# IMAGE PARTITION NEAR AN IDEMPOTENT

TANUSHREE BISWAS

Abstract. Some of the classical results of Ramsey Theory can be naturally stated in terms of image partition regularity of matrices. There have been many significant results of image partition regular matrices as well as image partition regular matrices *near zero*. Here, we are investigating image partition regularity near an idempotent of an arbitrary Hausdorff semitopological semigroup $(T, +)$ and a dense subsemigroup $S$ of $T$. We describe some combinatorial applications on finite as well as infinite image partition regular matrices based on the Central Sets Theorem near an idempotent of $T$.

## 1. Introduction

One of the most famous results of the field of Ramsey Theory was given by van der Waerden in the year 1927, guaranteeing monochromatic arithmetic progressions.

**Theorem 1.1** (van der Waerden)**.** *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$ be partitioned into finitely many cells. Then, for any arbitrary length $l$, there exist $i \in \{1, 2, \ldots, r\}$ and $a, d \in \mathbb{N}$ such that*

$$a,\, a+d,\, \ldots,\, ld \in A_i.$$

*Proof.* See [13] □

These above classical results of Ramsey Theory can be also represented in a following way . Given $u, v \in \mathbb{N}$ and a $u \times v$ matrix $M$ with non-negative integer entries, whenever $\mathbb{N}$ is finitely colored there must exist some $\vec{x} \in \mathbb{N}^v$ such that the entries of $M\vec{x}$ are monochromatic. The arithmetic progression $\{a, a+d, a+2d, a+3d\}$ in the van der Waerden's Theorem for $l = 3$ is precisely the set of entries of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} a \\ d \end{pmatrix}.$$

Also, Schur's Theorem [12] and Hilbert's Theorem [8] (for $k = 3$) can also be written in the above manner using the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

*Key words and phrases.* Ramsey theory, Central sets near an idempotent, Image partition near an idempotent.

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

respectively.

These matrices are called *image partition regular matrices.* We give a proper definition below.

**Definition 1.2.** Let $u, v \in \mathbb{N}$, and let $M$ be a $u \times v$ matrix . We say $M$ to be an image partition regular matrix over $\mathbb{N}$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \cup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $M\vec{x} \in A_i^u$.

Image partition regularity is one of the most important concepts of Ramsey Theory. Similarly, image partition regularity near zero has been defined over a dense subsemigroup $((0, \infty), +)$. Further, in [9], some algebraic results were obtained on the Stone-Čech compactification of the ultrafilters near zero defined over a dense subsemigroup $((0, \infty), +)$. Based on the results obtained in [9], the authors in [3] investigated various combinatorial properties partition regularity near zero. In this article, we are trying to generalize the notion of *near zero* into an arbitrary idempotent using the idea of Stone-Čech compactification. We first give a brief description of algebraic structure of $\beta T$.

Let $(T, +)$, be a discrete space. We get the Stone-Čech compactification of $T$ by taking the points of $\beta T$ to be the ultrafilters on $T$, identifying the principal ultrafilters with the points of $T$ and thus pretending that $T \subseteq \beta T$. Given $A \subseteq T$, $A = \overline{A} = \{p \in \beta T : A \in p\}$ forms a basis of open (close) sets for a topology on $\beta T$. The operation $+$ on $T$ has a natural extension to the Stone-Čech compactification $\beta T$ of $T$ and under this extended operation, $(\beta T, +)$ is a compact right topological semigroup (meaning that, for any $p \in \beta T$, the function $\rho_p : \beta T \to \beta T$ defined by $\rho_p(q) = q+p$ is continuous) with $T$ contained in its topological center (meaning that, for any $x \in T$, the function $\lambda_x : \beta T \to \beta T$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta T$ and $A \subseteq T$, $A \in p + q$ if and only if $\{x \in T : -x + A \in q\} \in p$, where $-x + A = \{y \in T : x + y \in A\}$.

The set $0^+$ of all nonprincipal ultrafilters on $T = ((0, \infty), +)$ that converge to 0 (0 being the idempotent in $T = ((0, \infty), +)$ ) is a semigroup under the restriction of the usual operation $'+'$ on $\beta T$ , the Stone-Čech compactification of the discrete semigroup $T = ((0, \infty), +)$ . The set $0^+$ is nothing but ultrafilters near 0. In [9], N. Hindman and I. Leader characterized the smallest ideal of $(0^+, +)$, its closure, and those sets *central* in $(0^+, +)$ , that is, those sets which are members of minimal idempotents in $(0^+, +)$. They derived new combinatorial applications of those sets that are central in $(0^+, +)$. Further, in [3] and [4], De, Hindman and Strauss worked out some more and much more elaborated applications of central sets.

Later, Akbari Tootkaboni, M and Vahed, T, in [1] defines a semigroup of ultrafilters near an arbitrary idempotent (instead of taking the particular 0). We give basic definitions and results which will be required throughout the paper.

**Definition 1.3.** Let $(T, +)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$. Let $E(T)$ be the set consisting of idempotents of $T$. Then

(a) Given $e \in E(T)$, $e^* = \left\{p \in \beta T_d \colon e \in \cap_{A \in p} \overline{A}\right\}$
(b) Given $x \in T$, $x_S^* = \left\{p \in \beta S_d \colon x \in \cap_{A \in p} \overline{A}\right\}$

In [1], Akbari Tootkaboni, M and Vahed, T, defined ultrafilters near a point and showed that if $S$ is a dense subsemigroup of a semitopological semigroup $(T, +)$and if $e \in T$ is an idempotent, then $e_S^*$ forms a compact right topological semigroup.

As a compact right topological semigroup, $e_S^*$ has a smallest two-sided ideal [2, Theorem 1.3.11] that contains idempotents [6, Corollary 2.10]. They characterized the members of the smallest ideal of $(e^*, +)$, its closure and also those subsets of $T$ that have idempotents in $(e^*, +)$ in their closure. They also defined the sets central near $e$, and derived various combinatorial applications from it.

Before moving further, we need some basic definitions and results based on which our article stands. Note that throughout this paper, $(T, +)$ denotes a Hausdorff semitopological semigroup and $S$ is a dense subsemigroup of $T$. $E(T)$ denotes the collection of all idempotents in $T$ . For every $x \in T$, $\tau_x$ denotes the collection of all neighborhoods of $x$ , where a set $U$ is called a neighborhood of $x \in T$ if $x \in int_T(U)$. For a set $A$, we write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of $A$ and $\mathcal{P}(A)$ denote the collection of all subsets of $A$.

