arXiv:0803.2412v2 [math.NT] 19 Mar 2008

RESULTS ABOUT PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$ AND RELATED EXPONENTIALS SUMS

JORGEN CHERLY

1. EXPONENTIAL SUMS AND RANK OF PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

RÉSUMÉ. Soit $\mathbb{K}$ le corps des séries de Laurent formelles $\mathbb{F}_2((T^{-1}))$. Nous calculons en particulier des sommes exponentielles dans $\mathbb{K}$ de la forme $\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)$ où t est dans la boule unité de $\mathbb{K}$, en démontrant qu'elles dépendent seulement du rang de matrices persymétriques avec des entrées dans $\mathbb{F}_2$ qui leur sont associées. (Une matrice $[\alpha_{i,j}]$ est persymétrique si $\alpha_{i,j}=\alpha_{r,s}$ pour i+j = r+s). En outre nous établissons des propriétes de rang d'une partition de matrices persymétriques. Nous utilisons ces résultats pour calculer le nombre Γ_i de matrices persymétriques sur $\mathbb{F}_2$ de rang i. Nous retrouvons en particulier une formule générale donnée par D.E.Daykin. Notre démonstration est, comme indiqué, très différente, puisqu'elle se fonde sur les propriétés de rang d'une partition de matrices persymétriques. Nous montrons également que le nombre R de représentations dans $\mathbb{F}_2[T]$ de 0 comme une somme de formes quadratiques associées aux sommes exponentielles $\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)$ est donné par une intégrale étendue à la boule unité et est une combinaison linéaire des Γ_i. Nous calculons alors explicitement le nombre R. Des résultats similaires sont également obtenus pour les $\mathbb{K}$- espaces vectoriels de dimension n+1. Nous terminons notre article en calculant explicitement le nombre de matrices de rang i de la forme $\left[\frac{A}{B}\right]$, où A est persymétrique.

ABSTRACT. Let $\mathbb{K}$ be the field of Laurent Series $\mathbb{F}_2((T^{-1}))$.
We compute in particular exponential sums in $\mathbb{K}$ of the form $\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)$ where t is in the unit interval of $\mathbb{K}$, by showing that they only depend on the rank of some associated persymmetric matrices with entries in $\mathbb{F}_2$. (A matrix $[\alpha_{i,j}]$ is persymmetric if $\alpha_{i,j}=\alpha_{r,s}$ for i+j = r+s). Besides we establish rank properties of a partition of persymmetric matrices. We use these results to compute the number Γ_i of persymmetric matrices over $\mathbb{F}_2$ of rank i. We recover in this particular a general formula given by D. E. Daykin. Our proof is as indicated very different since it relies on rank properties of a partition of persymmetric matrices. We also prove that the number R of representations in $\mathbb{F}_2[T]$ of 0 as a sum of some quadratic forms associated to the exponential sums $\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)$ is given by an integral over the unit interval, and is a linear combination of the $\Gamma_i's$. We then compute explicitly the number R. Similar results are also obtained for n+1 dimensional $\mathbb{K}$ - vector spaces.We finish the paper by computing explicitely the number of rank i matrices of the form $\left[\frac{A}{B}\right]$, where A is persymmetric.

1.1. **An outline of the main results.**

Theorem 1.1. *The number* $\Gamma_i^{s\times k}$ *of persymmetric* $s\times k$ *matrices over* $\mathbb{F}_2$

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_k & \alpha_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{s-1} & \alpha_s & \alpha_{s+1} & \dots & \alpha_{k+s-3} & \alpha_{k+s-2} \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \dots & \alpha_{k+s-2} & \alpha_{k+s-1} \end{pmatrix}$$

of rank i *is given by*

$$\begin{cases} 1 & if \quad i=0, \\ 3\cdot 2^{2(i-1)} & if \quad 1\le i\le s-1, \\ 2^{k+s-1}-2^{2s-2} & if \quad i=s\ (s\le k). \end{cases}$$

Remark 1.2. David E. Daykin has already proved this result over any finite field $\mathbb{F}$ with the number 2 in the formula replaced by$|\mathbb{F}|$, and the number 3 replaced by $|\mathbb{F}|^2-1$. Our proof is different and proper to the finite field with two elements.

Theorem 1.3. *Let* $(j_1,j_2,j_3,j_4)\in\mathbb{N}^4$, *then*

$$\#\Big(\begin{array}{c|c} j_1 & j_2 \\ \hline j_3 & j_4 \end{array}\Big)_{\mathbb{P}/\mathbb{P}_{k+s-1}} = \begin{cases} 1 & \text{if } j_1=j_2=j_3=j_4=0, \\ 2^{2j-1} & \text{if } j_1=j_2=j_3=j\ , j_4\in\{j,j+1\}\ , 1\le j\le s-1, \\ 2^{2j-3} & \text{if } j_1=j-2\ , j_2=j_3=j-1\ , j_4=j\ , 2\le j\le s, \\ 2^{k+s-1}-2^{2s-1} & \text{if } j_1=j_2=s-1, j_3=j_4=s, \\ 0 & \text{otherwise}, \end{cases}$$

Theorem 1.4. *Let* $h_{s,k}(t)=h(t)$ *be the quadratic exponential sum in* $\mathbb{P}$ *defined by*

$$t\in\mathbb{P}\longmapsto \sum_{degY\le k-1}\ \sum_{degZ\le s-1} E(tYZ)\in\mathbb{Z}.$$

Then

$$h(t)=2^{k+s-r(D_{s\times k}(t))}$$

and

$$\int_{\mathbb{P}} h^q(t)dt = 2^{(q-1)(k+s)+1}\sum_{i=0}^{s}\Gamma_i^{s\times k}2^{-qi}.$$

Let R *denote the number of solutions* $(Y_1,Z_1,\dots,Y_q,Z_q)$ *of the polynomial equation*

$$Y_1Z_1+Y_2Z_2+\ldots+Y_qZ_q=0$$

satisfying the degree conditions

$$degY_i\le k-1,\quad degZ_i\le s-1\quad for 1\le i\le q.$$

Then

$$R=\int_{\mathbb{P}} h^q(t)dt$$

Theorem 1.5. *Let $g_{s,k}(t) = g(t)$ be the quadratic exponential sum in $\mathbb{P}$ defined by*

$$t \in \mathbb{P} \longmapsto \sum_{degY=k-1} \sum_{degZ=s-1} E(tYZ) \in \mathbb{Z}.$$

Then

$$g(t) = \begin{cases} 2^{s+k-j-2} & if \quad r(D_{(s-1)\times(k-1)}(t)) = r(D_{s\times(k-1)}(t)) = r(D_{(s-1)\times k}(t)) = r(D_{s\times k}(t)) = j, \\ -2^{s+k-j-2} & if \quad r(D_{(s-1)\times(k-1)}(t)) = r(D_{s\times(k-1)}(t)) = r(D_{(s-1)\times k}(t)) = j \text{ and } r(D_{s\times k}(t)) = j+1, \\ 0 & if \quad otherwise, \end{cases}$$

and

$$\int_{\mathbb{P}} g^{2q}(t)dt = 2^{(s+k-2)(2q-1)} \cdot \sum_{j=0}^{s-1} \#\left(\begin{array}{c|c} j & j \\ \hline j & j \end{array}\right)_{\mathbb{P}/\mathbb{P}_{k+s-1}} \cdot 2^{-2qj}.$$

Theorem 1.6. *Let $g_{m,k}(t,\eta) = g(t,\eta)$ be the exponential sum in $\mathbb{P}\times\mathbb{P}$ defined by*

$$(t,\eta) \in \mathbb{P}\times\mathbb{P} \longmapsto \sum_{degY\leq k-1} \sum_{degZ\leq m} E(tYZ) \sum_{degU=0} E(\eta YU) \in \mathbb{Z}.$$

Then

$$g(t,\eta) = \begin{cases} 2^{k+m+1-r(D_{(1+m)\times k}(t))} & \text{if } r(D_{(1+m)\times k}(t)) = r(D^{\left[1+\frac{1}{m}\right]\times k}(t,\eta)), \\ 0 & \text{otherwise }, \end{cases}$$

and

$$\int_{\mathbb{P}}\int_{\mathbb{P}} g^q(t,\eta)dtd\eta = 2^{q(k+m+1)-2k-m} \sum_{i=0}^{inf(k,1+m)} \sigma_{i,i}^{\left[1+\frac{1}{m}\right]\times k} 2^{-iq}.$$

Theorem 1.7. *Let $f_{m,k}(t,\eta) = f(t,\eta)$ be the exponential sum in $\mathbb{P}\times\mathbb{P}$ defined by*

$$(t,\eta) \in \mathbb{P}\times\mathbb{P} \longmapsto \sum_{degY\leq k-1} \sum_{degZ\leq m} E(tYZ) \sum_{degU\leq 0} E(\eta YU) \in \mathbb{Z}.$$

Then

$$f(t,\eta) = 2^{k+m+2-r(r(D^{\left[1+\frac{1}{m}\right]\times k}(t,\eta))}$$

and

$$\int_{\mathbb{P}}\int_{\mathbb{P}} f^q(t,\eta)dtd\eta = 2^{q(k+m+2)-2k-m} \sum_{i=0}^{inf(k,2+m)} \Gamma_i^{\left[1+\frac{1}{m}\right]\times k} 2^{-iq}.$$

Theorem 1.8. *We have the following formula for all $0 \leq i \leq \inf(k, 2+m)$*

$$\Gamma_i^{\left[1+\frac{1}{m}\right]\times k} = (2^k - 2^{i-1}) \cdot \Gamma_{i-1}^{(1+m)\times k} + 2^i \Gamma_i^{(1+m)\times k}.$$

The case k = 2

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1+m\end{smallmatrix}\right]\times 2}=\begin{cases}1 & \text{if } i=0,\\ 9 & \text{if } i=1,\\ 2^{4+m}-10 & \text{if } i=2.\end{cases}$$

The case m = 0 , k ≥ 2

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\\ 3\cdot(2^k-1) & \text{if } i=1,\\ 2^{2k}-3\cdot 2^k+2 & \text{if } i=2.\end{cases}$$

The case m = 1 , k ≥ 3

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1+1\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\\ 2^k+5 & \text{if } i=1,\\ 11\cdot(2^k-1) & \text{if } i=2,\\ 2^{2k+1}-3\cdot 2^{k+2}+2^4 & \text{if } i=3.\end{cases}$$

The case 3 ≤ k ≤ 1 + m

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1+m\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\\ 2^k+5 & \text{if } i=1,\\ 3\cdot 2^{k+2i-4}+21\cdot 2^{3i-5} & \text{if } 2\le i\le k-1,\\ 2^{2k+m}-5\cdot 2^{3k-5} & \text{if } i=k.\end{cases}$$

The case 2 ≤ m ≤ k − 2

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1+m\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\\ 2^k+5 & \text{if } i=1,\\ 3\cdot 2^{k+2i-4}+21\cdot 2^{3i-5} & \text{if } 2\le i\le m,\\ 11\cdot[2^{k+2m-2}-2^{3m-2}] & \text{if } i=m+1,\\ 2^{2k+m}-3\cdot 2^{k+2m}+2^{3m+1} & \text{if } i=m+2.\end{cases}$$

Theorem 1.9. *Let* $\Gamma_i^{\left[\begin{smallmatrix}n\\1+m\end{smallmatrix}\right]\times k}$ *denote the number of matrices of the form* $\left[\frac{A}{B}\right]$ *of rank* i *such that* A *is a* $(1+m)\times k$ *persymmetric matrix and* B *is a* $n\times k$ *matrix over* $\mathbb{F}_2$*, and where* $\Gamma_i^{(1+m)\times k}$ *denotes the number of* $(1+m)\times k$ *persymmetric matrices over* $\mathbb{F}_2$ *of rank* i*.*

Then $\Gamma_i^{\left[\begin{smallmatrix}n\\1+m\end{smallmatrix}\right]\times k}$ *expressed as a linear combination of the* $\Gamma_{i-j}^{(1+m)\times k}$ *is equal to*

$$\sum_{j=0}^{n}2^{(n-j)\cdot(i-j)}a_j^{(n)}\prod_{l=1}^{j}(2^k-2^{i-l})\cdot\Gamma_{i-j}^{(1+m)\times k}\quad for\quad 0\le i\le inf(k,n+m+1)$$

where

$$a_j^{(n)}=\sum_{s=0}^{j-1}(-1)^s\prod_{l=0}^{j-(s+1)}\frac{2^{n+1}-2^l}{2^{j-s}-2^l}\cdot 2^{s(n-j)+\frac{s(s+1)}{2}}+(-1)^j\cdot 2^{jn-\frac{j(j-1)}{2}}\quad for\quad 1\le j\le n-1.$$

