

ON SCATTERING FOR CMV MATRICES

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ABSTRACT. Adamjan-Arov [1] (Lax–Phillips [5]) model space is considered as a scattering representation space for a CMV matrix [7] in context of new (extended Marchenko–Faddeev [4]) scattering theory developed in [6, 8, 9]. That is, there exists a basis in which the multiplication by independent variable is a CMV matrix. This basis as well as Verblunski coefficients are computed explicitly in terms of Nehari interpolation [2, 3]. Asymptotically the Verblunski coefficients go to zero. Moreover, relations between the basis and wandering subspaces are established. Transformation from scattering representation to spectral representation is given.

1. SPACE L^R , ITS SUBSPACES AND FUNCTIONAL MODELS

Let $R(t)$ be a given Szegő contractive function R on the unite circle \mathbb{T}

$$(1.1) \quad |R(t)| \leq 1, \quad \log(1 - |R|) \in L^1.$$

Then there exists a unique outer function T such that

$$|T|^2 = 1 - |R|^2, \quad T(0) > 0.$$

We consider the space L^R of vector functions on \mathbb{T} with the following weighted inner product on it

$$(1.2) \quad \left\langle \begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix}^{-1} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\rangle = \left\langle \frac{1}{|T|^2} \begin{bmatrix} 1 & -\bar{R} \\ -R & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \right\rangle.$$

Since the weight matrix is greater than $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and greater than $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, the entries f_1 and f_2 , in particular, belong to L^2 . The set of functions

$$\begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

($g_1 \in L^2$ and $g_2 \in L^2$) is dense in L^R . We denote by U_R the (unitary) operator of multiplication by independent variable t on L_R . We consider the following subspaces in L^R

$$(1.3) \quad \check{H}_{n,m} = \text{Clos} \left\{ \begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix} \begin{bmatrix} t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix} \right\},$$

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where n and m are integers. The subspaces appear in the context of the Nehari problem if the Fourier coefficients of R with indices from $-\infty$ to $-(n+m+1)$ are specified. Note that

$$\check{\mathcal{H}}_{n,m} \supseteq \check{\mathcal{H}}_{n,m+1} \quad \text{and} \quad \check{\mathcal{H}}_{n,m} \supseteq \check{\mathcal{H}}_{n+1,m},$$

and that due to the Szegő property (1.1) of R , $\check{\mathcal{H}}_{n,m+1}$ and $\check{\mathcal{H}}_{n+1,m}$ are of codimension one in $\check{\mathcal{H}}_{n,m}$. Define subspaces $\check{\Delta}_{n,m}$ and $\check{\tilde{\Delta}}_{n,m}$ of $\check{\mathcal{H}}_{n,m}$ by

$$(1.4) \quad \check{\Delta}_{n,m} = \check{\mathcal{H}}_{n,m} \ominus \check{\mathcal{H}}_{n+1,m}, \quad \check{\tilde{\Delta}}_{n,m} = \check{\mathcal{H}}_{n,m} \ominus \check{\mathcal{H}}_{n,m+1}.$$

U_R maps $\check{\mathcal{H}}_{n,m+1}$ onto $\check{\mathcal{H}}_{n+1,m}$ and, therefore, $\check{\mathcal{H}}_{n,m}^\perp \oplus \check{\tilde{\Delta}}_{n,m}$ onto $\check{\mathcal{H}}_{n,m}^\perp \oplus \check{\Delta}_{n,m}$.

Let $\Delta = \check{\tilde{\Delta}} = \mathbb{C}$ with unitary identification maps

$$\check{i}_{n,m}: \Delta \mapsto \check{\Delta}_{n,m}, \quad \check{\tilde{i}}_{n,m}: \check{\tilde{\Delta}} \mapsto \check{\tilde{\Delta}}_{n,m}.$$

They are defined up to multiplying by unitary constants and we will choose specific normalizations later. We denote by $\check{K}_{n,m}$ and $\check{\tilde{K}}_{n,m}$ the images of 1 under $\check{i}_{n,m}$ and $\check{\tilde{i}}_{n,m}$, respectively and we will get "explicit" formulas for the vectors $\check{K}_{n,m}$ and $\check{\tilde{K}}_{n,m}$ using a construction motivated by the solution of the Nehari problem. Define a unitary operator

$$\check{U}_{0,n,m} = \begin{bmatrix} A_{0,n,m} & B_{0,n,m} \\ C_{0,n,m} & 0 \end{bmatrix} : \begin{bmatrix} \check{\mathcal{H}}_{n,m} \\ \Delta \end{bmatrix} \mapsto \begin{bmatrix} \check{\mathcal{H}}_{n,m} \\ \check{\tilde{\Delta}} \end{bmatrix},$$

using the decompositios

$$\check{U}_{0,n,m} : \begin{bmatrix} \check{\mathcal{H}}_{n,m+1} \oplus \check{\tilde{\Delta}}_{n,m} \\ \Delta \end{bmatrix} \mapsto \begin{bmatrix} \check{\mathcal{H}}_{n+1,m} \oplus \check{\Delta}_{n,m} \\ \check{\tilde{\Delta}} \end{bmatrix}$$

by

$$(1.5) \quad \check{U}_{0,n,m} : \begin{bmatrix} h_{n,m+1} + \check{\tilde{K}}_{n,m} \cdot \beta \\ \alpha \end{bmatrix} \mapsto \begin{bmatrix} U_R h_{n,m+1} + \check{K}_{n,m} \cdot \alpha \\ \beta \end{bmatrix}$$

and a unitary operator $\check{U}_{1,n,m} : \check{\mathcal{H}}_{n,m}^\perp \oplus \check{\tilde{\Delta}} \rightarrow \check{\mathcal{H}}_{n,m}^\perp \oplus \Delta$ by

$$\check{U}_{1,n,m} = \begin{bmatrix} A_{1,n,m} & B_{1,n,m} \\ C_{1,n,m} & D_{1,n,m} \end{bmatrix} : \begin{bmatrix} h_{n,m}^\perp \\ \beta \end{bmatrix} \mapsto \begin{bmatrix} (h_{n,m}^\perp)' \\ \alpha \end{bmatrix},$$

such that

$$U_R : h_{n,m}^\perp + \check{\tilde{K}}_{n,m} \cdot \beta \mapsto (h_{n,m}^\perp)' + \check{K}_{n,m} \cdot \alpha$$

Then U_R is the feedback coupling of $\check{U}_{0,n,m}$ with $\check{U}_{1,n,m}$ given by the formula

$$(1.6) \quad U_R = \begin{bmatrix} A_{0,n,m} + B_{0,n,m} D_{1,n,m} C_{0,n,m} & B_{0,n,m} C_{1,n,m} \\ B_{1,n,m} C_{0,n,m} & A_{1,n,m} \end{bmatrix}.$$

In addition to $\check{U}_{0,n,m}$ we consider it's unitary dilation $\check{\mathcal{U}}_{0,n,m}$ that acts on

$$\mathcal{L}_{n,m} = \dots \Delta \oplus \Delta \oplus \check{\mathcal{H}}_{n,m} \oplus \check{\tilde{\Delta}} \oplus \check{\tilde{\Delta}} \dots$$

Subspaces of L_R

$$\begin{bmatrix} 1 \\ R \end{bmatrix} L^2 \quad \text{and} \quad \begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} L^2$$

can be embedded isometrically into $\mathcal{L}_{n,m}$ as follows. Since

$$\begin{bmatrix} 1 \\ R \end{bmatrix} t^n H_+^2 \subset \check{\mathcal{H}}_{n,m} \subset \mathcal{L}_{n,m}$$

and

$$\begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} \bar{t}^m H_-^2 \subset \check{\mathcal{H}}_{n,m} \subset \mathcal{L}_{n,m},$$

we can define the embedding as

$$\check{i}'_{n,m} : \begin{bmatrix} 1 \\ R \end{bmatrix} t^j H_+^2 \rightarrow \check{\mathcal{U}}_{0,n,m}^{j-n} \begin{bmatrix} 1 \\ R \end{bmatrix} t^n H_+^2, \quad j \in \mathbb{Z}$$

and

$$\check{i}''_{n,m} : \begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} \bar{t}^j H_-^2 \rightarrow \check{\mathcal{U}}_{0,n,m}^{m-j} \begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} \bar{t}^m H_-^2, \quad j \in \mathbb{Z}.$$

They are well defined isometric mappings from dense linear subsets of $\begin{bmatrix} 1 \\ R \end{bmatrix} L^2$ and $\begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} L^2$, respectively. Therefore, they extend by continuity.