**Definition 1.4.** Let $\{x_n\}$ be a sequence in $S$. Then

$$\mathrm{FS}\left(\langle x_n \rangle_{n=1}^{\infty}\right) = \left\{\sum_{n \in F} x_n \colon F \text{ is a finite nonempty subset of } \mathbb{N}\right\}$$

is called *finite sum* of the sequence $x_n$.

**Definition 1.5.** Let $(T, +)$ be a semitopological semigroup.

(a) Let $\mathcal{B}$ be a local base at the point $x \in T$. We say $\mathcal{B}$ has the finite cover property if $\{V \in \mathcal{B} \colon y \in V\}$ is finite for each $y \in T \setminus \{x\}$.
(b) Let $S$ be a dense subsemigroup of $T$ and $e \in E(T)$. Let $\{x_n\}$ be a sequence in $S$. We say $\sum_{n=1}^{\infty} x_n$ converges near $e$ if for each $U \in \tau_e$ there exists $m \in \mathbb{N}$ such that $FS\left(\langle x_n \rangle_{n=k}^{l}\right) \subseteq U$ for each $l > k \geq m$.
(c) Let $\mathcal{B} = \{U_n \colon n \in \mathbb{N}\}$ be a countable local base at the point $x \in T$ such that $U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$, $U_{n+1} + U_{n+1} \subseteq U_n$, and for each sequence $\{x_n\}_{n=1}^{\infty}$ if $x_n \in U_n$ for each $n \in \mathbb{N}$ then $\sum_{n=1}^{\infty} x_n$ converges near $x$. Then we say $\mathcal{B}$ is a countable local base for convergence at the point $x \in T$ .
(d) Let $\mathcal{B}$ be a local base at the point $x \in T$ . If $\mathcal{B}$ satisfies in conditions $(a)$ and $(c)$ then $\mathcal{B}$ is called a countable local base that has the finite cover property for convergence at the point $x$. For simplicity we say $\mathcal{B}$ has the **F** property at the point $x$.

**Theorem 1.6.** *Let $(T, +)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$, $e \in E(T)$ and $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ the **F** property at the point $e$. Then there exists $p = p+p$ in $e_S^*$ with $A \in p$ if and only if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $S$ such that $\lim_{n\to\infty} x_n = e$, $\sum_{n=1}^{\infty} x_n$ converges near $e$ and $FS\left(\langle x_n \rangle_{n=1}^{\infty}\right) \subseteq A$.*

*Proof.* [1, Theorem 3.2]. □

**Definition 1.7.** Let $S$ be a dense subsemigroup of $(T,+)$ and $e\in E(T)$.

(a) $K$ is the smallest ideal of $e_S^*$.
(b) A subset $B$ of $S$ is *syndetic near* $e$ if and only if for each $U\in\tau_e$ , there exist some $F\in\mathcal{P}_f(U\cap S)$ and some $V\in\tau_e$ such that $S\cap V\subseteq\cup_{t\in F}(-t+B)$ .
(c) A set $A\subseteq S$ is central near $e$ if and only if there is some idempotent $p\in K$ with $A\in p$.

**Definition 1.8.** Let $(T,+)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$has the **F** property at the point $e\in E(T)$.

(a) $\Phi=\{f\colon\mathbb{N}\to\mathbb{N}\colon\forall n\in\mathbb{N},\,f(n)\leq n\}$.
(b) $\mathcal{Y}=\left\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\right\rangle_{i=1}^{\infty}$such that for each $i\in\mathbb{N}$, $\langle y_{i,t}\rangle_{t=1}^{\infty}\in S$ is a sequence for which $\sum_{t=1}^{\infty}y_{i,t}$ converges.
(c) Given $Y=\left\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\right\rangle_{i=1}^{\infty}\in\mathcal{Y}$ and $A\subseteq S$, $A$ is a $J_Y-set$ near $e$ if and only if $\forall n\in\mathbb{N}$ there exist $a\in U_n$ and $H\in\mathcal{P}_f(\mathbb{N})$ such that $min\,H_n\geq n$ and for each $i\in\{1,2,\ldots,n\}$, $a+\sum_{t\in H}y_{i,t}\in A$.
(d) Given $Y\in\mathcal{Y}$, $J_Y=\{p\in e_S^*\colon\forall A\in p,\,A\,is\,a\,J_Y-set\,near\,e\}$.
(e) $J=\cap_{Y\in\mathcal{Y}}J_Y$.

**Lemma 1.9.** *Let $(T,+)$ be a commutative semitopological semigroup and $S$ be a dense subsemigroup of $T$ . Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$ has the **F** property at the point $e\in E(T)$. Let $Y\in\mathcal{Y}$. Then $K\subseteq J_Y$.*

*Proof.* [1, Lemma 4.2]. □

Due to the above 1.9, the authors, in [1] were able to show that the sets that are central near an idempotent satisfy a version of the Frustenberg's Central Sets Theorem [7] which is as follows.

**Theorem 1.10.** *Let $(T,+)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$has the **F** property at the point $e\in E(T)$ and $A$ be a central set near $e$. Suppose that $Y=\left\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\right\rangle_{i=1}^{\infty}\in\mathcal{Y}$ . Then there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$in $S$ $\langle H_n\rangle_{n=1}^{\infty}$in $\mathcal{P}_f(\mathbb{N})$ such that*

(a) *for each $n\in\mathbb{N}$, $a_n\in U_n$ and $max\,H_n<min\,H_{n+1}$.*
(b) *for each $f\in\Phi$, $FS\left(\left\langle a_n+\sum_{t\in H_n}y_{f(n),t}\right\rangle_{n=1}^{\infty}\right)\subseteq A$.*

*Proof.* See Theorem 4.11 of [9]. □

Akbari Tootkaboni, M and Vahed, T gave an example of one of many possible combinatorial applications based on 1.10.

**Corollary 1.11.** *Let $(T,+)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$. Let $\mathcal{B}=\{U_n\}_{n=1}^{\infty}$has the **F** property at the point $e\in E(T)$ and $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $S$ such that $\lim_{n\to\infty}x_n=e$. Assume $r\in\mathbb{N}$ and $S=\cup_{i=1}^{r}A_i$. Then there is some $i\in\{1,2,\ldots,r\}$ such that for every $U\in\tau_e$ and every $l\in\mathbb{N}$ there is an arithmetic progression $\{a,a+d,\ldots,a+ld\}\subseteq A_i\cap U$ with increment $d\in FS(\langle x_n\rangle_{n=1}^{\infty})$.*

*Proof.* [9, Corollary 5.1]. □

In our research article, we try to bring out some combinatorial properties of image partition regularity near an idempotent of a semitopological semigroup.