We set

$$a_0^{(n)}=a_n^{(n)}=1$$

$$and\quad \Gamma_{i-j}^{(1+m)\times k}=0\quad if\quad i-j\notin\{0,1,2,\dots,inf(k,1+m)\}.$$

Corollary 1.10. *We have the following formulas for n = 1,2,3,4,5 :*

$$\Gamma_i^{\left[\begin{smallmatrix}1\\1+m\end{smallmatrix}\right]\times k} = 2^i\Gamma_i^{(1+m)\times k} + (2^k - 2^{i-1})\cdot\Gamma_{i-1}^{(1+m)\times k} \quad for \quad 0 \le i \le inf(k, 2+m),$$

$$\Gamma_i^{\left[\begin{smallmatrix}2\\1+m\end{smallmatrix}\right]\times k} = 2^{2i}\Gamma_i^{(1+m)\times k} + 3\cdot 2^{i-1}(2^k - 2^{i-1})\cdot\Gamma_{i-1}^{(1+m)\times k}$$
$$+ (2^k - 2^{i-1})(2^k - 2^{i-2})\cdot\Gamma_{i-2}^{(1+m)\times k} \quad for \quad 0 \le i \le inf(k, 3+m),$$

$$\Gamma_i^{\left[\begin{smallmatrix}3\\1+m\end{smallmatrix}\right]\times k} = 2^{3i}\Gamma_i^{(1+m)\times k} + 7\cdot 2^{(i-1)2}(2^k - 2^{i-1})\cdot\Gamma_{i-1}^{(1+m)\times k}$$
$$+ 7\cdot 2^{i-2}(2^k - 2^{i-1})(2^k - 2^{i-2})\cdot\Gamma_{i-2}^{(1+m)\times k}$$
$$+ (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})\Gamma_{i-3}^{(1+m)\times k} \quad for \quad 0 \le i \le inf(k, 4+m),$$

$$\Gamma_i^{\left[\begin{smallmatrix}4\\1+m\end{smallmatrix}\right]\times k} = 2^{4i}\Gamma_i^{(1+m)\times k} + 15\cdot 2^{(i-1)3}(2^k - 2^{i-1})\cdot\Gamma_{i-1}^{(1+m)\times k}$$
$$+ 35\cdot 2^{2i-4}(2^k - 2^{i-1})(2^k - 2^{i-2})\cdot\Gamma_{i-2}^{(1+m)\times k}$$
$$+ 15\cdot 2^{i-3}(2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})\Gamma_{i-3}^{(1+m)\times k}$$
$$+ (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4})\Gamma_{i-4}^{(1+m)\times k}$$
$$for \quad 0 \le i \le inf(k, 5+m),$$

$$\Gamma_i^{\left[\begin{smallmatrix}5\\1+m\end{smallmatrix}\right]\times k} = 2^{5i}\Gamma_i^{(1+m)\times k} + 31\cdot 2^{(i-1)4}(2^k - 2^{i-1})\cdot\Gamma_{i-1}^{(1+m)\times k}$$
$$+ 155\cdot 2^{3i-6}(2^k - 2^{i-1})(2^k - 2^{i-2})\cdot\Gamma_{i-2}^{(1+m)\times k}$$
$$+ 155\cdot 2^{2i-6}(2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})\Gamma_{i-3}^{(1+m)\times k}$$
$$+ 31\cdot 2^{i-4}(2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4})\Gamma_{i-4}^{(1+m)\times k}$$
$$+ (2^k - 2^{i-1})(2^k - 2^{i-2})(2^k - 2^{i-3})(2^k - 2^{i-4})(2^k - 2^{i-5})\Gamma_{i-5}^{(1+m)\times k}$$
$$for \quad 0 \le i \le inf(k, 6+m).$$

Theorem 1.11. *Let $f_{m,k}(t, \eta_1, \eta_2, \dots, \eta_n)$ be the exponential sum in $\mathbb{P}^{n+1}$ defined by* $(t, \eta_1, \eta_2, \dots, \eta_n) \in \mathbb{P}^{n+1} \longrightarrow$

$$\sum_{degY \le k-1} \sum_{degZ \le m} E(tYZ) \sum_{degU_1 \le 0} E(\eta_1 Y U_1) \sum_{degU_2 \le 0} E(\eta_2 Y U_2) \dots \sum_{degU_n \le 0} E(\eta_n Y U_n).$$

Set

$$(t, \eta_1, \eta_2, \dots, \eta_n) = \big(\sum_{i\ge 1}\alpha_i T^{-i}, \sum_{i\ge 1}\beta_{1i}T^{-i}, \dots, \sum_{i\ge 1}\beta_{ni}T^{-i}\big) \in \mathbb{P}^{n+1}.$$

Then

$$f_{m,k}(t, \eta_1, \eta_2, \dots, \eta_n) = 2^{k+m+n+1-r(D^{\left[\begin{smallmatrix}n\\1+m\end{smallmatrix}\right]\times k}(t,\eta_1,\eta_2,\dots,\eta_n))}$$

where

$$D^{\left[\begin{smallmatrix}n\\1+m\end{smallmatrix}\right]\times k}(t, \eta_1, \eta_2, \dots, \eta_n)$$

denotes the following $(1+n+m)\times k$ *matrix*

$$\left(\begin{array}{cccccc} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_k & \alpha_{k+1} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \alpha_{1+m} & \alpha_{2+m} & \alpha_{3+m} & \dots & \alpha_{k+m-1} & \alpha_{k+m} \\ \hline \beta_{11} & \beta_{12} & \beta_{13} & \dots & \beta_{1k-1} & \beta_{1k} \\ \beta_{21} & \beta_{22} & \beta_{23} & \dots & \beta_{2k-1} & \beta_{2k} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & \beta_{nk-1} & \beta_{nk} \end{array}\right).$$

Then the number denoted by $R_q(n,k,m)$ *of solutions*

$$(Y_1, Z_1, U_1^{(1)}, U_2^{(1)}, \dots, U_n^{(1)}, Y_2, Z_2, U_1^{(2)}, U_2^{(2)}, \dots, U_n^{(2)}, \dots Y_q, Z_q, U_1^{(q)}, U_2^{(q)}, \dots, U_n^{(q)})$$

of the polynomial equations

$$\left\{\begin{array}{c} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0 \\ Y_1U_1^{(1)} + Y_2U_1^{(2)} + \ldots + Y_qU_1^{(q)} = 0 \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + \ldots + Y_qU_2^{(q)} = 0 \\ \vdots \\ Y_1U_n^{(1)} + Y_2U_n^{(2)} + \ldots + Y_qU_n^{(q)} = 0 \end{array}\right.$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degZ_i \leq m, \quad degU_j^i \leq 0, \quad for \quad 1 \leq j \leq n \quad 1 \leq i \leq q$$

is equal to the following integral over the unit interval in $\mathbb{K}^{n+1}$

$$\int_{\mathbb{P}^{n+1}} f_{m,k}^q(t, \eta_1, \eta_2, \dots, \eta_n) dt d\eta_1 d\eta_2 \dots d\eta_n.$$

Observing that $f_{m,k}(t, \eta_1, \eta_2, \dots, \eta_n)$ *is constant on cosets of* $\mathbb{P}_{k+m} \times \mathbb{P}_k^n$, *the above integral is equal to*

$$2^{q(k+m+n+1)-(n+1)k-m} \sum_{i=0}^{inf(k,n+1+m)} \Gamma_i^{\left[1+\frac{n}{m}\right]\times k} 2^{-iq} = R_q(n,k,m)$$

Example. The number $R_q(0,k,m)$ of solutions $(Y_1, Z_1, \dots, Y_q, Z_q)$ of the polynomial equation

$$Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degZ_i \leq m \leq k-1 \quad for 1 \leq i \leq q.$$

is equal to the following integral

$$\int_{\mathbb{P}} \Big[\sum_{degY \leq k-1} \sum_{degZ \leq m} E(tYZ) \Big]^q dt = 2^{(q-1)(k+m+1)+1} \sum_{i=0}^{1+m} \Gamma_i^{(1+m)\times k} 2^{-qi}$$

$$= \begin{cases} 2^k + 2^{1+m} - 1 & if \quad q = 1, \\ 2^{2k} + 3 \cdot (m+1) \cdot 2^{k+m} & if \quad q = 2, \\ 2^{(q-1)(k+m+1)+1} \left[1 + 3\frac{1-2^{(2-q)m}}{2^q - 2^2} + (2^{k+m} - 2^{2m}) 2^{-q(1+m)} \right] & if \quad 3 \leq q. \end{cases}$$

Example. The number $\Gamma_i^{\left[\begin{smallmatrix}1\\1+2\end{smallmatrix}\right]\times 3}$ of rank i matrices of the form

$$\left(\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_3 & \alpha_4 & \alpha_5 \\ \hline \beta_1 & \beta_2 & \beta_3 \end{array} \right)$$

is equal to

$$\begin{cases} 1 & \text{if } i = 0, \\ 13 & \text{if } i = 1, \\ 66 & \text{if } i = 2, \\ 176 & \text{if } i = 3. \end{cases}$$

The number $R_q(1,3,2)$ of solutions $(Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0 \\ Y_1U_1 + Y_2U_2 + \ldots + Y_qU_q = 0 \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 2, \quad degZ_i \leq 2, \quad degU_i \leq 0 \quad for \quad 1 \leq i \leq q$$

is equal to the following integral

$$\int_{\mathbb{P}} \int_{\mathbb{P}} \Big[\sum_{degY \leq 2} \sum_{degZ \leq 2} E(tYZ) \sum_{degU \leq 0} E(\eta YU) \Big]^q dt d\eta = 2^{7q-8} \sum_{i=0}^{3} \Gamma_i^{\left[\begin{smallmatrix}1\\1+2\end{smallmatrix}\right]\times 3} 2^{-iq}$$

$$= 2^{4q-8} \cdot \big[2^{3q} + 13 \cdot 2^{2q} + 66 \cdot 2^q + 176 \big].$$

Example. The number $\Gamma_i^{\left[\begin{smallmatrix}5\\1+2\end{smallmatrix}\right]\times 4}$ of rank i matrices of the form

$$\left(\begin{array}{cccc} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \hline \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & \beta_{33} & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & \beta_{44} \\ \beta_{51} & \beta_{52} & \beta_{53} & \beta_{54} \end{array} \right)$$

is equal to

$$\begin{cases} 1 & \text{if } i=0, \\ 561 & \text{if } i=1, \\ 65670 & \text{if } i=2, \\ 3731208 & \text{if } i=3, \\ 63311424 & \text{if } i=4. \end{cases}$$

The number $R_3(5,4,2)$ of solutions

$$(Y_1, Z_1, U_1^{(1)}, U_2^{(1)}, U_3^{(1)}, U_4^{(1)}, U_5^{(1)}, Y_2, Z_2, U_1^{(2)}, U_2^{(2)}, U_3^{(2)}, U_4^{(2)}, U_5^{(2)}, Y_3, Z_3, U_1^{(3)}, U_2^{(3)}, U_3^{(3)}, U_4^{(3)}, U_5^{(3)})$$

of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + Y_3Z_3 = 0, \\ Y_1U_1^{(1)} + Y_2U_1^{(2)} + Y_3U_1^{(3)} = 0, \\ Y_1U_2^{(1)} + Y_2U_2^{(2)} + Y_3U_2^{(3)} = 0, \\ Y_1U_3^{(1)} + Y_2U_3^{(2)} + Y_3U_3^{(3)} = 0, \\ Y_1U_4^{(1)} + Y_2U_4^{(2)} + Y_3U_4^{(3)} = 0, \\ Y_1U_5^{(1)} + Y_2U_5^{(2)} + Y_3U_5^{(3)} = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 3, \quad degZ_i \leq 2, \quad degU_i \leq 0 \quad for \quad 1 \leq j \leq 5 \quad 1 \leq i \leq 3$$

is equal to the following integral over the unit interval in $\mathbb{K}^6$

$$\int_{\mathbb{P}^6} f_{2,4}^3(t, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5) dt d\eta_1 d\eta_2 d\eta_3 d\eta_4 d\eta_5 = 2^{10} \cdot \sum_{i=0}^{4} \Gamma_i^{\left[\begin{smallmatrix} 5 \\ 1+2 \end{smallmatrix}\right] \times 4} 2^{-i3} = 24413824.$$