We consider four scales

$$\begin{aligned} \Psi_{0,n,m} &: \Delta \rightarrow \check{i}_{n,m} \Delta, & \tilde{\Psi}_{0,n,m} &: \tilde{\Delta} \rightarrow \check{\mathcal{U}}_{0,n,m} \tilde{i}_{n,m} \tilde{\Delta}, \\ \Psi'_{0,n,m} &: E' \rightarrow \check{i}'_{n,m} \begin{bmatrix} 1 \\ R \end{bmatrix} E' = \check{\mathcal{U}}_{0,n,m}^{-n} \begin{bmatrix} 1 \\ R \end{bmatrix} t^n E', \\ \Psi''_{0,n,m} &: E'' \rightarrow \check{i}''_{n,m} \begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} E'' = \check{\mathcal{U}}_{0,n,m}^{m+1} \begin{bmatrix} \overline{R} \\ 1 \end{bmatrix} \bar{t}^{m+1} E''. \end{aligned}$$

We also define the characteristic (scattering) measure of $\check{\mathcal{U}}_{0,n,m}$ with respect to the vector scale

$$(1.7) \quad \begin{bmatrix} \Psi_{0,n,m}^* \\ \tilde{\Psi}_{0,n,m}^* \\ \Psi_{0,n,m}'^* \\ \Psi_{0,n,m}''^* \end{bmatrix} E_{\check{\mathcal{U}}_{0,n,m}} \begin{bmatrix} \Psi_{0,n,m} & \tilde{\Psi}_{0,n,m} & \Psi'_{0,n,m} & \Psi''_{0,n,m} \end{bmatrix},$$

where $E_{\check{\mathcal{U}}_{0,n,m}}$ is the spectral measure of the unitary operator $\check{\mathcal{U}}_{0,n,m}$. It is absolutely continuous with respect to the Lebesgue measure. Its density can be formally defined as (precisely it is the

boundary value of a harmonic function)

$$(1.8) \quad \begin{aligned} \check{\Sigma}_{0,n,m} &:= \begin{bmatrix} \Psi_{0,n,m}^* \\ \check{\Psi}_{0,n,m}^* \\ \Psi_{0,n,m}^{\prime*} \\ \Psi_{0,n,m}^{\prime\prime*} \end{bmatrix} \sum_{k=-\infty}^{\infty} t^k \check{\mathcal{U}}_{0,n,m}^*{}^k \begin{bmatrix} \Psi_{0,n,m} & \check{\Psi}_{0,n,m} & \Psi'_{0,n,m} & \Psi''_{0,n,m} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \check{b}_{n,m} & \check{s}_{1,n,m} & 0 \\ \bar{\check{b}}_{n,m} & 1 & 0 & \bar{\check{s}}_{2,n,m} \\ \bar{\check{s}}_{1,n,m} & 0 & 1 & \bar{\check{s}}_{0,n,m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n,m} & 1 \end{bmatrix} : \begin{bmatrix} \Delta \\ \check{\Delta} \\ E' \\ E'' \end{bmatrix} \rightarrow \begin{bmatrix} \Delta \\ \check{\Delta} \\ E' \\ E'' \end{bmatrix}. \end{aligned}$$

Since the scale $\begin{bmatrix} \Psi'_{0,n,m} & \Psi''_{0,n,m} \end{bmatrix}$ is $*$ -cyclic for $\check{\mathcal{U}}_{0,n,m}$, we have

$$\begin{bmatrix} 1 & \check{b}_{n,m} \\ \bar{\check{b}}_{n,m} & 1 \end{bmatrix} = \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \bar{\check{s}}_{2,n,m} \end{bmatrix} \begin{bmatrix} 1 & \bar{\check{s}}_{0,n,m} \\ \check{s}_{0,n,m} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \bar{\check{s}}_{1,n,m} & 0 \\ 0 & \check{s}_{2,n,m} \end{bmatrix}.$$

Therefore, the matrix

$$\check{S}_{n,m} = \begin{bmatrix} \check{b}_{n,m} & \check{s}_{1,n,m} \\ \check{s}_{2,n,m} & \check{s}_{0,n,m} \end{bmatrix} : \begin{bmatrix} \check{\Delta} \\ E' \end{bmatrix} \rightarrow \begin{bmatrix} \Delta \\ E'' \end{bmatrix}.$$

is unitary almost everywhere on \mathbb{T} .

The entries of $\check{S}_{n,m}$ have these properties: $\check{b}_{n,m}$ is analytic and $\check{b}_{n,m}(0) = 0$ (although it is not important to us, note that $\check{b}_{n,m}$ is a characteristic function of the colligation $\check{U}_{0,n,m}^*$); $\check{s}_{1,n,m} = \bar{t}^n \check{a}_{n,m}$ and $\check{s}_{2,n,m} = \bar{t}^m \check{a}_{n,m}$, where $\check{a}_{n,m}$ is an outer function. We can fix now normalization of $\check{i}_{n,m}$ and $\check{\bar{i}}_{n,m}$ (in a unique way) such that $\check{a}_{n,m}(0) > 0$. Note that if $n + m = n' + m'$ then

$$U_R^{n'-n} : \check{\mathcal{H}}_{n,m} \rightarrow \check{\mathcal{H}}_{n',m'}$$

Therefore, unitary colligations $\check{U}_{0,n,m}$ and $\check{U}_{0,n',m'}$ are unitarily equivalent. Hence, (under the above normalization of $\check{i}_{n,m}$ and $\check{\bar{i}}_{n,m}$) we have $\check{b}_{n,m} = \check{b}_{n',m'}$, consequently, $\check{a}_{n,m} = \check{a}_{n',m'}$. We also get

$$\check{s}_{0,n,m} = -\frac{\check{s}_{2,n,m}}{\bar{\check{s}}_{1,n,m}} \check{b}_{n,m} = -\bar{t}^{m+n} \frac{\check{a}_{n,m}}{\bar{\check{a}}_{n,m}} \check{b}_{n,m} = -\bar{t}^{m'+n'} \frac{\check{a}_{n',m'}}{\bar{\check{a}}_{n',m'}} \check{b}_{n',m'} = \check{s}_{0,n',m'}$$

Let $\check{\omega}_{n,m}$ be the characteristic function of $\check{U}_{1,n,m}^*$, $\check{\omega}_{n,m}(\zeta) : \Delta \rightarrow \check{\Delta}$. Analogous to the above is that $\check{\omega}_{n,m} = \check{\omega}_{n',m'}$ if $n + m = n' + m'$. In view of the above remarks sometimes we will use the notations $\check{b}_{n+m}, \check{a}_{n+m}, \check{s}_{0,n+m}, \check{\omega}_{n+m}$ instead of the former ones. We can also write $\check{s}_{1,n,m} = \bar{t}^n \check{a}_{n+m}$ and $\check{s}_{2,n,m} = \bar{t}^m \check{a}_{n+m}$. Since U_R is the feedback coupling of $\check{U}_{0,n,m}$ with $\check{U}_{1,n,m}$ we get that

$$R = \check{s}_{0,n+m} + \bar{t}^{n+m} \frac{\check{a}_{n+m}^2 \check{\omega}_{n+m}}{1 - \check{\omega}_{n+m} \check{b}_{n+m}} = \bar{t}^{n+m} \cdot \frac{\check{a}_{n+m}}{\bar{\check{a}}_{n+m}} \cdot \frac{\check{\omega}_{n+m} - \bar{\check{b}}_{n+m}}{1 - \check{\omega}_{n+m} \check{b}_{n+m}}.$$

To formulate one more crucial property of $\check{\Sigma}_{0,n,m}$ we need a functional model of $\check{\mathcal{U}}_{0,n,m}$. The Fourier representation is defined as

$$\begin{bmatrix} \Psi_{0,n,m}^* \\ \tilde{\Psi}_{0,n,m}^* \\ \Psi_{0,n,m}' \\ \Psi_{0,n,m}'' \end{bmatrix} \sum_{k=-\infty}^{\infty} t^k \check{\mathcal{U}}_{0,n,m}^{*k} : \mathcal{L}_{n,m} \rightarrow L^{\check{\Sigma}_{0,n,m}}.$$