## 2. Finite Matrices

First, we shall investigate on some image partition regularity near $e$ on finite matrices. Then we will move on some infinite ones.

**Definition 2.1.** Let $S$ be a dense subsemigroup of a semitopological semigroup $(T, +)$. Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$has the **F** property at the point $e \in E(T)$. Let $M$ be a $u \times v \in \mathbb{N}$ with entries from $\mathbb{Q}$. Then $M$ is image partition regular over $S$ near $e$ (abbreviated $IPR/S_e$) if and only if, whenever $S = \cup_{i=1}^{r} A_i$ for some $r \in \mathbb{N}$, there is some $i \in \{1, 2, \ldots, r\}$ such that for every $U \in \tau_e$ there exists $\vec{x} \in S^v$ with $\lim_{n\to\infty} x_n = e$ for which $M\vec{x} \in A_i \cap U$.

**Lemma 2.2.** *Let $A$ be a $u \times v$ matrix which is image partition regular over $\mathbb{N}$. Then it is image partition regular near any $e \in E(T)$ over any dense subsemigroup $S$ of semitopological semigroup $T$ .*

*Proof.* Let $r \in \mathbb{N}$ and $S = \bigcup_{j=1}^{r} A_j$ . By a standard compactness arguement pick $n \in \mathbb{N}$ such that whenever $\{1, 2, 3, \ldots, n\} = \bigcup_{j=1}^{r} D_j$, there exist $\overrightarrow{x} \in \{1, 2, 3, \ldots, n\}^v$ and $i \in \{1, 2, 3, \ldots, r\}$ such that $Ax \in (D_i)^u$. Pick

$$z \in \left\{S \bigcap U :\ U \in \mathcal{T}_e\right\}.$$

For $i \in \{1, 2, \ldots, n\}$, let

$$D_i = \left\{t \in \{1, 2, 3, \ldots, n\} :\ z + z + z + z(t - times) \in A_i \bigcap U\right\}.$$

Thus, picking $i \in \{1, 2, 3, \ldots, r\}$ and $\overrightarrow{x} \in \{1, 2, 3, \ldots, n\}^v$ such that $A\overrightarrow{x} \in (D_i)^u$ and assuming $\overrightarrow{y} = z\overrightarrow{x}$, we get $A\overrightarrow{y} \in A_i \bigcap U$. □

*Remark* 2.3. This is to note that the converse of the above lemma also holds good. This can be verified by the Corollary 1.6 of [5].

**Definition 2.4.** First entries condition. Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with integer entries. Then

(a) $A$ satisfies the first entries condition if and only if each row of $A$ is not 0 and whenever $i, j \in \{1, 2, \ldots, u\}$ and

$$t = min\{k \in \{1, 2, \ldots, v\} :\ a_{i,k} \neq 0\} = min\{k \in \{1, 2, \ldots, v\} :\ a_{j,k} \neq 0\}$$

one has $a_{i,t} = a_{j,t} > 0$.

(b) A number $c$ is a first entry of $A$ if and only if for some $i \in \{1, 2, \ldots, u\}$ and some $t \in \{1, 2, \ldots, v\}$, $t = min\{k \in \{1, 2, \ldots, v\} :\ a_{i,k} \neq 0\}$ and $c = a_{i,t}$ .

**Definition 2.5.** Let $S$ be a semigroup and let $A \subseteq S$. Then $A$ is an $IP^*$set if and only if for every $IP$ set $B \subseteq S$, $A \cap B \neq \emptyset$.

**Lemma 2.6.** *Let $S$ be a dense subsemigroup of a semitopological semigroup $(T, +)$. Let $M$ be a $u \times v$ matrix, $u, v \in \mathbb{N}$, with entries from $\omega$ satisfying the first entries condition. Assume that for every first entry $c$ of $M$, $cS$ is an $IP^\star$-set. Let $C \subseteq S$ be a $e$-*central *set for any $e \in E(T)$. Then there exist sequences $\langle x_{1,t}\rangle_{t=1}^{\infty}$, $\langle x_{2,t}\rangle_{t=1}^{\infty}$, $\cdots$, $\langle x_{v,t}\rangle_{t=1}^{\infty}$ in $S$ such that for each $i \in \{1, 2, \ldots, v\}$, $lim_{t\to\infty} x_{i,t} = e$ and for each $F \in \mathcal{P}_f(\mathbb{N})$, $M\vec{x}_F \in C^u$, where $\vec{x}_F$ is*

$$\vec{x}_F = \begin{pmatrix} \sum_F x_{1,\,n} \\ \sum_F x_{2,\,n} \\ \vdots \\ \sum_F x_{v,\,n} \end{pmatrix}.$$

*Proof.* Let $v = 1$. Then by deleting repeated rows, we get $M = (c)$ for some $c \in \mathbb{N}$. Given $cS$ being IP$^\star$, $cS \cap C$ is $e$-central set. So, there is some idempotent $p \in e_S^*$ for which $cS \cap C \in p$. By Theorem 3.2 of [1], there exist some sequence $(y_n)_{n=1}^\infty$ in $S$ such that $\lim_{n\to\infty} y_n = e$, $\sum_{n=1}^\infty y_n$ converges $e$ and $FS\left((y_n)_{n=1}^\infty\right) \subseteq cS \cap C$. Since each $y_n \in cS \cap C$, we choose $y_n = cx_{1,n}$ for each $n \in \mathbb{N}$ and $x_{1,n} \in S$. Hence, we get our required $(x_{1,n})_{n=1}^\infty$ such that $\sum_{n=1}^\infty x_{1,\,n}$ converges near $e$ for $v = 1$.

Now, we assume that the result holds for $v$ and prove it for $v + 1$. Let $M$ be a $u \times (v + 1)$ matrix satisfying the assumptions of our result. By adding additional rows, if necessary, we may assume that there is some $l \in \{1, 2, \ldots, u - 1\}$ and some $c \in \mathbb{N}$ such that for each $j \in \{1, 2, \ldots, u\}$,

$$b_{j,\,1} = \begin{matrix} 0 & if\, j \leq l \\ c & if\, j > l \end{matrix}$$

Let $D$ be the $l \times v$ matrix defined by $d_{j,\,i} = b_{j,\,i+1}$ and so $D$ satisfies the first entries condition and all the first entries of $M$ are the first entries of $D$. Hence, by induction, choose sequences $(x_{1,\,t})_{t=1}^\infty$, $(x_{2,\,t})_{t=1}^\infty$, $\cdots$, $(x_{v,\,t})_{t=1}^\infty$ in $S$ such that $\sum_{t=1}^\infty x_{i,\,t}$ converges near $e$ $\forall i \in \{1, 2, \ldots, v\}$. For each $j \in \{l + 1, l + 2, \ldots, u\}$ and each $t \in \mathbb{N}$, let $y_{j,\,t} = \sum_{i=2}^{v+1} b_{j,\,i} \cdot x_{i-1,\,t}$. Note that since each $\sum_{t=1}^\infty x_{i,\,t}$ converges near $e$ $\forall i \in \{1, 2, \ldots, v\}$, $\sum_{t=1}^\infty y_{j,\,t}$ also converges near $e$. For $j \in \mathbb{N} \setminus \{l + 1, l + 2, \ldots, u\}$and $t \in \mathbb{N}$, let $y_{i,\,t} = y_{u,\,t}$. Let $Y = \left((y_{i,\,t})_{t=1}^\infty\right)_{i=1}^\infty$ . Note that using the Definition 4.1 of [1], we get $Y \in \mathcal{Y}$.