2. EXPONENTIAL SUMS AND RANK OF DOUBLE PERSYMMETRIC MATRICES OVER $\mathbb{F}_2$

RÉSUMÉ. Soit $\mathbb{K}^2$ le $\mathbb{K}$ - espace vectoriel de dimension 2 où $\mathbb{K}$ dénote le corps des séries de Laurent formelles $\mathbb{F}_2((T^{-1}))$. Nous calculons en particulier des sommes exponentielles (dans $\mathbb{K}^2$) de la forme
$\sum_{\deg Y\leq k-1}\sum_{\deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)$ où (t,η) est dans la boule unité de $\mathbb{K}^2$.
Nous démontrons qu'elles dépendent uniquement du rang de matrices doubles persymétriques avec des entrées dans $\mathbb{F}_2$, c'est-à-dire des matrices de la forme $\left[\frac{A}{B}\right]$ où A est une matrice $s\times k$ persymétrique et B une matrice $(s+m)\times k$ persymétrique (une matrice $[\alpha_{i,j}]$ est persymétrique si $\alpha_{i,j}=\alpha_{r,s}$ pour i+j = r+s). En outre, nous établissons plusieurs formules concernant des propriétés de rang de partitions de matrices doubles persymétriques, ce qui nous conduit à une formule récurrente du nombre $\Gamma_i^{\left[\substack{s\\s+m}\right]\times k}$ des matrices de rang i de la forme $\left[\frac{A}{B}\right]$. Nous déduisons de cette formule récurrente que si $0\leq i\leq \inf(s-1,k-1)$, le nombre $\Gamma_i^{\left[\substack{s\\s+m}\right]\times k}$ dépend uniquement de i. D'autre part, si $i\geq s+1, k\geq i$, $\Gamma_i^{\left[\substack{s\\s+m}\right]\times k}$ peut être calculé à partir du nombre $\Gamma_{s'+1}^{\left[\substack{s'\\s'+m'}\right]\times k'}$ de matrices de rang (s'+1) de la forme $\left[\frac{A'}{B'}\right]$ où A' est une matrice $s'\times k'$ persymétrique et B' une matrice $(s'+m')\times k'$ persymétrique, où s', m' et k' dépendent de i, s, m et k. La preuve de ce résultat est basée sur une formule (donnée dans [4]) du nombre de matrices de rang i de la forme $\left[\frac{A}{b_-}\right]$ où A est persymétrique et b_- une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous montrons également que le nombre R de représentations dans $\mathbb{F}_2[T]$ des équations polynomiales

$$\begin{cases} YZ+Y_1Z_1+\ldots+Y_{q-1}Z_{q-1}=0\\ YU+Y_1U_1+\ldots+Y_{q-1}U_{q-1}=0\end{cases}$$

associées aux sommes exponentielles
$\sum_{deg Y\leq k-1}\sum_{deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)$ est donné par une intégrale sur la boule unité de $\mathbb{K}^2$ et est une combinaison linéaire de $\Gamma_i^{\left[\substack{s\\s+m}\right]\times k}$ pour $i\geq 0$.
Nous pouvons alors calculer explicitement le nombre R.

ABSTRACT. Let $\mathbb{K}^2$ be the 2-dimensional vectorspace over $\mathbb{K}$ where $\mathbb{K}$ denotes the field of Laurent Series $\mathbb{F}_2((T^{-1}))$. We compute in particular exponential sums, (in $\mathbb{K}^2$) of the form
$\sum_{\deg Y\leq k-1}\sum_{\deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)$ where (t,η) is in the unit interval of $\mathbb{K}^2$. We show that they only depend on the rank of some associated double persymmetric matrices with entries in $\mathbb{F}_2$, that is matrices of the form $\left[\frac{A}{B}\right]$ where A is a $s\times k$ persymmetric matrix and B a $(s+m)\times k$ persymmetric matrix. (A matrix $[\alpha_{i,j}]$ is persymmetric if $\alpha_{i,j}=\alpha_{r,s}\ for\ i+j=r+s$). Besides, we establish several formulas concerning rank properties of partitions of double persymmetric matrices, which leads to a recurrent formula for the number $\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}$ of rank i matrices of the form $\left[\frac{A}{B}\right]$. We deduce from the recurrent formula that if $0\leq i\leq \inf(s-1,k-1)$ then $\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}$ depends only on i. On the other hand, if $i\geq s+1, k\geq i,\ \Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}$ can be computed from the number $\Gamma_{s'+1}^{\left[\begin{smallmatrix}s'\\s'+m'\end{smallmatrix}\right]\times k'}$ of rank (s'+1) matrices of the form $\left[\frac{A'}{B'}\right]$ where A' is a $s'\times k'$ persymmetric matrix and B' a $(s'+m')\times k'$ persymmetric matrix, where s', m' and k' depend on i, s, m and k. The proof of this result is based on a formula (given in [4]) of the number of rank i matrices of the form $\left[\frac{A}{b_-}\right]$ where A is persymmetric and b_- a one-row matrix with entries in $\mathbb{F}_2$. We also prove that the number R of representations in $\mathbb{F}_2[T]$ of the polynomial equations

$$\begin{cases} YZ+Y_1Z_1+\ldots+Y_{q-1}Z_{q-1}=0\\ YU+Y_1U_1+\ldots+Y_{q-1}U_{q-1}=0\end{cases}$$

associated to the exponential sums
$\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)$ is given by an integral over the unit interval of $\mathbb{K}^2$, and is a linear combination of the $\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}\ for\ i\geq 0$. We can then compute explicitly the number R.

2.1. **An outline of the main results.**

Theorem 2.1. *Let q be a rational integer ≥ 1, then*

(2.1)

$$g_{k,s,m}(t,\eta)=g(t,\eta)=\sum_{degY\leq k-1}\sum_{degZ\leq s-1}E(tYZ)\sum_{degU\leq s+m-1}E(\eta YU)=2^{2s+m+k-r(D^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}(t,\eta))},$$

(2.2)

$$\int_{\mathbb{P}\times\mathbb{P}}g^q_{k,s,m}(t,\eta)dtd\eta=2^{(2s+m+k)(q-1)}\cdot 2^{-k+2}\cdot\sum_{i=0}^{\inf(2s+m,k)}\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}\cdot 2^{-qi}.$$

Theorem 2.2. *Let $s\geq 2,\ m\geq 0,\ k\geq 1$ and $0\leq i\leq \inf(2s+m,k)$. Then we have the following recurrent formula for the number $\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\end{smallmatrix}\right]\times k}$ of rank i matrices of the*

form $\left[\frac{A}{B}\right]$,

$$\left(\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \dots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \dots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\hline
\beta_1 & \beta_2 & \beta_3 & \dots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \beta_4 & \dots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \beta_{m+3} & \dots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \beta_{s+m+1} & \dots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \beta_{s+m+2} & \dots & \beta_{s+m+k-2} & \beta_{s+m+k-1}
\end{array}\right).$$

such that A *is a* $s \times k$ *persymmetric matrix and* B *a* $(s+m) \times k$ *persymmetric matrix with entries in* $\mathbb{F}_2$:

(2.3)

$$\Gamma_i^{\left[\substack{s \\ s+m}\right] \times k} = 2 \cdot \Gamma_{i-1}^{\left[\substack{s-1 \\ s-1+(m+1)}\right] \times k} + 4 \cdot \Gamma_{i-1}^{\left[\substack{s \\ s+(m-1)}\right] \times k} - 8 \cdot \Gamma_{i-2}^{\left[\substack{s-1 \\ s-1+m}\right] \times k} + \Delta_i^{\left[\substack{s \\ s+m}\right] \times k}$$

where the remainder $\Delta_i^{\left[\substack{s \\ s+m}\right] \times k}$ *is equal to*

(2.4)
$$\sigma_{i,i,i}^{\left[\substack{\substack{s-1 \\ \frac{s+m-1}{\alpha_{s-}}} \\ \beta_{s+m-}}\right] \times k} - 3 \cdot \sigma_{i-1,i-1,i-1}^{\left[\substack{\substack{s-1 \\ \frac{s+m-1}{\alpha_{s-}}} \\ \beta_{s+m-}}\right] \times k} + 2 \cdot \sigma_{i-2,i-2,i-2}^{\left[\substack{\substack{s-1 \\ \frac{s+m-1}{\alpha_{s-}}} \\ \beta_{s+m-}}\right] \times k}.$$

Recall that

$$\sigma_{i,i,i}^{\left[\substack{\substack{s-1 \\ \frac{s+m-1}{\alpha_{s-}}} \\ \beta_{s+m-}}\right] \times k}$$

is equal to the cardinality of the following set

$$\left\{(t,\eta) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \mid r(D^{\left[\substack{s-1 \\ s-1+m}\right] \times k}(t,\eta)) = r(D^{\left[\substack{s \\ s+m-1}\right] \times k}(t,\eta)) = r(D^{\left[\substack{s \\ s+m}\right] \times k}(t,\eta)) = i\right\}.$$

Theorem 2.3. *Let* $s \geq 2$ *and* $m \geq 0$, *we have in the following two cases :*
The case $1 \leq k \leq 2s+m-2$

$$\sigma_{i,i,i}^{\left[\begin{smallmatrix}\frac{s}{s+m}\\ \alpha_{s-}\\ \beta_{s+m-}\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\quad k\geq 1,\\ 4\cdot\Gamma_i^{\left[\frac{s-1}{s-1+m}\right]\times i}-\Gamma_{i+1}^{\left[\frac{s-1}{s-1+m}\right]\times(i+1)} & \text{if } 1\leq i\leq k-1,\\ 4\cdot\Gamma_k^{\left[\frac{s-1}{s-1+m}\right]\times k} & \text{if } i=k.\end{cases}\tag{2.5}$$

The case $k\geq 2s+m-2$

$$\sigma_{i,i,i}^{\left[\begin{smallmatrix}\frac{s}{s+m}\\ \alpha_{s-}\\ \beta_{s+m-}\end{smallmatrix}\right]\times k}=\begin{cases}1 & \text{if } i=0,\\ 4\cdot\Gamma_i^{\left[\frac{s-1}{s-1+m}\right]\times i}-\Gamma_{i+1}^{\left[\frac{s-1}{s-1+m}\right]\times(i+1)} & \text{if } 1\leq i\leq 2s+m-3,\\ 4\cdot\Gamma_{2s+m-2}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-2)} & \text{if } i=2s+m-2.\end{cases}\tag{2.6}$$

Theorem 2.4. *The remainder $\Delta_i^{\left[\frac{s}{s+m}\right]\times k}$ in the recurrent formula is equal to*

(2.7)

$$\begin{cases}1 & \text{if } i=0,\ k\geq 1,\\ 4\cdot\Gamma_1^{\left[\frac{s-1}{s-1+m}\right]\times 1}-\Gamma_2^{\left[\frac{s-1}{s-1+m}\right]\times 2} & \text{if } i=1,\ k\geq 2,\\ 4\cdot\Gamma_1^{\left[\frac{s-1}{s-1+m}\right]\times 1}-3 & \text{if } i=1,\ k=1,\\ 7\cdot\Gamma_2^{\left[\frac{s-1}{s-1+m}\right]\times 2}-12\cdot\Gamma_1^{\left[\frac{s-1}{s-1+m}\right]\times 1}-\Gamma_3^{\left[\frac{s-1}{s-1+m}\right]\times 3}+2 & \text{if } i=2,\ k\geq 3,\\ 7\cdot\Gamma_2^{\left[\frac{s-1}{s-1+m}\right]\times 2}-12\cdot\Gamma_1^{\left[\frac{s-1}{s-1+m}\right]\times 1}+2 & \text{if } i=2,\ k=2,\\ 7\cdot\Gamma_i^{\left[\frac{s-1}{s-1+m}\right]\times i}-14\cdot\Gamma_{i-1}^{\left[\frac{s-1}{s-1+m}\right]\times(i-1)}+8\cdot\Gamma_{i-2}^{\left[\frac{s-1}{s-1+m}\right]\times(i-2)}-\Gamma_{i+1}^{\left[\frac{s-1}{s-1+m}\right]\times(i+1)} & \text{if } 3\leq i\leq 2s+m-3, k\geq i+1,\\ 7\cdot\Gamma_i^{\left[\frac{s-1}{s-1+m}\right]\times i}-14\cdot\Gamma_{i-1}^{\left[\frac{s-1}{s-1+m}\right]\times(i-1)}+8\cdot\Gamma_{i-2}^{\left[\frac{s-1}{s-1+m}\right]\times(i-2)} & \text{if } 3\leq i\leq 2s+m-3,\ k=i,\\ 7\cdot\Gamma_{2s+m-2}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-2)}-14\cdot\Gamma_{2s+m-3}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-3)}+8\cdot\Gamma_{2s+m-4}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-4)} & \text{if } i=2s+m-2,\ k\geq i,\\ -14\cdot\Gamma_{2s+m-2}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-2)}+8\cdot\Gamma_{2s+m-3}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-3)} & \text{if } i=2s+m-1,\ k\geq i,\\ 8\cdot\Gamma_{2s+m-2}^{\left[\frac{s-1}{s-1+m}\right]\times(2s+m-2)} & \text{if } i=2s+m,\ k\geq i.\end{cases}$$

Theorem 2.5. *We have*

(2.8)

$$\Delta_i^{\left[\frac{s}{s+m}\right]\times k}=\Delta_i^{\left[\frac{s}{s+m}\right]\times(i+1)}\qquad \text{for } i\in[0,2s+m-3],\ k\geq i+1,$$

