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# Discrete Torsion, Quotient Stacks, and String Orbifolds

**Lecture at Wisconsin-Madison**

**Workshop on Mathematical Aspects of Orbifold String Theory**

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# 1 Introduction

In this talk we shall describe two separate topics: discrete torsion (a previously-mysterious degree of freedom in string orbifolds, of which we shall give a reasonably complete understanding), and the relation between quotient stacks and string orbifolds (where, unlike discrete torsion, we shall only set up basics, and will emphasize that much work remains to be done).

Discrete torsion is a degree of freedom that appears in describing string orbifolds. Historically discrete torsion has been considered very mysterious. However, in the first part of this note we shall outline recent work [1, 2, 3, 4] that de-mystifies it. To be brief, we shall argue that discrete torsion is the choice of equivariant structure (*i.e.*, orbifold group action) on the $B$ field, and from this derive the classification by $H^2(G, U(1))$, Vafa's twisted sector phases, Douglas's projectivized D-brane actions (*i.e.*, projectivized equivariant K-theory), and analogues for other tensor field potentials. In a nutshell, discrete torsion has a natural understanding that has nothing to do with conformal field theory, Riemann surfaces, or any other baggage of perturbative string theory, and we shall outline this understanding.

In the second part of the talk we shall discuss the relation between string orbifolds and quotient stacks, and in particular we shall outline how a string orbifold is precisely a sigma model on a quotient stack, a description that clarifies the physics of string orbifolds. For physicists, this notion is a radical conceptual shift: although string orbifolds are described in terms of group actions on covers, physicists have historically assumed that this was merely scaffolding. Physicists speak of string orbifolds as describing strings on quotient spaces decorated with some sort of quantum or 'stringy' behavior at the singularities; for example, physicists often speak of string orbifolds as describing strings on quotient spaces suitably decorated with $B$ fields [11, 12, 13], or as strings on some resolution of the quotient space (because of massless moduli which can often be interpreted as Kähler moduli). In particular, no physicist has ever claimed[1] that string orbifolds describe strings on any sort of stack, or (equivalently) that any formal geometric meaning assigned to the group-actions-on-covers scaffolding had any physical relevance. To make matters even more confusing, in practice physicists often implicitly assume that string orbifolds describe strings on quotient spaces, and ignore any quantum effects at singularities. For example, string moduli spaces are constructed around this assumption, and they play an important role in understanding many of the duality symmetries that have been of interest to physicists over the last decade.

For mathematicians who are acquainted with quotient stacks, the idea that string orbifold conformal field theories (CFT's) coincide with CFT's for strings compactified on quotient stacks surely seems much more natural. After all, among other things, quotient stacks are an overcomplicated way to describe group actions on covers, the language used in string

[1] After all, one can use group actions on covers to describe quotient spaces as well as quotient stacks, and quotient spaces are far less exotic.

orbifolds. This formal similarity might even lead someone who was not acquainted with the physics literature to claim that they 'know' string orbifolds are strings on quotient stacks. However, for such a statement to be true implies that string orbifold CFT's coincide with CFT's for strings on quotient stacks, and even assuming that the notion of compactification on stacks is sensible, there is a tremendous amount of work that must be done to justify this. Put another way, such a statement implicitly assumes that the extra structure of a 'generalized space,' as possessed by a stack, is physically relevant. Any competent physicist would observe that not only is the physics lore apparently contradictory[2], but one would need to understand string compactification on stacks before such a statement could really be justified, and furthermore, such a statement fails several basic physical consistency conditions. In more detail:

- Before one can claim that string orbifold CFT's coincide with CFT's for strings compactified on *quotient* stacks, one must first check whether the notion of string compactification on stacks is even sensible, something that was not considered by physicists at all until very recently [5]. One way to do this (which we shall describe the first stages of) is to first write down the classical action for a sigma model on a stack, and understand basic consequences of that notion, such as the massless spectrum of that sigma model. To be certain that such a classical action can be consistently quantized, there are global considerations that must be taken into account. One must resolve apparent physical contradictions, such as the fact that the massless spectrum of such sigma models is *not* given by cohomology of the target. Countless questions, ranging from "how does one make sense of anomalies in this context" (and other nontrivial global issues) to "can one do QFT on a stack, now viewed as spacetime," must be answered before the matter can be considered to be completely settled. After one has settled at least some of these issues, one can then check explicitly whether, in fact, CFT's for strings compactified on quotient stacks really do coincide with CFT's for string orbifolds.