Since $\check{\mathcal{U}}_{0,n,m}$ goes to multiplication by t under this transform and due to (1.7) we get that

$$\begin{bmatrix} 1 \\ R \end{bmatrix} t^n H_+^2 \rightarrow \begin{bmatrix} 1 & \check{b}_{n+m} & \check{s}_{1,n,m} & 0 \\ \check{b}_{n+m} & 1 & 0 & \check{s}_{2,n,m} \\ \check{s}_{1,n,m} & 0 & 1 & \check{s}_{0,n+m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^m H_-^2 \rightarrow \begin{bmatrix} 1 & \check{b}_{n+m} & \check{s}_{1,n,m} & 0 \\ \check{b}_{n+m} & 1 & 0 & \check{s}_{2,n,m} \\ \check{s}_{1,n,m} & 0 & 1 & \check{s}_{0,n+m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \bar{t}^m H_-^2 \end{bmatrix}.$$

Thus,

$$(1.9) \quad \check{\mathcal{H}}_{n,m} \rightarrow \text{Clos} \begin{bmatrix} 1 & \check{b}_{n+m} & \check{s}_{1,n,m} & 0 \\ \check{b}_{n+m} & 1 & 0 & \check{s}_{2,n,m} \\ \check{s}_{1,n,m} & 0 & 1 & \check{s}_{0,n+m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}.$$

Also

$$\dots \Delta \oplus \Delta \rightarrow \begin{bmatrix} 1 & \check{b}_{n+m} & \check{s}_{1,n,m} & 0 \\ \check{b}_{n+m} & 1 & 0 & \check{s}_{2,n,m} \\ \check{s}_{1,n,m} & 0 & 1 & \check{s}_{0,n+m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} H_-^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\tilde{\Delta} \oplus \tilde{\Delta} \dots \rightarrow \begin{bmatrix} 1 & \check{b}_{n+m} & \check{s}_{1,n,m} & 0 \\ \check{b}_{n+m} & 1 & 0 & \check{s}_{2,n,m} \\ \check{s}_{1,n,m} & 0 & 1 & \check{s}_{0,n+m} \\ 0 & \check{s}_{2,n,m} & \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ H_+^2 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that

$$(1.10) \quad \check{K}_{n,m} \rightarrow \begin{bmatrix} 1 \\ \check{b}_{n+m} \\ \check{s}_{1,n,m} \\ 0 \end{bmatrix} = \check{\Sigma}_{0,n,m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$(1.11) \quad \tilde{K}_{n,m} \rightarrow \begin{bmatrix} \check{b}_{n+m} \\ 1 \\ 0 \\ \check{s}_{2,n,m} \end{bmatrix} \bar{t} = \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ \bar{t} \\ 0 \\ 0 \end{bmatrix}.$$

Thus, we arrive at a crucial property of the entries of the matrix $\check{\Sigma}_{0,n,m}$:

$$(1.12) \quad \begin{bmatrix} 1 \\ \bar{b}_{n+m} \\ \bar{s}_{1,n,m} \\ 0 \end{bmatrix} \in \text{Clos} \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \bar{s}_{2,n,m} \\ 1 & \bar{s}_{0,n+m} \\ \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}$$

and

$$(1.13) \quad \begin{bmatrix} \check{b}_{n+m} \\ 1 \\ 0 \\ \check{s}_{2,n,m} \end{bmatrix} \bar{t} \in \text{Clos} \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \bar{s}_{2,n,m} \\ 1 & \bar{s}_{0,n+m} \\ \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}.$$

In other notations

$$(1.14) \quad \check{\Sigma}_{0,n,m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \text{Clos} \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix},$$

and

$$(1.15) \quad \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ \bar{t} \\ 0 \\ 0 \end{bmatrix} \in \text{Clos} \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}.$$

Note that since the scale $\begin{bmatrix} \Psi'_{0,n,m} & \Psi''_{0,n,m} \end{bmatrix}$ is *-cyclic

$$L^{\check{\Sigma}_{0,n,m}} = \text{Clos} \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \bar{s}_{2,n,m} \\ 1 & \bar{s}_{0,n+m} \\ \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} L^2 \\ L^2 \end{bmatrix} = \text{Clos} \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ 0 \\ L^2 \\ L^2 \end{bmatrix}$$

and $L^{\check{s}_{0,n+m}}$ is mapped unitarily onto $L^{\check{\Sigma}_{0,n,m}}$ by

$$\begin{bmatrix} f' \\ f'' \end{bmatrix} \rightarrow \begin{bmatrix} \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \check{s}_{2,n,m} \end{bmatrix} \begin{bmatrix} 1 & \check{s}_{0,n+m} \\ \check{s}_{0,n+m} & 1 \end{bmatrix}^{-1} \begin{bmatrix} f' \\ f'' \end{bmatrix} \\ \begin{bmatrix} f' \\ f'' \end{bmatrix} \end{bmatrix}$$

However, this does not imply (1.14) and (1.15).

A functional model for $\check{U}_{1,n,m}^*$ (using the scales $\Psi_{n,m}$ and $\tilde{\Psi}_{n,m}$ defined below) is the "multiplication by \check{t} " : $\mathcal{H}^{\check{\omega}_{n+m}} \oplus \Delta \rightarrow \mathcal{H}^{\check{\omega}_{n+m}} \oplus \tilde{\Delta}$, where $\mathcal{H}^{\check{\omega}_{n+m}}$ is a Hilbert space of vector functions

$$\begin{bmatrix} h_{-}^{\check{\omega}_{n+m}} \\ h_{+}^{\check{\omega}_{n+m}} \end{bmatrix}, \quad h_{-}^{\check{\omega}_{n+m}} \in H_{-}^2, \quad h_{+}^{\check{\omega}_{n+m}} \in H_{+}^2$$

with the norm

$$\left\| \begin{bmatrix} h_{-}^{\check{\omega}_{n+m}} \\ h_{+}^{\check{\omega}_{n+m}} \end{bmatrix} \right\|^2 = \left\langle \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix}^{-1} \begin{bmatrix} h_{-}^{\check{\omega}_{n+m}} \\ h_{+}^{\check{\omega}_{n+m}} \end{bmatrix}, \begin{bmatrix} h_{-}^{\check{\omega}_{n+m}} \\ h_{+}^{\check{\omega}_{n+m}} \end{bmatrix} \right\rangle.$$

$\check{U}_{1,n,m}$ is realized as the "multiplication by t ", respectively.

We consider four scales associated to U_R

$$\begin{aligned} \Psi_{n,m} : \Delta &\rightarrow \check{i}_{n,m}\Delta, & \tilde{\Psi}_{n,m} : \tilde{\Delta} &\rightarrow \mathcal{U}_R \tilde{i}_{n,m} \tilde{\Delta}, \\ \Psi' : E' &\rightarrow \begin{bmatrix} 1 \\ R \end{bmatrix} E', & \Psi'' : E'' &\rightarrow \begin{bmatrix} R \\ 1 \end{bmatrix} E''. \end{aligned}$$

We consider the characteristic (scattering) measure of \mathcal{U}_R with respect to the vector scale

$$(1.16) \quad \begin{bmatrix} \Psi_{n,m}^* \\ \tilde{\Psi}_{n,m}^* \\ \Psi'^* \\ \Psi''^* \end{bmatrix} E_{\mathcal{U}_R} \begin{bmatrix} \Psi_{n,m} & \tilde{\Psi}_{n,m} & \Psi' & \Psi'' \end{bmatrix},$$

where $E_{\mathcal{U}_R}$ is the spectral measure of the unitary operator U_R . It is absolutely continuous with respect to the Lebesgue measure (this follows from *-cyclicity of the scale $\begin{bmatrix} \Psi' & \Psi'' \end{bmatrix}$). Its density can be formally defined as (precisely it is the boundary value of a harmonic function)

$$\check{\Sigma}_{n,m} := \begin{bmatrix} \Psi_{n,m}^* \\ \tilde{\Psi}_{n,m}^* \\ \Psi'^* \\ \Psi''^* \end{bmatrix} \sum_{k=-\infty}^{\infty} t^k \mathcal{U}_R^{*k} \begin{bmatrix} \Psi_{n,m} & \tilde{\Psi}_{n,m} & \Psi' & \Psi'' \end{bmatrix}.$$