(2.9)

$$\Delta_i^{\left[\frac{s}{s+m}\right]\times k}=\Delta_i^{\left[\frac{s}{s+m}\right]\times i}\qquad \text{for } i\in\{2s+m-2,2s+m-1,2s+m\},\ k\geq i.$$

Theorem 2.6. *We have for all $m \geq 0$*

$$\Gamma_j^{\left[\substack{s\\s+m}\right]\times(k+1)} - \Gamma_j^{\left[\substack{s\\s+m}\right]\times k} = 0 \quad \text{if} \quad 0 \leq j \leq s-1,\ k > j. \tag{2.10}$$

We have in the cases $m \in \{0,1\}$

(2.11)
$$\Gamma_{s+j}^{\left[\substack{s\\s}\right]\times(k+1)} - \Gamma_{s+j}^{\left[\substack{s\\s}\right]\times k} = \begin{cases} 3\cdot 2^{k+s-1} & \text{if } j=0,\ k>s,\\ 21\cdot 2^{k+s+3j-4} & \text{if } 1\leq j\leq s-1,\ k>s+j,\\ 3\cdot 2^{2k+2s-2} - 3\cdot 2^{k+4s-4} & \text{if } j=s,\ k>2s, \end{cases}$$

(2.12)
$$\Gamma_{s+j}^{\left[\substack{s\\s+1}\right]\times(k+1)} - \Gamma_{s+j}^{\left[\substack{s\\s+1}\right]\times k} = \begin{cases} 2^{k+s-1} & \text{if } j=0,\ k>s,\\ 11\cdot 2^{k+s-1} & \text{if } j=1,\ k>s+1,\\ 21\cdot 2^{k+s+3j-5} & \text{if } 2\leq j\leq s,\ k>s+j,\\ 3\cdot 2^{2k+2s-1} - 3\cdot 2^{k+4s-2} & \text{if } j=s+1,\ k>2s+1. \end{cases}$$

In the case $m \geq 2$

$$\Gamma_{s+j}^{\left[\substack{s\\s+m}\right]\times(k+1)} - \Gamma_{s+j}^{\left[\substack{s\\s+m}\right]\times k} = \begin{cases} 2^{k+s-1} & \text{if } j=0,\ k>s,\\ 3\cdot 2^{k+s+2j-3} & \text{if } 1\leq j\leq m-1,\ k>s+j,\\ 11\cdot 2^{k+s+2m-3} & \text{if } j=m,\ k>s+m, \end{cases} \tag{2.13}$$

(2.14)
$$\Gamma_{s+m+j}^{\left[\substack{s\\s+m}\right]\times(k+1)} - \Gamma_{s+m+j}^{\left[\substack{s\\s+m}\right]\times k} = \begin{cases} 21\cdot 2^{k+s+2m+3j-4} & \text{if } 1\leq j\leq s-1,\ k>s+m+j,\\ 3\cdot 2^{2k+2s+m-2} - 3\cdot 2^{k+4s+2m-4} & \text{if } j=s,\ k>2s+m. \end{cases}$$

Theorem 2.7. *We have for $m \geq 1$*

$$\Gamma_{s+j}^{\left[\substack{s\\s+m}\right]\times k} = 8^{j-1}\cdot \Gamma_{s+1}^{\left[\substack{s\\s+(m-(j-1))}\right]\times(k-(j-1))} \qquad \text{if } 1\leq j\leq m,\ k\geq s+j, \tag{2.15}$$

(2.16)
$$\Gamma_{s+1}^{\left[\substack{s\\s+(m-(j-1))}\right]\times(s+1)} = 2^{4s+(m-(j-1))} - 3\cdot 2^{3s-1} + 2^{2s-1} \qquad \text{if } 1\leq j\leq m,\ k=s+j,$$

(2.17)
$$\Gamma_{s+1}^{\left[\substack{s\\s+(m-(j-1))}\right]\times(k-(j-1))} = 3\cdot 2^{k-j+s} + 21\cdot[2^{3s-1} - 2^{2s-1}] \qquad \text{if } 1\leq j\leq m-1,\ k>s+j,$$

(2.18)
$$\Gamma_{s+1}^{\left[\substack{s\\s+1}\right]\times(k-(m-1))} = 11\cdot 2^{k-m+s} + 21\cdot 2^{3s-1} - 11\cdot 2^{2s-1} \quad \text{if } j=m,\ k>s+m.$$

Theorem 2.8. *We have for $m \geq 0$*

(2.19)
$$\Gamma_{s+m+1+j}^{\left[\begin{smallmatrix} s \\ s+m \end{smallmatrix}\right]\times k} = 8^{2j+m} \cdot \Gamma_{s-j+1}^{\left[\begin{smallmatrix} s-j \\ s-j \end{smallmatrix}\right]\times(k-m-2j)} \qquad \text{if} \quad 0 \leq j \leq s-1,\ k \geq s+m+1+j,$$

(2.20)
$$\Gamma_{s-j+1}^{\left[\begin{smallmatrix} s-j \\ s-j \end{smallmatrix}\right]\times(s-j+1)} = 2^{4s-4j} - 3 \cdot 2^{3s-3j-1} + 2^{2s-2j-1} \qquad \text{if} \quad 0 \leq j \leq s-1,\ k = s+m+1+j,$$

(2.21)
$$\Gamma_{s-j+1}^{\left[\begin{smallmatrix} s-j \\ s-j \end{smallmatrix}\right]\times(k-m-2j)} = 21 \cdot [2^{k-m-3j+s-1} + 2^{3s-3j-1} - 5 \cdot 2^{2s-2j-1}] \quad \text{if } 0 \leq j \leq s-2,\ k > s+m+1+j,$$

(2.22)
$$\Gamma_{2}^{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\times(k-m-2s+2)} = 2^{2(k-m)-4s+4} - 3 \cdot 2^{k-m-2s+2} + 2 \qquad \text{if} \quad j = s-1,\ k > 2s+m.$$

Theorem 2.9. *We have*

(2.23)
$$\Gamma_{i}^{\left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0,\ k \geq 1, \\ 21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if } 1 \leq i \leq s-1,\ k > i, \\ 3 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-4} - 27 \cdot 2^{2s-3} & \text{if } i = s,\ k > s, \\ 21 \cdot [2^{k-2s+3i-4} + 2^{3i-4} - 5 \cdot 2^{4i-2s-5}] & \text{if } s+1 \leq i \leq 2s-1,\ k > i, \\ 2^{2k+2s-2} - 3 \cdot 2^{k+4s-4} + 2^{6s-5} & \text{if } i = 2s,\ k > 2s. \end{cases}$$

Theorem 2.10. *We have*

(2.24)
$$\Gamma_{i}^{\left[\begin{smallmatrix} s \\ s \end{smallmatrix}\right]\times i} = \begin{cases} 2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \leq i \leq s, \\ 2^{2s+2i-2} - 3 \cdot 2^{3i-4} + 2^{4i-2s-5} & \text{if } s+1 \leq i \leq 2s. \end{cases}$$

Theorem 2.11. *We have*

(2.25)
$$\Gamma_{i}^{\left[\begin{smallmatrix} s \\ s+1 \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0,\ k \geq 1, \\ 21 \cdot 2^{3i-4} - 3 \cdot 2^{2i-3} & \text{if } 1 \leq i \leq s-1,\ k > i, \\ 2^{k+s-1} + 21 \cdot 2^{3s-4} - 11 \cdot 2^{2s-3} & \text{if } i = s,\ k > s, \\ 11 \cdot 2^{k+s-1} + 21 \cdot 2^{3s-1} - 53 \cdot 2^{2s-1} & \text{if } i = s+1,\ k > s+1, \\ 21 \cdot [2^{k-2s+3i-5} + 2^{3i-4} - 5 \cdot 2^{4i-2s-6}] & \text{if } s+2 \leq i \leq 2s,\ k > i, \\ 2^{2k+2s-1} - 3 \cdot 2^{k+4s-2} + 2^{6s-2} & \text{if } i = 2s+1,\ k > 2s+1. \end{cases}$$

Theorem 2.12. *We have*

$$\Gamma_i^{\left[\substack{s\\s+1}\right]\times i} = \begin{cases} 2^{2s+2i-1} - 3\cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \le i \le s+1, \\ 2^{2s+2i-2} - 3\cdot 2^{3i-4} + 2^{4i-2s-6} & \text{if } s+2 \le i \le 2s+1. \end{cases} \tag{2.26}$$

Theorem 2.13. *We have for $m \ge 2$*

(2.27)

$$\Gamma_i^{\left[\substack{s\\s+m}\right]\times k} = \begin{cases} 1 & \text{if } i = 0,\ k \ge 1, \\ 21\cdot 2^{3i-4} - 3\cdot 2^{2i-3} & \text{if } 1 \le i \le s-1,\ k > i, \\ 2^{k+s-1} + 21\cdot 2^{3s-4} - 11\cdot 2^{2s-3} & \text{if } i = s,\ k > s, \\ 3\cdot 2^{k-s+2i-3} + 21\cdot [2^{3i-4} - 2^{3i-s-4}] & \text{if } s+1 \le i \le s+m-1,\ k > i, \\ 11\cdot 2^{k+s+2m-3} + 21\cdot 2^{3s+3m-4} - 53\cdot 2^{2s+3m-4} & \text{if } i = s+m,\ k > s+m, \\ 21\cdot [2^{k-2s+3i-m-4} + 2^{3i-4} - 5\cdot 2^{4i-2s-m-5}] & \text{if } s+m+1 \le i \le 2s+m-1,\ k > i, \\ 2^{2k+2s+m-2} - 3\cdot 2^{k+4s+2m-4} + 2^{6s+3m-5} & \text{if } i = 2s+m,\ k > 2s+m. \end{cases}$$

Theorem 2.14. *We have for $m \ge 2$*

(2.28)

$$\Gamma_i^{\left[\substack{s\\s+m}\right]\times i} = \begin{cases} 2^{2s+2i+m-2} - 3\cdot 2^{3i-4} + 2^{2i-3} & \text{if } 1 \le i \le s+1, \\ 2^{2s+2i+m-2} - 3\cdot 2^{3i-4} + 2^{3i-s-4} & \text{if } s+2 \le i \le s+m+1, \\ 2^{2s+2i+m-2} - 3\cdot 2^{3i-4} + 2^{4i-2s-m-5} & \text{if } s+m+2 \le i \le 2s+m. \end{cases}$$

Theorem 2.15. *We denote by $R_q(k,s,m)$ the number of solutions $(Y_1, Z_1, U_1, \ldots, Y_q, Z_q, U_q)$ of the polynomial equations*

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0, \\ Y_1U_1 + Y_2U_2 + \ldots + Y_qU_q = 0, \end{cases}$$

satisfying the degree conditions

$$deg Y_i \le k-1, \quad deg Z_i \le s-1, \quad deg U_i \le s+m-1 \quad for \quad 1 \le i \le q.$$

Then

$$\begin{aligned} R_q(q,k,s,m) &= \int_{\mathbb{P}\times\mathbb{P}} g_{k,s,m}^q(t,\eta)dtd\eta \\ &= 2^{(2s+m+k)(q-1)} \cdot 2^{-k+2} \cdot \sum_{i=0}^{\inf(2s+m,k)} \Gamma_i^{\left[\substack{s\\s+m}\right]\times k} \cdot 2^{-qi}. \end{aligned} \tag{2.29}$$

Example. Let q = 3, k = 4, s = 3, m = 2. Then

$$\Gamma_i^{\left[\substack{3\\3+2}\right]\times 4} = \begin{cases} 1 & \text{if } i = 0, \\ 9 & \text{if } i = 1, \\ 78 & \text{if } i = 2, \\ 648 & \text{if } i = 3, \\ 15648 & \text{if } i = 4. \end{cases}$$

Hence the number $R_3(4,3,2)$ of solutions
$(Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + Y_3Z_3 = 0, \\ Y_1U_1 + Y_2U_2 + Y_3U_3 = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 3, \quad degZ_i \leq 2, \quad degU_i \leq 4 \quad for \quad 1 \leq i \leq 3$$

is equal to

$$\begin{aligned} &\int_{\mathbb{P}\times\mathbb{P}} g^3_{4,3,2}(t,\eta)dtd\eta = 2^{22} \cdot \sum_{i=0}^{4} \Gamma_i^{\left[\begin{smallmatrix}3\\3+2\end{smallmatrix}\right]\times 4} \cdot 2^{-3i} \\ &= 2^{22} \cdot [1 + 9 \cdot 2^{-3} + 78 \cdot 2^{-6} + 648 \cdot 2^{-9} + 15648 \cdot 2^{-12}] \\ &= 35356672. \end{aligned}$$

