Hypersurfaces of two space forms and conformally flat hypersurfaces

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Abstract

We address the problem of determining the hypersurfaces $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ with dimension $n \geq 3$ of a pseudo-Riemannian space form of dimension $n+1$, constant curvature c and index $s \in \{0,1\}$ for which there exists another isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$. For $n \geq 4$, we provide a complete solution by extending results for $s = 0 = \tilde{s}$ by do Carmo and Dajczer [3] and by Dajczer and the second author [5]. Our main results are for the most interesting case $n = 3$, and these are new even in the Riemannian case $s = 0 = \tilde{s}$. In particular, we characterize the solutions that have dimension $n = 3$ and three distinct principal curvatures. We show that these are closely related to conformally flat hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures, and we obtain a similar characterization of the latter that improves a theorem by Hertrich-Jeromin [8]. We also derive a Ribaucour transformation for both classes of hypersurfaces, which gives a process to produce a family of new elements of those classes, starting from a given one, in terms of solutions of a linear system of PDE's. This enables us to construct explicit examples of three-dimensional solutions of the problem, as well as new explicit examples of three-dimensional conformally flat hypersurfaces that have three distinct principal curvatures.

We denote by $\mathbb{Q}_s^N(c)$ a pseudo-Riemannian space form of dimension N, constant sectional curvature c and index $s \in \{0,1\}$, that is, $\mathbb{Q}_s^N(c)$ is either a Riemannian or Lorentzian space-form of constant curvature c, corresponding to $s = 0$ or $s = 1$, respectively. By a hypersurface $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ we always mean an isometric immersion of a *Riemannian* manifold M^n of dimension n into $\mathbb{Q}_s^{n+1}(c)$, thus f is a *space-like* hypersurface if $s = 1$.

One of the main purposes of this paper is to address the following

Problem $*$: For which hypersurfaces $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ of dimension $n \geq 3$ does there exist another isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$?

This problem was studied for $s = 0 = \tilde{s}$ and $n \geq 4$ by do Carmo and Dajczer in [3], and by Dajczer and the second author in [5]. Some partial results in the most interesting case $n = 3$ were also obtained in [5]. Including Lorentzian ambient space forms in our study of Problem $*$ was motivated by our investigation in [2] of submanifolds of codimension two and constant curvature $c \in (0, 1)$ of $\mathbb{S}^5 \times \mathbb{R}$, which turned out to be related to hypersurfaces $f\colon M^3 \to \mathbb{S}^4$ for which M^3 also admits an isometric immersion into the Lorentz space $\mathbb{R}_1^4 = \mathbb{Q}_1^4(0)$.

We first state our results for the case $n \geq 4$. The next one extends a theorem due to do Carmo and Dajczer [3] in the case $s = 0 = \tilde{s}$. Here and in the sequel, for $s, \tilde{s} \in \{0, 1\}$ we denote $\epsilon = -2s + 1$ and $\tilde{\epsilon} = -2\tilde{s} + 1$.

Theorem 1. *Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ be a hypersurface of dimension $n \geq 4$. If there exists another isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$ with $\tilde{c} \neq c$, then $c < \tilde{c}$ if $s = 0$ and $\tilde{s} = 1$ (respectively, $c > \tilde{c}$ if $s = 1$ and $\tilde{s} = 0$) and f has a principal curvature λ of multiplicity at least $n - 1$ everywhere. Moreover, at any $x \in M^n$ the following holds:*

- *(i) if $\lambda = 0$ or f is umbilical with $c + \epsilon\lambda^2 \neq \tilde{c}$, then $\tilde{f}$ is umbilical;*
- *(ii) if f is umbilical and $c + \epsilon\lambda^2 = \tilde{c}$, then 0 is a principal curvature of $\tilde{f}$ with multiplicity at least $n - 1$;*
- *(iii) if $\lambda \neq 0$ with multiplicity $n - 1$, then $\tilde{f}$ has also a principal curvature $\tilde{\lambda}$ with the same eigenspace as λ.*

Thus, Problem $*$ has no solutions if $n \geq 4$ and either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$ or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, while, in the remaining cases, having a principal curvature of multiplicity at least $n-1$ is a necessary condition for a solution. In those cases, having a principal curvature of *constant* multiplicity n or $n - 1$ is also sufficient for simply connected hypersurfaces.

Theorem 2. *Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$, be an isometric immersion of a simply connected Riemannian manifold. Assume that f has a principal curvature λ of (constant) multiplicity either $n-1$ or n. Then M^n admits an isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$, unless $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or*

$c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, and assertions (i)-(iii) in Theorem 1 hold. Moreover, $\tilde{f}$ is unique up to congruence except in case (ii).

The next result, proved by Dajczer and the second author in [5] when $s = 0 = \tilde{s}$, shows how any solution $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$, of Problem $*$ arises.

Theorem 3. *Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ and $\tilde{f}\colon M^n \to \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})$, $n \geq 4$, be isometric immersions with, say, $c > \tilde{c}$. If $s = 0$, assume that $\tilde{s} = 0$. Then, for $s = \tilde{s}$ (respectively, $s = 1$ and $\tilde{s} = 0$), there exist, locally on an open dense subset of M^n, isometric embeddings*

$$H\colon \mathbb{Q}_s^{n+1}(\tilde{c}) \to \mathbb{Q}_s^{n+2}(\tilde{c}) \quad \text{and} \quad i\colon \mathbb{Q}_s^{n+1}(c) \to \mathbb{Q}_s^{n+2}(\tilde{c})$$

(respectively, $H\colon \mathbb{Q}_s^{n+1}(c) \to \mathbb{Q}_s^{n+2}(c)$ and $i\colon \mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c}) \to \mathbb{Q}_s^{n+2}(c)$), with i umbilical, and an isometry

$$\Psi\colon \bar{M}^n := H(\mathbb{Q}_s^{n+1}(\tilde{c})) \cap i(\mathbb{Q}_s^{n+1}(c)) \to M^n$$

(respectively, $\Psi\colon \bar{M}^n := H(\mathbb{Q}_s^{n+1}(c)) \cap i(\mathbb{Q}_{\tilde{s}}^{n+1}(\tilde{c})) \to M^n$) such that

$$f \circ \Psi = i^{-1}|_{\bar{M}^n} \quad \text{and} \quad \tilde{f} \circ \Psi = H^{-1}|_{\bar{M}^n}.$$

(respectively, $f \circ \Psi = H^{-1}|_{\bar{M}^n}$ and $\tilde{f} \circ \Psi = i^{-1}|_{\bar{M}^n}$).

Theorem 3 explains the existence of a principal curvature λ of multiplicity at least $n-1$ for a solution $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$, $n \geq 4$, of Problem $*$: the (images by f of the) leaves of the distribution on M^n given by the eigenspaces of λ are the intersections with $i(\mathbb{Q}_s^{n+1}(\tilde{c}))$ of the (images by H of the) relative nullity leaves of H, which have dimension at least n.

Next we consider Problem $*$ for hypersurfaces of dimension $n = 3$. The following result provides the solutions in two ("dual") special cases.

Theorem 4. *Let $f\colon M^3 \to \mathbb{Q}_s^4(c)$ be a hypersurface for which there exists an isometric immersion $\tilde{f}\colon M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $\tilde{c} \neq c$.*

(a) Assume that f has a principal curvature of multiplicity two. If either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or if $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, then f is a rotation hypersurface whose profile curve is a $\tilde{c}$-helix in a totally geodesic surface $\mathbb{Q}_s^2(c)$ of $\mathbb{Q}_s^4(c)$ and $\tilde{f}$ is a generalized cone over a surface with constant curvature in an umbilical hypersurface $\mathbb{Q}_{\tilde{s}}^3(\bar{c})$ of $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, $\bar{c} \geq \tilde{c}$. Otherwise, either the same conclusion holds or f and $\tilde{f}$ are locally given on an open dense subset as described in Theorem 3.

(b) *If one of the principal curvatures of f is zero, then f is a generalized cone over a surface with constant curvature in an umbilical hypersurface $\mathbb{Q}^3_s(\bar{c})$ of $\mathbb{Q}^4_s(c)$, $\bar{c} \geq c$, and $\tilde{f}$ is a rotation hypersurface whose profile curve is a c-helix in a totally geodesic surface $\mathbb{Q}^2_{\tilde{s}}(\tilde{c})$ of $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$.*

By a *generalized cone* over a surface $g\colon M^2 \to \mathbb{Q}^3_s(\bar{c})$ in an umbilical hypersurface $\mathbb{Q}^3_s(\bar{c})$ of $\mathbb{Q}^4_s(c)$, $\bar{c} \geq c$, we mean the hypersurface parametrized by (the restriction to the subset of regular points of) the map $G\colon M^2 \times \mathbb{R} \to \mathbb{Q}^4_s(c)$ given by

$$G(x,t) = \exp_{g(x)}(t\xi(g(x))),$$

where ξ is a unit normal vector field to the inclusion $i\colon \mathbb{Q}^3_s(\bar{c}) \to \mathbb{Q}^4_s(c)$ and exp is the exponential map of $\mathbb{Q}^4_s(c)$. A *c-helix* in $\mathbb{Q}^2_s(\tilde{c}) \subset \mathbb{R}^3_{s+\epsilon_0}$ with respect to a unit vector $v \in \mathbb{R}^3_{s+\epsilon_0}$ is a unit-speed curve $\gamma\colon I \to \mathbb{Q}^2_s(\tilde{c}) \subset \mathbb{R}^3_{s+\epsilon_0}$ such that the height function $\gamma_v = \langle \gamma, v\rangle$ satisfies $\gamma_v'' + c\gamma_v = 0$. Here $\epsilon_0 = 0$ or 1, corresponding to $\tilde{c} > 0$ or $\tilde{c} < 0$, respectively.

In order to deal with the generic case of Problem $*$ for hypersurfaces of dimension 3, we need to recall the notion of holonomic hypersurfaces. We call a hypersurface $f\colon M^n \to \mathbb{Q}^{n+1}_s(c)$ *holonomic* if M^n carries global orthogonal coordinates $(u_1, \ldots, u_n)$ such that the coordinate vector fields $\partial_j = \dfrac{\partial}{\partial u_j}$ are everywhere eigenvectors of the shape operator A of f. Set $v_j = \|\partial_j\|$, and define $V_j \in C^\infty(M)$, $1 \leq j \leq n$, by $A\partial_j = v_j^{-1}V_j\partial_j$. Thus, the first and second fundamental forms of f are

$$I = \sum_{i=1}^n v_i^2 du_i^2 \quad \text{and} \quad II = \sum_{i=1}^n V_i v_i du_i^2. \tag{1}$$

Set $v = (v_1, \ldots, v_n)$ and $V = (V_1, \ldots, V_n)$. We call (v, V) the pair associated to f. The next result is well known.

Proposition 5. *The triple (v, h, V), where $h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i}$, satisfies the system of PDE's*

$$\begin{cases} (i)\dfrac{\partial v_i}{\partial u_j} = h_{ji}v_j, \qquad (ii)\dfrac{\partial h_{ik}}{\partial u_j} = h_{ij}h_{jk}, \\ (iii)\dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + h_{ki}h_{kj} + \epsilon V_iV_j + cv_iv_j = 0, \\ (iv)\dfrac{\partial V_i}{\partial u_j} = h_{ji}V_j, \quad 1 \leq i \neq j \neq k \neq i \leq n. \end{cases} \tag{2}$$

Conversely, if (v, h, V) is a solution of (2) on a simply connected open subset $U \subset \mathbb{R}^n$, with $v_i \neq 0$ everywhere for all $1 \leq i \leq n$, then there exists a holonomic hypersurface $f\colon U \to \mathbb{Q}_s^{n+1}(c)$ whose first and second fundamental forms are given by (1).

The following characterization of hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures that are solutions of Problem $*$ is one of the main results of the paper.

Theorem 6. *Let $f\colon M^3 \to \mathbb{Q}_s^4(c)$ be a simply connected holonomic hypersurface whose associated pair (v, V) satisfies*

$$\sum_{i=1}^{3} \delta_i v_i^2 = \hat{\epsilon}, \quad \sum_{i=1}^{3} \delta_i v_i V_i = 0 \quad \text{and} \quad \sum_{i=1}^{3} \delta_i V_i^2 = C := \tilde{\epsilon}(c - \tilde{c}), \tag{3}$$

where $\hat{\epsilon}, \tilde{\epsilon} \in \{-1, 1\}$, $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ either if $\hat{\epsilon} = 1$ or if $\hat{\epsilon} = -1$ and $C > 0$, and $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$ if $\hat{\epsilon} = -1$ and $C < 0$. Then M^3 admits an isometric immersion into $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, which is unique up to congruence.

Conversely, if $f\colon M^3 \to \mathbb{Q}_s^4(c)$ is a hypersurface with three distinct principal curvatures for which there exists an isometric immersion $\tilde{f}\colon M^3 \to \mathbb{Q}_{\tilde{s}}^4(\tilde{c})$ with $\tilde{c} \neq c$, then f is locally a holonomic hypersurface whose associated pair (v, V) satisfies (3).

As we shall make precise in the sequel, the class of hypersurfaces that are solutions of Problem $*$ is closely related to that of conformally flat hypersurfaces of $\mathbb{Q}_s^{n+1}(c)$, that is, isometric immersions $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ of conformally flat manifolds. Recall that a Riemannian manifold M^n is *conformally flat* if each point of M^n has an open neighborhood that is conformally diffeomorphic to an open subset of Euclidean space $\mathbb{R}^n$. First, for $n \geq 4$ we have the following extension of a result due to E. Cartan when $s = 0$.

Theorem 7. *Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ be a hypersurface of dimension $n \geq 4$. Then M^n is conformally flat if and only if f has a principal curvature of multiplicity at least $n - 1$.*

It was already known by E. Cartan that the "only if" assertion in the preceding result is no longer true for $n = 3$ and $s = 0$. The study of conformally flat hypersurfaces by Cartan was taken up by Hertrich-Jeromin [8], who showed that a conformally flat hypersurface $f\colon M^3 \to \mathbb{Q}^4(c)$ with three distinct principal curvatures admits locally principal coordinates (u_1, u_2, u_3)

such that the induced metric $ds^2 = \sum_{i=1}^3 v_i^2 du_i^2$ satisfies, say, $v_2^2 = v_1^2 + v_3^2$. The next result states that conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures are characterized by the existence of such principal coordinates under some additional conditions.

Theorem 8. *Let $f\colon M^3 \to \mathbb{Q}_s^4(c)$ be a holonomic hypersurface whose associated pair (v, V) satisfies*

$$\sum_{i=1}^3 \delta_i v_i^2 = 0, \quad \sum_{i=1}^3 \delta_i v_i V_i = 0 \quad \textit{and} \quad \sum_{i=1}^3 \delta_i V_i^2 = 1, \tag{4}$$

where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. Then M^3 is conformally flat.

Conversely, any conformally flat hypersurface $f\colon M^3 \to \mathbb{Q}_s^4(c)$ with three distinct principal curvatures is locally a holonomic hypersurface whose associated pair (v, V) satisfies (4).

It is amazing that the class of holonomic Euclidean hypersurfaces of any dimension n whose associated pair (v, V) satisfies the conditions

$$\sum_{i=1}^n \delta_i v_i^2 = K_1 \quad \text{and} \quad \sum_{i=1}^n \delta_i V_i^2 = K_2,$$

where $K_1, K_2 \in \mathbb{R}$ and $\delta_i \in \{-1, 1\}$ for $1 \leq i \leq n$, was considered by Bianchi [1] almost one century ago, his interest on such hypersurfaces relying on the fact that they satisfy many of the properties of constant curvature surfaces an their parallel surfaces in $\mathbb{R}^3$. In particular, a Ribaucour transformation for that class was sketched in Bianchi's paper.