Now by applying Theorem 4.3 of [1], choose sequences $(a_n)_{n=1}^\infty$ in $S$ and $(H_n)_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that

(a) for each $n \in \mathbb{N}$, $a_n \in U_n$ and $maxH_n < minH_{n+1}$, and
(b) for each $f \in \phi$, $FS\left(\left\langle a_n + \sum_{t\in H_n} y_{f(n),\,t}{}_{n=1}^\infty\right\rangle\right) \subseteq cS \cap C$.

By considering the function $f$ to be $f(n) = n$ for every $n \in \mathbb{N}$ , we can presume that for each $j \in \{l+1, l+2, \ldots, u\}$ and each $F \in \mathcal{P}_f(\mathbb{N})$ one has $\sum_{n\in F}\left(a_n + \sum_{t\in H_n} y_{j,\,t}\right) \in cS \cap C$. For each $n \in \mathbb{N}$, let $z_{1,\,n}$ be the elements satisfying the equation $a_n = z_{1,\,n} + z_{1,\,n} + \cdots z_{1,\,n}$ $(c - times)$. For each $n \in \mathbb{N}$ and $i \in \{2, 3, \ldots, v + 1\}$, let $z_{i,\,n} = \sum_{t\in H_n} x_{i-1,\,t}$. We claim that the sequences $(z_{1,\,t})_{t=1}^\infty$, $(z_{2,\,t})_{t=1}^\infty$, $\cdots$, $(z_{v+1,\,t})_{t=1}^\infty$ are as required.

For each $i \in \{1, 2, \ldots, v + 1\}$, $lim_{t\to\infty} z_{i,\,t} = e$ holds. Let $j \in \{1, 2, \ldots, u\}$ and $F \in \mathcal{P}_f(\mathbb{N})$. We show that

$$\sum_{i=1}^{v+1} b_{j,\,i} \cdot \sum_{n\in F} z_{i,\,n} \in B$$

Let $G = \cup_{n\in F} H_n$. First, assume $j \in \{1, 2, \ldots, l\}$ . Then,

$$\sum_{i=1}^{v+1} b_{j,\,i} \cdot \sum_{n\in F} z_{i,\,n} = \sum_{i=1}^{v+1} b_{j,\,i} \cdot \sum_{n\in F}\left(\sum_{t\in H_n} x_{i-1,\,t}\right) = \sum_{i=1}^{v} d_{j,\,i} \cdot \sum_{t\in G} x_{i,\,t} \in C$$

by induction hypothesis.

Next, let $j \in \{l+1, l+2, \ldots, u\}$. Then

$$\begin{aligned}\sum_{i=1}^{v+1} b_{j,i} \cdot \sum_{n \in F} z_{i,n} &= c \cdot \sum_{n \in F} z_{1,n} + \sum_{i=1}^{v+1} b_{j,i} \cdot \sum_{n \in F} \left( \sum_{t \in H_n} x_{i-1,t} \right) \\ &= \sum_{n \in F} \left( a_n + \sum_{i=2}^{v+1} b_{j,i} \cdot \sum_{t \in H_n} x_{i-1,t} \right)\end{aligned}$$

and so

$$\sum_{i=1}^{v+1} b_{j,i} \cdot \sum_{n \in F} z_{i,n} = \sum_{n \in F} \left( a_n + \sum_{t \in H_n} y_{j,t} \right)$$

□

**Theorem 2.7.** *Let $M$ be $u \times v$ matrix with entries from $\omega$. Then the following are equivalent.*

(a) *$M$ is $IPR/S$.*
(b) *$M$ is $IPR/S_e$, $e \in E(T)$.*
(c) *For every central set $C$ near any $e \in E(T)$, there exists $\overrightarrow{x} \in S^v$ such that $M\overrightarrow{x} \in C^u$.*

*Proof.* (b) $\Longrightarrow$ (a) is obvious. (c) $\Longrightarrow$ (b) is obvious too. (b) $\Longrightarrow$ (c) Given an idempotent $e \in E(T)$, let $C$ be a central set near that $e$. Let $M$ be $IPR/S_e$. By 2.3, we can say $M$ is $IPR/\mathbb{N}$. Then by a similar result of [11, Theorem 4.1], there exist $v \times m$ matrix $G$ some $m \in \{1, 2, \ldots, u\}$ such that $B = MG$ where $B$ is a $u \times m$ first entry matrix. Then by the above lemma 2.6, there exist some $\overrightarrow{y} \in S^m$ such that $B\vec{y} \in C^u$ and hence we get some $\vec{x} \in S^v$ such that $M\vec{x} \in C^u$. □

## 3. Infinite Matrices

Combinatorial properties of infinite matrices, on the other hand, is substantially different from that of finite matrices.

**Definition 3.1.** Let $S$ be a dense subsemigroup of a semitopological semigroup $(T, +)$. Let $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$ has the **F** property at the point $e \in E(T)$. Let $M$ be a $\omega \times \omega$ with entries from $\mathbb{Q}$. Then $M$ is image partition regular over $S$ near $e$ (abbreviated $IPR/S_e$) if and only if, whenever $S = \bigcup_{i=1}^{r} A_i$ for some $r \in \mathbb{N}$, there is some $i \in \{1, 2, \ldots, r\}$ such that for every $U \in \tau_e$ there exists $\vec{x} \in S^v$ with $\lim_{n\to\infty} x_n = e$ for which $M\vec{x} \in A_i \cap U$.