Example. Let q = 4, k = 6, s = 5, m = 0. Then

$$\Gamma_i^{\left[\begin{smallmatrix}5\\5\end{smallmatrix}\right]\times 6} = \begin{cases} 1 & \text{if } i = 0, \\ 9 & \text{if } i = 1, \\ 78 & \text{if } i = 2, \\ 648 & \text{if } i = 3, \\ 5280 & \text{if } i = 4, \\ 42624 & \text{if } i = 5, \\ 999936 & \text{if } i = 6. \end{cases}$$

Hence the number $R_4(6,5,0)$ of solutions
$(Y_1, Z_1, U_1, Y_2, Z_2, U_2, Y_3, Z_3, U_3, Y_4, Z_4, U_4)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + Y_3Z_3 + Y_4Z_4 = 0, \\ Y_1U_1 + Y_2U_2 + Y_3U_3 + Y_4U_4 = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 5, \quad degZ_i \leq 4, \quad degU_i \leq 4 \quad for \quad 1 \leq i \leq 4$$

is equal to

$$\begin{aligned} &\int_{\mathbb{P}\times\mathbb{P}} g^4_{4,5,0}(t,\eta)dtd\eta \\ &= 2^{44} \cdot \sum_{i=0}^{6} \Gamma_i^{\left[\begin{smallmatrix}5\\5\end{smallmatrix}\right]\times 6} \cdot 2^{-4i} \\ &= 2^{44} \cdot [1 + 9 \cdot 2^{-4} + 78 \cdot 2^{-8} + 648 \cdot 2^{-12} + 5280 \cdot 2^{-16} + 42624 \cdot 2^{-20} + 999936 \cdot 2^{-24}] \\ &= 37014016 \cdot 2^{20}. \end{aligned}$$

Example. The fraction of square double persymmetric $\begin{bmatrix} s \\ s+m \end{bmatrix} \times (2s+m)$ matrices which are invertible is equal to $\dfrac{\Gamma_{2s+m}^{\begin{bmatrix} s \\ s+m \end{bmatrix} \times (2s+m)}}{\sum_{i=0}^{2s+m} \Gamma_i^{\begin{bmatrix} s \\ s+m \end{bmatrix} \times (2s+m)}} = \dfrac{3}{8}$.

3. Exponential sums and rank of triple persymmetric matrices over $\mathbb{F}_2$

Abstract. Notre travail concerne une généralisation des résultats obtenus dans :
Exponential sums and rank of double persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1937.

Soit $\mathbb{K}^3$ le $\mathbb{K}$ - espace vectoriel de dimension 3 où $\mathbb{K}$ dénote le corps des séries de Laurent formelles $\mathbb{F}_2((T^{-1}))$. Nous calculons en particulier des sommes exponentielles (dans $\mathbb{K}^3$) de la forme
$\sum_{\deg Y\leq k-1}\sum_{\deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)\sum_{\deg V\leq s+m+l-1}E(\xi YV)$
où (t,η,ξ) est dans la boule unité de $\mathbb{K}^3$.
Nous démontrons qu'elles dépendent uniquement du rang de matrices triples persymétriques avec des entrées dans $\mathbb{F}_2$, c'est-à-dire des matrices de la forme $\begin{bmatrix}A\\ \frac{B}{C}\end{bmatrix}$ où A est une matrice $s\times k$ persymétrique , B une matrice $(s+m)\times k$ persymétrique et C une matrice $(s+m+l)\times k$ persymétrique (une matrice $[\alpha_{i,j}]$ est persymétrique si $\alpha_{i,j}=\alpha_{r,s}$ pour i+j = r+s). En outre, nous établissons plusieurs formules concernant des propriétés de rang de partitions de matrices triples persymétriques, ce qui nous conduit à une formule récurrente du nombre $\Gamma_i^{\left[\begin{smallmatrix}s\\ s+m\\ s+m+l\end{smallmatrix}\right]\times k}$ des matrices de rang i de la forme $\begin{bmatrix}A\\ \frac{B}{C}\end{bmatrix}$ Nous déduisons de cette formule récurrente que si $0\leq i\leq \inf(s-1,k-1)$, le nombre $\Gamma_i^{\left[\begin{smallmatrix}s\\ s+m\\ s+m+l\end{smallmatrix}\right]\times k}$ dépend uniquement de i. D'autre part, si $i\geq 2s+m+1, k\geq i$, $\Gamma_i^{\left[\begin{smallmatrix}s\\ s+m\\ s+m+l\end{smallmatrix}\right]\times k}$ peut être calculé à partir du nombre $\Gamma_{2s'+m'+1}^{\left[\begin{smallmatrix}s'\\ s'+m'\\ s'+m'+l'\end{smallmatrix}\right]\times k'}$ de matrices de rang (2s'+m'+1) de la forme $\begin{bmatrix}A'\\ \frac{B'}{C'}\end{bmatrix}$ où A' est une matrice $s'\times k'$ persymétrique, B' une matrice $(s'+m')\times k'$ persymétrique et C' une matrice $(s'+m'+l')\times k'$ où s', m',l' et k' dépendent de i, s, m,l et k. La preuve de ce résultat est basée sur une formule du nombre de matrices de rang i de la forme $\begin{bmatrix}A\\ b_-\end{bmatrix}$ où A est une matrice double persymétrique et b_- une matrice ligne avec entrées dans $\mathbb{F}_2$. Nous montrons également que le nombre R de représentations dans $\mathbb{F}_2[T]$ des équations polynomiales

$$\begin{cases} YZ+Y_1Z_1+\ldots+Y_{q-1}Z_{q-1}=0\\ YU+Y_1U_1+\ldots+Y_{q-1}U_{q-1}=0\\ YV+Y_1V_1+\ldots+Y_{q-1}V_{q-1}=0\end{cases}$$

associées aux sommes exponentielles
$\sum_{deg Y\leq k-1}\sum_{deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)\sum_{\deg V\leq s+m+l-1}E(\xi YV)$
est donné par une intégrale sur la boule unité de $\mathbb{K}^3$ et est une combinaison linéaire de $\Gamma_i^{\left[\begin{smallmatrix}s\\ s+m\\ s+m+l\end{smallmatrix}\right]\times k}$ pour $i\geq 0$. Nous pouvons alors calculer explicitement le nombre R. Notre article est, pour des raisons de longueur, limité au cas $m\geqslant 0,\ l=0$.

Abstract. Our work concerns a generalization of the results obtained in : Exponential sums and rank of double persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1937.

Let $\mathbb{K}^3$ be the 3-dimensional vectorspace over $\mathbb{K}$ where $\mathbb{K}$ denotes the field of Laurent Series $\mathbb{F}_2((T^{-1}))$. We compute in particular exponential sums, (in $\mathbb{K}^3$) of the form
$\sum_{\deg Y\leq k-1}\sum_{\deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)\sum_{\deg V\leq s+m+l-1}E(\xi YV)$
where (t,η,ξ) is in the unit interval of $\mathbb{K}^3$. We show that they only depend on the rank of some associated triple persymmetric matrices with entries in $\mathbb{F}_2$, that is matrices of the form $\left[\begin{smallmatrix}A\\ \hline B\\ \hline C\end{smallmatrix}\right]$ where A is a $s\times k$ persymmetric matrix, B a $(s+m)\times k$ persymmetric matrix and C is a $(s+m+l)\times k$ persymmetric matrix (A matrix $[\alpha_{i,j}]$ is persymmetric if $\alpha_{i,j}=\alpha_{r,s}\ for\ i+j=r+s$). Besides, we establish several formulas concerning rank properties of partitions of triple persymmetric matrices, which leads to a recurrent formula for the number $\Gamma_i^{\left[\begin{smallmatrix}s\\ \hline s+m\\ \hline s+m+l\end{smallmatrix}\right]\times k}$ of rank i matrices of the form $\left[\begin{smallmatrix}A\\ \hline B\\ \hline C\end{smallmatrix}\right]$ We deduce from the recurrent formula that if $0\leq i\leq \inf(s-1,k-1)$ then $\Gamma_i^{\left[\begin{smallmatrix}s\\ \hline s+m\\ \hline s+m+l\end{smallmatrix}\right]\times k}$ depends only on i. On the other hand, if $i\geq 2s+m+1, k\geq i$, $\Gamma_i^{\left[\begin{smallmatrix}s\\ \hline s+m\\ \hline s+m+l\end{smallmatrix}\right]\times k}$ can be computed from the number $\Gamma_{2s'+m'+1}^{\left[\begin{smallmatrix}s'\\ \hline s'+m'\\ \hline s'+m'+l'\end{smallmatrix}\right]\times k'}$ of rank (2s'+m'+1) matrices of the form $\left[\begin{smallmatrix}A'\\ \hline B'\\ \hline C'\end{smallmatrix}\right]$ where A' is a $s'\times k'$ persymmetric matrix, B' a $(s'+m')\times k'$ persymmetric matrix and C' a $(s'+m'+l')\times k'$ persymmetric matrix , where s', m',l' and k' depend on i, s, m,l and k. The proof of this result is based on a formula of the number of rank i matrices of the form $\left[\begin{smallmatrix}A\\ \hline b_-\end{smallmatrix}\right]$ where A is double persymmetric and b_- a one-row matrix with entries in $\mathbb{F}_2$. We also prove that the number R of representations in $\mathbb{F}_2[T]$ of the polynomial equations

$$\begin{cases} YZ+Y_1Z_1+\ldots+Y_{q-1}Z_{q-1}=0\\ YU+Y_1U_1+\ldots+Y_{q-1}U_{q-1}=0\\ YV+Y_1V_1+\ldots+Y_{q-1}V_{q-1}=0\end{cases}$$

associated to the exponential sums
$\sum_{deg Y\leq k-1}\sum_{deg Z\leq s-1}E(tYZ)\sum_{\deg U\leq s+m-1}E(\eta YU)\sum_{\deg U\leq s+m+l-1}E(\xi YV)$
is given by an integral over the unit interval of $\mathbb{K}^3$, and is a linear combination of the $\Gamma_i^{\left[\begin{smallmatrix}s\\ \hline s+m\\ \hline s+m+l\end{smallmatrix}\right]\times k}$ pour $i\geq 0$. We can then compute explicitly the number R.Our article is for reasons of length limited to the case $m\geqslant 0,\ l=0$

3.1. **A recurrent formula for the number of rank i matrices of the form $\left[\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right]$, where A, B and C are persymmetric matrices over $\mathbb{F}_2$,.**

Lemma 3.1. *Let $s \geq 2,\ m \geq 0,\ l \geq 0,\ k \geq 1$ and $0 \leq i \leq \inf(3s+2m+l, k)$. Then we have the following recurrent formula for the number $\Gamma_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k}$ of rank i matrices of the form $\left[\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right]$ such that A is a $s \times k$ persymmetric matrix over $\mathbb{F}_2$, B a $(s+m) \times k$ persymmetric matrix and C a $(s+m+l) \times k$ persymmetric matrix*

(3.1)

$$\begin{aligned}
&\Gamma_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k} \\
&= \left[2 \cdot \Gamma_{i-1}^{\left[\begin{smallmatrix} s-1 \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k} + 4 \cdot \Gamma_{i-1}^{\left[\begin{smallmatrix} s \\ s+m-1 \\ s+m+l \end{smallmatrix}\right]\times k} + 8 \cdot \Gamma_{i-1}^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l-1 \end{smallmatrix}\right]\times k}\right] - \left[8 \cdot \Gamma_{i-2}^{\left[\begin{smallmatrix} s-1 \\ s+m-1 \\ s+m+l \end{smallmatrix}\right]\times k} + 16 \cdot \Gamma_{i-2}^{\left[\begin{smallmatrix} s-1 \\ s+m \\ s+m+l-1 \end{smallmatrix}\right]\times k} + 32 \cdot \Gamma_{i-2}^{\left[\begin{smallmatrix} s \\ s+m-1 \\ s+m+l-1 \end{smallmatrix}\right]\times k}\right] \\
&+ 64 \cdot \Gamma_{i-3}^{\left[\begin{smallmatrix} s-1 \\ s+m-1 \\ s+m+l-1 \end{smallmatrix}\right]\times k} + \Delta_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k}
\end{aligned}$$

where

(3.2)

$$\Delta_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k} = \sigma_{i,i,i,i}^{\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]\times k} - 7 \cdot \sigma_{i-1,i-1,i-1,i-1}^{\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]\times k} + 14 \cdot \sigma_{i-2,i-2,i-2,i-2}^{\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]\times k} - 8 \cdot \sigma_{i-3,i-3,i-3,i-3}^{\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]\times k}$$