It follows from Theorems 6 and 8 that, in order to produce hypersurfaces of $\mathbb{Q}_s^4(c)$ that are either conformally flat or admit an isometric immersion into $\mathbb{Q}_s^4(\tilde{c})$ with $\tilde{c} \neq c$, one must start with solutions (v, h, V) on an open simply connected subset $U \subset \mathbb{R}^3$ of the same system of PDE's, namely, the one obtained by adding to system (2) (for $n = 3$) the equations

$$\delta_i \frac{\partial v_i}{\partial u_i} + \delta_j h_{ij} v_j + \delta_k h_{ik} v_k = 0 \tag{5}$$

and

$$\delta_i \frac{\partial V_i}{\partial u_i} + \delta_j h_{ij} V_j + \delta_k h_{ik} V_k = 0, \quad 1 \leq i \neq j \neq k \neq i \leq 3, \tag{6}$$

with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. Such system has the first integrals

$$\sum_{i=1}^{3} \delta_i v_i^2 = K_1, \quad \sum_{i=1}^{3} \delta_i v_i V_i = K_2 \ \text{ and } \ \sum_{i=1}^{3} \delta_i V_i^2 = K_3.$$

If initial conditions at some point are chosen so that $K_1 = 1$ (respectively, $K_1 = 0$), $K_2 = 0$ and $K_3 = \epsilon(c - \tilde{c})$ (respectively, $K_3 = 1$), then the corresponding solutions give rise to hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures that can be isometrically immersed into $\mathbb{Q}_s^4(\tilde{c})$ (respectively, are conformally flat).

Our characterizations in Theorems 6 and 8 of hypersurfaces of $\mathbb{Q}_s^4(c)$ with three distinct principal curvatures that admit an isometric immersion into $\mathbb{Q}_{\tilde{s}}^4(\tilde{c})$, with $c \neq \tilde{c}$, or are conformally flat, respectively, allow us to derive a Riabaucour transformation for both classes of hypersurfaces. In particular, it yields the following process to generate a family of new elements of such classes from a given one. We denote by $i\colon \mathbb{Q}_s^4(c) \to \mathbb{R}^5_{s+\epsilon_0}$ an umbilical inclusion, where $\epsilon_0 = 0$ or 1, corresponding to $c > 0$ or $c < 0$, respectively.

Theorem 9. *If $f\colon M^3 \to \mathbb{Q}_s^4(c)$ is a holonomic hypersurface whose associated pair (v, V) satisfies (3) (respectively, (4)), then the linear system of PDE's*

$$\begin{cases} (i)\dfrac{\partial \varphi}{\partial u_i} = v_i \gamma_i, \quad (ii)\dfrac{\partial \gamma_j}{\partial u_i} = h_{ji}\gamma_i, \qquad i \neq j, \\ (iii)\dfrac{\partial \gamma_i}{\partial u_i} = (v_i - v_i')\psi - \sum_{j\neq i} h_{ji}\gamma_j + \beta V_i - c\varphi v_i, \\ (iv)\epsilon\dfrac{\partial \beta}{\partial u_i} = -V_i\gamma_i, \\ (v)\dfrac{\partial \log \psi}{\partial u_i} = -\dfrac{\gamma_i v_i'}{\varphi}, \quad (vi)\dfrac{\partial v_i'}{\partial u_j} = h_{ji}' v_j', \qquad i \neq j, \\ (vii)\delta_i \dfrac{\partial v_i'}{\partial u_i} + \delta_j h_{ij}' v_j' + \delta_k h_{ik}' v_k' = 0, \end{cases} \tag{7}$$

where

$$h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i} \quad \textit{and} \quad h_{ij}' = h_{ij} + (v_j' - v_j)\frac{\gamma_i}{\varphi}, \tag{8}$$

is completely integrable and has the first integrals

$$\sum_i {\gamma_i}^2 + \epsilon\beta^2 + c\varphi^2 - 2\varphi\psi = K_1 \in \mathbb{R} \tag{9}$$

and

$$\delta_1 v_1'^2 + \delta_2 v_2'^2 + \delta_3 v_3'^2 = K_2 \in \mathbb{R}. \tag{10}$$

Let $(\gamma_1, \gamma_2, \gamma_3, v_1', v_2', v_3', \varphi, \psi, \beta)$ *be a solution of (7) with initial conditions at some point chosen so that* $K_1 = 0$ *and* $K_2 = \hat{\epsilon}$ *(respectively,* $K_2 = 0$*), and so that the function*

$$\Omega = \varphi \sum_{j=1}^{3} \delta_j v_j' V_j - \epsilon\beta \left(K_2 - \sum_{j=1}^{3} \delta_j v_j v_j' \right), \tag{11}$$

with $K_2 = \hat{\epsilon}$ *(respectively,* $K_2 = 0$*), vanishes at that point. Then, the map* $F' \colon M^3 \to \mathbb{R}^5_{s+\epsilon_0}$*, given in terms of* $F = i \circ f$ *by*

$$F' = F - \frac{1}{\psi} \left(\sum_i \gamma_i F_* e_i + \beta i_* \xi + c\varphi F \right), \tag{12}$$

where ξ *is a unit normal vector field* ξ *to* f *and* $e_i = v_i^{-1}\partial_i$*,* $1 \leq i \leq 3$*, satisfies* $F' = i \circ f'$*, where* $f' \colon M^3 \to \mathbb{Q}^4_s(c)$ *is a holonomic hypersurface whose associated pair* (v', V')*, with*

$$V_i' = V_i + (v_i - v_i')\frac{\epsilon\beta}{\varphi},$$

also satisfies (3) (respectively, (4)).

Explicit examples of hypersurfaces of $\mathbb{Q}^4_s(c)$ with three distinct principal curvatures that admit an isometric immersion into $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ with $c \neq \tilde{c}$, as well as of conformally flat hypersurfaces of $\mathbb{R}^4_s$ with three distinct principal curvatures, are constructed in Section 6 by means of Theorem 9.

As a special consequence of Theorem 9, it follows that hypersurfaces $f \colon M^3 \to \mathbb{Q}^4_s(c)$ that can be isometrically immersed into $\mathbb{R}^4_{\tilde{s}}$ arise in families of parallel hypersurfaces.

Corollary 10. *Let* $f \colon M^3 \to \mathbb{Q}^4_s(c)$ *be a holonomic hypersurface whose associated pair* (v, V) *satisfies (3) with* $\tilde{c} = 0$*. Then any parallel hypersurface* $f_t \colon M^3 \to \mathbb{Q}^4_s(c)$ *to* f *has also the same property.*

It was already shown in [5] for $s = 0 = \tilde{s}$ that, unlike the case of dimension $n \geq 4$, among hypersurfaces $f \colon M^n \to \mathbb{Q}^{n+1}_s(c)$ of dimension $n = 3$ with three distinct principal curvatures, the classes of solutions of Problem $*$ and

conformally flat hypersurfaces are distinct. Moreover, it was observed that their intersection contains the generalized cones over surfaces with constant curvature in an umbilical hypersurface $\mathbb{Q}^3_s(\bar{c})$ of $\mathbb{Q}^4_s(c)$, $\bar{c} \geq c$. Our last result states that such intersection contains no other elements.

Theorem 11. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a conformally flat hypersurface with three distinct principal curvatures. If M^3 admits an isometric immersion into $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$, $\tilde{c} \neq c$, then f is a generalized cone over a surface with constant curvature in an umbilical hypersurface $\mathbb{Q}^3_s(\bar{c})$ of $\mathbb{Q}^4_s(c)$, $\bar{c} \geq c$.*

1 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1: Let $i\colon \mathbb{Q}^{n+1}_s(c) \to \mathbb{Q}^{n+2}_{s+\epsilon_0}(\tilde{c})$ be an umbilical inclusion, where $\epsilon_0 = 0$ or 1, corresponding to $c > \tilde{c}$ or $c < \tilde{c}$, respectively, and set $\hat{f} = i \circ f$. Then, the second fundamental forms α and $\hat{\alpha}$ of f and $\hat{f}$, respectively, are related by

$$\hat{\alpha} = i_*\alpha + \sqrt{|c - \tilde{c}|}\langle\ ,\ \rangle\xi, \tag{13}$$

where ξ is one of the unit vector fields that are normal to i.

For a fixed point $x \in M^n$, define $W^3(x) := N_{\hat{f}}M(x) \oplus N_{\tilde{f}}M(x)$, and endow $W^3(x)$ with the inner product

$$\langle\!\langle(\xi + \tilde{\xi}, \eta + \tilde{\eta})\rangle\!\rangle_{W^3(x)} := \langle \xi, \eta\rangle_{N_{\hat{f}}M(x)} - \langle \tilde{\xi}, \tilde{\eta}\rangle_{N_{\tilde{f}}M(x)},$$

which has index $(s + \epsilon_0) + (1 - \tilde{s})$.

Now define a bilinear form $\beta_x\colon T_xM \times T_xM \to W^3(x)$ by

$$\beta_x = \hat{\alpha}(x) \oplus \tilde{\alpha}(x),$$

where $\hat{\alpha}(x)$ and $\tilde{\alpha}(x)$ are the second fundamental forms of $\hat{f}$ and $\tilde{f}$, respectively, at x. Notice that $\mathcal{N}(\beta_x) \subset \mathcal{N}(\hat{\alpha}(x)) = \{0\}$ by (13). On the other hand, it follows from the Gauss equations of $\hat{f}$ and $\tilde{f}$ that β_x is flat with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$, that is,

$$\langle\!\langle\beta_x(X, Y), \beta_x(Z, W)\rangle\!\rangle = \langle\!\langle\beta_x(X, W), \beta_x(Z, Y)\rangle\!\rangle$$

for all $X, Y, Z, W \in T_xM$. Thus, if $\langle\!\langle\ ,\ \rangle\!\rangle$ is positive definite, which is the case when $s = 0$, $\tilde{s} = 1$ and $\epsilon_0 = 0$, that is, $c > \tilde{c}$, we obtain a contradiction with Corollary 1 of [9], according to which one has the inequality

$$\dim \mathcal{N}(\beta_x) \geq n - \dim W(x) = n - 3 > 0. \tag{14}$$

The same contradiction is reached by applying the preceding inequality to $-\langle\langle\ ,\ \rangle\rangle$ when $s=1$, $\tilde{s}=0$ and $c<\tilde{c}$, in which case $\langle\langle\ ,\ \rangle\rangle$ is negative definite. Therefore, such cases can not occur, which proves the first assertion.

In all other cases, the index of $\langle\langle\ ,\ \rangle\rangle$ is either 1 or 2. Thus, by applying Corollary 2 in [9] to $\langle\langle\ ,\ \rangle\rangle$ in the first case and to $-\langle\langle\ ,\ \rangle\rangle$ in the latter, we obtain that $\mathcal{S}(\beta_x)$ must be degenerate, for otherwise the inequality (14) would still hold, and then we would reach a contradiction as before.

Since $\mathcal{S}(\beta_x)$ is degenerate, there exist $\zeta\in N_{\hat{f}}M(x)$ and $\tilde{N}\in N_{\tilde{f}}M(x)$ such that $(0,0)\neq(\zeta,\tilde{N})\in\mathcal{S}(\beta_x)\cap\mathcal{S}(\beta_x)^{\perp}$. In particular, from $0=\langle\langle\zeta+\tilde{N},\zeta+\tilde{N}\rangle\rangle$ it follows that $\langle\tilde{N},\tilde{N}\rangle=\langle\zeta,\zeta\rangle$. Thus, either $\tilde{N}=0$ and $\zeta\in\mathcal{S}(\hat{\alpha}(x))\cap\mathcal{S}(\hat{\alpha}(x))^{\perp}$, or we can assume that $\langle\tilde{N},\tilde{N}\rangle=\tilde{\epsilon}=\langle\zeta,\zeta\rangle$.

The former case occurs precisely when f is umbilical at x with a principal curvature λ with respect to one of the unit normal vectors N to f, satisfying

$$\epsilon\lambda^2+c-\tilde{c}=0,$$

in which case $N_{\hat{f}}M(x)$ is a Lorentzian two-plane and $\zeta=\lambda i_*N+\sqrt{|c-\tilde{c}|}\xi$ is a light-like vector that spans $\mathcal{S}(\hat{\alpha}(x))$. In this case, all sectional curvatures of M^n at x are equal to $\tilde{c}$ by the Gauss equation of f, and hence $\tilde{f}$ has 0 as a principal curvature at x with multiplicity at least $n-1$ by the Gauss equation of $\tilde{f}$.

Now assume that $\langle\tilde{N},\tilde{N}\rangle=\tilde{\epsilon}=\langle\zeta,\zeta\rangle$. Then, from

$$0=\langle\langle\beta,\zeta+\tilde{N}\rangle\rangle=\langle\hat{\alpha},\zeta\rangle-\langle\tilde{\alpha},\tilde{N}\rangle,$$

we obtain that $A^{\hat{f}}_{\zeta}=A^{\tilde{f}}_{\tilde{N}}$. Let $\zeta^{\perp}\in N_{\hat{f}}M(x)$ be such that $\{\zeta,\zeta^{\perp}\}$ is an orthonormal basis of $N_{\hat{f}}M(x)$. The Gauss equations for $\hat{f}$ and $\tilde{f}$ imply that

$$\langle A^{\hat{f}}_{\zeta^{\perp}}X,Y\rangle\langle A^{\hat{f}}_{\zeta^{\perp}}Z,W\rangle=\langle A^{\hat{f}}_{\zeta^{\perp}}X,W\rangle\langle A^{\hat{f}}_{\zeta^{\perp}}Z,Y\rangle$$

for all $X,Y,Z,W\in T_xM$, which is equivalent to $\dim\mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}})\geq n-1$. Since $A^{\hat{f}}_{\xi}=\delta\sqrt{|c-\tilde{c}|}I$ by (13), with $\delta=(c-\tilde{c})/|c-\tilde{c}|$, it follows that the restriction to $\mathcal{N}(A^{\hat{f}}_{\zeta^{\perp}})$ of all shape operators $A^{\hat{f}}_{\eta}$, $\eta\in N_{\hat{f}}M(x)$, is a multiple of the identity tensor. In particular, this is the case for $A^{\hat{f}}_{i_*N}=A^{f}_{N}$, where N is one of the unit normal vector fields to f, hence f has a principal curvature λ at x with multiplicity at least $n-1$.

Moreover, if $\lambda = 0$ then $\zeta^\perp$ must coincide with i_*N, and hence ζ with ξ, up to signs. Therefore $A^{\tilde{f}}_{\tilde{N}} = A^{\hat{f}}_{\xi}$, up to sign, hence $\tilde{f}$ is umbilical at x. If f is umbilical at x and $c + \epsilon\lambda^2 \neq \tilde{c}$, then $A_{\zeta^\perp} = 0$ and $A^{\tilde{f}}_{\tilde{N}} = A^{\hat{f}}_{\zeta}$ is a (nonzero) constant multiple of the identity tensor. Finally, if $\lambda \neq 0$ has multiplicity $n-1$, then we must have $\zeta^\perp \neq i_*N$ and $\dim \mathcal{N}(A^{\hat{f}}_{\zeta^\perp}) = n-1$, hence $\mathcal{N}(A^{\hat{f}}_{\zeta^\perp})$ is an eigenspace of $A^{\hat{f}}_{\zeta} = A^{\tilde{f}}_{\tilde{N}}$. $\square$

Proof of Theorem 2: Suppose first that f is umbilical, with a (constant) principal curvature λ. If $c + \epsilon\lambda^2 = \tilde{c}$, then M^n has constant curvature $\tilde{c}$, hence it admits isometric immersions into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$ having 0 as a principal curvature with multiplicity at least $n-1$. Otherwise, by the assumption there exists $\tilde{\lambda} \neq 0$ such that $c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\tilde{\lambda}^2$. Hence $c + \epsilon\lambda^2 = \tilde{c} + \tilde{\epsilon}\tilde{\lambda}^2$, thus $\tilde{A} = \tilde{\lambda} I$ satisfies the Gauss and Codazzi equation for an (umbilical) isometric immersion into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$.