**Lemma 3.2.** *Let $A$ be finite image partition regular matrix over $\mathbb{N}$ and $B$ be infinite image partition regular matrix near an idempotent $e$ over any dense subsemigroup $S$ of semitopological semigroup $T$. Then*

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

*is image partition regular near the idempotent $e$.*

*Proof.* Let $A$ be a $u \times v$ matrix. Let $r \in \mathbb{N}$ and let $\mathbb{N}$ be $r$-colored. By compactness arguement, we can always assume that whenever the set $\{1, 2, 3, \ldots, n\}$

is partitioned into $r$-cells, there exists $\overrightarrow{x} \in \mathbb{N}^v$ such that the entries of $A\overrightarrow{x}$ are monochromatic. Let $S = \bigcup_{j=1}^{r} A_j$ be $r$-colored by $\varphi$.

Next, let $S$ be partitioned into $r^n$colors via $\psi$, where $\psi(x) = \psi(y)$ if and only if for all $t \in \{1, 2, 3, \ldots, n\}$, $\varphi(tx) = \varphi(ty)$. Pick $\overrightarrow{y} \in S^\omega$ such that the entries of $B\overrightarrow{y}$ are monochromatic and are contained in some open subset $U$, $U \in \mathcal{T}_e$, with respect to $\psi$. Pick an entry $a \in \{S \bigcap U : U \in \mathcal{T}_e\}$ of $B\overrightarrow{y}$. Define $\gamma\colon \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, r\}$ by $\gamma(i) = \varphi(ia)$. For $i \in \{1, 2, \ldots, n\}$, let us set $D_i = \{t \in \{1, 2, 3, \ldots, n\} : a + a + \ldots + a(t - \text{times}) \in A_i \bigcap U\}$. Chosing $n \in \mathbb{N}$ large enough so that whenever we partition the set $\{1, 2, 3, \ldots, n\}$ into $r$ cells we can pick $i \in \{1, 2, 3, \ldots, r\}$ and $\overrightarrow{x} \in \{1, 2, 3, \ldots, n\}^v$ such that $A\overrightarrow{x} \in (D_i)^u$ and assuming $\overrightarrow{u} = \overrightarrow{x}a$, we get $A\overrightarrow{u} \in A_i \bigcap U$. Choose an entry $i$ of $A\overrightarrow{x}$ and let $j = \gamma(i)$.

Let $\vec{z} = \begin{pmatrix} a\vec{x} \\ i\vec{y} \end{pmatrix}$. We claim that for any row $\vec{w}$ of $\begin{pmatrix} A & O \\ O & B \end{pmatrix}$, $\varphi(\vec{w} \cdot \vec{z}) = j$. First assume that $\vec{w}$ is a row of $\begin{pmatrix} M & O \end{pmatrix}$, so that $\vec{w} = \{\vec{s}\}^\frown \vec{0}$, where $\vec{s}$ is a row of $M$. Then $\vec{w} \cdot \vec{z} = \vec{s} \cdot (a\vec{x}) = a(\vec{s} \cdot \vec{x})$. Therefore $\varphi(\vec{w} \cdot \vec{z}) = \varphi(a(\vec{s} \cdot \vec{x})) = \gamma(\vec{s} \cdot \vec{x}) = j$.

Next assume that $\vec{w}$ is a row of $\begin{pmatrix} O & N \end{pmatrix}$, so that $\vec{w} = \vec{0}^\frown \vec{s}$ where $\vec{s}$ is a row of $N$. Then $\vec{w} \cdot \vec{z} = i(\vec{s} \cdot \vec{y})$. Now $\psi(\vec{s} \cdot \vec{y}) = \psi(a)$. So $\varphi(i(\vec{s} \cdot \vec{y})) = \varphi(ia) = \gamma(i) = j$. □

3.1. **Insertion Matrices.** Here, we shall prove a class of infinite matrices, called *Insertion matrix*, which are image partition regular near an idempotent over $S$ , where $S$ is a semitopological semigroup.

**Definition 3.3.** Let $\gamma, \delta \in \omega \cup \{\omega\}$ and let $C$ be a $\gamma \times \delta$ matrix with finitely many nonzero entries in each row. For each $t < \delta$ , let $B_t$ be a finite matrix of dimension $u_t \times v_t$ .

Let $R = \{(i, j) : i < \gamma \,\text{and}\, j \in \times_{t<\delta} \{0, 1, \ldots, u_t - 1\}\}$. Given $t < \delta$ and $k \in \{0, 1, \ldots, u_t - 1\}$, denote by $\vec{b}_k^{(t)}$ the $k^{th}$ row of $B_t$. Then $D$ is an insertion matrix of $\langle B_t \rangle_{t<\delta}$ into $C$ if and only if the rows of $D$ are all of the form $c_{i,0} \cdot \vec{b}_{j(0)}^{(0)} \frown c_{i,1} \cdot \vec{b}_{j(1)}^{(1)} \frown \ldots$ where $(i, j) \in R$.

**Example 3.4.** If

$$C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 1 & 1 \\ 5 & 7 \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix},$$

then the insertion matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ 10 & 14 & 0 & 1 \\ 10 & 14 & 3 & 3 \end{pmatrix}$$

that is,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 5 & 7 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ 10 & 14 & 0 & 1 \\ 10 & 14 & 3 & 3 \end{pmatrix}$$

is an insertion matrix of $\langle B_t \rangle_{t<2}$ into $C$.

We shall now turn our attention to the Milliken-Taylor matrices with entries from $\omega$ which are one of the main sources of infinite image partition regular matrices.

**Definition 3.5.** Let $\overrightarrow{a} = \langle a_1, a_2, \ldots , a_k \rangle$ be a finite sequence in $\mathbb{N}$, and let $\vec{x} = \langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $S$. Then the *Milliken-Taylor system* determined by $\overrightarrow{a}$ and $\vec{x}$, abbreviated as MT $(\overrightarrow{a}, \vec{x})$ is as follows:

$$\left\{ \sum_{t=1}^{k} a_t \cdot \sum_{n \in F_t} x_n : F_t \in \mathcal{P}_f(\omega), \text{ and if } t < k, \text{ then } \max F_t < \min F_{t+1} \right\}.$$

Let $\vec{a}$ has adjacent repeated entries and let $\vec{c}$ is obtained from $\vec{a}$ by deleting such repetitions, then for any infinite sequence $\vec{x}$, one has MT $(\vec{a}, \vec{x}) \subseteq$ MT $(\vec{c}, \vec{x})$, so it suffices to consider sequences $\vec{c}$ without adjacent repeated entries.

**Definition 3.6.** Let $\vec{a}$ be a finite or infinite sequence in $\omega$ with only finitely many nonzero entries. Then $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting all zeroes and then deleting all adjacent repeated entries. The sequence $c(\vec{a})$ is the compressed form of $\vec{a}$. If $\vec{a} = c(\vec{a})$, then $\vec{a}$ is a compressed sequence.

