,i.e. $\alpha_\nu = \alpha^a_\nu \tau^a$, τ^a being the generators of $SO(3)$, $a, b, c = 1, 2, 3$. Putting this in ($20b$) gives (Λ now is unity)

$$[D^\mu(\alpha_\mu \phi)]_a = -(\partial_{\phi^a} V)\Lambda \tag{24}$$

Problem therefore reduces to finding solutions of (23) and (24). These will be discussed elsewhere. Eq.(23) is like a *master equation* for obtaining solutions of classical Y-M theory incorporating duality.

5. Conclusion

We have given an alternative way to understand duality using the approach of spacetime dependent lagrangians coupled to the Schwarz postulate [11] that whenever fields in a lagrangian are not defined everywhere one has monopole solutions. Our results indicate that some analogue of the holographic principle may be operative even at length scales far larger than the Planck scale in theories which incorporate duality invariance in the equations of motion. The advantages of our method are

(1)It gives a procedure for obtaining *new* solutions to classical Yang-Mills theory incorporating duality. In particular, the solutions can accommodate a new vector field. It can be shown that none of the known aspects of the Georgi-Glashow lagrangian are violated by the presence of this solution.

Details of such solutions and applications to a full quantum theory will be discussed elsewhere.

(2)There is a possibility of a generalisation of the Bogomolny equations (refer to equation (23)). Usual Bogomolny equations seem to be the simplest first steps towards realisation of duality. This is partly evident in that the Higgs' vacuum condition has been obtained without specifying $V(\phi)$. Spontaneous symmetry breaking has neither been invoked nor contradicted. Λ is just required to be finite.

(3)The same analysis and criteria are valid for both abelian and non-abelian dualities. *The Dirac-string solutions, the Dirac quantisation condition and the t'Hooft-Polyakov monopole solutions all follow from the same underlying principle.* This was not possible before [1b,1c].

(4)When Λ is a constant, all known lagrangians exhibiting duality become accessible. When Λ is not a constant but some finite well defined function everywhere, then a plethora of new lagrangians whose equations of motion exhibit duality can be constructed (at the expense of Poincare invariance).

(5)Finally, our method indicates that some flavour of the holographic principle can be obtained in certain gauge theories *even at length scales very much larger than those of quantum gravity.* This aspect has never been revealed before in any study using standard methods.

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