Assume now that f has principal curvatures λ and μ of multiplicities $n-1$ and 1, respectively, with corresponding eigenbundles E_λ and E_μ. If $\lambda = 0$, then M^n has constant curvature c, hence it admits an umbilical isometric immersion into $\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$. From now on, assume that $\lambda \neq 0$. Then, one can check that the Codazzi equations for f are equivalent to the fact that E_λ and E_μ are umbilical distributions with mean curvature normals η and ζ, respectively, satisfying

$$\eta = \frac{(\nabla\lambda)_{E_\mu}}{\lambda - \mu} \quad \text{and} \quad \zeta = \frac{(\nabla\mu)_{E_\lambda}}{\mu - \lambda}.$$

By the assumption, there exist $\tilde{\lambda}, \tilde{\mu} \in C^\infty(M)$ such that

$$c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\tilde{\lambda}^2 \quad \text{and} \quad c - \tilde{c} + \epsilon\lambda\mu = \tilde{\epsilon}\tilde{\lambda}\tilde{\mu}.$$

Moreover, the first of the preceding equations implies that $\tilde{\lambda} \neq 0$ everywhere, and hence $\tilde{\lambda}$ and $\tilde{\mu}$ are unique if $\tilde{\lambda}$ is chosen to be positive. From both equations we obtain that

$$\epsilon\lambda^2 - \tilde{\epsilon}\tilde{\lambda}^2 = \epsilon\lambda\mu - \tilde{\epsilon}\tilde{\lambda}\tilde{\mu}, \qquad \epsilon\lambda\nabla\lambda = \tilde{\epsilon}\tilde{\lambda}\nabla\tilde{\lambda}$$

and

$$\epsilon((\nabla\lambda)\mu + \lambda\nabla\mu) = \tilde{\epsilon}((\nabla\tilde{\lambda})\tilde{\mu} + \tilde{\lambda}\nabla\tilde{\mu}).$$

It follows that

$$\frac{(\nabla\tilde{\lambda})_{E_\mu}}{\tilde{\lambda} - \tilde{\mu}} = \frac{(\nabla\lambda)_{E_\mu}}{\lambda - \mu} \tag{15}$$

and similarly,

$$\frac{(\nabla\tilde{\mu})_{E_\lambda}}{\tilde{\mu}-\tilde{\lambda}} = \frac{(\nabla\mu)_{E_\lambda}}{\mu-\lambda}. \tag{16}$$

Let $\tilde{A}$ be the endomorphism of TM with eigenvalues $\tilde{\lambda}$, $\tilde{\mu}$ and corresponding eigenbundles E_λ and E_μ, respectively. Since

$$c+\epsilon\lambda^2 = \tilde{c}+\tilde{\epsilon}\tilde{\lambda}^2 \quad \text{and} \quad c+\epsilon\lambda\mu = \tilde{c}+\tilde{\epsilon}\tilde{\lambda}\tilde{\mu},$$

the Gauss equations for an isometric immersion $\tilde{f}\colon M^n \to \mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c})$ are satisfied by $\tilde{A}$. It follows from (15) and (16) that $\tilde{A}$ also satisfies the Codazzi equations. □

Proof of Theorem 3: Since we are assuming that $c > \tilde{c}$, there exist umbilical inclusions $i\colon \mathbb{Q}^{n+1}_s(c) \to \mathbb{Q}^{n+2}_s(\tilde{c})$ and $i\colon \mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c}) \to \mathbb{Q}^{n+2}_s(c)$ for $(s,\tilde{s}) = (1,0)$. If $s = \tilde{s}$ (respectively, $(s,\tilde{s}) = (1,0)$), set $\hat{f} = i\circ f$ (respectively, $\hat{f} = i\circ\tilde{f}$). Then, one can use the existence of normal vector fields $\zeta \in \Gamma(N_{\hat{f}}M)$ and $\tilde{N} \in \Gamma(N_{\hat{f}}M)$ satisfying $\langle\zeta,\zeta\rangle = \tilde{\epsilon} = \langle\tilde{N},\tilde{N}\rangle$ and $A^{\hat{f}}_\zeta = A^{\tilde{f}}_{\tilde{N}}$ and argue as in the proof of Theorem 3 in [5]. One obtains that there exists an open dense subset $U \subset M^n$, each point of which has an open neighborhood $V \subset M^n$ such that $\hat{f}|_V$ (respectively, $f|_V$) is a composition $\hat{f}|_V = H\circ\tilde{f}|_V$ (respectively, $f|_V = H\circ\hat{f}|_V$) with an isometric embedding $H\colon W \subset \mathbb{Q}^{n+1}_s(\tilde{c}) \to \mathbb{Q}^{n+2}_s(\tilde{c})$ (respectively, $H\colon W \subset \mathbb{Q}^{n+1}_s(c) \to \mathbb{Q}^{n+2}_s(c)$), with $\tilde{f}(V) \subset W$ (respectively, $\hat{f}(V) \subset W$). Set $\bar{M}^n = H(W)\cap i(\mathbb{Q}^{n+1}_s(c))$ (respectively, $\bar{M}^n = H(W)\cap i(\mathbb{Q}^{n+1}_{\tilde{s}}(\tilde{c}))$). Then $i\circ f|_V = H\circ\tilde{f}|_V\colon V \to \bar{M}^n$ (respectively, $H\circ f|_V = i\circ\tilde{f}|_V\colon V\to\bar{M}^n$) is an isometry. Let $\Psi\colon \bar{M}^n \to V$ be the inverse of this isometry. Then $f\circ\Psi = i^{-1}|_{\bar{M}^n}$ and $\tilde{f}\circ\Psi = H^{-1}|_{\bar{M}^n}$ (respectively, $f\circ\Psi = H^{-1}|_{\bar{M}^n}$ and $\tilde{f}\circ\Psi = i^{-1}|_{\bar{M}^n}$), where i^{-1} and H^{-1} denote the inverses of the maps i and H, respectively, regarded as maps onto their images. □

2 Proof of Theorem 4

Before going into the proof of Theorem 4, we establish a basic fact that will also be used in the proof of Theorem 6 in the next section.

Lemma 12. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ and $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ be hypersurfaces with $c \neq \tilde{c}$. Then, at each point $x \in M^3$ there exists an orthonormal basis $\{e_1,e_2,e_3\}$ of T_xM^3 that simultaneously diagonalizes the second fundamental forms of f and $\tilde{f}$.*

Proof. Define $i\colon \mathbb{Q}^4_s(c) \to \mathbb{Q}^5_{s+\epsilon_0}(\tilde{c})$ and $\hat{f}$, as well as $W^3(x)$, $\langle\!\langle\ ,\ \rangle\!\rangle_{W^3(x)}$ and β_x for each $x \in M^n$, as in the proof of Theorem 1. If $\mathcal{S}(\beta_x)$ is degenerate for all $x \in M^3$, we conclude as in the case $n \geq 4$ that the assertions in Theorem 1 hold, hence the statement is clearly true in this case.

Suppose now that $\mathcal{S}(\beta_x)$ is nondegenerate at $x \in M^3$. Then the inequality

$$\dim \mathcal{S}(\beta_x) \geq \dim T_xM - \dim \mathcal{N}(\beta_x)$$

holds by Corollary 2 in [9]. Since $\mathcal{N}(\beta_x) = \{0\}$, the right-hand-side is equal to $\dim T_xM = 3 = \dim W^3(x)$, hence we must have equality in the above inequality. By Theorem $2-b$ in [9], there exists an orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ of $W^3(x)$ and a basis $\{\theta^1, \theta^2, \theta^3\}$ of T^*_xM such that

$$\beta = \sum_{j=1}^{3} \theta^j \otimes \theta^j \xi_j.$$

In particular, if $i \neq j$ then $\beta(e_i, e_j) = 0$ for the dual basis $\{e_1, e_2, e_3\}$ of $\{\theta^1, \theta^2, \theta^3\}$. It follows that $\{e_1, e_2, e_3\}$ diagonalyzes both $\hat{\alpha}$ and $\tilde{\alpha}$, and therefore both α and $\tilde{\alpha}$, in view of (13). It also follows from (13) that

$$0 = \langle \hat{\alpha}(e_i, e_j), \xi \rangle = \sqrt{|c - \tilde{c}|}\langle e_i, e_j \rangle, \ \ i \neq j,$$

hence the basis $\{e_1, e_2, e_3\}$ is orthogonal. □

Lemma 13. *Under the assumptions of Lemma 12, let $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 be the principal curvatures of f and $\tilde{f}$ correspondent to e_1, e_2 and e_3, respectively.*

(a) Assume that f has a principal curvature of multiplicity two, say, that $\lambda_1 = \lambda_2 := \lambda$. If either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$, then

$$c - \tilde{c} + \epsilon\lambda\lambda_3 = 0, \quad \mu_3 = 0 \ \ \textit{and} \ \ c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\mu_1\mu_2.$$

Otherwise, either the same conclusion holds or

$$\mu_1 = \mu_2 := \mu, \ \ c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\mu^2 \ \ \textit{and} \ \ c - \tilde{c} + \epsilon\lambda\lambda_3 = \tilde{\epsilon}\mu\mu_3.$$

(b) Assume, say, that $\lambda_3 = 0$. Then $\mu_1 = \mu_2 := \mu$,

$$c - \tilde{c} + \epsilon\lambda_1\lambda_2 = \tilde{\epsilon}\mu^2 \tag{17}$$

and

$$c - \tilde{c} = \tilde{\epsilon}\mu\mu_3. \tag{18}$$

Proof. By the Gauss equations for f and $\tilde{f}$, we have

$$c + \epsilon\lambda_i\lambda_j = \tilde{c} + \tilde{\epsilon}\mu_i\mu_j, \;\; 1 \leq i \neq j \leq 3. \tag{19}$$

(a) If $\lambda_1 = \lambda_2 := \lambda$, then the preceding equations are

$$c + \epsilon\lambda^2 = \tilde{c} + \tilde{\epsilon}\mu_1\mu_2, \tag{20}$$

$$c + \epsilon\lambda\lambda_3 = \tilde{c} + \tilde{\epsilon}\mu_1\mu_3 \tag{21}$$

and

$$c + \epsilon\lambda\lambda_3 = \tilde{c} + \tilde{\epsilon}\mu_2\mu_3. \tag{22}$$

The two last equations yield

$$\mu_3(\mu_1 - \mu_2) = 0,$$

hence either $\mu_3 = 0$ or $\mu_1 = \mu_2$. In view of (20), the second possibility can not occur if either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$. Thus, in these cases we must have $\mu_3 = 0$, and then $c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\mu_1\mu_2$ and $c - \tilde{c} + \epsilon\lambda\lambda_3 = 0$ by (21) and (22).

Otherwise, either the same conclusion holds or $\mu_1 = \mu_2 := \mu$, and then $c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\mu^2$ and $c - \tilde{c} + \epsilon\lambda\lambda_3 = \tilde{\epsilon}\mu\mu_3$ by (21) and (22).

(b) If $\lambda_3 = 0$, then equations (19) become

$$c - \tilde{c} + \epsilon\lambda_1\lambda_2 = \tilde{\epsilon}\mu_1\mu_2, \tag{23}$$

$$c - \tilde{c} = \tilde{\epsilon}\mu_1\mu_3 \tag{24}$$

and

$$c - \tilde{c} = \tilde{\epsilon}\mu_2\mu_3 \tag{25}$$

Since $\mu_3 \neq 0$ by (24) or (25), these equations imply that $\mu_1 = \mu_2 := \mu$, and we obtain (18). Equation (17) then follows from (23). □

Proof of Theorem 4: Assume that f has a principal curvature of multiplicity two, say, $\lambda_1 = \lambda_2 := \lambda$. Suppose first that either $c > \tilde{c}$, $s = 0$ and $\tilde{s} = 1$, or $c < \tilde{c}$, $s = 1$ and $\tilde{s} = 0$. Then, it follows from Lemma 13 that

$$c - \tilde{c} + \epsilon\lambda\lambda_3 = 0, \;\; \mu_3 = 0 \;\text{ and }\; c - \tilde{c} + \epsilon\lambda^2 = \tilde{\epsilon}\mu_1\mu_2. \tag{26}$$

In particular, we must have $\lambda \neq 0$ by the first of the preceding equations, whereas the last one implies that $\mu_1\mu_2 \neq 0$. Then, it is well known that

E_λ is a spherical distribution, that is, it is umbilical and its mean curvature normal $\eta = \nu e_3$ satisfies $e_1(\nu) = 0 = e_2(\nu)$. In particular, a leaf σ of E_λ has constant sectional curvature $\nu^2 + \epsilon\lambda^2 + c = \nu^2 + \tilde{\epsilon}\mu_1\mu_2 + \tilde{c}$. Denoting by ∇ and $\tilde{\nabla}$ the connections on M^3 and $\tilde{f}^*T\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$, respectively, we have

$$\tilde{\nabla}_{e_i}\tilde{f}_*e_3 = \tilde{f}_*\nabla_{e_i}e_3 = -\nu\tilde{f}_*e_i, \quad 1 \leq i \leq 2,$$

hence $\tilde{f}(\sigma)$ is contained in an umbilical hypersurface $\mathbb{Q}^3_{\tilde{s}}(\bar{c})$ of $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ with constant curvature $\bar{c} = \tilde{c} + \nu^2$ and $\tilde{f}_*e_3$ as a unit normal vector field.

Moreover, $E_\lambda^\perp = E_{\mu_3}$ is the relative nullity distribution of $\tilde{f}$. Thus, it is totally geodesic, and in fact its integral curves are mapped by $\tilde{f}$ into geodesics of $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$. It follows that $\tilde{f}(M^3)$ is contained in a generalized cone over $\tilde{f}(\sigma)$.

On the other hand, it is not hard to extend the proof of Theorem 4.2 in [4] to the case of Lorentzian ambient space forms, and conclude that f is a rotation hypersurface in $\mathbb{Q}^4_s(c)$. This means that there exist subspaces $P^2 \subset P^3 = P^3_{s+\epsilon_0}$ in $\mathbb{R}^5_{s+\epsilon_0} \supset \mathbb{Q}^4_s(c)$ with $P^3 \cap \mathbb{Q}^4_s(c) \neq \emptyset$, where $\epsilon_0 = 0$ or $\epsilon_0 = 1$, corresponding to $c > 0$ or $c < 0$, respectively, and a regular curve γ in $\mathbb{Q}^2_s(c) = P^3 \cap \mathbb{Q}^4_s(c)$ that does not meet P^2, such that $f(M^2)$ is the union of the orbits of points of γ under the action of the subgroup of orthogonal transformations of $\mathbb{R}^5_{s+\epsilon_0}$ that fix pointwise P^2. If P^2 is nondegenerate, then f can be parameterized by

$$f(s,u) = (\gamma_1(s)\phi_1(u), \gamma_1(s)\phi_2(u), \gamma_1(s)\phi_3(u), \gamma_4(s), \gamma_5(s)),$$

with respect to an orthonormal basis $\{e_1, \ldots, e_5\}$ of $\mathbb{R}^5_{s+\epsilon_0}$ satisfying the conditions in either (i) or (ii) below, according to whether the induced metric on P^2 has index $s + \epsilon_0$ or $s + \epsilon_0 - 1$, respectively:

(i) $\langle e_i, e_i\rangle = 1$ for $1 \leq i \leq 3$, $\langle e_{3+j}, e_{3+j}\rangle = \epsilon_j$ for $1 \leq j \leq 2$, and (ϵ_1, ϵ_2) equal to either $(1,1)$, $(1,-1)$ or $(-1,-1)$, corresponding to $s + \epsilon_0 = 0$, 1 or 2, respectively.

(ii) $\langle e_1, e_1\rangle = -1$, $\langle e_i, e_i\rangle = 1$ for $2 \leq i \leq 4$ and $\langle e_5, e_5\rangle = \bar{\epsilon}$, where $\bar{\epsilon} = 1$ or $\bar{\epsilon} = -1$, corresponding to $s + \epsilon_0 = 1$ or 2, respectively.

In both cases, we have $P^2 = \text{span}\{e_4, e_5\}$, $P^3 = \text{span}\{e_1, e_4, e_5\}$, $u = (u_1, u_2)$, $\gamma(s) = (\gamma_1(s), \gamma_4(s), \gamma_5(s))$ a unit–speed curve in $\mathbb{Q}^2_s(c) \subset P^3$ and $\phi(u) = (\phi_1(u), \phi_2(u), \phi_3(u))$ an orthogonal parameterization of the unit sphere $\mathbb{S}^2 \subset (P^2)^\perp$ in case (i) and of the hyperbolic plane $\mathbb{H}^2 \subset (P^2)^\perp$ in case (ii). Accordingly, the hypersurface is said to be of spherical or hyperbolic type.

If P^2 is degenerate, then f is a rotation hypersurface of parabolic type parameterized by

$$f(s,u) = (\gamma_1(s), \gamma_1(s)u_1, \gamma_1(s)u_2, \gamma_4(s) - \frac{1}{2}\gamma_1(s)(u_1^2 + u_2^2), \gamma_5(s)),$$

with respect to a pseudo–orthonormal basis $\{e_1, \ldots, e_5\}$ of $\mathbb{R}^5_{s+\epsilon_0}$ such that $\langle e_1, e_1\rangle = 0 = \langle e_4, e_4\rangle$, $\langle e_1, e_4\rangle = 1$, $\langle e_2, e_2\rangle = 1 = \langle e_3, e_3\rangle$ and $\langle e_5, e_5\rangle = -2(s+\epsilon_0) + 3$, where $\gamma(s) = (\gamma_1(s), \gamma_4(s), \gamma_5(s))$ is a unit–speed curve in $\mathbb{Q}^2_s(c) \subset P^3 = \mathrm{span}\{e_1, e_4, e_5\}$.