For example, if $\vec{a} = \langle 0, 1, 0, 0, 1, 2, 0, 2, 0, 0, \ldots \rangle$, then $c(\vec{a}) = \langle 1, 2 \rangle$.

**Definition 3.7.** Let $\overrightarrow{a}$ be a compressed sequence in $\mathbb{N}$. A Milliken-Taylor matrix determined by $\overrightarrow{a}$ is $\omega \times \omega$ matrix $A$ such that the rows of $A$ are all possible rows with finitely many nonzero entries and its compressed form is equal to $\overrightarrow{a}$.

Note that the set of entries of $A\vec{x}$ is precisely MT $(\vec{a}, \vec{x})$, where $A$ is a Milliken-Taylor matrix whose each row have compressed form $\vec{a}$ and $\vec{x}$ is an infinite sequence in $S$.

**Theorem 3.8.** *Let $(T, +)$ be a commutative semitopological semigroup and $S$ be a dense subsemigroup of $T$. Let $\mathcal{B}$ have the* **F** *property at the point $e \in E(T)$. Let $\vec{a} = \langle a_0, a_1, ..., a_l \rangle$ be a compressed sequence in $\mathbb{N}$ with $l > 0$ , let $C$ be a Milliken-Taylor matrix for $\vec{a}$ , and for each $t < \omega$ , let $B_t$ be a $u_t \times v_t$ finite matrix with entries from $\omega$ which is image partition regular matrix over $\mathbb{N}$. Then any insertion matrix of $\langle B_t \rangle_{t<\omega}$ into $C$ is image partition regular near idempotent ie $IPR/S_e$.*

*Proof.* Pick by Lemma 2.3 of [1]some minimal idempotent $p$ of $e_S^*$. Let $q = a_0 \cdot p + a_1 \cdot p + \cdots + a_l \cdot p$. Then by Lemma 2.5 of [5], $q \in e_S^*$. Let $\mathcal{G}$ be a finite partition of $S$ and pick $A \in \mathcal{G}$ such that $A \in q$.

Now $\{x \in S : -x + A \in a_1.p + ..... + a_l.p\} \in a_0.p$ so that $D_0 = \{x \in S : -a_0.x + A \in a_1.p + ..... + a_l.p\} \in p$. Then $D_0^{\star} \in p$ (as used in Theorem 2.6 of [5]).

Let $\alpha_0 = 0$ and inductively let $\alpha_{n+1} = \alpha_n + v_n$.

So pick by theorem 2.7 , $x_0, x_1, \ldots\ldots, x_{\alpha_1 - 1} \in S$ such that

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1-1} \end{pmatrix} \in (D_0^*)^{u_0}$$

Let $H_0$ be the set of entries of

$$B_0 \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\alpha_1-1} \end{pmatrix}$$

Inductively, let $n \in \mathbb{N}$ and assume that we have chosen $\langle x_t \rangle_{t=0}^{\alpha_n - 1}$ in $S$ , $\langle D_k \rangle_{k=0}^{n-1}$ in $p$ , and $\langle H_k \rangle_{k=0}^{n-1}$ in the set $\mathcal{P}_f(\mathbb{N})$ of finite nonempty subsets of $\mathbb{N}$ such that for $r \in \{0, 1, ..., n-1\}$,

**(I)** $H_r$ is the set of entries of

$$B_r \begin{pmatrix} x_{\alpha_r} \\ x_{\alpha_r+1} \\ \vdots \\ x_{\alpha_{r+1}-1} \end{pmatrix},$$

**(II)** if $\emptyset \neq F \subseteq \{0, 1, ...., r\}$, $k = minF$, and for each $t \in F$, $y_t \in H_t$, then $\sum_{t \in F} y_t \in D_k^\star$

**(III)** if $r < n-1$, then $D_{r+1} \subseteq D_r$;

**(IV)** if $m \in \{0, 1, ...., l-1\}$ and $F_0, F_1, ...., F_m$ are all nonempty subsets of $\{0, 1, ..., r\}$, for each $i \in \{0, 1, ..., m-1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $-\sum_{t=0}^m a_i . \sum_{t \in F_i} y_t + A \in a_{m+1}.p + a_{m+2}.p + ... + a_l.p$;

**(V)** if $r < n-1$ $F_0, F_1, ...., F_{l-1}$ are nonempty subsets of the set $\{0, 1, ..., r\}$, for each $i \in \{0, 1, ..., m-1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $D_{r+1} \subseteq a_l^{-1}(-\sum_{t=0}^{l-1} a_i . \sum_{t \in F_i} y_t + A)$; and

**(VI)** if $r < n-1$, $m \in \{0, 1, ...., l-2\}$, $F_0, F_1, ...., F_m$ are nonempty subsets of $\{0, 1, ..., r\}$, for each $i \in \{0, 1, ..., m-1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $D_{r+1} \subseteq \{x \in \mathbb{R} : -a_{m+1}.x + (-\sum_{t=0}^m a_i . \sum_{t \in F_i} y_t + A) \in a_{m+2}.p + a_{m+3}.p + ... + a_l.p\}$.

At $n = 1$ , hypotheses **(I)**, **(II)**, and **(IV)** hold directly while **(III)**, **(V)**, and **(VI)** are vacuous.

For $m \in \{0, 1, ..., l-1\}$, let $G_m = \{\sum_{t=0}^m a_i . \sum_{t \in F_i} y_t \ : \ F_0, F_1, ...., F_m$ are nonempty subsets of $\{0, 1, ..., n-1\}$, for each $i \in \{0, 1, ..., m-1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t\}$.

For $k \in \{0, 1, ..., n-1\}$, let

$E_k = \{\sum_{t \in F} y_t$: if $\phi \neq F \subseteq \{0, 1, ...., r\}$, $k = minF$, and for each $t \in F$, $y_t \in H_t\}$.

Given $b \in E_k$, we have that $b \in D_k^\star$ by hypothesis **(II)** and so $-b + D_k^\star \in p$. If $d \in G_{l-1}$, then by **(IV)**, $-d + A \in a_l.p$ so that $a_l^{-1}(-d + A) \in p$. If $m \in \{0, 1, ..., l-2\}$ and $d \in G_m$, then by **(IV)**, $-d + A \in a_{m+1}.p + a_{m+2}.p + ... + a_l.p$ so that

$\{x \in S : -a_{m+1}.x + (-d + A) \in a_{m+2}.p + a_{m+3}.p + ... + a_l.p\} \in p$.