- The statement that a string orbifold CFT coincides with the CFT for a string compactified on a quotient stack (assuming this is a sensible notion) has nontrivial physical implications, which must be checked for consistency. Put another way, one cannot consistently consider string orbifolds in isolation from the rest of string theory. For example, in constructing moduli spaces of string vacua (used to justify items from mirror symmetry to string/string duality) physicists have always assumed that the deformation theory of a string orbifold is the same as that of a quotient space. If string orbifolds do not describe strings on quotient spaces, then one must explain how the deformation theory arguments used by physicists can possibly have been consistent.

In short, not only does the idea that string orbifolds coincide with string compactification on

[2] After all, if string orbifolds really do describe strings on resolutions, then they cannot possibly also describe strings on quotient stacks – clearly, a choice must be made.

quotient stacks naively contradict the physics lore, but a tremendous amount of very basic work must be done to begin to justify such claims, and such claims even appear to yield physical contradictions. We shall describe some of the basic work needed to justify such claims, and resolve some (but not all) of the contradictions, but much work remains to be done. In particular, it must be said that, at present, there is still a good chance that string orbifolds do *not* describe strings on quotient stacks.

Another question one might ask is, why bother? If one is only using quotient stacks as a highly overcomplicated means of describing group actions on covers, then there is hardly a point. However, we shall argue later that the idea that a string orbifold CFT coincides with the CFT for a string compactified on a quotient stack (if indeed this is a sensible notion) has highly nontrivial physical implications. One aspect is that this gives a new geometric way of understanding certain physical properties of string orbifolds. Another aspect, as mentioned above, is that this calls into question the arguments physicists have used to construct moduli spaces of string vacua, essential to understand string dualities of all types.

As we have previously written about quotient stacks for a physics audience [5], here we shall speak to a mathematics audience. We shall describe some of the basics needed by physicists to make sense of the notion that string orbifolds describe strings compactifieid on quotient stacks, and shall outline the work that remains to be done before quotient stacks can be universally accepted in the physics community as being genuinely relevant to string orbifolds.

## 2 Lightning review of string orbifolds

Before describing either discrete torsion or the relationship between string orbifolds and quotient stacks, we shall take a few moments to review string orbifolds. A string orbifold [7, 8] is simply a sigma model with the action of a discrete group on the target space gauged. Since a string orbifold is a gauged string sigma model, let us take a moment to review sigma models.

A sigma model with target space $X$ and base space[3] $Y$ is a weighted sum over maps $Y \longrightarrow X$, as schematically illustrated for two-dimensional $Y$ in figure 1. (The 'sum' in question is a sum in the sense of path integrals.)

For example, a sigma model describing the propagation of a point particle on $X$ is a sum over maps from a one-dimensional $Y$ (one-dimensional because it encodes the 'worldline' of

[3] For standard sigma models, $Y$ is assumed a manifold with either a Lorentzian or Riemannian metric. Depending upon whether one wants to describe all of spacetime, or just a factor in a spacetime of the form $\mathbf{R}^4 \times Y$, for example, one can consider either case of Lorentzian or Riemannian signature. Analogous issues arise for worldsheet metrics. We shall ignore this issue in the remainder of this section.

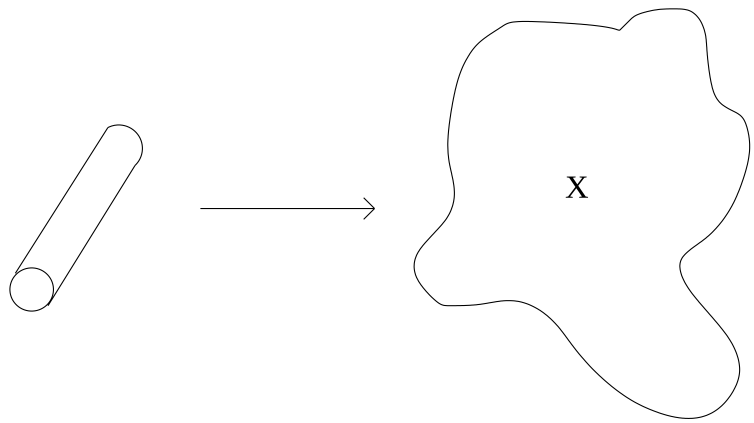


Figure 1: String sigma models sum over "worldsheets" swept out by strings in $X$, as shown for a free string, whose worldsheet is the cylinder $S^1 \times \mathbf{R}$.

a point particle moving in spacetime) into $X$, weighted by $\exp(iS)$, where $S$ is known as the classical action and has the form

$$S \sim \int dt (\phi^* G_{\mu\nu}) \frac{d\phi^\mu}{dt} \frac{d\phi^\nu}{dt} + \cdots \tag{1}$$

Such a sigma model is one description of the quantum mechanics of a point particle on $X$ [6].