Recall that

$$\sigma_{i,i,i,i}^{\left[\begin{smallmatrix} \alpha \\ \beta \\ \gamma \end{smallmatrix}\right]\times k}$$

denotes the cardinality of the following set

$$\begin{aligned}
&\Big\{(t,\eta,\xi) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \times \mathbb{P}/\mathbb{P}_{k+s+m+l-1} \mid r(D^{\left[\begin{smallmatrix} s-1 \\ s+m-1 \\ s+m+l-1 \end{smallmatrix}\right]\times k}(t,\eta,\xi)) = i \\
&r(D^{\left[\begin{smallmatrix} s \\ s+m-1 \\ s+m+l-1 \end{smallmatrix}\right]\times k}(t,\eta,\xi)) = i, \quad r(D^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l-1 \end{smallmatrix}\right]\times k}(t,\eta,\xi)) = i, \quad r(D^{\left[\begin{smallmatrix} s \\ s+m \\ s+m+l \end{smallmatrix}\right]\times k}(t,\eta,\xi)) = i\Big\}
\end{aligned}$$

3.2. **An outline of the main results in the case $m = l = 0$.**

Theorem 3.2. *The number* $\Gamma_i^{\left[\begin{smallmatrix}s\\s\\s\end{smallmatrix}\right]\times k}$ *of triple persymmetric* $3s\times k$ *matrices over* $\mathbb{F}_2$

$$\left(\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{k-1} & \alpha_k\\
\alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_k & \alpha_{k+1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\alpha_{s-1} & \alpha_s & \alpha_{s+1} & \dots & \alpha_{s+k-3} & \alpha_{s+k-2}\\
\alpha_s & \alpha_{s+1} & \alpha_{s+2} & \dots & \alpha_{s+k-2} & \alpha_{s+k-1}\\
\hline
\beta_1 & \beta_2 & \beta_3 & \dots & \beta_{k-1} & \beta_k\\
\beta_2 & \beta_3 & \beta_4 & \dots & \beta_k & \beta_{k+1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\beta_{s-1} & \beta_s & \beta_{s+1} & \dots & \beta_{s+k-3} & \beta_{s+k-2}\\
\beta_s & \beta_{s+1} & \beta_{s+2} & \dots & \beta_{s+k-2} & \beta_{s+k-1}\\
\hline
\gamma_1 & \gamma_2 & \gamma_3 & \dots & \gamma_{k-1} & \gamma_k\\
\gamma_2 & \gamma_3 & \gamma_4 & \dots & \gamma_k & \gamma_{k+1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\gamma_{s-1} & \gamma_s & \gamma_{s+1} & \dots & \gamma_{s+k-3} & \gamma_{s+k-2}\\
\gamma_s & \gamma_{s+1} & \gamma_{s+2} & \dots & \gamma_{s+k-2} & \gamma_{s+k-1}
\end{array}\right)$$

of rank i *is given by*

(3.3)

$$\begin{cases}
1 & \text{if } i=0,\\
105\cdot 2^{4i-6}-21\cdot 2^{3i-5} & \text{if } 1\le i\le s-1,\ k\ge i+1,\\
7\cdot 2^{k+s-1}-7\cdot 2^{2s}+105\cdot 2^{4s-6}-21\cdot 2^{3s-5} & \text{if } i=s,\ k\ge s+1,\ s\ge 1,\\
147\cdot(5\cdot 2^{j-1}-1)\cdot 2^{k+s+3j-6} & \\
+21\cdot\left[5\cdot 2^{4s+4j-6}-2^{3s+3j-5}-(155\cdot 2^{j-1}-35)\cdot 2^{2s+4j-7}\right] & \text{if } i=s+j,\ 1\le j\le s-1,\ k\ge s+j+1,\\
7\cdot 2^{2k+2s-2}+21\cdot\left[35\cdot 2^{k+5s-7}-39\cdot 2^{k+4s-6}\right] & \\
+7\cdot\left[15\cdot 2^{8s-6}-465\cdot 2^{7s-8}+349\cdot 2^{6s-7}\right] & \text{if } i=2s,\ k\ge 2s+1,\\
105\cdot\left(2^{2k+2s+4j-2}+7\cdot 2^{k+5s+4j-3}-31\cdot 2^{k+4s+5j-3}\right) & \\
+105\cdot\left(2^{8s+4j-2}-31\cdot 2^{7s+5j-3}+93\cdot 2^{6s+6j-3}\right) & \text{if } \ i=2s+1+j,\ k\ge 2s+2+j,\\
 & 0\le j\le s-2,\\
2^{3k+3s-3}-7\cdot 2^{2k+6s-6}+7\cdot 2^{k+9s-8}-2^{12s-9} & \text{if } i=3s,\ k\ge 3s
\end{cases}$$

(3.4)

$$\Gamma_i^{\left[\begin{smallmatrix}s\\s\\s\end{smallmatrix}\right]\times i}=\begin{cases}
2^{3s+3i-3}-7\cdot 2^{4i-6}+3\cdot 2^{3i-5} & \text{if } \ 1\le i\le s+1,\\
2^{6s+3j-3}+7\cdot 2^{2s+5j-8}-7\cdot 2^{2s+4j-7}-7\cdot 2^{4s+4j-6}+3\cdot 2^{3s+3j-5} & \text{if } \ i=s+j,\ 1\le j\le s+1,\\
2^{9s+3j}-7\cdot 2^{8s+4j-2}+7\cdot 2^{7s+5j-3}-2^{6s+6j-3} & \text{if } \ i=2s+1+j,\ 0\le j\le s-1,
\end{cases}$$

We have for $0\le j\le s-2,\quad k\ge 2s+2+j$

$$\Gamma_{2s+1+j}^{\left[\begin{smallmatrix}s\\s\\s\end{smallmatrix}\right]\times k}=16^{3j}\cdot\Gamma_{2(s-j)+1}^{\left[\begin{smallmatrix}s-j\\s-j\\s-j\end{smallmatrix}\right]\times(k-3j)} \tag{3.5}$$

We have for $0\le j\le s-1$.

$$\Gamma_{2s+1+j}^{\left[\begin{smallmatrix}s\\s\\s\end{smallmatrix}\right]\times(2s+1+j)}=16^{3j}\cdot\Gamma_{2(s-j)+1}^{\left[\begin{smallmatrix}s-j\\s-j\\s-j\end{smallmatrix}\right]\times(2(s-j)+1)} \tag{3.6}$$

We have for $k \geq 3s$.

$$\Gamma_{3s}^{\left[\begin{smallmatrix} s \\ s \\ s \end{smallmatrix}\right]\times k} = 16^{3(s-1)} \cdot \Gamma_{3}^{\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]\times(k-3(s-1))} \tag{3.7}$$

Theorem 3.3. *Let q be a rational integer ≥ 1, then*

(3.8)

$$g_{k,s}(t,\eta,\xi) = g(t,\eta,\xi) = \sum_{degY \leq k-1} \sum_{degZ \leq s-1} E(tYZ) \sum_{degU \leq s-1} E(\eta YU) \sum_{degV \leq s-1} E(\eta YV) = 2^{3s+k-r(D^{\left[\begin{smallmatrix} s \\ s \\ s \end{smallmatrix}\right]\times k}(t,\eta,\xi))},$$

(3.9)

$$\int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g^q(t,\eta,\xi)dtd\eta d\xi = 2^{(3s+k)q} \cdot 2^{-3k-3s+3} \cdot \sum_{i=0}^{\inf(3s,k)} \Gamma_{i}^{\left[\begin{smallmatrix} s \\ s \\ s \end{smallmatrix}\right]\times k} \cdot 2^{-qi}.$$

Theorem 3.4. *We denote by $R_q(k,s)$ the number of solutions $(Y_1, Z_1, U_1, V_1, \ldots, Y_q, Z_q, U_q, V_q)$ of the polynomial equations*

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0, \\ Y_1U_1 + Y_2U_2 + \ldots + Y_qU_q = 0, \\ Y_1V_1 + Y_2V_2 + \ldots + Y_qV_q = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degZ_i \leq s-1, \quad degU_i \leq s-1, \quad degV_i \leq s-1 \quad for \quad 1 \leq i \leq q.$$

Then

(3.10)

$$R_q(k,s) = \int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g_{k,s}^q(t,\eta,\xi)dtd\eta d\xi = 2^{(3s+k)q} \cdot 2^{-3k-3s+3} \cdot \sum_{i=0}^{\inf(3s,k)} \Gamma_{i}^{\left[\begin{smallmatrix} s \\ s \\ s \end{smallmatrix}\right]\times k} \cdot 2^{-qi}.$$

Example. s = 1, $k \geq i+1$ for $0 \leq i \leq 2$

$$\Gamma_{i}^{\left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0 \\ 7 \cdot (2^k - 1) & \text{if } i = 1 \\ 7 \cdot (2^k - 1) \cdot (2^k - 2) & \text{if } i = 2 \\ 2^{3k} - 7 \cdot 2^{2k} + 7 \cdot 2^{k+1} - 2^3 & \text{if } i = 3,\ k \geq 3 \end{cases}$$

Example. s = 2, $k \geq i+1$ for $0 \leq i \leq 5$

$$\Gamma_{i}^{\left[\begin{smallmatrix} 2 \\ 2 \\ 2 \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 7 \cdot 2^{k+1} + 266 & \text{if } i = 2 \\ 147 \cdot 2^{k+1} + 1344 & \text{if } i = 3 \\ 7 \cdot 2^{2k+2} + 651 \cdot 2^{k+2} - 22624 & \text{if } i = 4 \\ 105 \cdot 2^{2k+2} - 315 \cdot 2^{k+5} + 53760 & \text{if } i = 5 \\ 2^{3k+3} - 7 \cdot 2^{2k+6} + 7 \cdot 2^{k+10} - 32768 & \text{if } i = 6,\ k \geq 6 \end{cases}$$

Example. s = 2, k = 6.

The number $\Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]\times 6}$ of rank i matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{pmatrix}$$

is equal to

$$\begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 1162 & \text{if } i = 2 \\ 20160 & \text{if } i = 3 \\ 258720 & \text{if } i = 4 \\ 1128960 & \text{if } i = 5 \\ 688128 & \text{if } i = 6 \end{cases}$$

The number of solutions
$(Y_1, Z_1, U_1, V_1, \ldots, Y_q, Z_q, U_q, V_q)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0, \\ Y_1U_1 + Y_2U_2 + \ldots + Y_qU_q = 0, \\ Y_1V_1 + Y_2V_2 + \ldots + Y_qV_q = 0, \end{cases}$$

satisfying the degree conditions

$$deg Y_i \leq 5, \quad deg Z_i \leq 1, \quad deg U_i \leq 1, \quad deg V_i \leq 1 \quad for \quad 1 \leq i \leq q.$$

is equal to

$$R_q(6,2) = \int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g_{6,2}^q(t,\eta,\xi)dtd\eta d\xi = 2^{12q-21} \cdot \sum_{i=0}^{6} \Gamma_i^{\left[\begin{smallmatrix}2\\2\\2\end{smallmatrix}\right]\times 6} \cdot 2^{-qi}$$
$$= 2^{12q-21} \cdot \left(1 + 21 \cdot 2^{-q} + 1162 \cdot 2^{-2q} + 20160 \cdot 2^{-3q} + 258720 \cdot 2^{-4q} + 1128960 \cdot 2^{-5q} + 688128 \cdot 2^{-6q}\right)$$
$$= 2^{6q-21} \cdot \left(2^{6q} + 21 \cdot 2^{5q} + 1162 \cdot 2^{4q} + 20160 \cdot 2^{3q} + 258720 \cdot 2^{2q} + 1128960 \cdot 2^{q} + 688128\right)$$

Example. s = 3, $k \geq i + 1$ for $0 \leq i \leq 8$

$$\Gamma_i^{\left[\begin{smallmatrix}3\\3\\3\end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 378 & \text{if } i = 2 \\ 7 \cdot 2^{k+2} + 5936 & \text{if } i = 3 \\ 147 \cdot 2^{k+2} + 84672 & \text{if } i = 4 \\ 147 \cdot 9 \cdot 2^{k+3} + 959616 & \text{if } i = 5 \\ 7 \cdot 2^{2k+4} + 2121 \cdot 2^{k+6} + 5863424 & \text{if } i = 6 \\ 105 \cdot 2^{2k+4} + 2625 \cdot 2^{k+9} - 92897280 & \text{if } i = 7 \\ 105 \cdot 2^{2k+8} - 315 \cdot 2^{k+14} + 220200960 & \text{if } i = 8 \\ 2^{3k+6} - 7 \cdot 2^{2k+12} + 7 \cdot 2^{k+19} - 134217728 & \text{if } i = 9,\ k \geq 9 \end{cases}$$

Example. s = 3, k = 5, q = 3.