Thus we have that $D_n \in p$, where

$D_n = D_{n-1} \cap \bigcap_{k=0}^{n-1} \bigcap_{b \in E_k} (-b + D_k^\star) \cap \bigcap_{d \in G_{l-1}} a_l^{-1}(-d + A)$
$\cap \bigcap_{m=0}^{l-2} \bigcap_{d \in G_m} \{x \in S : -a_{m+1}.x + (-d + A) \in a_{m+2}.p + a_{m+3}.p + ... + a_l.p\}$.
(Here, if say $l = 1$ or $n < l$ , we are using the convention that $\bigcap \emptyset = \mathbb{N}$)
Pick, again by Theorem 2.7 , $x_{\alpha_n}, x_{\alpha_n+1}, ....., x_{\alpha_{n+1}-1} \in S$ such that

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix} \in (D_n^*)^{u_n} .$$

Let $H_n$ be the set of entries of

$$B_n \begin{pmatrix} x_{\alpha_n} \\ x_{\alpha_n+1} \\ \vdots \\ x_{\alpha_{n+1}-1} \end{pmatrix}$$

Then hypotheses **(I)**, **(III)**, **(V)**, and **(VI)** hold directly.

To verify hypothesis **(II)**, let $\emptyset \neq F \subseteq \{0, 1, ...., r\}$, let $k = minF$, and for each $t \in F$, let $y_t \in H_t$. If $n$ does not belong to $F$, then $\sum_{t \in F} y_t \in D_k^\star$ by hypothesis **(II)** at $n - 1$ , so assume that $n \in F$. If $F = \{n\}$, then we have that $y_n \in D_n^\star$ directly so assume that $F \neq \{n\}$. Let $b = \sum_{t \in F \setminus \{n\}} y_t$. Then $b \in E_k$ and so $y_n \in -b + D_k^\star$ and thus $b + y_n \in D_k^\star$ as required.

To verify hypothesis **(IV)**, let $m \in \{0, 1, ...., l-1\}$ and $F_0, F_1, ...., F_m$ are nonempty subsets of $\{0, 1, ..., n\}$, such that for each $i \in \{0, 1, ..., m - 1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$. If $m = 0$ , then $\sum_{t \in F_0} y_t \in D_0^\star$ by **(II)** and **(III)** so that $-a_0. \sum_{t \in F_0} y_t + A \in a_1.p + a_2.p + ... + a_l.p$ as required.

So assume that $m > 0$. Let $k = minF_m$ and $j = maxF_{m-1}$. Then
$\sum_{t \in F_m} y_t \in D_k^\star$ by **(II)**
$\subseteq D_{j+1}$ by **(III)**
$\subseteq \{x \in S : -a_m.x + (-\sum_{t=0}^{m-1} a_i. \sum_{t \in F_i} y_t + A) \in a_{m+1}.p + a_{m+2}.p + ... + a_l.p\}$
by **(VI)**
as required.

The induction being complete, we claim that whenever $F_0, F_1, ...., F_l$ are nonempty subsets of $\omega$ such that for each $i \in \{0, 1, ..., l - 1\}$, $maxF_i < minF_{i+1}$, and for each $t \in \bigcup_{i=0}^m F_i$, $y_t \in H_t$, then $-\sum_{i=0}^l a_i. \sum_{t \in F_i} y_t \in A$. To see this , let $k = minF_l$ and let $j = maxF_{l-1}$. Then $\sum_{t \in F_l} y_t \in D_k^\star \subseteq D_{j+1} \subseteq a_l^{-1}(-\sum_{t=0}^{l-1} a_i. \sum_{t \in F_i} y_t + A)$ by hypothesis **(V)**, and so $\sum_{i=0}^l a_i. \sum_{t \in F_i} y_t \in A$ as claimed.

Let $Q$ be an insertion matrix of $\langle B_t \rangle_{t<\omega}$ into $C$. We claim that all entries of $Q\vec{x}$ are in $A$. To see this , let $\gamma < \omega$ be given and let $j \in \times_{t<\omega}\{0, 1, ..., u_t - 1\}$, so that $c_{\gamma,0}.\vec{b}_{j(0)}^{(0)} \frown c_{\gamma,1}.\vec{b}_{j(1)}^{(1)} \frown$ ...is a row of $Q$ , say row $\delta$. For each $t \in \{0, 1, ..., m\}$ , let $y_t = \sum_{k=0}^{v_t-1} b_{j(t),k}^{(t)}.x_{\alpha_t+k}$ (so that $y_t \in H_t$). Therefore we have $\sum_{q=0}^{\infty} q_{\delta,s}.x_s = \sum_{t=0}^m c_{\gamma,t}.y_t$. Choose nonempty subsets $F_0, F_1, ...., F_l$ of $\{0, 1, ..., m\}$ such that for each $i \in \{0, 1, ..., l - 1\}$, $maxF_i < minF_{i+1}$, and for each $t \in F_i$, $c_{\gamma,t} = a_i$. (One can do this because $C$ is a Milliken -Taylor matrix for $\vec{a}$.) Then $\sum_{t=0}^m c_{\gamma,t}.y_t = \sum_{i=0}^l a_i. \sum_{t \in F_i} y_t \in A$. □

## 4. Some infinite Centrally image partition regularity of matrices near any idempotent

**Definition 4.1.** Let $M$ be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$. Then $M$ is a segmented image partition regular matrix if and only if

(a) no row of $M$ is row is $\vec{0}$;
(b) for each $i \in \omega$, $\{j \in \omega : a_{i,j} \neq 0\}$ is finite; and
(c) there is an increasing sequence $\langle \alpha_n \rangle_{n=0}^{\infty}$ in $\omega$ such that $\alpha_0 = 0$ and for each $n \in \omega$,
$\{\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle : i \in \omega\} \setminus \{\vec{0}\}$
is empty or is the set of rows of a finite image partition regular matrix.

If each of these finite image partition regular matrices is a first entries matrix, then $M$ is a segmented first entries matrix. If also the first nonzero entry of each $\langle a_{i,\alpha_n}, a_{i,\alpha_n+1}, a_{i,\alpha_n+2}, \ldots, a_{i,\alpha_{n+1}-1} \rangle$, if any, is 1, then $M$ is a monic segmented first entries matrix.

**Definition 4.2.** Let $M$ be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$ and let $S$ be a dense subsemigroup of a semitopological semigroup $(T, +)$. Given an idempotent $e \in E(T)$, we say $M$ is *centrally image partition regular near* $e$ if and only if whenever $C$ is a central set near $e$ in $S$, there exists $\vec{x} \in S^{\omega}$ such that $M\vec{x} \in C^{\omega}$.

Next, we show that segmented image partition regular matrices hold good for central image partition regularity too.