For another example, a sigma model describing the propagation of a string on $X$, as illustrated in figure 1, is a sum over maps from a two-dimensional $Y$ (two-dimensional because it encodes the path swept out by the string over time), known as the worldsheet, into $X$, weighted by $\exp(iS)$, where $S$ is known as the classical action and has the form

$$S \sim \int d^2\sigma \, (\phi^* G_{\mu\nu}) \, h^{\alpha\beta} \frac{\partial \phi^\mu}{\partial \sigma^\alpha} \frac{\partial \phi^\nu}{\partial \sigma^\beta} + \cdots \tag{2}$$

Just as a sigma model for a point particle describes the quantum mechanics of a point particle on $X$, a sigma model for a string describes the 'stringy quantum mechanics' of a string on $X$. Formally one can continue to write sigma models in higher dimensions, but above two dimensions they become less well-behaved.

Now, a physical orbifold is obtained by starting with a sigma model on some space $X$, and 'gauging' the action of a discrete group $G$ on $X$. To 'gauge' the action of a discrete group means that fields in the sigma model differing by the action of $G$ should be identified – we impose an equivalence relation on the 'field space' that our 'path integral' integrates over, and integrate over equivalence classes. (It should be noted that the gauging we describe is a process performed by physicists to build one class of physical theories from another class; gauging a symmetry is not a physical process, but rather a process performed by physicists.)

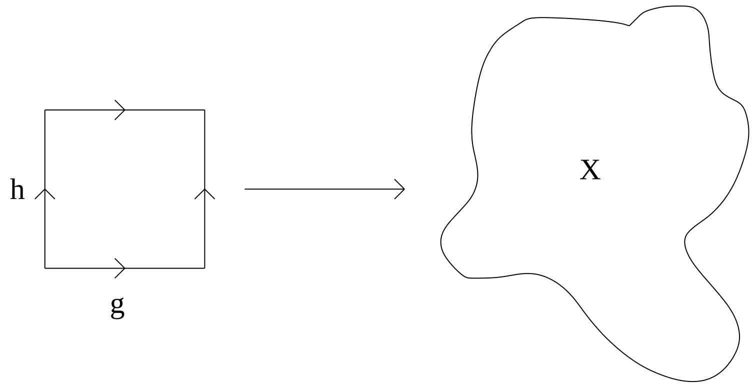


Figure 2: A contribution to the $(g,h)$ twisted sector of a string orbifold $[X/G]$ on $T^2$

In practice, what effect does this 'gauging' have on a sigma model? The answer is that in the new physical theory obtained by gauging, fields on the base space $Y$, such as the map into $X$, need no longer be well-defined over $Y$, but only well-defined up to the action of $X$ – we are allowed to introduce branch cuts. Also, as part of the gauging, we are required to sum over all possible choices of branch cuts. For example, if $Y = T^2$, then instead of summing over maps $T^2 \longrightarrow X$, we sum over maps as illustrated in figure 2, where branch cuts have been introduced on the $T^2$. Maps with nontrivial branch cuts are known as contributions to a 'twisted sector.'

More formally, each twisted sector contribution should be thought of as, a $G$-equivariant map from the total space of a principal $G$-bundle on $Y$ to $X$, restricted to a particular lift of $Y$ to the total space of the bundle. Such a lift introduces branch cuts, as shown in figure 2. Then, our path integral sum is a sum over equivalence classes of bundles and $G$-equivariant maps.

If we let $Z_{(g,h)}$ denote a (path-integral-type) sum over contributions to the $(g,h)$ twisted sector on $T^2$, weighted as above by[4] $\exp(iS)$, then naively we are led to believe that the path integral sum (known as a 'partition function') for a string orbifold on $T^2$ has the form

$$Z(T^2) \;=\; \sum_{\substack{g,h \in G \\ gh=hg}} Z_{(g,h)}$$

The expression above is almost correct, except for one small subtlety. If we sum over *all* possible twisted-sector maps of the form illustrated in figure 2, then we actually overcount by $|G|$. After all, as mentioned earlier, we only wish to sum over equivalence classes of bundles

[4]Note that we can only consider group actions for which the action $S$ is well-defined on the twisted sector illustrated in figure 2. This one of several constraints on possible choices of groups and group actions.