In each case, one can compute the principal curvatures of f as in [4] and check that the first equation in (26) is satisfied if and only if $\gamma_1'' + \tilde{c}\gamma_1 = 0$, that is, γ is a $\tilde{c}$–helix in $\mathbb{Q}^2_s(c) \subset \mathbb{R}^3_{s+\epsilon_0}$.

Under the remaining possibilities for $c, \tilde{c}, s$ and $\tilde{s}$, either the same conclusions hold or the bilinear form β_x in the proof of Theorem 1 is everywhere degenerate, in which case there exist normal vector fields $\zeta \in \Gamma(N_{\hat{f}}M)$ and $\tilde{N} \in \Gamma(N_{\tilde{f}}M)$ satisfying $\langle \zeta, \zeta\rangle = \tilde{\epsilon} = \langle \tilde{N}, \tilde{N}\rangle$ and $A^{\hat{f}}_{\zeta} = A^{\tilde{f}}_{\tilde{N}}$, and we obtain as before that f and $\tilde{f}$ are locally given on an open dense subset as described in Theorem 3.

Finally, if one of the principal curvatures of f is zero, then the preceding argument applies with the roles of f and $\tilde{f}$ interchanged. □

3 Proof of Theorem 6

Proof of Theorem 6: Let (v, V) be the pair associated to f. Define

$$\tilde{V}_j = (-1)^{j+1}\delta_j(v_iV_k - v_kV_i), \;\; 1 \le i \ne j \ne k \le 3, \;\; i < k. \tag{27}$$

Then $\tilde{V} = (\tilde{V}_1, \tilde{V}_2, \tilde{V}_3)$ is the unique vector in $\mathbb{R}^3$, up to sign, such that $(v, |C|^{-1/2}V, |C|^{-1/2}\tilde{V})$ is an orthonormal basis of $\mathbb{R}^3$ with respect to the inner product

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3)\rangle = \sum_{i=1}^{3} \delta_i x_i y_i. \tag{28}$$

Therefore, the matrix $D = (v, |C|^{-1/2}V, |C|^{-1/2}\tilde{V})$ satisfies $D\delta D^t = \delta$, where $\delta = \mathrm{diag}(\hat{\epsilon}, C/|C|, -\hat{\epsilon}C/|C|)$. It follows that

$$\hat{\epsilon}v_iv_j + C/|C|^2V_iV_j - \hat{\epsilon}C/|C|^2\tilde{V}_i\tilde{V}_j = 0, \;\; 1 \le i \ne j \le 3.$$

Multiplying by ϵC and using that $\hat{\epsilon}\epsilon = \tilde{\epsilon}$ and $\hat{\epsilon}\epsilon C = \hat{\epsilon}\epsilon\tilde{\epsilon}(c - \tilde{c}) = c - \tilde{c}$ we obtain

$$(c - \tilde{c})v_i v_j + \epsilon V_i V_j - \tilde{\epsilon}\tilde{V}_i\tilde{V}_j = 0,$$

or equivalently,

$$cv_i v_j + \epsilon V_i V_j = \tilde{c}v_i v_j + \tilde{\epsilon}\tilde{V}_i\tilde{V}_j.$$

Substituting the preceding equation into (v) yields

$$\frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ji}}{\partial u_j} + h_{ki}h_{kj} + \tilde{\epsilon}\tilde{V}_i\tilde{V}_j + \tilde{c}v_i v_j = 0.$$

On the other hand, differentiating (27) and using equations (i)–(iv) yields

$$\frac{\partial \tilde{V}_j}{\partial u_i} = h_{ij}\tilde{V}_i, \;\; 1 \leq i \neq j \leq 3.$$

It follows from Proposition 5 that there exists a hypersurface $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ whose first and second fundamental forms are

$$I = \sum_{i=1}^{3} v_i^2 du_i^2 \;\; \text{and} \;\; II = \sum_{i=1}^{3} \tilde{V}_i v_i du_i^2,$$

thus M^3 admits an isometric immersion into $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$.

Conversely, let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a hypersurface for which there exists an isometric immersion $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$. By Lemma 12, there exists an orthonormal frame $\{e_1, e_2, e_3\}$ of M^3 of principal directions of both f and $\tilde{f}$. Let $\lambda_1, \lambda_2, \lambda_3$ and μ_1, μ_2, μ_3 be the principal curvatures of f and $\tilde{f}$ correspondent to e_1, e_2 and e_3, respectively. Assume that $\lambda_1 < \lambda_2 < \lambda_3$, and that the unit normal vector field to f has been chosen so that $\lambda_1 < 0$. The Gauss equations for f and $\tilde{f}$ yield

$$c + \epsilon\lambda_i\lambda_j = \tilde{c} + \tilde{\epsilon}\mu_i\mu_j, \;\; 1 \leq i \neq j \leq 3.$$

Thus

$$\mu_i\mu_j = C + \hat{\epsilon}\lambda_i\lambda_j, \;\; C = \tilde{\epsilon}(c - \tilde{c}), \;\; 1 \leq i \neq j \leq 3. \tag{29}$$

It follows that

$$\mu_j^2 = \frac{(C + \hat{\epsilon}\lambda_j\lambda_i)(C + \hat{\epsilon}\lambda_j\lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}, \;\; 1 \leq j \neq i \neq k \neq j \leq 3. \tag{30}$$

The Codazzi equations for f and $\tilde{f}$ are, respectively,

$$\begin{aligned} e_i(\lambda_j) &= (\lambda_i - \lambda_j)\langle \nabla_{e_j} e_i, e_j\rangle, \ i \neq j, & (31)\\ (\lambda_j - \lambda_k)\langle \nabla_{e_i} e_j, e_k\rangle &= (\lambda_i - \lambda_k)\langle \nabla_{e_j} e_i, e_k\rangle, \ \ i \neq j \neq k. & (32)\end{aligned}$$

and

$$\begin{aligned} e_i(\mu_j) &= (\mu_i - \mu_j)\langle \nabla_{e_j} e_i, e_j\rangle, \ i \neq j, & (33)\\ (\mu_j - \mu_k)\langle \nabla_{e_i} e_j, e_k\rangle &= (\mu_i - \mu_k)\langle \nabla_{e_j} e_i, e_k\rangle, \ \ i \neq j \neq k. & (34)\end{aligned}$$

Multiplying (34) by μ_j and using (30) and (32) we obtain

$$\hat{\epsilon} C \frac{(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}\langle \nabla_{e_i} e_j, e_k\rangle = 0, \ i \neq j \neq k.$$

Since the principal curvatures λ_1, λ_2 and λ_3 are distinct, it follows that

$$\langle \nabla_{e_i} e_j, e_k\rangle = 0, \ \ 1 \leq i \neq j \neq k \neq i \leq 3. \tag{35}$$

Computing $2\mu_j e_i(\mu_j)$, first by differentiating (30) and then by multiplying (33) by $2\mu_j$, and using (31), (29) and (30), we obtain

$$\begin{aligned} (C + \hat{\epsilon}\lambda_j\lambda_k)(\lambda_k - \lambda_j)e_i(\lambda_i) + (C + \hat{\epsilon}\lambda_i\lambda_k)(\lambda_k - \lambda_i)e_i(\lambda_j) \\ +(C + \hat{\epsilon}\lambda_i\lambda_j)(\lambda_i - \lambda_j)e_i(\lambda_k) = 0. \end{aligned} \tag{36}$$

Now let $\{\omega_1, \omega_2, \omega_3\}$ be the dual frame of $\{e_1, e_2, e_3\}$, and define the one-forms γ_j, $1 \leq j \leq 3$, by

$$\gamma_j = \sqrt{\delta_j \frac{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}}\omega_j, \ \ 1 \leq j \neq i \neq k \neq j \leq 3,$$

where $\delta_j = y_j/|y_j|$ for $y_j = \frac{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}{C + \hat{\epsilon}\lambda_i\lambda_k}$.

By (30), either all the three numbers $C + \hat{\epsilon}\lambda_j\lambda_i$, $C + \hat{\epsilon}\lambda_j\lambda_k$ and $C + \hat{\epsilon}\lambda_i\lambda_k$ are positive or two of them are negative and the remaining one is positive. Hence there are four possible cases:

(I) $C + \hat{\epsilon}\lambda_i\lambda_j > 0$, $1 \leq i \neq j \leq 3$.

(II) $C + \hat{\epsilon}\lambda_1\lambda_2 < 0$, $C + \hat{\epsilon}\lambda_1\lambda_3 < 0$ and $C + \hat{\epsilon}\lambda_2\lambda_3 > 0$.

(III) $C + \hat{\epsilon}\lambda_1\lambda_2 > 0$, $C + \hat{\epsilon}\lambda_1\lambda_3 < 0$ and $C + \hat{\epsilon}\lambda_2\lambda_3 < 0$.

(IV) $C + \hat{\epsilon}\lambda_1\lambda_2 < 0$, $C + \hat{\epsilon}\lambda_1\lambda_3 > 0$ and $C + \hat{\epsilon}\lambda_2\lambda_3 < 0$.

Notice that $(\delta_1, \delta_2, \delta_3)$ equals $(1, -1, 1)$ in case (I), $(1, 1, -1)$ in case (II), $(-1, 1, 1)$ in case (III) and $(-1, -1, -1)$ in case (IV). It is easily checked that one must have $\hat{\epsilon} = -1$ and $C < 0$ in case (IV), whereas in the remaining cases either $\hat{\epsilon} = 1$ or $\hat{\epsilon} = -1$ and $C > 0$. Therefore, $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$ if $\hat{\epsilon} = -1$ and $C < 0$, and in the remaining cases we may assume that $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ after possibly reordering the coordinates.

We claim that (36) are precisely the conditions for the one-forms γ_j, $1 \leq j \leq 3$, to be closed. To prove this, set $x_j = \sqrt{\delta_j y_j}$, $1 \leq j \leq 3$, so that $\gamma_j = x_j \omega_j$. It follows from (35) that

$$d\gamma_j(e_i, e_k) = e_i\gamma_j(e_k) - e_k\gamma_j(e_i) - \gamma_j([e_i, e_k]) = 0.$$

On the other hand, using (31) we obtain

$$\begin{aligned} d\gamma_j(e_i, e_j) &= e_i\gamma_j(e_j) - e_j\gamma_j(e_i) - \gamma_j([e_i, e_j]) \\ &= e_i(x_j) + x_j\langle \nabla_{e_j} e_i, e_j\rangle \\ &= e_i(x_j) + x_j \frac{e_i(\lambda_j)}{\lambda_i - \lambda_j}, \end{aligned}$$

hence closedness of γ_j is equivalent to

$$e_i(x_j) = \frac{x_j}{\lambda_j - \lambda_i} e_i(\lambda_j), \quad 1 \leq i \neq j \leq 3. \tag{37}$$

We have

$$e_i(x_j) = e_i((\delta_j y_j)^{1/2}) = \frac{1}{2}(\delta_j y_j)^{-1/2}\delta_j e_i(y_j) = \frac{\delta_j}{2x_j} e_i(y_j),$$

thus (37) is equivalent to

$$\frac{2x_j^2}{\lambda_j - \lambda_i} e_i(\lambda_j) = \delta_j e_i(y_j),$$

or yet, to

$$2(\lambda_j - \lambda_k)e_i(\lambda_j) = e_i(y_j)(C + \hat{\epsilon}\lambda_i\lambda_k).$$

The preceding equation is in turn equivalent to

$$\begin{aligned}2(\lambda_j-\lambda_k)(C+\hat{\epsilon}\lambda_i\lambda_k)e_i(\lambda_j) &= (e_i(\lambda_j)-e_i(\lambda_i)(\lambda_j-\lambda_k)(C+\hat{\epsilon}\lambda_i\lambda_k)\\ &+(\lambda_j-\lambda_i)(e_i(\lambda_j)-e_i(\lambda_k))(C+\hat{\epsilon}\lambda_i\lambda_k)\\ &-(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)(\hat{\epsilon}(e_i(\lambda_i)\lambda_k+\lambda_i e_i(\lambda_k)),\end{aligned}$$

which is the same as (36).

Therefore, each point $x \in M^3$ has an open neigborhood V where one can find functions $u_j \in C^\infty(V)$, $1 \leq j \leq 3$, such that $du_j = \gamma_j$, and we can choose V so small that $\Phi = (u_1, u_2, u_3)$ is a diffeomorphism of V onto an open subset $U \subset \mathbb{R}^3$, that is, (u_1, u_2, u_3) are local coordinates on V. From $\delta_{ij} = du_j(\partial/\partial u_i) = x_j\omega_j(\partial/\partial u_i)$ it follows that $\partial/\partial u_i = v_i e_i$, with $v_i = x_i^{-1}$.

Now notice that

$$\sum_{j=1}^{3}\delta_j v_j^2 = \sum_{i,k\neq j=1}^{3}\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)} = \hat{\epsilon},$$

$$\sum_{j=1}^{3}\delta_j v_j V_j = \sum_{j=1}^{3}\delta_j\lambda_j v_j^2 = \sum_{i,k\neq j=1}^{3}\lambda_j\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)} = 0$$

and

$$\sum_{j=1}^{3}\delta_j V_j^2 = \sum_{j=1}^{3}\delta_j\lambda_j^2 v_j^2 = \sum_{i,k\neq j=1}^{3}\lambda_j^2\frac{C+\hat{\epsilon}\lambda_i\lambda_k}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)} = C.$$

It follows that the pair (v, V) satisfies (3). □

4 Proof of Theorem 7

Before starting the proof of Theorem 7, recall that the *Weyl tensor* of a Riemannian manifold M^n is defined by

$$\begin{aligned}\langle C(X,Y)Z,W\rangle &= \langle R(X,Y)Z,W\rangle - L(X,W)\langle Y,Z\rangle - L(Y,Z)\langle X,W\rangle\\ &+L(X,Z)\langle Y,W\rangle + L(Y,W)\langle X,Z\rangle\end{aligned}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, where L is the *Schouten tensor* of M^n, which is given in terms of the Ricci tensor and the scalar curvature s by

$$L(X,Y) = \frac{1}{n-2}(\mathrm{Ric}\ (X,Y) - \frac{1}{2}ns\langle X,Y\rangle).$$

It is well-known that, if $n \geq 4$, then the vanishing of the Weyl tensor is a necessary and sufficient condition for M^n to be conformally flat.