**Theorem 4.3.** *Let $S$ be a dense subsemigroup of $T$ such that for any $c \in \mathbb{N}$,$cS = \{c - times\, s \colon s \in S\}$ is an $IP^*$set. Let $M$ be a segmented image partition regular matrix over $\mathbb{N}$. Then $M$ is* centrally image partition regular *for any $e-$ central set.*

*Proof.* Let $\vec{c}_0, \vec{c}_1, \vec{c}_2, \ldots$ denote the columns of $M$. Let $\langle \alpha_n \rangle_{n=0}^{\infty}$ be as in the definition of a segmented image partition regular matrix. For each $n \in \omega$, let $M_n$ be the matrix whose columns are $\vec{c}_{\alpha_n}, \vec{c}_{\alpha_n+1}, \ldots, \vec{c}_{\alpha_{n+1}-1}$. Then the set of non-zero rows of $M_n$ is finite and, if nonempty, is the set of rows of a finite image partition regular matrix. Let $B_n = (M_0\ M_1 \ldots M_n)$.

Lemma 2.3 of [1] shows that $e_S^*$ is a compact right topological semigroup so that we can choose an minimal idempotent $p \in e_S^*$. Let $C \subseteq S$ such that $C \in p$. Let $C^* = \{x \in C : -x + C \in p\}$.Then $C^* \in p$. We claim that $-x + C^* \in p$. This can be proved as a consequence of Theorem 4.12 of [10]. Now the set of non-zero rows of $M_n$ is finite and, if nonempty, is the set of rows of a finite image partition regular matrix over $\mathbb{N}$. As a result of Theorem 15.24 of [10], there exist $m \in \mathbb{N}$ and a $u \times m$ matrix $D$ with entries from $\omega$ which satisfies the first entries condition such that given any $\overrightarrow{y} \in \mathbb{N}^m$ there is some $\overrightarrow{x} \in \mathbb{N}$ with $M_n \overrightarrow{x} = D\overrightarrow{y}$. Then by using Theorem 2.6, we can choose $\vec{x}^{(0)} \in S^{\alpha_1 - \alpha_0}$ such that, if $\vec{y} = M_0 \vec{x}^{(0)}$, then $y_i \in C^*$ for every $i \in \omega$ for which the $i^{th}$ row of $M_0$ is non-zero. We now make the inductive assumption that, for some $m \in \omega$, we have chosen $\vec{x}^{(0)}, \vec{x}^{(1)}, \ldots, \vec{x}^{(1)}$ such that $\vec{x}^{(i)} \in S^{\alpha_{i+1} - \alpha_i}$ for every $i \in \{0, 1, 2, \ldots, m\}$, and, if □

$$\vec{y} = B_m \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ . \\ . \\ . \\ \vec{x}^{(m)} \end{pmatrix},$$

then $y_j \in C^*$ for every $j \in \omega$ for which the $j^{th}$ row of $B_m$ is non-zero.

Let $D = \{j \in \omega : \text{row } j \text{ of } B_{m+1} \text{ is not } \vec{0}\}$ and note that for each $j \in \omega$, $-y_j + C^* \in p$. Again by the previous arguement, we can choose $\vec{x}^{(m+1)} \in S^{\alpha_{m+2}-\alpha_{m+1}}$ such that, if $\vec{z} = M_{m+1}\vec{x}^{(m+1)}$, then $z_j \in \bigcap_{t \in D}(-y_t + C^*)$ for every $j \in D$.

Thus we can choose an infinite sequence $\langle \vec{x}^{(i)} \rangle_{i \in \omega}$ such that, for every $i \in \omega$, $\vec{x}^{(i)} \in S^{\alpha_{i+1}-\alpha_i}$, and, if

$$\vec{y} = B_i \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ . \\ . \\ . \\ \vec{x}^{(i)} \end{pmatrix},$$

then $y_j \in C^*$ for every $j \in \omega$ for which the $j^{th}$ row of $B_i$ is non-zero.

Let

$$\vec{x} = \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ \vec{x}^{(2)} \\ \vdots \end{pmatrix}$$

and let $\vec{y} = M\vec{x}$. We note that, for every $j \in \omega$, there exists $m \in \omega$ such that $y_j$ is the $j^{th}$ entry of

$$B_i \begin{pmatrix} \vec{x}^{(0)} \\ \vec{x}^{(1)} \\ . \\ . \\ . \\ \vec{x}^{(i)} \end{pmatrix}$$

whenever $i > m$. Thus all the entries of $\vec{y}$ are in $C^*$.

## References


1. M Akbari Tootkaboni and T Vahed, *The semigroup of ultrafilters near an idempotent of a semitopological semigroup*, Topology and its Applications **159** (2012), no. 16, 3494–3503.
2. John F Berglund, Hugo Dietrich Junghenn, and Paul Milnes, *Analysis on semigroups: Function spaces, compactifications, representations*, Wiley New York, 1989.
3. D. De and N. Hindman, *Image partition regularity near zero*, Discrete Math. **309** (2009), no. 10, 3219–3232.
4. D. De, N. Hindman, and D. Strauss, *Sets central with respect to certain subsemigroups of* $\beta S_d$, Topology Proceedings, vol. 33, 2009, pp. 55–79.

5. D. De and R. Paul, *Universally image partition regularity*, Electron. J. Combin. **15** (2008), no. 1, R141.
6. Robert Ellis, Robert Ellis, Robert Ellis, and Robert Ellis, *Lectures on topological dynamics*, WA Benjamin New York, 1969.
7. Harry Furstenberg and H Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, vol. 2, Princeton University Press Princeton, 1981.
8. David Hilbert, *Über die irreduzibilität ganzer rationaler funktionen mit ganzzahligen koeffizienten*, Algebra· Invariantentheorie· Geometrie, Springer, 1933, pp. 264–286.
9. N. Hindman and I. Leader, *The semigroup of ultrafilters near 0*, Semigroup Forum, vol. 59, Springer, 1999, pp. 33–55.
10. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, vol. 27, Walter de Gruyter, 1998.
11. Neil Hindman, *Image partition regularity over the reals*, New York J. Math **9** (2003), 79–91.
12. I Schur, *Uber die kongruenz(mod)+≡ m m m x y z p*, Jahresbericht der Deutchen Mathematiker Vereinigung **25** (1916), 114–117.
13. Bartel Leendert van der Waerden, *van der: Beweis einer baudetschen vermutung*, Nieuw Arch. Wiskd **15** (1927), 212–216.

Department of Mathematics, University of Kalyani, Kalyani-741235, West Bengal, India

*E-mail address*: tanushreebiswas87@gmail.com