and $G$-equivariant maps from the total space of the bundle to $X$. For each such bundle and $G$-equivariant map, there are $|G|$ twisted-sector maps, corresponding to $|G|$ distinct lifts, so summing over twisted sector maps overcounts by $|G|$, and we find that the correct expression for the path integral sum (*i.e.* the partition function) is given by

$$Z(T^2) \;=\; \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} Z_{(g,h)} \tag{3}$$

Now, although we have been talking about string orbifolds, and string orbifolds on $T^2$, the same remarks apply not only to other Riemann surfaces, but to $Y$'s of any dimension. Again, for $Y$ of dimension greater than two, the notion of a sigma model is not well-defined beyond the classical limit, but one can write down such gauged sigma models classically in any number of dimensions.

A few notes on expression (3) are in order.

1. First, note that $Z_{(1,1)}$ is the same as the partition function for the original (ungauged) sigma model into $X$.
2. Second, note that once one introduces some nontrivial twisted sectors into the theory on $T^2$, consistency with modular invariance forces one to sum over all possible twisted sectors. After all, under the transformation $\tau \mapsto \tau+1$, a twisted sector $(g,h) \mapsto (gh,h)$, so clearly modular transformations mix twisted sectors.
3. Third, some remarks on the Hamiltonian description of orbifolds are in order. From expression (3), as a string propagates around the loop, it comes back to itself, but meets the operator
$$\frac{1}{|G|} \sum_g g$$
This is a projection operator, and it projects onto $G$-invariant states – only $G$-invariant string states are allowed to propagate.

What can we do with this physical theory? Given a string sigma model on a target space $X$, the Euler characteristic of the target space $X$ can be obtained by evaluating the partition function for the theory on $T^2$ in a limit of the worldsheet metric. Now, our 'string orbifold' theory is no longer a sigma model on any particular space, however one can play the same formal game to recover the so-called 'stringy orbifold Euler characteristic' [7, 8, 9, 10]

$$\chi_G(X) \;=\; \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} e\left( X^{<g,h>} \right) \tag{4}$$

$$= \sum_{[g]} e\left(X^g/C(g)\right) \quad (5)$$

where $X^{<g,h>}$ denotes the subset of $X$ invariant under both $g, h \in G$, $e(X)$ denotes the Euler characteristic of $X$, $[g]$ denotes the conjugacy class of $g \in G$, and $C(g)$ denotes the centralizer of $g \in G$.

In the special cases when the quotient space $X/G$ admits a crepant resolution, the orbifold Euler characteristic above agrees[5] with the Euler characteristic of the crepant resolution. More generally, the physical theory has massless modes which correspond to Kähler moduli (when the quotient space admits crepant resolutions)i, and the conformal field theory behaves as though it describes a string on a smooth space. For such reasons, physicists have historically often claimed that string orbifolds appeared to be describing strings on some sort of resolution of the quotient space.

More generally, how do physicists interpret string orbifolds? String orbifolds were originally created in an attempt to describe strings on quotient spaces. However, they are not sigma models on quotient spaces. Also, as described above, they have certain physical properties which suggest that they might have some sort of interpretation as strings on resolutions (though in general resolutions need neither exist nor be unique). More recently, it has been suggested [11, 12, 13] that there is an accompanying $B$ field decoration, which ties into other physical characteristics of string orbifold conformal field theories. To make matters even more confusing, in practice physicists often implicitly assume that string orbifolds describe strings on quotient spaces[6]. For example, string moduli spaces are constructed with the implicit assumption that the deformation theory of a string orbifold coincides with that of a quotient space.

One interpretation that most physicists would agree upon is that string orbifolds describe strings on quotient spaces, but with some sort of 'stringy' behavior located at the singularities, which has the effect of somehow resolving the singularities. Since, in a sigma model, massless modes are identifiable with cohomology of the target space, physicists have believed that twist fields should have some understanding as some unknown cohomology of the quotient space[7], referred to as 'orbifold cohomology.' It was hypothesized that knowledge of such an orbifold cohomology, of the form suggested by physics, would shed light on the physics underlying string orbifold CFT's, by giving a better understanding of the stringy phenomena taking place at the quotient singularities.

[5] Moreover, the Euler characteristic is independent of the choice of crepant resolution, when more than one exists.

[6] Also see, for example, [14, 15] for a random sampling of some recent prominent physics papers making this assumption in different contexts.

[7] To be contrasted with a cohomology constructed from group actions on covers. Although string orbifolds are phrased in terms of group actions on covers, the point here is that they naively seem to predict the existence of a cohomology theory directly on quotient spaces.