Using feedback decomposition (1.6) it can be expressed as

$$(1.17) \quad \check{\Sigma}_{n,m} = \begin{bmatrix} \check{\Sigma}_{n,m}^{(11)} & \check{\Sigma}_{n,m}^{(12)} \\ \check{\Sigma}_{n,m}^{(21)} & \Sigma^{(22)} \end{bmatrix} : \begin{bmatrix} \Delta \\ \tilde{\Delta} \\ E' \\ E'' \end{bmatrix} \rightarrow \begin{bmatrix} \Delta \\ \tilde{\Delta} \\ E' \\ E'' \end{bmatrix},$$

where

$$(1.18) \quad \Sigma_{n,m}^{(11)} = \frac{1}{2} \left(\frac{I + \begin{bmatrix} 0 & \check{b}_{n+m} \\ \check{\omega}_{n+m} & 0 \end{bmatrix}}{I - \begin{bmatrix} 0 & \check{b}_{n+m} \\ \check{\omega}_{n+m} & 0 \end{bmatrix}} + \frac{I + \begin{bmatrix} 0 & \bar{\omega}_{n+m} \\ \bar{b}_{n+m} & 0 \end{bmatrix}}{I - \begin{bmatrix} 0 & \bar{\omega}_{n+m} \\ \bar{b}_{n+m} & 0 \end{bmatrix}} \right) : \begin{bmatrix} \Delta \\ \tilde{\Delta} \end{bmatrix} \rightarrow \begin{bmatrix} \Delta \\ \tilde{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \frac{1 + \check{b}_{n+m} \check{\omega}_{n+m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} & \frac{\check{b}_{n+m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} \\ \frac{\check{\omega}_{n+m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} & \frac{1}{2} \frac{1 + \check{\omega}_{n+m} \check{b}_{n+m}}{1 - \check{\omega}_{n+m} \check{b}_{n+m}} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \frac{1 + \bar{\omega}_{n+m} \bar{b}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} & \frac{\bar{\omega}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} \\ \frac{\bar{b}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} & \frac{1}{2} \frac{1 + \bar{b}_{n+m} \bar{\omega}_{n+m}}{1 - \bar{b}_{n+m} \bar{\omega}_{n+m}} \end{bmatrix},$$

$$(1.19) \quad \Sigma_{n,m}^{(12)} = \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix} \begin{bmatrix} \frac{\check{s}_{1,n,m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} & 0 \\ 0 & \frac{\bar{s}_{2,n,m}}{1 - \bar{b}_{n+m} \bar{\omega}_{n+m}} \end{bmatrix} : \begin{bmatrix} E' \\ E'' \end{bmatrix} \rightarrow \begin{bmatrix} \Delta \\ \tilde{\Delta} \end{bmatrix},$$

$$(1.20) \quad \Sigma_{n,m}^{(21)} = \Sigma_{n,m}^{(12)*} \\ = \begin{bmatrix} \frac{\bar{s}_{1,n,m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} & 0 \\ 0 & \frac{\check{s}_{2,n,m}}{1 - \check{\omega}_{n+m} \check{b}_{n+m}} \end{bmatrix} \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix} : \begin{bmatrix} \Delta \\ \tilde{\Delta} \end{bmatrix} \rightarrow \begin{bmatrix} E' \\ E'' \end{bmatrix},$$

$$(1.21) \quad \Sigma^{(22)} = \begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix} : \begin{bmatrix} E' \\ E'' \end{bmatrix} \rightarrow \begin{bmatrix} E' \\ E'' \end{bmatrix}.$$

Since the scale $\begin{bmatrix} \Psi'_{n,m} & \Psi''_{n,m} \end{bmatrix}$ is $*$ -cyclic for \mathcal{U}_R , we have

$$(1.22) \quad \Sigma_{n,m}^{(11)} = \Sigma_{n,m}^{(12)} \Sigma^{(22)^{-1}} \Sigma_{n,m}^{(21)},$$

and L^R is mapped unitarily onto $L^{\check{\Sigma}_{n,m}}$ by

$$(1.23) \quad \begin{bmatrix} f' \\ f'' \end{bmatrix} \rightarrow \begin{bmatrix} \Sigma_{n,m}^{(12)} \Sigma^{(22)^{-1}} \begin{bmatrix} f' \\ f'' \end{bmatrix} \\ \begin{bmatrix} f' \\ f'' \end{bmatrix} \end{bmatrix}.$$

It follows from general feedback loading arguments that

$$\text{Clos } \check{\Sigma}_{0,n,m} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix} = \text{Clos } \begin{bmatrix} \check{s}_{1,n,m} & 0 \\ 0 & \check{s}_{2,n,m} \\ 1 & \check{s}_{0,n+m} \\ \check{s}_{0,n+m} & 1 \end{bmatrix} \begin{bmatrix} t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}$$

is unitarily mapped onto $\check{\mathcal{H}}_{n,m}$ by means of multiplication by

$$\begin{bmatrix} 0 & \frac{\check{s}_{1,n,m}\check{\omega}_{n+m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} & 1 & 0 \\ \frac{\check{s}_{2,n,m}\check{\omega}_{n+m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} & 0 & 0 & 1 \end{bmatrix}.$$

This fact also follows from the next formula since Fourier coefficients of $\check{s}_{0,n,m}$ and R from $-\infty$ to $-(n+m+1)$ agree:

$$(1.24) \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \check{\Sigma}_{n,m} = \begin{bmatrix} 0 & \frac{\check{s}_{1,n,m}\check{\omega}_{n+m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} & 1 & 0 \\ \frac{\check{s}_{2,n,m}\check{\omega}_{n+m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} & 0 & 0 & 1 \end{bmatrix} \check{\Sigma}_{0,n,m}.$$

In particular, this implies, in view of (1.10), (1.11) and (1.24), that

$$(1.25) \quad \check{K}_{n,m} = \begin{bmatrix} \frac{\check{s}_{1,n,m}}{1-\check{\omega}_{n+m}\check{b}_{n+m}} \\ \frac{\check{s}_{2,n,m}\check{\omega}_{n+m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} \end{bmatrix} = \check{\Sigma}_{n,m}^{(21)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$(1.26) \quad \check{\tilde{K}}_{n,m} = \begin{bmatrix} \frac{\check{s}_{1,n,m}\check{\omega}_{n+m}}{1-\check{\omega}_{n+m}\check{b}_{n+m}} \\ \frac{\check{s}_{2,n,m}}{1-\check{b}_{n+m}\check{\omega}_{n+m}} \end{bmatrix} \bar{t} = \check{\Sigma}_{n,m}^{(21)} \begin{bmatrix} 0 \\ \bar{t} \end{bmatrix}.$$

At the same time $\mathcal{H}^{\check{\omega}_{n+m}}$ is unitarily mapped onto $\mathcal{H}_{n,m}^\perp$ by the formula

$$(1.27) \quad \check{\mathcal{H}}_{n,m}^\perp = \begin{bmatrix} \frac{\check{s}_{1,n,m}}{1-\check{\omega}_{n+m}\check{b}_{n+m}} & 0 \\ 0 & \frac{\check{s}_{2,n,m}}{1-\check{\omega}_{n+m}\check{b}_{n+m}} \end{bmatrix} \mathcal{H}^{\check{\omega}_{n+m}}.$$

We can also write (unitarily equivalent) $L^{\check{\Sigma}_{n,m}}$ realization for L^R using (1.23). In this realization $\mathcal{H}_{n,m}$ looks as

$$\text{Clos } \begin{bmatrix} \Sigma_{n,m}^{(12)} \\ \Sigma_{n,m}^{(22)} \end{bmatrix} \begin{bmatrix} t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix} = \text{Clos } \check{\Sigma}_{n,m} \begin{bmatrix} 0 \\ 0 \\ t^n H_+^2 \\ \bar{t}^m H_-^2 \end{bmatrix}.$$

Since the factor in (1.27) is $\check{\Sigma}_{n,m}^{(21)} \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix}^{-1}$, then $L^{\check{\Sigma}_{n,m}}$ realization of $\mathcal{H}_{n,m}^\perp$ looks as

$$\begin{bmatrix} \Sigma_{n,m}^{(11)} \\ \Sigma_{n,m}^{(21)} \end{bmatrix} \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix}^{-1} \mathcal{H}^{\check{\omega}_{n+m}}.$$

In view of (1.25), (1.26) and (1.22)

$$(1.28) \quad \check{K}_{n,m} \rightarrow \begin{bmatrix} \check{\Sigma}_{n,m}^{(11)} \\ \check{\Sigma}_{n,m}^{(21)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \check{\Sigma}_{n,m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{K}_{n,m} \rightarrow \begin{bmatrix} \check{\Sigma}_{n,m}^{(11)} \\ \check{\Sigma}_{n,m}^{(21)} \end{bmatrix} \begin{bmatrix} 0 \\ \bar{t} \end{bmatrix} = \check{\Sigma}_{n,m} \begin{bmatrix} 0 \\ \bar{t} \\ 0 \\ 0 \end{bmatrix}.$$

We can see from the later formulas that in this realization $\check{K}_{n,m}$ reproduces the 0 Fourier coefficient of the first entry in $L^{\check{\Sigma}_{n,m}}$ and $\tilde{K}_{n,m}$ reproduces the -1 Fourier coefficient of the second entry in $L^{\check{\Sigma}_{n,m}}$.