Proof of Theorem 7: Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ be a conformally flat hypersurface of dimension $n \geq 4$. For a fixed point $x \in M^n$, choose a unit normal vector $N \in N_x^f M$ and let $A = A_N\colon T_xM \to T_xM$ be the shape operator of f with respect to N. Let W^3 be a vector space endowed with the Lorentzian inner product $\langle\!\langle\, , \rangle\!\rangle$ given by

$$\langle\!\langle (a,b,c),(a',b',c')\rangle\!\rangle = \epsilon(-aa' + bb' + \epsilon cc').$$

Define a bilinear form $\beta\colon T_xM \times T_xM \to W^3$ by

$$\beta(X,Y) = (L(X,Y) + \frac{1}{2}(1-c)\langle X,Y\rangle, L(X,Y) - \frac{1}{2}(1+c)\langle X,Y\rangle, \langle AX,Y\rangle).$$

Note that $\beta(X,X) \neq 0$ for all $X \neq 0$. Moreover,

$$\begin{aligned}\langle\!\langle \beta(X,Y),\beta(Z,W)\rangle\!\rangle - \langle\!\langle \beta(X,W),\beta(Z,Y)\rangle\!\rangle &= -L(X,Y)\langle Z,W\rangle \\ -L(Z,W)\langle X,Y\rangle + L(X,W)\langle Z,Y\rangle &+ L(Z,Y)\langle X,W\rangle + c\langle (X\wedge Z)W,Y\rangle \\ +\epsilon\langle (AX\wedge AZ)W,Y\rangle &= \langle C(X,Z)W,Y\rangle = 0.\end{aligned}$$

Thus β is flat with respect to $\langle\!\langle\, , \rangle\!\rangle$. We claim that $S(\beta)$ must be degenerate. Otherwise, we would have

$$0 = \dim\ker\beta \geq n - \dim S(\beta) > 0,$$

a contradiction. Now let $\zeta \in S(\beta)\cap S(\beta)^\perp$ and choose a pseudo-orthonormal basis ζ,η,ξ of W^3 with $\langle\!\langle \zeta,\zeta\rangle\!\rangle = 0 = \langle\!\langle \eta,\eta\rangle\!\rangle$, $\langle\!\langle \zeta,\eta\rangle\!\rangle = 1 = \langle\!\langle \xi,\xi\rangle\!\rangle$ and $\langle\!\langle \xi,\zeta\rangle\!\rangle = 0 = \langle\!\langle \xi,\eta\rangle\!\rangle$. Then

$$\beta = \phi\zeta + \psi\xi,$$

where $\phi = \langle\!\langle \beta,\eta\rangle\!\rangle$ and $\psi = \langle\!\langle \beta,\xi\rangle\!\rangle$. Flatness of β implies that $\dim\ker\psi = n-1$. We claim that $\ker\psi$ is an eigenspace of A. Given $Z \in \ker\psi$ we have

$$\beta(Z,X) = \phi(Z,X)\zeta \tag{38}$$

for all $X \in T_xM$. Let $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ be the canonical basis of W and write $\zeta = \sum_{j=1}^3 a_j e_j$. Then (38) gives

$$L(Z,X) + \frac{1}{2}(1-c)\langle Z,X\rangle = a_1\phi(Z,X)$$

and

$$L(Z,X) - \frac{1}{2}(1+c)\langle Z,X\rangle = a_2\phi(Z,X).$$

Subtracting the second of the preceding equations from the first yields

$$\langle Z,X\rangle = (a_1 - a_2)\phi(Z,X),$$

which implies that $a_1 - a_2 \neq 0$ and

$$\phi(Z,X) = \frac{1}{a_1 - a_2}\langle Z,X\rangle.$$

Moreover, we also obtain from (38) that

$$\langle AZ,X\rangle = a_3\phi(Z,X) = \frac{a_3}{a_1 - a_2}\langle Z,X\rangle,$$

which proves our claim. □

5 Proof of Theorem 8

First recall that a necessary and sufficient condition for a three-dimensional Riemannian manifold M^3 to be conformally flat is that its Schouten tensor L be a *Codazzy tensor*, that is,

$$(\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where

$$(\nabla_X L)(Y,Z) = X(L(Y,Z)) - L(\nabla_X Y, Z) - L(Y, \nabla_X Z).$$

Proof of Theorem 8: Let $f\colon\ M^3 \to \mathbb{Q}^4_s(c)$ be a holonomic hypersurface whose associated pair (v,V) satisfies (4). Then $v = (v_1, v_2, v_3)$ is a null vector with respect to the Lorentzian inner product $\langle\ ,\ \rangle$ given by (28), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, and $V = (V_1, V_2, V_3)$ is a unit space-like vector orthogonal to v. Thus, we may write

$$V = \frac{\rho}{v_2}v + \frac{\lambda}{v_2}(-v_3, 0, v_1), \quad \lambda = \pm 1,$$

for some $\rho \in C^\infty(M)$, which is equivalent to

$$V_1 = \frac{1}{v_2}(V_2 v_1 - \lambda v_3) \quad \text{and} \quad V_3 = \frac{1}{v_2}(V_2 v_3 + \lambda v_1). \tag{39}$$

The eigenvalues μ_1, μ_2 and μ_3 of the Schouten tensor L are given by

$$2\mu_j = c + \epsilon(\lambda_i\lambda_j + \lambda_k\lambda_j - \lambda_i\lambda_k), \;\; 1 \leq j \leq 3,$$

where λ_j, $1 \leq j \leq 3$, are the principal curvatures of f. Define

$$\phi_j = v_j(\lambda_i\lambda_j + \lambda_k\lambda_j - \lambda_i\lambda_k), \;\; 1 \leq j \leq 3. \tag{40}$$

That L is a Codazzi tensor is then equivalent to the equations

$$\frac{\partial \phi_j}{\partial u_i} = h_{ij}\phi_i, \;\; 1 \leq i \neq j \leq 3. \tag{41}$$

Replacing $\lambda_j = \frac{V_j}{v_j}$ in (40) and using (39) we obtain

$$\phi_1 = \frac{1}{v_2^2}(-2\lambda V_2 v_3 + (V_2^2 - 1)v_1), \qquad \phi_2 = \frac{1}{v_2}(V_2^2 + 1)$$

and

$$\phi_3 = \frac{1}{v_2^2}((V_2^2 - 1)v_3 + 2\lambda V_2 v_1).$$

It is now a straightforward computation to verify (41) by using equations (i) and (iv) of system (2) together with equations (5) and (6).

Conversely, assume that $f\colon M^3 \to \mathbb{Q}_s^4(c)$ is an isometric immersion with three distinct principal curvatures $\lambda_1 < \lambda_2 < \lambda_3$ of a conformally flat manifold. Let $\{e_1, e_2, e_3\}$ be a correspondent orthonormal frame of principal directions. Then $\{e_1, e_2, e_3\}$ also diagonalyzes the Schouten tensor L, and the correspondent eigenvalues are

$$2\mu_j = \epsilon(\lambda_i\lambda_j + \lambda_j\lambda_k - \lambda_i\lambda_k) + c, \;\; 1 \leq j \leq 3. \tag{42}$$

The Codazzi equations for f and L are, respectively,

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\langle \nabla_{e_j} e_i, e_j\rangle, \quad i \neq j, \tag{43}$$

$$(\lambda_j - \lambda_k)\langle \nabla_{e_i} e_j, e_k\rangle = (\lambda_i - \lambda_k)\langle \nabla_{e_j} e_i, e_k\rangle, \quad i \neq j \neq k. \tag{44}$$

and

$$
\begin{aligned}
e_i(\mu_j) &= (\mu_i - \mu_j)\langle \nabla_{e_j} e_i, e_j\rangle, \quad i \neq j, && (45)\\
(\mu_j - \mu_k)\langle \nabla_{e_i} e_j, e_k\rangle &= (\mu_i - \mu_k)\langle \nabla_{e_j} e_i, e_k\rangle, \quad i \neq j \neq k. && (46)
\end{aligned}
$$

Substituting (42) into (46), and using (44), we obtain

$$(\lambda_i - \lambda_j)(\lambda_j - \lambda_k)\langle \nabla_{e_i} e_j, e_k\rangle = 0, \quad i \neq j \neq k. \tag{47}$$

Since λ_1, λ_2 and λ_3 are pairwise distinct, it follows that

$$\langle \nabla_{e_i} e_j, e_k\rangle = 0, \quad 1 \leq i \neq j \neq k \neq i \leq 3. \tag{48}$$

Differentiating (42) with respect to e_i, we obtain

$$2e_i(\mu_j) = \epsilon[(\lambda_i + \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_j - \lambda_i)e_i(\lambda_k)]. \tag{49}$$

On the other hand, it follows from (31), (45) and (42) that

$$e_i(\mu_j) = \epsilon\lambda_k e_i(\lambda_j). \tag{50}$$

Hence

$$(\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_i - \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_i)e_i(\lambda_k) = 0. \tag{51}$$

Now let $\{\omega_1, \omega_2, \omega_3\}$ be the dual frame of $\{e_1, e_2, e_3\}$, and define the one-forms γ_j, $1 \leq j \leq 3$, by

$$\gamma_j = \sqrt{\delta_j(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}\,\omega_j, \quad 1 \leq j \neq i \neq k \neq j \leq 3, \tag{52}$$

where $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$. One can check that (51) are precisely the conditions for the one-forms γ_j, $1 \leq j \leq 3$, to be closed.

Therefore, each point $x \in M^3$ has an open neigborhood V where one can find functions $u_j \in C^\infty(V)$, $1 \leq j \leq 3$, such that $du_j = \gamma_j$, and we can choose V so small that $\Phi = (u_1, u_2, u_3)$ is a diffeomorphism of V onto an open subset $U \subset \mathbb{R}^3$, that is, (u_1, u_2, u_3) are local coordinates on V. From $\delta_{ij} = du_j(\partial_i) = x_j\omega_j(\partial_i)$ it follows that $\partial_j = v_j e_j$, $1 \leq j \leq 3$, with

$$v_j = \sqrt{\frac{\delta_j}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}}$$

.

Now notice that

$$\sum_{j=1}^{3}\delta_j v_j^2=\sum_{i,k\neq j=1}^{3}\frac{1}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=0,$$

$$\sum_{j=1}^{3}\delta_j v_j V_j=\sum_{j=1}^{3}\delta_j\lambda_j v_j^2=\sum_{i,k\neq j=1}^{3}\frac{\lambda_j}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=0$$

and

$$\sum_{j=1}^{3}\delta_j V_j^2=\sum_{j=1}^{3}\delta_j\lambda_j^2 v_j^2=\sum_{i,k\neq j=1}^{3}\frac{\lambda_j^2}{(\lambda_j-\lambda_i)(\lambda_j-\lambda_k)}=1.$$

It follows that (v,V) satisfies (4). □

6 The Ribaucour transformation

Two immersions $f\colon M^n\to\mathbb{R}^{n+p}_s$ and $f'\colon M^n\to\mathbb{R}^{n+p}_s$ are said to be related by a Ribaucour transformation if $|f-f'|\neq 0$ everywhere and there exist a vector bundle isometry $\mathcal{P}\colon f^*T\mathbb{R}^{n+p}_s\to f'^*T\mathbb{R}^{n+p}_s$, a tensor $D\in\Gamma(T^*M\otimes TM)$, which is symmetric with respect to the induced metrics, and a nowhere vanishing $\delta\in\Gamma(f^*T\mathbb{R}^{n+p}_s)$ such that

(a) $\mathcal{P}(Z)-Z=\langle\delta,Z\rangle(f-f')$ for all $Z\in\Gamma(f^*T\mathbb{R}^{n+p}_s)$;

(b) $\mathcal{P}\circ f_*\circ D=f'_*$.

Given an immersion $f\colon M^n\to\mathbb{Q}^{n+p}_s(c)$, with $c\neq 0$, let $F=i\circ f\colon M^n\to\mathbb{R}^{n+p+1}_{s+\epsilon_0}$, where $\epsilon_0=0$ or 1 corresponding to $c>0$ or $c<0$, respectively, and $i\colon\mathbb{Q}^{n+p}_s(c)\to\mathbb{R}^{n+p+1}_{s+\epsilon_0}$ denotes an umbilical inclusion. An immersion $f'\colon M^n\to\mathbb{Q}^{n+p}_s(c)$ is said to be a Ribaucour transform of f with data $(\mathcal{P},D,\delta)$ if $F'=i\circ f'\colon M^n\to\mathbb{R}^{n+p+1}_{s+\epsilon_0}$ is a Ribaucour transform of F with data $(\hat{\mathcal{P}},D,\hat{\delta})$, where $\hat{\delta}=\delta-cF$ and $\hat{\mathcal{P}}\colon F^*T\mathbb{R}^{n+p+1}_{s+\epsilon_0}\to F'^*T\mathbb{R}^{n+p+1}_{s+\epsilon_0}$ is the extension of $\mathcal{P}$ such that $\hat{\mathcal{P}}(F)=F'$. The next result was proved in [7].

Theorem 14. *Let $f\colon M^n\to\mathbb{Q}^{n+p}_s(c)$ be an isometric immersion of a simply connected Riemannian manifold and let $f'\colon M^n\to\mathbb{Q}^{n+p}_s(c)$ be a Ribaucour transform of f with data $(\mathcal{P},D,\delta)$. Then there exist $\varphi\in C^\infty(M)$ and $\hat{\beta}\in\Gamma(N_fM)$ satisfying*

$$\alpha_f(\nabla\varphi,X)+\nabla^\perp_X\hat{\beta}=0\ \textit{for all}\ X\in TM \tag{53}$$

such that $F' = i \circ f'$ *and* $F = i \circ f$ *are related by*

$$F' = F - 2\nu\varphi\mathcal{G}, \tag{54}$$

where $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ *and* $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$. *Moreover,*

$$\hat{\mathcal{P}} = I - 2\nu\mathcal{G}\mathcal{G}^*, \quad D = I - 2\nu\varphi\Phi \quad \text{and} \quad \hat{\delta} = -\varphi^{-1}\mathcal{G}, \tag{55}$$

where $\Phi = Hess\,\varphi + c\varphi I - A^f_{\hat{\beta}}$. *Conversely, given* $\varphi \in C^\infty(M)$ *and* $\hat{\beta} \in \Gamma(N_fM)$ *satisfying (53) such that* $\varphi\nu \neq 0$ *everywhere, let* $U \subset M^n$ *be an open subset where the tensor* D *given by* (55) *is invertible, and let* $F'\colon U \to \mathbb{R}^{n+p+1}_{s+\epsilon_o}$ *be defined by* (54). *Then* $F' = i \circ f'$, *where* f' *is a Ribaucour transform of* f. *Moreover, the second fundamental forms of* f *and* f' *are related by*

$$\tilde{A}^{f'}_{\mathcal{P}\xi} = D^{-1}(A^f_\xi + 2\nu\langle\hat{\beta}, \xi\rangle\Phi), \ \text{for any } \xi \in \Gamma(N_fM). \tag{56}$$

We now derive from Theorem 14 a Ribaucour transformation for holonomic hypersurfaces, in a form that is slightly different from the one in [6]. For that we need the following.

Proposition 15. *Let* $f\colon M^n \to \mathbb{Q}^{n+1}_s(c)$ *be a holonomic hypersurface with associated pair* (v, V). *Then, the linear system of PDE's*

$$\begin{cases} (i)\dfrac{\partial\varphi}{\partial u_i} = v_i\gamma_i, \quad (ii)\dfrac{\partial\gamma_j}{\partial u_i} = h_{ji}\gamma_i, \qquad i \neq j, \\ (iii)\dfrac{\partial\gamma_i}{\partial u_i} = (v_i - v_i')\psi - \sum_{j\neq i} h_{ji}\gamma_j + \beta V_i - c\varphi v_i, \\ (iv)\epsilon\dfrac{\partial\beta}{\partial u_i} = -V_i\gamma_i, \ \ \epsilon = -2s + 1, \\ (v)\dfrac{\partial\log\psi}{\partial u_i} = -\dfrac{\gamma_i v_i'}{\varphi}, \quad (vi)\dfrac{\partial v_i'}{\partial u_j} = h'_{ji}v'_j, \qquad i \neq j, \end{cases} \tag{57}$$

with h_{ij} *and* h'_{ij} *given by (8), is completely integrable and has the first integral*

$$\sum_i {\gamma_i}^2 + \epsilon\beta^2 + c\varphi^2 - 2\varphi\psi = K \in \mathbb{R}. \tag{58}$$

Proof. A straightforward computation. □

Theorem 16. *Let $f\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ be a holonomic hypersurface with associated pair (v, V). If $f'\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ is a Ribaucour transform of f, then there exists a solution $(\gamma, v', \varphi, \psi, \beta)$ of (57) satisfying*

$$\sum_i {\gamma_i}^2 + \epsilon\beta^2 + c\varphi^2 - 2\varphi\psi = 0 \tag{59}$$

such that $F' = i \circ f'$ and $F = i \circ f$ are related by

$$F' = F - \frac{1}{\psi}\left(\sum_i \gamma_i F_* e_i + \beta i_* \xi + c\varphi F\right), \tag{60}$$

where ξ is a unit normal vector field to f and $e_i = v_i^{-1}\partial_i$, $1 \le i \le n$.

Conversely, given a solution $(\gamma, v', \varphi, \psi, \beta)$ of (57) satisfying (59) on an open subset $U \subset M^n$ where v_i' is positive for $1 \le i \le n$, then F' defined by (60) is an immersion such that $F' = i \circ f'$, where f' is a Ribaucour transform of f whose associated pair is (v', V'), with

$$V_i' = V_i + (v_i - v_i')\frac{\epsilon\beta}{\varphi}, \;\; 1 \le i \le n. \tag{61}$$

Proof. Let $f'\colon M^n \to \mathbb{Q}_s^{n+1}(c)$ be a Ribaucour transform of f. By Theorem 14, there exist $\varphi \in C^\infty(M)$ and $\hat{\beta} \in \Gamma(N_f M)$ satisfying (53) such that F' is given by (54), where $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ and $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$.