The number $\Gamma_i^{\left[\begin{smallmatrix}3\\3\\3\end{smallmatrix}\right]\times 5}$ of rank i matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 \\ \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \end{pmatrix}$$

is equal to

$$\begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 378 & \text{if } i = 2 \\ 6832 & \text{if } i = 3 \\ 103488 & \text{if } i = 4 \\ 1986432 & \text{if } i = 5 \end{cases}$$

The number of solutions
$(Y_1, Z_1, U_1, V_1, Y_2, Z_2, U_2, V_2, Y_3, Z_3, U_3, V_3)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + Y_3Z_3 = 0, \\ Y_1U_1 + Y_2U_2 + Y_3U_3 = 0, \\ Y_1V_1 + Y_2V_2 + Y_3V_3 = 0, \end{cases}$$

satisfying the degree conditions

$$deg Y_i \leq 4, \quad deg Z_i \leq 2, \quad deg U_i \leq 2, \quad deg V_i \leq 2 \quad for \quad 1 \leq i \leq 3.$$

is equal to

$$R_3(5,3) = \int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g^3_{5,3}(t,\eta,\xi)dtd\eta d\xi = 2^{33} \cdot \sum_{i=0}^{5} \Gamma_i^{\left[\begin{smallmatrix}3\\3\\3\end{smallmatrix}\right]\times 5} \cdot 2^{-3i}$$
$$= 2^{33} \cdot \left(1 + 21 \cdot 2^{-3} + 378 \cdot 2^{-6} + 6832 \cdot 2^{-9} + 103488 \cdot 2^{-12} + 1986432 \cdot 2^{-15}\right) = 3563904 \times 2^{18}$$

3.3. **An outline of the main results in the case $m = 1,\ l = 0$.**

Theorem 3.5. *We have*

(3.11)

$$\Gamma_i^{\left[\begin{smallmatrix} s \\ s+1 \\ s+1 \end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0, \\ 105\cdot 2^{4i-6} - 21\cdot 2^{3i-5} & \text{if } 1 \le i \le s-1,\ k \ge i+1, \\ 2^{k+s-1} - 2^{2s} + 105\cdot 2^{4s-6} - 21\cdot 2^{3s-5} & \text{if } i = s,\ k \ge s+1,\ s \ge 1, \\ 33\cdot 2^{k+s-1} + [105\cdot 2^{4s-2} - 21\cdot 2^{3s-2} - 69\cdot 2^{2s}] & \text{if } i = s+1,\ k \ge s+2, \\ 630\cdot 2^{k+s-1} + 21\cdot[5\cdot 2^{4s+2} - 2^{3s+1} - 65\cdot 2^{2s+1}] & \text{if } i = s+2,\ k \ge s+3, \\ 1365\cdot 2^{k+s+2} + 21\cdot[5\cdot 2^{4s+6} - 2^{3s+4} - 285\cdot 2^{2s+4}] & \text{if } i = s+3,\ k \ge s+4, \\ 2835\cdot 2^{k+s+5} + 21\cdot[5\cdot 2^{4s+10} - 2^{3s+7} - 595\cdot 2^{2s+8}] & \text{if } i = s+4,\ k \ge s+5, \\ 105\cdot(7\cdot 2^{j-2} - 1)\cdot 2^{k+s+3j-7} & \\ \quad +21\cdot\left[5\cdot 2^{4s+4j-6} - 2^{3s+3j-5} - 155\cdot 2^{2s+5j-10} + 25\cdot 2^{2s+4j-8}\right] & \text{if } i = s+j, \\ & 2 \le j \le s,\ k \ge s+j+1, \\ 3\cdot 2^{2s-1}\cdot(2^{2k} - 2^{4s+4}) + \left(735\cdot 2^{5s-5} - 393\cdot 2^{4s-4}\right)\cdot\left(2^k - 2^{2s+2}\right) & \\ \quad +21\cdot[5\cdot 2^{8s-2} + 2^{6s-4} - 15\cdot 2^{7s-5}] & \text{if } i = 2s+1,\ k \ge 2s+2, \\ 53\cdot 2^{2s-1}\cdot(2^{2k} - 2^{4s+6}) + \left(735\cdot 2^{5s-1} - 1629\cdot 2^{4s-1}\right)\cdot\left(2^k - 2^{2s+3}\right) & \\ \quad +21\cdot[5\cdot 2^{8s+2} + 3\cdot 2^{6s} - 15\cdot 2^{7s}] & \text{if } i = 2s+2,\ k \ge 2s+3, \\ 105\cdot\left(2^{2k+2s+4j+2} + 7\cdot 2^{k+5s+4j+3} - 31\cdot 2^{k+4s+5j+3}\right) & \\ \quad +105\cdot\left(2^{8s+4j+6} - 31\cdot 2^{7s+5j+5} + 93\cdot 2^{6s+6j+5}\right) & \text{if } \quad i = 2s+3+j,\ k \ge 2s+4+j, \\ & 0 \le j \le s-2, \\ 2^{3k+3s-1} - 7\cdot 2^{2k+6s-2} + 7\cdot 2^{k+9s-2} - 2^{12s-1} & \text{if } i = 3s+2,\ k \ge 3s+2 \end{cases}$$

(3.12)

$$\Gamma_i^{\left[\begin{smallmatrix} s \\ s+1 \\ s+1 \end{smallmatrix}\right]\times i} = \begin{cases} 2^{3s+3i-1} - 7\cdot 2^{4i-6} + 3\cdot 2^{3i-5} & \text{if } \ 1 \le i \le s+1, \\ 2^{6s+3j-1} - 7\cdot 2^{4s+4j-6} + 3\cdot 2^{3s+3j-5} & \\ \quad +7\cdot 2^{2s+5j-10} - 5\cdot 2^{2s+4j-8} & \text{if } \ i = s+j,\ 2 \le j \le s+3, \\ 2^{9s+2} + 7\cdot 2^{7s-5} - 7\cdot 2^{8s-2} + 7\cdot 2^{6s-4} & \text{if } \ i = 2s+1, \\ 2^{9s+5} + 7\cdot 2^{7s} - 7\cdot 2^{8s+2} + 2^{6s} & \text{if } \ i = 2s+2, \\ 2^{9s+8} + 7\cdot 2^{7s+5} - 7\cdot 2^{8s+6} - 2^{6s+5} & \text{if } \ i = 2s+3, \\ 2^{9s+3j+8} - 7\cdot 2^{8s+4j+6} + 7\cdot 2^{7s+5j+5} - 2^{6s+6j+5} & \text{if } \ i = 2s+3+j,\ 0 \le j \le s-1 \end{cases}$$

We have the following reduction formulas

$$\Gamma_{2s+2+j}^{\left[\begin{smallmatrix} s \\ s+1 \\ s+1 \end{smallmatrix}\right]\times k} = 16^{2j}\Gamma_{2s+2-j}^{\left[\begin{smallmatrix} s \\ s+(1-j) \\ s+(1-j) \end{smallmatrix}\right]\times(k-2j)} \qquad \text{if } 0 \le j \le 1 \tag{3.13}$$

$$\Gamma_{2s+3+j}^{\left[\begin{smallmatrix} s \\ s+1 \\ s+1 \end{smallmatrix}\right]\times k} = 16^{2+3j}\Gamma_{2(s-j)+1}^{\left[\begin{smallmatrix} s-j \\ s-j \\ s-j \end{smallmatrix}\right]\times(k-2-3j)} \qquad \text{if } 0 \le j \le s-1 \tag{3.14}$$

Example. We have for $s = 3$:

$$\Gamma_i^{\left[\begin{smallmatrix}3\\3+1\\3+1\end{smallmatrix}\right]\times k} = \begin{cases} 1 & \text{if } i = 0,\ k \geq 1, \\ 21 & \text{if } i = 1,\ k \geq 2, \\ 378 & \text{if } i = 2,\ k \geq 3, \\ 2^{k+2} + 6320 & \text{if } i = 3,\ k \geq 4, \\ 33 \cdot 2^{k+2} + 100416 & \text{if } i = 4,\ k \geq 5, \\ 630 \cdot 2^{k+2} + 1524096 & \text{if } i = 5,\ k \geq 6, \\ 1365 \cdot 2^{k+5} + 21224448 & \text{if } i = 6,\ k \geq 7, \\ 96 \cdot 2^{2k} + 163008 \cdot 2^{k+2} + 1029 \cdot 2^{18} & \text{if } i = 7,\ k \geq 8, \\ 1696 \cdot 2^{2k} + 2176512 \cdot 2^{k+2} + 5723 \cdot 2^{18} & \text{if } i = 8,\ k \geq 9, \\ 105 \cdot 2^{2k+8} + 2625 \cdot 2^{k+15} - 90720 \cdot 2^{18} & \text{if } i = 9,\ k \geq 10, \\ 105 \cdot 2^{2k+12} - 315 \cdot 2^{k+20} + 215040 \cdot 2^{18} & \text{if } i = 10,\ k \geq 11, \\ 2^{3k+8} - 7 \cdot 2^{2k+16} + 7 \cdot 2^{k+25} - 2^{35} & \text{if } i = 11.\ k \geq 11. \end{cases}$$

3.4. **An outline of the main results in the case $m \geqslant 2$, $l = 0$.**

Theorem 3.6. *The number* $\Gamma_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m \end{smallmatrix}\right] \times k}$ *of triple persymmetric* $(3s+2m) \times k$ *matrices over* $\mathbb{F}_2$ *of the form*

$$\left(\begin{array}{ccccc}
\alpha_1 & \alpha_2 & \dots & \alpha_{k-1} & \alpha_k \\
\alpha_2 & \alpha_3 & \dots & \alpha_k & \alpha_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{s-1} & \alpha_s & \dots & \alpha_{s+k-3} & \alpha_{s+k-2} \\
\alpha_s & \alpha_{s+1} & \dots & \alpha_{s+k-2} & \alpha_{s+k-1} \\
\hline
\beta_1 & \beta_2 & \dots & \beta_{k-1} & \beta_k \\
\beta_2 & \beta_3 & \dots & \beta_k & \beta_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{m+1} & \beta_{m+2} & \dots & \beta_{k+m-1} & \beta_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\beta_{s+m-1} & \beta_{s+m} & \dots & \beta_{s+m+k-3} & \beta_{s+m+k-2} \\
\beta_{s+m} & \beta_{s+m+1} & \dots & \beta_{s+m+k-2} & \beta_{s+m+k-1} \\
\hline
\gamma_1 & \gamma_2 & \dots & \gamma_{k-1} & \gamma_k \\
\gamma_2 & \gamma_3 & \dots & \gamma_k & \gamma_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{m+1} & \gamma_{m+2} & \dots & \gamma_{k+m-1} & \gamma_{k+m} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma_{s+m-1} & \gamma_{s+m} & \dots & \gamma_{s+m+k-3} & \gamma_{s+m+k-2} \\
\gamma_{s+m} & \gamma_{s+m+1} & \dots & \gamma_{s+m+k-2} & \gamma_{s+m+k-1}
\end{array}\right).$$

is given by

(3.15)

$$
\begin{cases}
1 & \text{if } i=0,\ k\geq 1\\
105\cdot 2^{4i-6}-21\cdot 2^{3i-5} & \text{if } 1\leq i\leq s-1,\ k\geq i+1,\\
2^{k+s-1}-2^{2s}+21\cdot\left(5\cdot 2^{4s-6}-2^{3s-5}\right) & \text{if } i=s,\ k\geq s+1\\
\\
(21\cdot 2^{j-1}-3)\cdot 2^{k+s+2j-4} & \\
+21\cdot\left(5\cdot 2^{4s+4j-6}-2^{3s+3j-5}-5\cdot 2^{2s+4j-6}+2^{2s+3j-5}\right) & \text{if } i=s+j,\ 1\leq j\leq m-1,\\
 & k\geq s+j+1\\
\\
\left(21\cdot 2^{s+3m-5}+45\cdot 2^{s+2m-4}\right)\cdot\left(2^{k}-2^{s+m+1}\right) & \\
+105\cdot 2^{4s+4m-6}-21\cdot 2^{3s+3m-5}-21\cdot 2^{2s+4m-6}+9\cdot 2^{2s+3m-5} & \text{if } i=s+m,\ k\geq s+m+1\\
\\
21\cdot\big[2^{k+s+3m+3j-5}+35\cdot 2^{k+s+2m+4j-7}-9\cdot 2^{k+s+2m+3j-6} & \\
+5\cdot 2^{4s+4m+4j-6}-2^{3s+3m+3j-5}-5\cdot 2^{2s+4m+4j-6} & \\
-155\cdot 2^{2s+3m+5j-8}+45\cdot 2^{2s+3m+4j-7}\big] & \text{if } i=s+m+j,\ 1\leq j\leq s-1,\\
 & k\geq s+m+j+1\\
\\
3\cdot 2^{2k+2s+m-2}+21\cdot 2^{k+4s+3m-5}+735\cdot 2^{k+5s+2m-7}-477\cdot 2^{k+4s+2m-6} & \\
+105\cdot 2^{8s+4m-6}-105\cdot 2^{6s+4m-6}-3255\cdot 2^{7s+3m-8}+1629\cdot 2^{6s+3m-7} & \text{if } i=2s+m,\ k\geq 2s+m+1\\
\\
21\cdot 2^{2k+2s+m+3j-2}+21\cdot 2^{k+4s+3m+3j-2}+735\cdot 2^{k+5s+2m+4j-3} & \\
-945\cdot 2^{k+4s+2m+4j-3}+105\cdot 2^{8s+4m+4j-2}-105\cdot 2^{6s+4m+4j-2} & \\
-3255\cdot 2^{7s+3m+5j-3}+3255\cdot 2^{6s+3m+5j-3} & \text{if } i=2s+m+1+j,\\
 & 0\leq j\leq m-2,\ k\geq 2s+m+2+j\\
\\
53\cdot 2^{2s-1}\cdot(2^{2k+4m-4}-2^{4s+8m-2}) & \\
+\left(735\cdot 2^{5s-1}-1629\cdot 2^{4s-1}\right)\cdot\left(2^{k+6m-6}-2^{2s+8m-5}\right) & \\
+21\cdot[5\cdot 2^{8s+8m-6}+3\cdot 2^{6s+8m-8}-15\cdot 2^{7s+8m-8}] & \text{if } i=2s+2m,\ k\geq 2s+2m+1\\
\\
105\cdot\left(2^{2k+2s+4m+4j-2}+7\cdot 2^{k+5s+6m+4j-3}-31\cdot 2^{k+4s+6m+5j-3}\right) & \\
+105\cdot\left(2^{8s+8m+4j-2}-31\cdot 2^{7s+8m+5j-3}+93\cdot 2^{6s+8m+6j-3}\right) & \text{if } i=2s+2m+1+j,\\
 & 0\leq j\leq s-2,\ k\geq 2s+2m+2+j\\
\\
2^{3k+2m+3s-3}-7\cdot 2^{2k+4m+6s-6}+7\cdot 2^{k+6m+9s-8}-2^{8m+12s-9} & \text{if } i=3s+2m,\ k\geq 3s+2m
\end{cases}
$$