Just in case we mention here that

$$\check{\Sigma}_{n,m} = \begin{bmatrix} \frac{1}{1-\bar{b}_{n+m}\check{\omega}_{n+m}} & \frac{\bar{\omega}_{n+m}}{1-\bar{b}_{n+m}\bar{\omega}_{n+m}} & 0 & 0 \\ \frac{\check{\omega}_{n+m}}{1-\bar{b}_{n+m}\check{\omega}_{n+m}} & \frac{1}{1-\bar{b}_{n+m}\bar{\omega}_{n+m}} & 0 & 0 \\ 0 & \frac{\bar{s}_{1,n,m}\bar{\omega}_{n+m}}{1-\bar{b}_{n+m}\bar{\omega}_{n+m}} & 1 & 0 \\ \frac{\bar{s}_{2,n,m}\check{\omega}_{n+m}}{1-\bar{b}_{n+m}\check{\omega}_{n+m}} & 0 & 0 & 1 \end{bmatrix} \check{\Sigma}_{0,n,m}.$$

2. VERBLUNSKY COEFFICIENTS

In this section we discuss recurrent relations between vectors $\check{K}_{n,m}, \tilde{K}_{n,m}$. Since the both pairs of vectors $\check{K}_{n,m}, \tilde{K}_{n+1,m}$ and $\tilde{K}_{n,m}, \check{K}_{n,m+1}$ form orthonormal bases for $H_{n+1,m+1} \ominus H_{n,m}$ they are related by a unitary matrix

$$(2.1) \quad \begin{bmatrix} \check{K}_{n,m} & \tilde{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{n,m} & \check{K}_{n,m+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{n,m} & \check{\rho}_{n,m} \\ \tilde{\rho}_{n,m} & \tilde{\alpha}_{n,m} \end{bmatrix}.$$

Note first that since the transformation matrix in (2.1) is unitary then either both $\check{\rho}_{n,m}$ and $\tilde{\rho}_{n,m}$ equal zero or both do not. If both equal zero then $\check{K}_{n,m}$ and $\tilde{K}_{n,m}$ are proportional, meaning

that $\mathcal{H}_{n+1,m} = \mathcal{H}_{n,m+1}$, which is impossible. Thus, $\check{\rho}_{n,m} \neq 0$ and $\check{\rho}_{n,m} \neq 0$. Using formulas (1.25) and (1.26) for $\check{K}_{n,m}$ and $\check{K}_{n,m}$ we can write (2.1) as

$$(2.2) \quad \begin{bmatrix} \frac{t^n \bar{a}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} & \frac{t^n \bar{a}_{n+m+1} \bar{\omega}_{n+m+1}}{1 - \bar{\omega}_{n+m+1} \bar{b}_{n+m+1}} \\ \frac{\bar{t}^m \check{a}_{n+m} \check{\omega}_{n+m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} & \frac{\bar{t}^{m+1} \check{a}_{n+m+1}}{1 - \check{b}_{n+m+1} \check{\omega}_{n+m+1}} \end{bmatrix} = \begin{bmatrix} \frac{t^{n-1} \bar{a}_{n+m} \bar{\omega}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} & \frac{t^n \bar{a}_{n+m+1}}{1 - \bar{\omega}_{n+m+1} \bar{b}_{n+m+1}} \\ \frac{\bar{t}^{m+1} \check{a}_{n+m}}{1 - \check{b}_{n+m} \check{\omega}_{n+m}} & \frac{\bar{t}^{m+1} \check{a}_{n+m+1} \check{\omega}_{n+m+1}}{1 - \check{b}_{n+m+1} \check{\omega}_{n+m+1}} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{n,m} & \check{\rho}_{n,m} \\ \check{\rho}_{n,m} & \check{\alpha}_{n,m} \end{bmatrix}.$$

Compare (1, 1) entries on the left and on the right, divide them by t^n and take the "zero" Fourier coefficient. We get

$$\bar{a}_{n+m}(0) = \bar{a}_{n+m+1}(0) \check{\rho}_{n,m}.$$

Therefore, according to our normalization,

$$\check{\rho}_{n,m} = \frac{\bar{a}_{n+m}(0)}{\bar{a}_{n+m+1}(0)} = \frac{\check{a}_{n+m}(0)}{\check{a}_{n+m+1}(0)} > 0.$$

Comparing (1, 2) entries the same way we get

$$\bar{a}_{n+m+1}(0) \bar{\omega}_{n+m+1}(0) = \bar{a}_{n+m+1}(0) \check{\alpha}_{n,m},$$

i.e.,

$$\check{\alpha}_{n,m} = \bar{\omega}_{n+m+1}(0).$$

Comparing (2, 1) entries, multiplying by t^{m+1} and taking "zero" Fourier coefficient we get

$$0 = \check{a}_{n+m}(0) \check{\alpha}_{n,m} + \check{a}_{n+m+1}(0) \check{\omega}_{n+m+1}(0) \check{\rho}_{n,m}.$$

Substituting the above formula for $\check{\rho}_{n,m}$ we obtain

$$\check{\alpha}_{n,m} = -\check{\omega}_{n+m+1}(0).$$

Comparing (2, 2) entries, multiplying by t^{m+1} and taking "zero" Fourier coefficient we get

$$\check{a}_{n+m+1}(0) = \check{a}_{n+m}(0) \check{\rho}_{n,m} + \check{a}_{n+m+1}(0) \check{\omega}_{n+m+1}(0) \check{\alpha}_{n,m}.$$

Since $\check{\omega}_{n+m+1}(0) = \bar{\alpha}_{n,m}$ and $1 - |\check{\alpha}_{n,m}|^2 = |\check{\rho}_{n,m}|^2$ (because the transformation matrix is unitary), we have

$$\check{a}_{n+m+1}(0) |\check{\rho}_{n,m}|^2 = \check{a}_{n+m}(0) \check{\rho}_{n,m}.$$

Using the above formula for $\check{\rho}_{n,m}$ we get

$$\check{\rho}_{n,m} = \frac{\check{a}_{n+m}(0)}{\check{a}_{n+m+1}(0)} = \check{\rho}_{n,m}.$$

Thus, (2.1) takes on the form

$$(2.3) \quad \begin{bmatrix} \check{K}_{n,m} & \check{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \check{K}_{n,m} & \check{K}_{n,m+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{n+m} & \check{\rho}_{n+m} \\ \check{\rho}_{n+m} & -\check{\alpha}_{n+m} \end{bmatrix},$$

where $\check{\rho}_{n+m} = \frac{\check{a}_{n+m}(0)}{\check{a}_{n+m+1}(0)} > 0$ and $\check{\alpha}_{n+m} = -\check{\omega}_{n+m+1}(0)$ (because the entries of the transformation matrix depend on the sum of the indices only, we changed the notations). Note also that $\check{\rho}_{n+m} = \sqrt{1 - |\check{\alpha}_{n+m}|^2}$ and $|\check{\alpha}_{n+m}| < 1$.