Write $\nabla\varphi = \sum_{i=1}^n \gamma_i e_i$, where $\gamma_i \in C^\infty(M)$, $1 \le i \le n$. Since $\partial_i = v_i e_i$, $1 \le i \le n$, this is equivalent to equation (i) of system (57). Now write $\hat{\beta} = \beta\xi$, where $\beta \in C^\infty(M)$. Then (53) can be written as

$$A\nabla\varphi = -\epsilon\nabla\beta, \tag{62}$$

which is equivalent, by taking inner products of both sides with ∂_i, to equation (iv) of system (57). On the other hand, equation (53) implies that

$$\mathcal{G}_* = F_*\Phi,$$

where $\Phi = \operatorname{Hess}\varphi + c\varphi I - A^f_{\hat{\beta}}$. Therefore Φ is a Codazzi tensor that satisfies

$$\alpha_f(\Phi X, Y) = \alpha_f(X, \Phi Y)$$

for all $X, Y \in TM$, that is, Φ has $\{e_1, \ldots, e_n\}$ as a diagonalyzing frame. Since

$$\Phi\partial_i = \left(\frac{\partial\gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + cv_i\varphi\right) e_i + \sum_{j\neq i}\left(\frac{\partial\gamma_j}{\partial u_i} - h_{ji}\gamma_i\right) e_j, \qquad (63)$$

equation (ii) of system (57) follows.

Now define $\psi \in C^\infty(M)$ by

$$2\varphi\psi = \langle \mathcal{G}, \mathcal{G}\rangle = \sum_i {\gamma_i}^2 + \epsilon\beta^2 + c\varphi^2.$$

Differentiating both sides with respect to u_i and using equations (i), (ii) and (iv) of (57) yields

$$\frac{\partial\gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + cv_i\varphi = v_i\psi + \frac{\varphi}{\gamma_i}\frac{\partial\psi}{\partial u_i}. \qquad (64)$$

Defining v_i' by (v), then (iii) follows from (64).

Finally, from $\dfrac{\partial^2\gamma_i}{\partial u_i\partial u_j} = \dfrac{\partial^2\gamma_i}{\partial u_j\partial u_i}$ we obtain

$$\frac{\partial}{\partial u_i}(h_{ij}\gamma_j) = \frac{\partial}{\partial u_j}\left((v_i - v_i')\psi - \sum_{k\neq i} h_{ki}\gamma_k + \beta V_i - c\varphi v_i\right),$$

thus

$$\begin{aligned}\frac{\partial h_{ij}}{\partial u_i}\gamma_j + h_{ij}\frac{\partial\gamma_j}{\partial u_i} = &\left(\frac{\partial v_i}{\partial u_j} - \frac{\partial v_i'}{\partial u_j}\right)\psi + (v_i - v_i')\frac{\partial\psi}{\partial u_j} - \frac{\partial h_{ji}}{\partial u_j}\gamma_j - h_{ji}\frac{\partial\gamma_j}{\partial u_j}\\ &-\frac{\partial h_{ki}}{\partial u_j}\gamma_k - h_{ki}\frac{\partial\gamma_k}{\partial u_j} + \frac{\partial\beta}{\partial u_j}V_i + \beta\frac{\partial V_i}{\partial u_j} - c\frac{\partial\varphi}{\partial u_j}v_i - c\varphi\frac{\partial v_i}{\partial u_j}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\frac{\partial h_{ij}}{\partial u_i} + \frac{\partial h_{ji}}{\partial u_j}\right)\gamma_j + h_{ij}h_{ji}\gamma_i = \psi h_{ji}v_j - \frac{\partial v_i'}{\partial u_j}\psi - (v_i - v_i')\frac{\gamma_j\psi}{\varphi}v_j'\\ &-h_{ji}(v_j - v_j')\psi + h_{ji}h_{ij}\gamma_i + h_{ji}h_{kj}\gamma_k - \beta h_{ji}V_j + c\varphi h_{ji}v_j - h_{kj}h_{ji}\gamma_k\\ &-h_{ki}h_{kj}\gamma_j - V_iV_j\gamma_j + \beta h_{ji}V_j - cv_iv_j\gamma_j - c\varphi h_{ji}v_j,\end{aligned}$$

which yields equation (vi) of (57).

Conversely, let F' be given by (60) in terms of a solution $(\gamma, v', \varphi, \psi, \beta)$ of (57) satisfying (59) on an open subset $U \subset M^n$ where v_i' is nowhere vanishing for $1 \leq i \leq n$. We have $\nabla\varphi = \sum_{i=1}^n \gamma_i e_i$ by equation (i) of (57). Defining $\hat{\beta} \in \Gamma(N_f M)$ by $\hat{\beta} = \beta\xi$, we can write F' as in (54), with $\mathcal{G} = F_*\nabla\varphi + i_*\hat{\beta} + c\varphi F$ and $\nu = \langle \mathcal{G}, \mathcal{G}\rangle^{-1}$. In view of (iv), equation (62) is satisfied, and hence so is (53). Thus $\mathcal{G}_* = F_* \circ \Phi$, where $\Phi = \operatorname{Hess}\varphi + c\varphi I - A^f_{\hat{\beta}}$.

It follows from (ii) and (63) that $\Phi\partial_i = B_i\partial_i$, where

$$B_i = v_i^{-1}\left(\frac{\partial\gamma_i}{\partial u_i} + \sum_{j\neq i} h_{ji}\gamma_j - \beta V_i + c v_i \varphi\right) = v_i^{-1}(v_i - v_i')\psi.$$

Using (iii) and (59) we obtain

$$D\partial_i = (1 - 2\nu\varphi B_i)\partial_i = (1 - 2\nu\varphi v_i^{-1}(v_i - v_i')\psi) = \frac{v_i'}{v_i}\partial_i.$$

Thus D is invertible wherever v_i' does not vanish for $1 \leq i \leq n$. It follows from Theorem 14 that the map F' defined by (60) is an immersion on U and that $F' = i \circ f'$, where f' is a Ribaucour transform of f. Moreover, we obtain from (56) that F', and hence f', is holonomic with $u_1, \ldots, u_n$ as principal coordinates. It also follows from (56) that

$$\frac{V_i'}{v_i'}\partial_i = A^{f'}\partial_i = \frac{v_i}{v_i'}\left(\frac{V_i}{v_i} + \frac{\epsilon\beta}{\varphi}\frac{v_i - v_i'}{v_i}\right)\partial_i,$$

which yields (61). □

6.1 The Ribaucour transformation for solutions of Problem $*$ and for conformally flat hypersurfaces.

We now specialize the Ribaucour transformation to the classes of hypersurfaces $f\colon M^3 \to \mathbb{Q}^4_s(c)$ that are either conformally flat or admit an isometric immersion into $\mathbb{Q}^4_{\tilde{s}}(\tilde{c})$ with $\tilde{c} \neq c$.

Proposition 17. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a holonomic hypersurface whose associated pair (v, V) satisfies (3) (respectively, (4)). Then, the linear system of PDE's obtained by adding the equation*

$$\delta_i \frac{\partial v_i'}{\partial u_i} + \delta_j h_{ij}' v_j' + \delta_k h_{ik}' v_k' = 0 \tag{65}$$

to system (57), where h'_{ij} is given by (8), is completely integrable and has (besides (58)) the first integral

$$\delta_1 v_1'^2 + \delta_2 v_2'^2 + \delta_3 v_3'^2 = K \in \mathbb{R}. \tag{66}$$

Moreover, the function

$$\Omega = \varphi \sum_{j=1}^{3} \delta_j v_j' V_j - \epsilon\beta \left(K - \sum_{j=1}^{3} \delta_j v_j v_j' \right) \tag{67}$$

satisfies

$$\frac{\partial \Omega}{\partial u_i} = \frac{\gamma_i}{\varphi}(v_i + v_i')\Omega. \tag{68}$$

In particular, if initial conditions for φ and β at $x_0 \in M^3$ are chosen so that Ω vanishes at x_0, then Ω vanishes everywhere.

Proof. The first two assertions follow from straightforward computations. To prove the last one, define $\rho = \sum_{i=1}^{3} \delta_i v_i' V_i$ and $\Theta = K - \sum_{i=1}^{3} \delta_i v_i' v_i$. We have

$$\begin{aligned}
\frac{\partial \rho}{\partial u_i} &= \delta_i \frac{\partial v_i'}{\partial u_i} V_i + \delta_i v_i' \frac{\partial V_i}{\partial u_i} + \textstyle\sum_{j \neq i} \delta_j \frac{\partial v_j'}{\partial u_i} V_j + \sum_{j \neq i} \delta_j v_j' \frac{\partial V_j}{\partial u_i} \\
&= \textstyle\sum_{j \neq i} \delta_j (h_{ij} - h'_{ij}) v_j' V_i - \sum_{j \neq i} \delta_j (h_{ij} - h'_{ij}) V_j v_i' \\
&= \textstyle\sum_{j \neq i} \delta_j (v_j' - v_j) V_j \frac{\gamma_i v_i'}{\varphi} - \sum_{j \neq i} \delta_j (v_j' - v_j) v_j' \frac{\gamma_i V_i}{\varphi} \\
&= \frac{v_i' \gamma_i}{\varphi} \left(\rho - \delta_i v_i' V_i + \delta_i v_i V_i \right) - \frac{V_i \gamma_i}{\varphi} \left(\Theta - \delta_i v_i'^2 + \delta_i v_i v_i' \right) \\
&= \frac{\gamma_i}{\varphi} (v_i' \rho - \Theta V_i)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial \Theta}{\partial u_i} &= -\delta_i \frac{\partial v_i}{\partial u_i} v_i' - \delta_i v_i \frac{\partial v_i'}{\partial u_i} - \sum_{j \neq i} \delta_j \frac{\partial v_j}{\partial u_i} v_j' - \sum_{j \neq i} \delta_j v_j \frac{\partial v_j'}{\partial u_i} \\
&= \left(\sum_{j \neq i} \delta_j v_j (h_{ij} - h_{ij}') \right) v_i' + \left(\sum_{j \neq i} \delta_j (h_{ij}' - h_{ij}) v_j' \right) v_i \\
&= \left(\sum_{j \neq i} \delta_j v_j (v_j - v_j') \right) \frac{\gamma_i v_i'}{\varphi} + \left(\sum_{j \neq i} \delta_j (v_j' - v_j) v_j' \right) \frac{v_i \gamma_i}{\varphi} \\
&= (\Theta - \delta_i v_i^2 + \delta_i v_i v_i')) \frac{\gamma_i v_i'}{\varphi} + (\Theta - \delta_i v_i'^2 + \delta_i v_i v_i')) \frac{v_i \gamma_i}{\varphi} \\
&= \frac{\gamma_i}{\varphi} (v_i + v_i') \Theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{\partial \Omega}{\partial u_i} &= \frac{\partial \varphi}{\partial u_i} \rho + \varphi \frac{\partial \rho}{\partial u_i} - \frac{\partial \epsilon \beta}{\partial u_i} \Theta - \epsilon \beta \frac{\partial \Theta}{\partial u_i} \\
&= v_i \gamma_i \rho + \varphi \frac{\gamma_i}{\varphi} (v_i' \rho - \Theta V_i) + V_i \gamma_i \Theta - \epsilon \beta \frac{\gamma_i}{\varphi} (v_i + v_i') \Theta \\
&= \rho \gamma_i (v_i + v_i') - \frac{\epsilon \beta \gamma_i}{\varphi} (v_i + v_i') \Theta \\
&= \frac{\gamma_i}{\varphi} (v_i + v_i') \Omega,
\end{aligned}
$$

which proves (68). The last assertion follows from (68) and the lemma below.

Lemma 18. *Let M^n be a connected manifold and let $\Omega \in C^\infty(M)$. Assume that there exists a smooth one-form ω on M^n such that $d\Omega = \omega\Omega$. If Ω vanishes at some point of M^n, then it vanishes everywhere.*

Proof. Given any smooth curve $\gamma \colon I \to M^n$ with $0 \in I$, denote $\lambda(s) = \omega(\gamma'(s))$. By the assumption we have

$$
(\Omega \circ \gamma)(t) = (\Omega \circ \gamma)(0) \exp \int_0^t \lambda(s) ds,
$$

and the conclusion follows from the connectedness of M^n. □

The next result contains Theorem 9 in the introduction.

Theorem 19. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a holonomic hypersurface whose associated pair (v, V) satisfies (3) (respectively, (4)) and $f'\colon M^3 \to \mathbb{Q}^4_s(c)$ a Ribaucour transform of f determined by a solution $(\gamma_1, \gamma_2, \gamma_3, v'_1, v'_2, v'_3, \varphi, \psi, \beta)$ of system (57). If the associated pair (v', V') of f' also satisfies (3) (respectively, (4)), then*

$$\Omega := \varphi \sum_{j=1}^{3} \delta_j v'_j V_j - \epsilon\beta \left(K - \sum_{j=1}^{3} \delta_j v_j v'_j \right) = 0, \tag{69}$$

with $K = \hat{\epsilon}$ (respectively, $K = 0$). Conversely, let $(\gamma_1, \gamma_2, \gamma_3, v'_1, v'_2, v'_3, \varphi, \psi, \beta)$ be a solution of the linear system of PDE's obtained by adding equation (65) to system (57). If (59), (69) and

$$\sum_{i=1}^{3} \delta_i {v'_i}^2 = K, \tag{70}$$

where $K = \hat{\epsilon}$ (respectively, $K = 0$), are satisfied at some point of M^3, then (they are satisfied at every point of M^3 and) the pair (v', V') associated to the Ribaucour transform of f determined by such a solution also satisfies (3) (respectively, (4)).

Proof. Let (v', V') be the pair associated to f'. Then, using conditions (3) (respectively, (4)), we obtain

$$\begin{aligned}
\sum_{j=1}^{3} \delta_j {V'_j}^2 - \sum_{j=1}^{3} \delta_j {V_j}^2 &= \sum_{j=1}^{3} \delta_j (V'_j - V_j)(V'_j + V_j) \\
&= \frac{\epsilon\beta}{\varphi} \sum_{j=1}^{3} \delta_j (v_j - v'_j) \left(2V_j + \frac{\epsilon\beta}{\varphi}(v_j - v'_j) \right) \\
&= \frac{\epsilon\beta}{\varphi} \left(2 \sum_{j=1}^{3} \delta_j V_j (v_j - v'_j) + \frac{\epsilon\beta}{\varphi} \sum_{j=1}^{3} \delta_j (v_j - v'_j)^2 \right) \\
&= \frac{\epsilon\beta}{\varphi^2} \left(-2\Omega + \epsilon\beta \left(\sum_{j=1}^{3} \delta_j {v'_j}^2 - K \right) \right),
\end{aligned} \tag{71}$$

where $K = \hat{\epsilon}$ (respectively, $K = 0$). If the pair (v', V') associated to f' satisfies (3) (respectively, (4)), then (70) holds, as well as

$$\sum_{j=1}^{3} \delta_j v'_j V'_j = 0 \tag{72}$$

and

$$\sum_{j=1}^{3} \delta_j {V'_j}^2 = C, \tag{73}$$

where $C = \tilde{\epsilon}(c - \tilde{c})$ (respectively, $C = 1$). It follows from (71) that (69) holds.

Conversely, let $(\gamma_1, \gamma_2, \gamma_3, v'_1, v'_2, v'_3, \varphi, \psi, \beta)$ be a solution of the linear system of PDE's obtained by adding equation (65) to system (57). If (59), (70) and (69) are satisfied at some point of M^n, then they are satisfied at every point of M^n by Proposition 17. Then, equations (70), (69) and (71) imply that (73) holds. On the other hand, using (61) we obtain

$$\begin{aligned}
\sum_{j=1}^{3} \delta_j v'_j V'_j &= \sum_{j=1}^{3} \delta_j v'_j V_j + \frac{\epsilon\beta}{\varphi} \sum_{j=1}^{3} \delta_j v'_j v_j - \frac{\epsilon\beta}{\varphi} \sum_{j=1}^{3} \delta_j {v'_j}^2 \\
&= \sum_{j=1}^{3} \delta_j v'_j V_j - \frac{\epsilon\beta}{\varphi}(K - \sum_{j=1}^{3} \delta_j v'_j v_j) \\
&= \varphi^{-1}\Omega = 0
\end{aligned}$$

by (70) and (69). Thus, the pair (v', V') associated to f' also satisfies (3) (respectively, (4)). □

6.2 Explicit three-dimensional solutions of Problem $*$

We now use Theorem 19 to compute explicit examples of pairs of isometric immersions $f\colon M^3 \to \mathbb{Q}^4_s(c)$ and $\tilde{f}\colon M^3 \to \mathbb{Q}^4_{\tilde{s}}(\tilde{c})$, $c \neq \tilde{c}$, with three distinct principal curvatures.