$$\Gamma_i^{\left[\begin{smallmatrix} s \\ s+m \\ s+m \end{smallmatrix}\right]\times i} \tag{3.16}$$

$$= \begin{cases} 2^{3s+2m+3i-3} - 7\cdot 2^{4i-6} + 3\cdot 2^{3i-5} & \text{if } 1 \le i \le s+1 \\ 2^{6s+2m+3j-3} - 7\cdot 2^{4s+4j-6} + 3\cdot 2^{3s+3j-5} & \\ \quad +(2^{j-1}-1)\cdot 2^{2s+3j-5} & \text{if } i = s+j,\ 1 \le j \le m+1 \\ 2^{6s+5m+3j-3} - 7\cdot 2^{4s+4m+4j-6} + 3\cdot 2^{3s+3m+3j-5} & \\ \quad +2^{2s+4m+4j-6} + 7\cdot 2^{2s+3m+5j-8} - 9\cdot 2^{2s+3m+4j-7} & \text{if } i = s+m+j,\ 1 \le j \le s-1 \\ 2^{9s+5m-3} - 7\cdot 2^{8s+4m-6} + 7\cdot 2^{7s+3m-8} + 3\cdot 2^{6s+3m-7} + 2^{6s+4m-6} & \text{if } i = 2s+m \\ 2^{9s+5m+2j} - 7\cdot 2^{8s+4m+4j-2} + 7\cdot 2^{7s+3m+5j-3} - 3\cdot 2^{6s+3m+5j-3} + 2^{6s+4m+4j-2} & \text{if } i = 2s+m+1+j, \\ & 0 \le j \le m-2 \\ 2^{9s+8m-3} - 7\cdot 2^{8s+8m-6} + 7\cdot 2^{7s+8m-8} - 2^{6s+8m-8} & \text{if } i = 2s+2m \\ 2^{9s+8m+3j} - 7\cdot 2^{8s+8m+4j-2} + 7\cdot 2^{7s+8m+5j-3} - 2^{6s+8m+6j-3} & \text{if } i = 2s+2m+1+j, \\ & 0 \le j \le s-2 \\ 21\cdot 2^{8m+12s-9} & \text{if } i = 3s+2m \end{cases}$$

We have for $0 \le j \le m-2,\ k \ge 2s+m+2+j$

$$\Gamma_{2s+m+1+j}^{\left[\begin{smallmatrix} s \\ s+m \\ s+m \end{smallmatrix}\right]\times k} = 16^{2j}\Gamma_{2s+1+(m-j)}^{\left[\begin{smallmatrix} s \\ s+(m-j) \\ s+(m-j) \end{smallmatrix}\right]\times(k-2j)} \tag{3.17}$$

We have for $j = m-1,\ k \ge 2s+2m+1$

$$\Gamma_{2s+2m}^{\left[\begin{smallmatrix} s \\ s+m \\ s+m \end{smallmatrix}\right]\times k} = 16^{2m-2}\Gamma_{2s+2}^{\left[\begin{smallmatrix} s \\ s+1 \\ s+1 \end{smallmatrix}\right]\times(k-2(m-1))} \tag{3.18}$$

We have for $0 \le j \le s-2,\ k \ge 2s+2m+2+j$

$$\Gamma_{2s+2m+1+j}^{\left[\begin{smallmatrix} s \\ s+m \\ s+m \end{smallmatrix}\right]\times k} = 16^{2m+3j}\Gamma_{2(s-j)+1}^{\left[\begin{smallmatrix} s-j \\ s-j \\ s-j \end{smallmatrix}\right]\times(k-2m-3j)} \tag{3.19}$$

We have for $j = s-1,\ k \geq 3s+2m$

$$\Gamma_{3s+2m}^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k} = 16^{2m+3s-3} \Gamma_3^{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times (k-2m-3s+3)} \tag{3.20}$$

(3.21)

Let q *be a rational integer* ≥ 1, *then*

$$g_{k,s,m}(t,\eta,\xi) = g(t,\eta,\xi) = \sum_{degY \leq k-1} \sum_{degZ \leq s-1} E(tYZ) \sum_{degU \leq s+m-1} E(\eta YU) \sum_{degV \leq s+m-1} E(\eta YV)$$

$$= 2^{3s+2m+k-r(D^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k}(t,\eta,\xi))}$$

and

$$\int_{\mathbb{P}^3} g^q(t,\eta,\xi) dt d\eta d\xi =$$

$$= \sum_{(t,\eta,\xi) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{\left(k+3s+2m-r(D^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k}(t,\eta,\xi))\right)q} \int_{\mathbb{P}_{k+s-1}} dt \int_{\mathbb{P}_{k+s+m-1}} d\eta \int_{\mathbb{P}_{k+s+m-1}} d\xi$$

$$r(D^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k}_{(t,\eta,\xi)=i}$$

$$= \sum_{i=0}^{\inf(k,3s+2m)} \sum_{(t,\eta,\xi) \in \mathbb{P}/\mathbb{P}_{k+s-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1} \times \mathbb{P}/\mathbb{P}_{k+s+m-1}} 2^{(k+3s+2m-i)q} \int_{\mathbb{P}_{k+s-1}} dt \int_{\mathbb{P}_{k+s+m-1}} d\eta \int_{\mathbb{P}_{k+s+m-1}} d\xi$$

$$= 2^{(k+3s+2m)q-(3k+3s+2m-3)} \sum_{i=0}^{\inf(k,3s+2m)} \Gamma_i^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k} \cdot 2^{-iq}$$

We denote by $R_q(k,s,m)$ *the number of solutions* $(Y_1, Z_1, U_1, V_1, \ldots, Y_q, Z_q, U_q, V_q)$ *of the polynomial equations*

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + \ldots + Y_qZ_q = 0, \\ Y_1U_1 + Y_2U_2 + \ldots + Y_qU_q = 0, \\ Y_1V_1 + Y_2V_2 + \ldots + Y_qV_q = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq k-1, \quad degZ_i \leq s-1, \quad degU_i \leq s+m-1, \quad degV_i \leq s+m-1 \quad for \quad 1 \leq i \leq q.$$

Then

(3.22)

$$R_q(k,s,m) = \int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g_{k,s,m}^q(t,\eta,\xi) dt d\eta d\xi = 2^{(k+3s+2m)q-(3k+3s+2m-3)} \sum_{i=0}^{\inf(k,3s+2m)} \Gamma_i^{\begin{bmatrix} s \\ s+m \\ s+m \end{bmatrix} \times k} \cdot 2^{-iq}$$

Example. We have for $s = 3,\ m = 4,\ k = 10$:

$$\Gamma_i^{\left[\begin{smallmatrix}3\\3+4\\3+4\end{smallmatrix}\right]\times 10} = \begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 378 & \text{if } i = 2 \\ 10416 & \text{if } i = 3 \\ 140352 & \text{if } i = 4, \\ 1994112 & \text{if } i = 5, \\ 29598720 & \text{if } i = 6 \\ 458661888 & \text{if } i = 7, \\ 109389 \cdot 2^{16} & \text{if } i = 8, \\ 213759 \cdot 2^{19} & \text{if } i = 9, \\ 2^{44} - 14273 \cdot 2^{23} & \text{if } i = 10 \end{cases}$$

Example. s = 3, m=4, k = 7, q = 3

The number $\Gamma_i^{\left[\begin{smallmatrix}3\\3+4\\3+4\end{smallmatrix}\right]\times 7}$ of rank i matrices of the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 & \alpha_9 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 \\ \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\ \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 \\ \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} \\ \beta_5 & \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} \\ \beta_6 & \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} \\ \beta_7 & \beta_8 & \beta_9 & \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 \\ \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 \\ \gamma_3 & \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 \\ \gamma_4 & \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} \\ \gamma_5 & \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\ \gamma_6 & \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} \\ \gamma_7 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} \end{pmatrix}$$

is equal to

$$\begin{cases} 1 & \text{if } i = 0 \\ 21 & \text{if } i = 1 \\ 378 & \text{if } i = 2 \\ 6832 & \text{if } i = 3 \\ 108096 & \text{if } i = 4, \\ 1714560 & \text{if } i = 5, \\ 27276288 & \text{if } i = 6 \\ 2^{35} - 3553 \cdot 2^{13} & \text{if } i = 7, \end{cases}$$

The number of solutions
$(Y_1, Z_1, U_1, V_1, Y_2, Z_2, U_2, V_2, Y_3, Z_3, U_3, V_3)$ of the polynomial equations

$$\begin{cases} Y_1Z_1 + Y_2Z_2 + Y_3Z_3 = 0, \\ Y_1U_1 + Y_2U_2 + Y_3U_3 = 0, \\ Y_1V_1 + Y_2V_2 + Y_3V_3 = 0, \end{cases}$$

satisfying the degree conditions

$$degY_i \leq 6, \quad degZ_i \leq 2, \quad degU_i \leq 6, \quad degV_i \leq 6 \quad for \quad 1 \leq i \leq 3.$$

is equal to

$$R_3(7,3,4) = \int_{\mathbb{P}\times\mathbb{P}\times\mathbb{P}} g^3_{7,3,4}(t,\eta,\xi)dtd\eta d\xi = 2^{37} \cdot \sum_{i=0}^{7} \Gamma_i^{\left[\begin{smallmatrix}3\\3+4\\3+4\end{smallmatrix}\right]\times 7} \cdot 2^{-3i}$$
$$= 2^{37} \cdot \big(1 + 21 \cdot 2^{-3} + 378 \cdot 2^{-6} + 6832 \cdot 2^{-9} + 108096 \cdot 2^{-12} + 1714560 \cdot 2^{-15}$$
$$+ 27276288 \cdot 2^{-18} + (2^{35} - 3553 \cdot 2^{13}) \cdot 2^{-21}\big) = 4243395 \cdot 2^{29}$$

Example. The fraction of square triple persymmetric $\left[\begin{smallmatrix}s\\s+m\\s+m\end{smallmatrix}\right] \times (3s+2m)$ matrices which are invertible is equal to $\dfrac{\Gamma_{3s+2m}^{\left[\begin{smallmatrix}s\\s+m\\s+m\end{smallmatrix}\right]\times(3s+2m)}}{\sum_{i=0}^{3s+2m}\Gamma_i^{\left[\begin{smallmatrix}s\\s+m\\s+m\end{smallmatrix}\right]\times(3s+2m)}} = \dfrac{21}{64}.$

References

[1] Cherly, Jorgen.
Exponential sums and rank of persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1306, 46 pp

[2] Cherly, Jorgen.
Exponential sums and rank of double persymmetric matrices over $\mathbf{F}_2$
arXiv : 0711.1937, 160 pp

[3] Cherly, Jorgen.
Exponential sums and rank of triple persymmetric matrices over $\mathbf{F}_2$
arXiv : 0803.1398, 233 pp

Département de Mathématiques, Université de Brest, 29238 Brest cedex 3, France
E-mail address: Jorgen.Cherly@univ-brest.fr
E-mail address: andersen69@wanadoo.fr