On the other hand the transformation matrix can be computed as

$$\begin{bmatrix} \check{K}_{n,m}^* \\ \check{K}_{n,m+1}^* \end{bmatrix} \begin{bmatrix} \check{K}_{n,m} & \check{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \check{\alpha}_{n+m} & \check{\rho}_{n+m} \\ \check{\rho}_{n+m} & -\check{\alpha}_{n+m} \end{bmatrix}.$$

We can compute $\langle \check{K}_{n,m}, \check{K}_{n,m} \rangle$ using $L^{\check{\Sigma}_{0,n,m}}$ representation (formulas (1.10) and (1.11)) or $L^{\check{\Sigma}_{n,m}}$ realization (formulas (1.28))

$$\begin{aligned} \check{\alpha}_{n+m} &= \langle \check{K}_{n,m}, \check{K}_{n,m} \rangle = \langle \check{\Sigma}_{0,n,m} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \check{t} \\ 0 \\ 0 \end{bmatrix} \rangle \\ &= ((2,1) \text{ entry of } \check{\Sigma}_{0,n,m})_{-1} = \frac{\check{b}_{n+m}}{\check{t}}(0). \end{aligned}$$

In particular we get

$$\check{\omega}_{n+m+1}(0) = -\frac{\check{b}_{n+m}}{\check{t}}(0).$$

Note that since $\check{\rho}_j \leq 1$, the sequence $\check{a}_j(0)$ is increasing. On the other hand, since \check{a}_j is a Schur class analytic function, $\check{a}_j(0) \leq 1$. Therefore, the limit of $\check{a}_j(0)$ exists and

$$0 < \lim_{j \rightarrow \infty} \check{a}_j(0) \leq 1.$$

In particular, this implies that

$$\lim_{j \rightarrow \infty} \check{\rho}_j = 1 \quad \text{and} \quad \lim_{j \rightarrow \infty} \check{\alpha}_j = 0.$$

3. A BASIS FOR L^R AND THE MATRIX OF U_R IN THIS BASIS

We consider a chain of subspaces in L^R

$$(3.1) \quad \dots \supset \mathcal{H}_{n-1,n-1} \supset \mathcal{H}_{n,n-1} \supset \mathcal{H}_{n,n} \supset \mathcal{H}_{n+1,n} \supset \mathcal{H}_{n+1,n+1} \supset \mathcal{H}_{n+2,n+1} \supset \dots$$

The first index increases first and then the second one increases. Every subsequent subspace is of codimension one in the preceding one. Since the union of subspaces (3.1) is dense in L^R , the sequence of vectors

$$(3.2) \quad \dots \check{K}_{n-1,n-1}, \check{K}_{n,n-1}, \check{K}_{n,n}, \check{K}_{n+1,n}, \check{K}_{n+1,n+1}, \check{K}_{n+2,n+1} \dots$$

forms an orthonormal basis for L^R .

We want to get the matrix of the operator U_R in the basis (3.2). Since U_R is a linear operator we have from (2.3)

$$(3.3) \quad \begin{bmatrix} U_R \check{K}_{n,n} & U_R \check{K}_{n+1,n} \end{bmatrix} = \begin{bmatrix} U_R \check{K}_{n,n} & U_R \check{K}_{n,n+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n} & \check{\rho}_{2n} \\ \check{\rho}_{2n} & -\check{\alpha}_{2n} \end{bmatrix}.$$

Since

$$\check{K}_{n,n} \in H_{n,n} \ominus H_{n,n+1}, \quad \check{K}_{n,n+1} \in H_{n,n+1} \ominus H_{n+1,n+1}$$

then

$$U_R \check{K}_{n,n} \in H_{n+1,n-1} \ominus H_{n+1,n}, \quad U_R \check{K}_{n,n+1} \in H_{n+1,n} \ominus H_{n+2,n}.$$

Due to normalization $\check{a}_k(0) > 0$ we have

$$U_R \check{K}_{n,n} = \check{K}_{n+1,n-1}, \quad U_R \check{K}_{n,n+1} = \check{K}_{n+1,n}.$$

Applying (2.3) with $n := n, m := n - 1$ we obtain

$$(3.4) \quad \check{K}_{n+1,n-1} = \begin{bmatrix} \check{K}_{n,n-1} & \check{K}_{n,n} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n-1} & \check{\rho}_{2n-1} \\ \check{\rho}_{2n-1} & -\check{\alpha}_{2n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Applying (2.3) with $n := n + 1, m := n$ we obtain

$$(3.5) \quad \check{K}_{n+1,n} = \begin{bmatrix} \check{K}_{n+1,n} & \check{K}_{n+1,n+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n+1} & \check{\rho}_{2n+1} \\ \check{\rho}_{2n+1} & -\check{\alpha}_{2n+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Substituting (3.4) and (3.5) to (3.3), we get

$$(3.6) \quad \begin{bmatrix} U_R \check{K}_{n,n} & U_R \check{K}_{n+1,n} \end{bmatrix} = \begin{bmatrix} \check{K}_{n,n-1} & \check{K}_{n,n} & \check{K}_{n+1,n} & \check{K}_{n+1,n+1} \end{bmatrix} \begin{bmatrix} \check{\rho}_{2n-1} & 0 \\ -\check{\alpha}_{2n-1} & 0 \\ 0 & \check{\alpha}_{2n+1} \\ 0 & \check{\rho}_{2n+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n} & \check{\rho}_{2n} \\ \check{\rho}_{2n} & -\check{\alpha}_{2n} \end{bmatrix}.$$

From this we see that the matrix of U_R is a CMV matrix.

4. VERBLUNSKY COEFFICIENTS AS SCHUR PARAMETERS

We summarize first results of the previous sections.

$$(4.1) \quad \begin{bmatrix} t\check{K}_{n,n} & t\check{K}_{n+1,n} \end{bmatrix} = \begin{bmatrix} \check{K}_{n,n-1} & \check{K}_{n,n} & \check{K}_{n+1,n} & \check{K}_{n+1,n+1} \end{bmatrix} \begin{bmatrix} \check{\rho}_{2n-1} & 0 \\ -\check{\alpha}_{2n-1} & 0 \\ 0 & \check{\alpha}_{2n+1} \\ 0 & \check{\rho}_{2n+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n} & \check{\rho}_{2n} \\ \check{\rho}_{2n} & -\check{\alpha}_{2n} \end{bmatrix},$$

where

$$(4.2) \quad \check{K}_{n,m} = \begin{bmatrix} \frac{t^n \bar{a}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} \\ \frac{\bar{t}^m \check{a}_{n+m} \check{\omega}_{n+m}}{1 - \bar{b}_{n+m} \check{\omega}_{n+m}} \end{bmatrix} = \check{\Sigma}_{n,m}^{(21)} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$(4.3) \quad \tilde{K}_{n,m} = \begin{bmatrix} \frac{t^n \bar{a}_{n+m} \bar{\omega}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} \\ \frac{\bar{t}^m \check{a}_{n+m}}{1 - \bar{b}_{n+m} \check{\omega}_{n+m}} \end{bmatrix} \quad \bar{t} = \check{\Sigma}_{n,m}^{(21)} \begin{bmatrix} 0 \\ \bar{t} \end{bmatrix},$$

$$\check{\alpha}_{n+m} = \frac{\bar{b}_{n+m}}{\bar{t}}(0) = -\check{\omega}_{n+m+1}(0),$$

$$\check{\rho}_{n+m} = \sqrt{1 - |\check{\alpha}_{n+m}|^2}.$$

Also $\check{\rho}_{n+m} = \frac{\check{a}_{n+m}(0)}{\check{a}_{n+m+1}(0)}$. Substitute all this in (4.1)