First notice that, if $c = 0$ (respectively, $c \neq 0$) and (v, h, V) is a solution of system (2) on a simply connected open subset $U \subset \mathbb{R}^3$ with $v_i \neq 0$ everywhere for $1 \leq i \leq 3$, then, in order to determine the corresponding immersion $f\colon U \to \mathbb{R}^4_s$ (respectively, $f\colon U \to \mathbb{Q}^4_s(c) \subset \mathbb{R}^5_{s+\epsilon_0}$, where $\epsilon_0 = c/|c|$), one has to integrate the system of PDE's

$$\begin{cases}
(i)\dfrac{\partial f}{\partial u_i} = v_i X_i, \quad (ii)\dfrac{\partial X_i}{\partial u_j} = h_{ij} X_j, \;\; i \neq j, \\
(iii)\dfrac{\partial X_i}{\partial u_i} = -\sum_{k \neq i} h_{ki} X_k + \epsilon V_i N - c v_i f, \\
(iv)\dfrac{\partial N}{\partial u_i} = -V_i X_i, \;\; 1 \leq i \leq 3,
\end{cases} \tag{74}$$

with initial conditions $X_1(u_0), X_2(u_0), X_3(u_0), N(u_0), f(u_0)$ at some point $u_0 \in U$ chosen so that the set $\{X_1(u_0), X_2(u_0), X_3(u_0), N(u_0)\}$ (respectively, $\{X_1(u_0), X_2(u_0), X_3(u_0), N(u_0), |c|^{1/2} f(u_0)\}$) is an orthonormal basis of $\mathbb{R}^4_s$ (respectively, $\mathbb{R}^5_{s+\epsilon_0}$).

The idea for the construction of explicit examples is to start with trivial solutions (v, h, V) of system (2). If $\hat{\epsilon} = 1$, one can start with the solution (v, h, V) of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (1, 0, 0)$, $h = 0$ and V is either $\sqrt{-C}(0, 1, 0)$ or $\sqrt{C}(0, 0, 1)$, corresponding to $C < 0$ or $C > 0$, respectively. If $\hat{\epsilon} = -1$ and $C > 0$, we may start with the solution (v, h, V) of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (0, 1, 0)$, $h = 0$ and $V = \sqrt{C}(0, 0, 1)$, whereas for $C < 0$ we take $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$, $v = (0, 0, 1)$, $h = 0$ and $V = \sqrt{-C}(1, 0, 0)$. Even though, for the corresponding solution (X_1, X_2, X_3, N, f) of system (74), the map $f\colon U \to \mathbb{Q}^4_s(c)$ is not an immersion, the map $f'\colon U \to \mathbb{Q}^4_s(c)$ obtained by applying Theorem 19 to it does define a hypersurface of $\mathbb{Q}^4_s(c)$, which is therefore a solution of Problem $*$.

In the following, we consider the case in which $\hat{\epsilon} = 1$ and $C < 0$, the others being similar. We take (v, h, V) as the solution of system (2), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$, for which $v = (1, 0, 0)$, $h = 0$ and $V = \sqrt{-C}(0, 1, 0)$.

If $c = 0$, the corresponding solution of system (74) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, \epsilon E_4, 0)$$

is given by

$$f = f(u_1) = u_1 E_1, \quad X_1 = E_1, \quad X_3 = E_3,$$
$$X_2 = \begin{cases} \cosh a u_2 E_2 + \sinh a u_2 E_4, & \text{if } \epsilon = -1, \\ \cos a u_2 E_2 + \sin a u_2 E_4, & \text{if } \epsilon = 1, \end{cases} \tag{75}$$

and

$$N = \begin{cases} -\sinh a u_2 E_2 - \cosh a u_2 E_4, & \text{if } \epsilon = -1, \\ -\sin a u_2 E_2 + \cos a u_2 E_4, & \text{if } \epsilon = 1, \end{cases} \tag{76}$$

where $a = \sqrt{-C}$. If $c \neq 0$, the corresponding solution of system (74) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, E_4, |c|^{-1/2} E_5)$$

is given by

$$f = f(u_1) = \begin{cases} \frac{1}{\sqrt{c}}(\cos \sqrt{c}\, u_1 E_5 + \sin \sqrt{c}\, u_1 E_1), & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}}(\cosh \sqrt{-c}\, u_1 E_5 + \sinh \sqrt{-c}\, u_1 E_1), & \text{if } c < 0, \end{cases} \tag{77}$$

$$X_1 = \begin{cases} -\sin\sqrt{c}\,u_1 E_5 + \cos\sqrt{c}\,u_1 E_1, \text{ if } c > 0, \\ \sinh\sqrt{-c}\,u_1 E_5 + \cosh\sqrt{-c}\,u_1 E_1, \text{ if } c < 0, \end{cases} \tag{78}$$

$X_3 = E_3$ and X_2, N as in (75) and (76), respectively.

We now solve system (7) for (v, h, V) as in the preceding paragraph. Notice that (9) and (10), with $K_1 = 0$ and $K_2 = 1$, reduce, respectively, to

$$2\varphi\psi = \sum_i \gamma_i^2 + \epsilon\beta^2 + c\varphi^2 \tag{79}$$

and

$$v'^2_2 = v'^2_1 + v'^2_3 - 1. \tag{80}$$

We also impose that

$$-a\varphi v_2' = \epsilon\beta(1 - v_1'), \tag{81}$$

which corresponds to the function Ω in (11) vanishing everywhere. It follows from equations (i), (ii) and (iv) of (7) that φ, γ_j and β depend only on u_1, u_j and u_2, respectively. Equation (iii) then implies that there exist smooth functions $\phi_i = \phi_i(u_i)$, $1 \leq i \leq 3$, such that

$$(\delta_{1i} - v_i')\psi = \phi_i. \tag{82}$$

Replacing (82) in (80) gives

$$\psi = \frac{\phi_1^2 - \phi_2^2 + \phi_3^2}{2\phi_1}. \tag{83}$$

Multiplying (81) by ψ and using (82) yields

$$a\varphi\phi_2 = \epsilon\beta\phi_1,$$

hence there exists $K \neq 0$ such that

$$\beta = \frac{\epsilon}{K}\phi_2 \quad \text{and} \quad \varphi = \frac{1}{Ka}\phi_1. \tag{84}$$

It follows from (i) and (iv) that

$$\gamma_1 = \frac{1}{Ka}\phi_1' \quad \text{and} \quad \gamma_2 = -\frac{1}{Ka}\phi_2' \tag{85}$$

where ϕ_i' stands for the derivative of ϕ_i (with respect to u_i). Using (v) for $i = 3$, (82) and the second equation in (84) we obtain that

$$\gamma_3 = \frac{1}{Ka}\phi_3'.$$

Then, it follows from (iii), (82), the first equation in (85) and the second one in (84) that

$$\phi_1'' = (Ka - c)\phi_1. \tag{86}$$

Similarly,

$$\phi_2'' = -(\epsilon a^2 + Ka)\phi_2 \quad \text{and} \quad \phi_3'' = Ka\phi_3. \tag{87}$$

Moreover, by (79) we must have

$$\phi_1'^2 - (Ka - c)\phi_1^2 + \phi_2'^2 + (\epsilon a^2 + Ka)\phi_2^2 + \phi_3'^2 - Ka\phi_3^2 = 0. \tag{88}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (86) and (87).

We compute explicitly the corresponding hypersurface given by (12) when $c = 0$, $\tilde{c} = 1$, $\epsilon = 1 = \tilde{\epsilon}$ and $K = 1$. In this case we have $C = -1$ and $a = 1$, hence equations (86) and (87) yield

$$\begin{cases} \phi_1 = A_{11}\cosh u_1 + A_{12}\sinh u_1, \\ \phi_2 = A_{21}\cos\sqrt{2}\,u_2 + A_{22}\sin\sqrt{2}\,u_2, \\ \phi_3 = A_{31}\cosh u_3 + A_{32}\sinh u_3, \end{cases}$$

where $A_{ij} \in \mathbb{R}$, $1 \leq i, j \leq 3$, satisfy

$$A_{12}^2 - A_{11}^2 + 2(A_{21}^2 + A_{22}^2) + A_{32}^2 - A_{31}^2 = 0,$$

in view of (88). Assuming, say, that

$$A_{12}^2 - A_{11}^2 < 0 \quad \text{and} \quad A_{32}^2 - A_{31}^2 < 0,$$

we may write $A_{11} = \rho_1\cosh\theta_1$, $A_{12} = \rho_1\sinh\theta_1$, $A_{21} = \rho_2\sin\theta_2$, $A_{22} = \rho_2\cos\theta_2$, $A_{31} = \rho_3\cosh\theta_3$ and $A_{32} = \rho_3\sinh\theta_3$ for some $\rho_i > 0$ and $\theta_i \in \mathbb{R}$, $1 \leq i \leq 3$. Then

$$\begin{cases} \phi_1 = \rho_1\cosh(u_1 + \theta_1), \\ \phi_2 = \rho_2\sin(\sqrt{2}\,u_2 + \theta_2), \\ \phi_3 = \rho_3\cosh(u_3 + \theta_3), \end{cases}$$

with

$$2\rho_2^2 = \rho_1^2 + \rho_3^2,$$

and we can assume that $\theta_i = 0$ after a suitable change $u_i \mapsto u_i + u_i^0$ of the coordinates u_i, $1 \leq i \leq 3$. Setting $\rho = \rho_2$, we can write $\rho_1 = \sqrt{2}\rho\cos\theta$ and $\rho_3 = \sqrt{2}\rho\sin\theta$ for some $\theta \in [0, 2\pi]$. Thus

$$\begin{cases} \phi_1 = \sqrt{2}\rho\cos\theta\cosh u_1, \\ \phi_2 = \rho\sin\sqrt{2}\,u_2, \\ \phi_3 = \sqrt{2}\rho\sin\theta\cosh u_3, \end{cases}$$

and the coordinate functions of the corresponding hypersurface $f'\colon U \to \mathbb{R}^4$ are

$$f_1' = u_1 - 2gh\cos\theta\sinh u_1, \quad f_2' = gh(2\cos\sqrt{2}u_2\cos u_2 + \sqrt{2}\sin\sqrt{2}u_2\sin u_2),$$

$$f_3' = -2gh\sin\theta\sinh u_3, \quad f_4' = gh(2\cos\sqrt{2}u_2\sin u_2 - \sqrt{2}\sin\sqrt{2}u_2\phi_2\cos u_2),$$

where

$$g = 2\cos\theta\cosh u_1$$

and

$$h^{-1} = 2\cos^2\theta\cosh^2 u_1 - \sin^2\sqrt{2}u_2 + 2\sin^2\theta\cosh^2 u_3.$$

To determine the immersion $\tilde{f}'\colon U \to \mathbb{S}^4$ that has the same induced metric as f', we start with the solution $(\tilde{v}, \tilde{h}, \tilde{V})$ of system (2), together with equations (5) and (6), with $(\delta_1, \delta_2, \delta_3) = (1, -1, 1)$ and c replaced by $\tilde{c} = 1$, for which $\tilde{v} = v = (1, 0, 0)$, $\tilde{h} = h = 0$ and $\tilde{V} = (0, 0, 1)$.

The corresponding solution $(\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{N}, \tilde{f})$ of system (74), with $\epsilon = \tilde{\epsilon} = 1$, $c = \tilde{c} = 1$ and initial conditions

$$(\tilde{X}_1(0), \tilde{X}_2(0), \tilde{X}_3(0), \tilde{N}(0), \tilde{f}(0)) = (E_1, E_2, E_3, E_4, E_5)$$

is given by

$$\tilde{f} = \tilde{f}(u_1) = \cos u_1 E_5 + \sin u_1 E_1, \tag{89}$$

$$\tilde{X}_1 = -\sin u_1 E_5 + \cos u_1 E_1, \quad \tilde{X}_2 = E_2, \tag{90}$$

$$\tilde{X}_3 = \cos u_3 E_3 + \sin u_3 E_4 \quad \text{and} \quad \tilde{N} = -\sin u_3 E_3 + \cos u_3 E_4. \tag{91}$$

Arguing as before, we solve system (7) together with equations (9) and (10), which now become

$$2\tilde{\varphi}\tilde{\psi} = \sum_i \tilde{\gamma}_i^2 + \tilde{\beta}^2 + \tilde{\varphi}^2 \tag{92}$$

and

$$\tilde{v'}_2^2 = \tilde{v'}_1^2 + \tilde{v'}_3^2 - 1. \tag{93}$$

We also impose that

$$\tilde{\varphi}\tilde{v}_3' = \tilde{\beta}(1 - \tilde{v}_1'), \tag{94}$$

which corresponds to the function Ω in (11) vanishing everywhere. We obtain

$$\tilde{\psi} = \frac{\tilde{\phi}_1^2 - \tilde{\phi}_2^2 + \tilde{\phi}_3^2}{2\tilde{\phi}_1}, \quad (\delta_{i1} - \tilde{v}_i')\tilde{\psi} = \tilde{\phi}_i, \tag{95}$$

$$\tilde{\beta} = \frac{1}{\tilde{K}}\tilde{\phi}_3, \quad \tilde{\varphi} = -\frac{1}{\tilde{K}}\tilde{\phi}_1, \tag{96}$$

$$\tilde{\gamma}_1 = -\frac{1}{\tilde{K}}\tilde{\phi}_1', \quad \tilde{\gamma}_2 = \frac{1}{\tilde{K}}\tilde{\phi}_2' \text{ and } \tilde{\gamma}_3 = -\frac{1}{\tilde{K}}\tilde{\phi}_3' \tag{97}$$

for some $\tilde{K} \in \mathbb{R}$, where the functions $\tilde{\phi}_i = \tilde{\phi}_i(u_i)$ satisfy

$$\tilde{\phi}_1'' = -(1 + \tilde{K})\tilde{\phi}_1, \quad \tilde{\phi}_2'' = \tilde{K}\tilde{\phi}_2 \quad \tilde{\phi}_3'' = -(1 + \tilde{K})\tilde{\phi}_3 \tag{98}$$

and

$$(\tilde{\phi}_1'^2 + (1 + \tilde{K})\tilde{\phi}_1^2) + (\phi_2'^2 - \tilde{K}\tilde{\phi}_2^2) + (\phi_3'^2 + (1 + \tilde{K})\tilde{\phi}_3^2) = 0. \tag{99}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (98). Notice also that, for $\tilde{K} = -2$, the two preceding equations coincide with (86), (87) and (88) for $1 = K = a = \epsilon$ and $c = 0$, hence the metrics induced by f' and $\tilde{f}' \colon U \to \mathbb{S}^4 \subset \mathbb{R}^5$ coincide by (82) and the second equation in (95). The coordinate functions of $\tilde{f}'$ are

$$\begin{array}{rcl} \tilde{f}_1' &=& \sin u_1 + gh(\cos\theta\cos u_1 \sinh u_1 + \cos\theta \sin u_1 \cosh u_1) \\ \tilde{f}_2' &=& -gh\cos\sqrt{2}u_2 \\ \tilde{f}_3' &=& gh(\sin\theta\cos u_3 \sinh u_3 - \sin\theta\sin u_3\cosh u_3) \\ \tilde{f}_4' &=& gh(\sin\theta\sin u_3\sinh u_3 + \sin\theta\cos u_3\cosh u_3) \\ \tilde{f}_5' &=& \cos u_1 + gh(\cos\theta\cos u_1\cosh u_1 - \cos\theta\sin u_1\sinh u_1) \end{array} \tag{100}$$

where

$$g = 2\cos\theta\cosh u_1$$

and

$$h^{-1} = 2\cos^2\theta\cos^2 u_1 - \sin^2\sqrt{2}u_2 + 2\sin^2\theta\cosh^2 u_3.$$

6.3 Examples of conformally flat hypersurfaces

One can also use Theorem 19 to compute explicit examples of conformally flat hypersurfaces $f\colon M^3 \to \mathbb{Q}^4_s(c)$ with three distinct principal curvatures. It suffices to consider the case $c = 0$, because any conformally flat hypersurface $f\colon M^3 \to \mathbb{Q}^4_s(c)$, $c \neq 0$, is the composition of a conformally flat hypersurface $g\colon M^3 \to \mathbb{R}^4_s$ with an "inverse stereographic projection".

We start with the trivial solution $v = (0, 1, 1)$, $V = (1, 0, 0)$ and $h = 0$ of system (2), for which the corresponding solution of system (74) with initial conditions

$$(X_1(0), X_2(0), X_3(0), N(0), f(0)) = (E_1, E_2, E_3, E_4, 0)$$

is given by

$$f = f(u_2, u_3) = u_2 E_2 + u_3 E_3, \quad X_2 = E_2, \quad X_3 = E_3,$$

$$X_1 = \begin{cases} \cosh u_1 E_1 + \sinh u_1 E_4, \text{ if } \epsilon = -1, \\ \cos u_1 E_1 + \sin u_1 E_4, \text{ if } \epsilon = 1, \end{cases} \tag{101}$$

and

$$N = \begin{cases} \sinh u_1 E_1 + \cosh u_1 E_4, \text{ if } \epsilon = -1, \\ -\sin u_1 E_1 + \cos u_1 E_4, \text{ if } \epsilon = 1. \end{cases} \tag{102}$$

Even though this solution does not correspond to a three-dimensional hypersurface, one can still apply Theorem 19. We solve system (7) for (v, h, V) as in the preceding paragraph. Equations (9) and (10), with $K_1 = 0 = K_2$, become

$$2\varphi\psi = \sum_i \gamma_i^2 + \epsilon\beta^2 \tag{103}$$

and

$$v'^2_2 = v'^2_1 + v'^2_3. \tag{104}$$

We also impose that

$$\varphi v'_1 = -\epsilon\beta\left(v'_3 - v'_2\right), \tag{105}$$

which corresponds to the function Ω in (11) vanishing everywhere. It follows from (iii) that

$$v'_1\psi = \beta - \frac{\partial\gamma_1}{\partial u_1}.$$

Since the right-hand-side of the preceding equation depends only on u_1 by (ii) and (iv), there exists a smooth function $\phi_1 = \phi_1(u_1)$ such that

$$v_1'\psi = \phi_1. \tag{106}$$

Similarly,

$$(1 - v_i')\psi = \phi_i \tag{107}$$

for some smooth functions $\phi_i = \phi_i(u_i)$, $2 \leq i \leq 3$. In particular,

$$(v_2' - v_3')\psi = \phi_3 - \phi_2. \tag{108}$$

Multiplying (105) by ψ and using (106) and (108) yields

$$\varphi = \frac{1}{K}(\phi_3 - \phi_2) \tag{109}$$

and

$$\beta = \frac{\epsilon}{K}\phi_1 \tag{110}$$

for some $K \in \mathbb{R}$. On the other hand, replacing (106) and (107) in (104) gives

$$\psi = \frac{\phi_1^2 - \phi_2^2 + \phi_3^2}{2(\phi_3 - \phi_2)}. \tag{111}$$

It follows from (i) and (109) that

$$\gamma_2 = -\frac{1}{K}\phi_2' \quad \text{and} \quad \gamma_3 = \frac{1}{K}\phi_3',$$

whereas (iv) and (110) yields

$$\gamma_1 = -\frac{1}{K}\phi_1'. \tag{112}$$

We obtain from (iii), (110) and (112) that

$$\phi_1'' = (K - \epsilon)\phi_1. \tag{113}$$

Similarly,

$$\phi_2'' = -K\phi_2 \quad \text{and} \quad \phi_3'' = K\phi_3. \tag{114}$$

Moreover, by (103) we must have

$$(\phi_1'^2 - (K - \epsilon)\phi_1^2) + (\phi_2'^2 + K\phi_2^2) + (\phi_3'^2 - K\phi_3^2) = 0. \tag{115}$$

Notice that each of the expressions under brackets in the preceding equation is constant, as follows from (113) and (114).

The conformally flat hypersurface given by (12) (with $c = 0$) has coordinate functions

$$f_1' = -\psi^{-1}(\phi_1' \cos u_1 + \phi_1 \sin u_1), \quad f_2' = u_2 - \psi^{-1}\phi_2',$$

$$f_3' = u_3 + \psi^{-1}\phi_3' \quad \text{and} \quad f_4' = \psi(\phi_1 \cos u_1 - \phi_1' \sin u_1),$$

with ψ as in (111). We compute them explicitly for the particular case $\epsilon = 1$ and $K < 0$, the others being similar. In this case we have

$$\begin{cases} \phi_1 = A_{11} \cos \sqrt{|K-1|}\, u_1 + A_{12} \sin \sqrt{|K-1|}\, u_1, \\ \phi_2 = A_{21} \cosh \sqrt{|K|}\, u_2 + A_{22} \sinh \sqrt{|K|}\, u_2, \\ \phi_3 = A_{31} \cos \sqrt{|K|}\, u_3 + A_{32} \sin \sqrt{|K|}\, u_3, \end{cases}$$

with $A_{ij} \in \mathbb{R}$ for $1 \leq i, j \leq 3$, and equation (115) reduces to

$$|K-1|(A_{11}^2 + A_{12}^2) + |K|(A_{22}^2 - A_{21}^2) + |K|(A_{31}^2 + A_{32}^2) = 0.$$

This implies that

$$A_{22}^2 - A_{21}^2 < 0,$$

hence we may write $A_{21} = \rho_2 \cosh \theta_2$ and $A_{22} = \rho_2 \sinh \theta_2$ for some $\rho_2 > 0$ and $\theta_2 \in \mathbb{R}$. We may also write $A_{11} = \rho_1 \cos \theta_1$, $A_{12} = \rho_1 \sin \theta_1$, $A_{31} = \rho_3 \cos \theta_3$ and $A_{32} = \rho_3 \sin \theta_3$ for some $\rho_1, \rho_3 > 0$ and $\theta_1, \theta_3 \in [0, 2\pi]$. Then

$$\begin{cases} \phi_1 = \rho_1 \cos(\sqrt{|K-1|}\, u_1 - \theta_1), \\ \phi_2 = \rho_2 \cosh(\sqrt{|K|}\, u_2 + \theta_2), \\ \phi_3 = \rho_3 \cos(\sqrt{|K|}\, u_3 - \theta_3), \end{cases}$$

with

$$|K|\rho_2^2 = |K-1|\rho_1^2 + |K|\rho_3^2,$$

and we can assume that $\theta_i = 0$ after a suitable change $u_i \mapsto u_i + u_i^0$ of the coordinates u_i, $1 \leq i \leq 3$. Setting $\rho = \rho_2$, we can write $\rho_1 = \sqrt{\frac{|K|}{|K-1|}}\rho \cos \theta$ and $\rho_3 = \rho \sin \theta$ for some $\theta \in [0, 2\pi]$. Thus

$$\begin{cases} \phi_1 = \sqrt{\frac{|K|}{|K-1|}}\rho \cos \theta \cos(\sqrt{|K-1|}\, u_1), \\ \phi_2 = \rho \cosh(\sqrt{|K|}\, u_2), \\ \phi_3 = \rho \sin \theta \cos(\sqrt{|K|}\, u_3). \end{cases}$$

For instance, for $K = -1$ we get the conformally flat hypersurface of $\mathbb{R}^4$ whose coordinate functions are

$$f_1' = 2\cos\theta gh(\sqrt{2}\cos\sqrt{2}u_1\sin u_1 - \sin\sqrt{2}u_1\cos u_1), \quad f_2' = u_2 + 4\sinh u_2 gh,$$

$$f_3' = u_3 + 4\sin\theta\sin u_3 gh, \quad f_4' = -2\cos\theta(\sin\sqrt{2}u_1\sin u_1 + \cos\sqrt{2}u_1\sin u_1)gh$$

where

$$g = \cosh u_2 - \sin\theta\cos u_3$$

and

$$h^{-1} = \cos^2\theta\cos^2\sqrt{2}u_1 - 2\cosh^2 u_2 + 2\sin^2\theta\cos^2 u_3.$$

6.4 Proof of Corollary 10

Given a hypersurface $f\colon M^3 \to \mathbb{Q}^4_s(c)$, set $\epsilon = -2s+1$, $\epsilon_c = c/|c|$ and $\check{\epsilon} = \epsilon\epsilon_c$. Let φ and ψ be defined by

$$(\varphi(t), \psi(t)) = \begin{cases} (\cos(\sqrt{|c|}t), \sin(\sqrt{|c|}t)), & \text{if } \check{\epsilon} = 1, \\ (\cosh(\sqrt{|c|}t), \sinh(\sqrt{|c|}t)), & \text{if } \check{\epsilon} = -1. \end{cases}$$

Then the family of parallel hypersurfaces $f_t\colon M^3 \to \mathbb{Q}^4_s(c) \subset \mathbb{R}^5_{s+\epsilon_0}$ to f is given by

$$i \circ f_t = \varphi(t) i \circ f + \frac{\psi(t)}{\sqrt{|c|}} i_* N,$$

where N is one of the unit normal vector fields to f and $i\colon \mathbb{Q}^4_s(c) \to \mathbb{R}^5_{s+\epsilon_0}$ is the inclusion, with $\epsilon_0 = 0$ or 1, corresponding to $c > 0$ or $c < 0$, respectively. We denote by M^3_t the manifold M^3 endowed with the metric induced by f_t.

Proposition 20. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a holonomic hypersurface. Then any parallel hypersurface $f_t\colon M^3_t \to \mathbb{Q}^4_s(c)$ to f is also holonomic and the pairs (v, V) and (v^t, V^t) associated to f and f_t, respectively, are related by*

$$\begin{cases} v_i^t = \varphi(t) v_i - \frac{\psi(t)}{\sqrt{|c|}} V_i \\ V_i^t = \check{\epsilon}\sqrt{|c|}\psi(t) v_i + \varphi(t) V_i. \end{cases} \tag{116}$$

In particular, $h^t_{ij} = h_{ij}$.

Proof. We have

$$f_{t*} = \varphi(t) f_* + \frac{\psi(t)}{\sqrt{|c|}} N_* = f_* \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right), \tag{117}$$

thus a unit normal vector field to f_t is

$$N_t = -\check{\epsilon}\sqrt{|c|}\psi(t) f + \varphi(t) N.$$

Then,

$$\begin{aligned} N_{t*} &= f_* \left(-\check{\epsilon}\sqrt{|c|}\psi(t) I - \varphi(t) A \right) \\ &= -f_{t*} \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right)^{-1} \left(\check{\epsilon}\sqrt{|c|}\psi(t) I + \varphi(t) A \right). \end{aligned}$$

which implies that

$$A_t = \left(\varphi(t) I - \frac{\psi(t)}{\sqrt{|c|}} A \right)^{-1} \left(\check{\epsilon}\sqrt{|c|}\psi(t) I + \varphi(t) A \right). \tag{118}$$

It follows from (117) and (118) that $\tilde{f}$ is also holonomic with associated pair given by (116). The assertion on h^t_{ij} follows from a straightforward computation. □

Proof of Corollary 10: Conditions (3) for (v^t, V^t) (with $\tilde{c} = 0$) follow immediately from those for (v, V). □

Remark 21. Given a hypersurface $f\colon M^3 \to \mathbb{Q}^4_s(c)$, it can be checked that the parallel hypersurfaces $f_t\colon M^3 \to \mathbb{Q}^4_s(c)$ correspond to the Ribaucour transforms of f determined by solutions $(\gamma_1, \gamma_2, \gamma_3, v_1', v_2', v_3', \varphi, \psi, \beta)$ of system (57) for which $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and φ, ψ and β are constants satisfying (59).

7 Proof of Theorem 11

For the proof of Theorem 11 we need the following preliminary fact, which was already observed in [5] for $s = 0$.

Lemma 22. *Let $f\colon M^3 \to \mathbb{Q}^4_s(c)$ be a hypersurface with three distinct principal curvatures λ_1, λ_2 and λ_3. Then, any two of the following three conditions imply the remaining one:*

(i) $(\lambda_j - \lambda_k)e_i(\lambda_i) + (\lambda_i - \lambda_k)e_i(\lambda_j) + (\lambda_j - \lambda_i)e_i(\lambda_k) = 0.$

(ii) $$(C + \hat{\epsilon}\lambda_j\lambda_k)(\lambda_k - \lambda_j)e_i(\lambda_i) + (C + \hat{\epsilon}\lambda_i\lambda_k)(\lambda_k - \lambda_i)e_i(\lambda_j) + (C + \hat{\epsilon}\lambda_i\lambda_j)(\lambda_i - \lambda_j)e_i(\lambda_k) = 0.$$

(iii) $e_i(\lambda_i\lambda_j) = 0.$

Proof. It is easily checked that (i) is equivalent to

$$(\lambda_k - \lambda_i)e_i(\lambda_i\lambda_j) = (\lambda_j - \lambda_i)e_i(\lambda_i\lambda_k), \quad 1 \leq i \neq j \neq k \neq i \leq 3,$$

whereas the difference between (ii) and (i) is equivalent to

$$\lambda_k(\lambda_k - \lambda_i)e_i(\lambda_i\lambda_j) = \lambda_j(\lambda_j - \lambda_i)e_i(\lambda_i\lambda_k), \quad 1 \leq i \neq j \neq k \neq i \leq 3,$$

and the statement follows easily. □

Proof of Theorem 11: By Theorem 8, f is locally a holonomic hypersurface whose associated pair (v, V) is given in terms of the principal curvatures λ_1, λ_2 and λ_3 of f by

$$v_j = \sqrt{\frac{\delta_j}{(\lambda_j - \lambda_i)(\lambda_j - \lambda_k)}} \quad \text{and} \quad V_j = \lambda_j v_j, \quad 1 \leq j \leq 3. \tag{119}$$

Moreover, we have seen in the proofs of Theorems 6 and 8 that conditions (i) and (ii) in Lemma 22 hold for λ_1, λ_2 and λ_3. Thus, also condition (iii) is satisfied. Assuming that $\lambda_j \neq 0$ for $1 \leq j \leq 3$, we can write

$$\lambda_i\lambda_j = \iota_k\phi_k^2, \quad \iota_k \in \{-1, 1\}, \quad 1 \leq i \neq j \neq k \neq i \leq 3, \tag{120}$$

for some positive smooth functions $\phi_k = \phi_k(u_k)$, $1 \leq k \leq 3$. It follows from (120) that

$$\lambda_j = \epsilon_j \frac{\phi_i\phi_k}{\phi_j}, \tag{121}$$

where $\epsilon_j = \frac{\lambda_j}{|\lambda_j|}$, $1 \leq j \leq 3$. We may suppose that $\lambda_1 < \lambda_2 < \lambda_3$, so that

$$\epsilon_k\phi_i^2 - \epsilon_i\phi_k^2 > 0, \quad 1 \leq i < k \leq 3.$$

Substituting (121) into (119), we obtain that

$$v_j = \frac{\phi_j}{\psi_i \psi_k}, \quad 1 \leq j \leq 3, \tag{122}$$

where

$$\psi_j = \sqrt{\epsilon_k \phi_i^2 - \epsilon_i \phi_k^2}$$

and

$$V_j = \lambda_j v_j = \epsilon_j \frac{\phi_i \phi_k}{\psi_i \psi_k}, \quad i, k \neq j, \quad i < k.$$

We obtain from (122) that

$$h_{ij} = \frac{1}{v_j} \frac{\partial v_j}{\partial u_i} = \frac{\psi_i \psi_k}{\phi_j} \frac{\phi_j}{\psi_i \psi_k^2} \left(-\frac{\partial \psi_k}{\partial u_i} \right) = -\frac{1}{\psi_k} \frac{\partial \psi_k}{\partial u_i}. \tag{123}$$

On the other hand, equation (iv) of system (2) yields

$$h_{ij} = \frac{1}{V_j} \frac{\partial V_j}{\partial u_i} = \frac{\psi_i \psi_k}{\phi_i \phi_k} \frac{\phi_k}{\psi_i \psi_k^2} \left(\frac{d\phi_i}{du_i} \psi_k - \phi_i \frac{\partial \psi_k}{\partial u_i} \right) = \frac{1}{\phi_i} \frac{d\phi_i}{du_i} - \frac{1}{\psi_k} \frac{\partial \psi_k}{\partial u_i}. \tag{124}$$

Comparying (123) and (124), we obtain that

$$\frac{d\phi_i}{du_i} = 0, \quad 1 \leq i \leq 3.$$

This implies that $\frac{\partial \psi_k}{\partial u_i} = 0$ for all $1 \leq i \neq k \leq 3$, and hence $h_{ij} = 0$ for all $1 \leq i \neq j \leq 3$. But then equation (ii) of system (2) gives

$$\epsilon \lambda_i \lambda_j + c = 0$$

for all $1 \leq i \neq j \leq 3$, which implies that $-\epsilon c > 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \sqrt{-\epsilon c}$, a contradiction. Thus, one of the principal curvatures must be zero, and the result follows from part $b)$ of Theorem 4. □

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