$$(4.4) \quad \begin{bmatrix} \frac{t^{n+1} \bar{a}_{2n}}{1 - \bar{\omega}_{2n} \bar{s}_{2n}} & \frac{t^{n+1} \bar{a}_{2n+1} \bar{\omega}_{2n+1}}{1 - \bar{\omega}_{2n+1} \bar{s}_{2n+1}} \\ \frac{\bar{t}^{n-1} \check{a}_{2n} \check{\omega}_{2n}}{1 - \bar{s}_{2n} \check{\omega}_{2n}} & \frac{\bar{t}^n \check{a}_{2n+1}}{1 - \bar{s}_{2n+1} \check{\omega}_{2n+1}} \end{bmatrix} = \begin{bmatrix} \frac{t^{n-1} \bar{a}_{2n-1} \bar{\omega}_{2n-1}}{1 - \bar{\omega}_{2n-1} \bar{s}_{2n-1}} & \frac{t^n \bar{a}_{2n}}{1 - \bar{\omega}_{2n} \bar{s}_{2n}} & \frac{t^n \bar{a}_{2n+1} \bar{\omega}_{2n+1}}{1 - \bar{\omega}_{2n+1} \bar{s}_{2n+1}} & \frac{t^{n+1} \bar{a}_{2n+2}}{1 - \bar{\omega}_{2n+2} \bar{s}_{2n+2}} \\ \frac{\bar{t}^n \check{a}_{2n-1}}{1 - \bar{s}_{2n-1} \check{\omega}_{2n-1}} & \frac{\bar{t}^n \check{a}_{2n} \check{\omega}_{2n}}{1 - \bar{s}_{2n} \check{\omega}_{2n}} & \frac{\bar{t}^{n+1} \check{a}_{2n+1}}{1 - \bar{s}_{2n+1} \check{\omega}_{2n+1}} & \frac{\bar{t}^{n+1} \check{a}_{2n+2} \check{\omega}_{2n+2}}{1 - \bar{s}_{2n+2} \check{\omega}_{2n+2}} \end{bmatrix} \begin{bmatrix} \check{\rho}_{2n-1} & 0 \\ -\bar{\check{\alpha}}_{2n-1} & 0 \\ 0 & \check{\alpha}_{2n+1} \\ 0 & \check{\rho}_{2n+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{2n} & \check{\rho}_{2n} \\ \check{\rho}_{2n} & -\bar{\check{\alpha}}_{2n} \end{bmatrix}.$$

Or returning to (2.3)

$$(4.5) \quad \begin{bmatrix} \check{K}_{n,m} & \tilde{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{n,m} & \check{K}_{n,m+1} \end{bmatrix} \begin{bmatrix} \check{\alpha}_{n+m} & \check{\rho}_{n+m} \\ \check{\rho}_{n+m} & -\bar{\check{\alpha}}_{n+m} \end{bmatrix}.$$

This can be rearranged as follows

$$(4.6) \quad \check{\rho}_{n+m} \begin{bmatrix} \check{K}_{n,m+1} & \tilde{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \check{K}_{n,m} & \tilde{K}_{n,m} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\check{\alpha}}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix}.$$

Substituting (4.2) and (4.3) in (4.6) we get

$$(4.7) \quad \check{\rho}_{n+m} \begin{bmatrix} \frac{t^n \bar{a}_{n+m+1}}{1 - \bar{\omega}_{n+m+1} \bar{b}_{n+m+1}} & \frac{t^n \bar{a}_{n+m+1} \bar{\omega}_{n+m+1}}{1 - \bar{\omega}_{n+m+1} \bar{b}_{n+m+1}} \\ \frac{\bar{t}^{m+1} \check{a}_{n+m+1} \check{\omega}_{n+m+1}}{1 - \bar{b}_{n+m+1} \check{\omega}_{n+m+1}} & \frac{\bar{t}^{m+1} \check{a}_{m+n+1}}{1 - \bar{s}_{m+n+1} \check{\omega}_{m+n+1}} \end{bmatrix} = \begin{bmatrix} \frac{t^n \bar{a}_{m+n}}{1 - \bar{\omega}_{m+n} \bar{s}_{m+n}} & \frac{t^{n-1} \bar{a}_{n+m} \bar{\omega}_{n+m}}{1 - \bar{\omega}_{n+m} \bar{b}_{n+m}} \\ \frac{\bar{t}^m \check{a}_{m+n} \check{\omega}_{m+n}}{1 - \bar{s}_{m+n} \check{\omega}_{m+n}} & \frac{\bar{t}^{m+1} \check{a}_{n+m}}{1 - \bar{b}_{n+m} \check{\omega}_{n+m}} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\check{\alpha}}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix},$$

or

$$(4.8) \quad \check{\rho}_{n+m} \Sigma_{n,m+1}^{(21)} = \Sigma_{n,m}^{(21)} \begin{bmatrix} 1 & 0 \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\alpha}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix}.$$

We can write $\Sigma_{n,m}^{(21)}$ as

$$(4.9) \quad \Sigma_{n,m}^{(21)} = \begin{bmatrix} t^n & 0 \\ 0 & \bar{t}^m \end{bmatrix} \Sigma_{n+m}^{(21)'}$$

where

$$(4.10) \quad \Sigma_{n+m}^{(21)'} = \begin{bmatrix} \bar{A}_{n+m} & 0 \\ 0 & A_{n+m} \end{bmatrix} \begin{bmatrix} 1 & \bar{\omega}_{n+m} \\ \check{\omega}_{n+m} & 1 \end{bmatrix}$$

and

$$(4.11) \quad A_{n+m} = \frac{\check{a}_{n+m}}{1 - \check{\omega}_{n+m} \bar{b}_{n+m}}.$$

Then (4.8) reads as

$$(4.12) \quad \check{\rho}_{n+m} \begin{bmatrix} 1 & 0 \\ 0 & \bar{t} \end{bmatrix} \Sigma_{n+m+1}^{(21)'} = \Sigma_{n+m}^{(21)'} \begin{bmatrix} 1 & 0 \\ 0 & \bar{t} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\alpha}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix}$$

and (4.7) reads as

$$(4.13) \quad \check{\rho}_{n+m} \begin{bmatrix} \bar{A}_{n+m+1} & \bar{A}_{n+m+1} \bar{\omega}_{n+m+1} \\ A_{n+m+1} \check{\omega}_{n+m+1} & A_{n+m+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_{n+m} & \bar{A}_{n+m} \bar{\omega}_{n+m} \bar{t} \\ A_{n+m} \check{\omega}_{n+m} t & A_{n+m} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\alpha}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix}.$$

The second row of this relation reads as

$$(4.14) \quad \check{\rho}_{n+m} \begin{bmatrix} A_{n+m+1} \check{\omega}_{n+m+1} & A_{n+m+1} \end{bmatrix} = \begin{bmatrix} A_{n+m} \check{\omega}_{n+m} t & A_{n+m} \end{bmatrix} \begin{bmatrix} 1 & -\bar{\alpha}_{n+m} \\ -\check{\alpha}_{n+m} & 1 \end{bmatrix},$$

or, comparing entries,

$$\check{\rho}_{n+m} A_{n+m+1} \check{\omega}_{n+m+1} = A_{n+m+1} (\check{\omega}_{n+m} t - \check{\alpha}_{n+m})$$

and

$$\check{\rho}_{n+m} A_{n+m+1} = A_{n+m+1} (1 - t \check{\omega}_{n+m} \bar{t} \bar{\alpha}_{n+m})$$

In particular, this implies that

$$(4.15) \quad \check{\omega}_{n+m+1} = \frac{t\check{\omega}_{n+m} - \check{\alpha}_{n+m}}{1 - t\check{\omega}_{n+m}\check{\alpha}_{n+m}}.$$

5. ASYMPTOTIC, CONVERGENCE, CMV BASIS AND SHIFT BASES

Space L^R has an additional structure which is responsible for the absolute continuity of the 2×2 scattering measure of the operator U_R and for the special form of its density $\begin{bmatrix} 1 & \bar{R} \\ R & 1 \end{bmatrix}$.

Consider the following two subspaces of L^R :

$$\mathcal{L}_1 = \text{Clos} \bigcup_{n=-\infty}^{\infty} \bigcap_{m=1}^{\infty} \mathcal{H}_{n,m} = \bigcup_{n=-\infty}^{\infty} \mathcal{H}_{n,\infty}, \quad \text{where} \quad \mathcal{H}_{n,\infty} = \bigcap_{m=1}^{\infty} \mathcal{H}_{n,m}$$

and

$$\mathcal{L}_2 = \text{Clos} \bigcup_{m=-\infty}^{\infty} \bigcap_{n=1}^{\infty} \mathcal{H}_{n,m} = \bigcup_{m=-\infty}^{\infty} \mathcal{H}_{\infty,m}, \quad \text{where} \quad \mathcal{H}_{\infty,m} = \bigcap_{n=1}^{\infty} \mathcal{H}_{n,m}.$$

Lemma 5.1.

$$\mathcal{H}_{n,\infty} = \begin{bmatrix} 1 \\ R \end{bmatrix} t^n H_+^2, \quad \mathcal{L}_1 = \begin{bmatrix} 1 \\ R \end{bmatrix} L^2$$

and

$$\mathcal{H}_{\infty,m} = \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^m H_-^2, \quad \mathcal{L}_2 = \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} L^2$$

Then \mathcal{L}_1 and \mathcal{L}_2 reduce U_R and it acts as a bilateral shift on \mathcal{L}_1 and on \mathcal{L}_2 . The corresponding wandering subspaces are defined as

$$\mathcal{H}_{n,\infty} \ominus \mathcal{H}_{n+1,\infty} = \left\{ \begin{bmatrix} 1 \\ R \end{bmatrix} t^n \right\} = \{e_n\}, \quad \text{and} \quad \mathcal{H}_{\infty,m} \ominus \mathcal{H}_{\infty,m+1} = \left\{ \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^{m+1} \right\} = \{d_m\}.$$

We start with proving some asymptotic properties of functions $\check{K}_{n,m}$ and $\check{\check{K}}_{n,m}$.

Lemma 5.2. *Let \mathcal{G} be a Hilbert space and \mathcal{G}_j be its closed subspaces such that*

$$\mathcal{G} \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset \dots$$

Let $\mathcal{G}_\infty = \bigcap_{j=1}^{\infty} \mathcal{G}_j$. Let P_j be orthogonal projection on \mathcal{G}_j . Then for every $g \in \mathcal{G}$

$$P_j g \rightarrow P_\infty g, \quad j \rightarrow \infty.$$

Theorem 5.3.

$$(5.1) \quad \check{K}_{n,m} \rightarrow \begin{bmatrix} 1 \\ R \end{bmatrix} t^n = e_n, \quad m \rightarrow \infty,$$

$$(5.2) \quad \check{\check{K}}_{n,m} \rightarrow \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^{m+1} = d_m, \quad n \rightarrow \infty$$

and

$$\check{a}_j(0) \rightarrow 1, \quad j \rightarrow \infty.$$

Proof. Note first that $\begin{bmatrix} 1 \\ R \end{bmatrix} t^n$ belongs to $\mathcal{H}_{n,m}$ and does not belong to $\mathcal{H}_{n+1,m}$. Using explicit formula (1.25) we compute

$$\langle \check{K}_{n,m}, \begin{bmatrix} 1 \\ R \end{bmatrix} t^n \rangle = (\bar{t}^n \check{K}_{n,m})_1(0) = \check{a}_{n+m}(0).$$

Therefore,

$$\begin{bmatrix} 1 \\ R \end{bmatrix} t^n - \check{a}_{n+m}(0) \check{K}_{n,m} = P_{\mathcal{H}_{n+1,m}} \begin{bmatrix} 1 \\ R \end{bmatrix} t^n.$$

By Lemma 5.2,

$$P_{\mathcal{H}_{n+1,m}} \begin{bmatrix} 1 \\ R \end{bmatrix} t^n \rightarrow P_{\mathcal{H}_{n+1,\infty}} \begin{bmatrix} 1 \\ R \end{bmatrix} t^n = 0, \quad m \rightarrow \infty.$$

Since

$$\left\| \begin{bmatrix} 1 \\ R \end{bmatrix} t^n - \check{a}_{n+m}(0) \check{K}_{n,m} \right\|^2 = 1 - \check{a}_{n+m}(0)^2,$$

we get that $\check{a}_{n+m}(0) \rightarrow 1$ as $m \rightarrow \infty$. Thus, we proved that

$$\check{a}_j(0) \rightarrow 1, \quad j \rightarrow \infty.$$

Using again formulas (1.25) and (1.26), we can see that

$$\left\| \begin{bmatrix} 1 \\ R \end{bmatrix} t^n - \check{K}_{n,m} \right\|^2 = 2 - 2\check{a}_{n+m}(0) \rightarrow 0, \quad m \rightarrow \infty$$

and

$$\left\| \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^{m+1} - \check{\check{K}}_{n,m} \right\|^2 = 2 - 2\check{a}_{n+m}(0) \rightarrow 0 \quad n \rightarrow \infty$$

□

Asymptotic of $\check{K}_{n,n}$ and $\check{\check{K}}_{n+1,n}$ can be obtained as a corollary of Theorem 5.3.

Corollary 5.4.

$$\left\| \begin{bmatrix} 1 \\ R \end{bmatrix} t^n - \check{K}_{n,n} \right\|^2 = 2 - 2\check{a}_{2n}(0) \rightarrow 0, \quad n \rightarrow \infty,$$

and

$$\left\| \begin{bmatrix} \bar{R} \\ 1 \end{bmatrix} \bar{t}^{n+1} - \tilde{K}_{n+1,n} \right\|^2 = 2 - 2\tilde{a}_{2n+1}(0) \rightarrow 0, \quad n \rightarrow \infty.$$

Remark 5.5. We observe that $\lim_{j \rightarrow \infty} \check{a}_j(0) = 1$ is equivalent to

$$(5.3) \quad \check{a}_n(0) = \prod_{j=n}^{\infty} \check{\rho}_j.$$

Thus, existence of the scattering function R implies convergence of the above product, which is equivalent to the convergence of the series

$$(5.4) \quad \sum_{j=n}^{\infty} (1 - \check{\rho}_j^2) = \sum_{j=n}^{\infty} |\check{\alpha}_j|^2 < \infty.$$

Using Theorem 5.3 we can get an expansion of e_n and d_m in the CMV basis constructed above in terms of the Verblunski coefficients. To this end we notice that

$$\check{K}_{n,n+2j} = U_R^{-j} \check{K}_{n+j,n+j}$$

and

$$\tilde{K}_{n+2j+1,n} = U_R^j \tilde{K}_{n+j+1,n+j}.$$

6. DIRECT SCATTERING FOR CMV MATRICES

Given Verblunsky coefficients and corresponding unitary CMV matrix in the basis (3.2), we want to obtain R . We assume that convergence (5.1) and (5.2) holds (which is equivalent to convergence of product (5.3), equivalently, of the sum (5.4)). Thus we obtain wandering systems of vectors e_n and d_m . The pair of vectors e_0 and d_0 is $*$ -cyclic (because the pair $\check{K}_{0,0}$ and $\tilde{K}_{1,0}$ is $*$ -cyclic ???). Then R is the Adamjan-Arov scattering function. Its harmonic continuation is computed as

$$R(z) = d_0^* \{ (I - zU_R^*)^{-1} + (I - \bar{z}U_R)^{-1} - I \} e_0.$$

7. SPECTRAL REPRESENTATION

Formula (3.6) implies, in particular, that pair of vectors $\check{K}_{n,n}$ and $\check{K}_{n+1,n}$ is $*$ -cyclic for the operator U_R . Indeed...Proof...Spectral measure of U_R is defined in a similar way but using the vectors $\check{K}_{n,n}$ and $\check{K}_{n+1,n}$ (this pair is $*$ -cyclic). It is easier to compute the spectral measure with respect to the vectors $\check{K}_{n,n}$ and $t\check{K}_{n,n}$. By formulas (1.25), (1.26) and (1.22) it is equal to $\check{\Sigma}_{n,n}^{(11)}$ which is defined by formula (1.18). Formula (2.3) implies that

$$\begin{bmatrix} \check{K}_{n,m} & \check{K}_{n+1,m} \end{bmatrix} = \begin{bmatrix} \check{K}_{n,m} & \check{K}_{n,m} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\bar{\alpha}_{n+m}}{\bar{\rho}_{n+m}} \\ 0 & \frac{1}{\bar{\rho}_{n+m}} \end{bmatrix}.$$

Thus the spectral measure with respect to $\check{K}_{n,n}$ and $\check{K}_{n+1,n}$ is given by

$$\begin{bmatrix} 1 & 0 \\ -\frac{\bar{\alpha}_{2n}}{\bar{\rho}_{2n}} & \frac{t}{\bar{\rho}_{2n}} \end{bmatrix} \check{\Sigma}_{n,n}^{(11)} \begin{bmatrix} 1 & -\frac{\bar{\alpha}_{2n}}{\bar{\rho}_{2n}} \\ 0 & \frac{t}{\bar{\rho}_{2n}} \end{bmatrix}.$$

Note that $\log \det \check{\Sigma}_{n,n} \in L^1$ because $\check{\Sigma}_{n,n}$ is factorized if $1 - |\check{\omega}_{n+m}|^2$ is factorized, which is the case since $1 - |R|^2$ is factorized.

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