## 3 Discrete torsion

In the beginning we mentioned that discrete torsion is a degree of freedom associated with string orbifolds. How does it enter?

Discrete torsion was originally discovered in the following fashion [17]. Start with a string orbifold partition function on, say, $T^2$, as we discussed earlier (equation (3)):

$$Z(T^2) \;=\; \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} Z_{(g,h)}$$

Now, weight the twisted sectors by phases, to obtain a new partition function:

$$Z'(T^2) \;=\; \frac{1}{|G|} \sum_{\substack{g,h \in G \\ gh=hg}} \epsilon(g,h)\, Z_{(g,h)}$$

$Z'$ is now the partition function of a theory "with discrete torsion." The phases $\epsilon(g,h)$ are heavily constrained by internal consistency conditions. After one does some work, one finds that one solution of the constraints is given by

$$\epsilon(g,h) \;=\; \frac{\omega(g,h)}{\omega(h,g)} \tag{6}$$

where the $\omega(g,h)$ are 2-cocycle (inhomogeneous) representatives of a class in the group cohomology group[8] $H^2(G, U(1))$. (For those readers who do not have group cohomology at their fingertips, this just means that the $\omega$ are maps $G \times G \longrightarrow U(1)$, obeying the cocycle condition

$$\omega(g_1 g_2, g_3)\, \omega(g_1, g_2) \;=\; \omega(g_1, g_2 g_3)\, \omega(g_2, g_3) \tag{7}$$

and with coboundaries defined by

$$\omega(g,h) \;\sim\; \omega'(g,h) \;\equiv\; f(gh)\, \omega(g,h)\, f(g)^{-1}\, f(h)^{-1}$$

for any map $f : G \longrightarrow U(1)$. Note that the phase (6) is invariant under coboundaries, *i.e.*, it descends to a well-defined map on group cohomology.)

To recap, discrete torsion is a degree of freedom in string orbifolds, measured by the group cohomology group $H^2(G, U(1))$, that corresponds to weighting twisted sector contributions to orbifold partition functions by phases.

[8] This group cohomology group is defined with trivial action on the coefficients. The same will be true of all group cohomology referenced in this lecture.

Historically discrete torsion has been extremely mysterious. It does not have an immediately obvious explanation, and so for a time it was viewed as something intrinsic to string theory or conformal field theory, a smoking gun for string theory distinguishing it from other possible theories of quantum gravity.

Since the original paper [17], there have been many papers written on discrete torsion. Rather than try to describe all of them, we shall only describe two followups that the physics community has deemed particularly important:

1. In [18], C. Vafa and E. Witten argued that turning on discrete torsion could obstruct supersymmetric moduli (moduli often naively identified with Calabi-Yau moduli).
2. In [19, 20], M. Douglas argued that turning on discrete torsion had the effect of projectivizing equivariant structures on D-brane worldvolumes (*i.e.*, projectivized equivariant K-theory), and these projectivized equivariant structures were related to the usual closed-string description of discrete torsion.

Also, there is a general belief that discrete torsion is intimately connected with the $B$ field, a (local) two-form tensor potential with a gauge symmetry closely analogous to that of connections on principal $U(1)$ bundles, namely $B \mapsto B + d\Lambda$ for any one-form $\Lambda$ is a symmetry of the theory. (More generally, a $B$ field is a higher-tensor analogue of a connection on a principal $U(1)$ bundle. Phrased yet more formally, the $B$ field is a connection on a gerbe, and readers are referred to [21] for a more thorough description.) The precise relationship between the discrete torsion and the $B$ field has been somewhat elusive in the past; however, any serious attempt to understand discrete torsion is certainly expected to explain the precise nature of this relationship, and whether the $B$ field itself is sufficient, or whether some conformal-field-theory-specific effects also play a role.

In this section we shall outline a purely mathematical description of discrete torsion. Specifically, we shall argue that:

*Discrete torsion is the choice of orbifold group action on the $B$ field.*

In other words, the $B$ field itself is sufficient to explain discrete torsion, one need not invoke any conformal-field-theory-specific effects, and more generally, string theory need not enter the discussion in any meaningful way. Technically, by considering orbifold group actions on $B$ fields one recovers not only discrete torsion, but also some other, more obscure degrees of freedom also associated with the $B$ field, but we shall concentrate on explaining discrete torsion.

In general, whenever you have a field with a gauge symmetry, specifying the group action on the underlying space does *not* suffice to specify the group action on the theory. After all, one can combine the group action with a gauge transformation. So, you *must* specify the orbifold group action on the fields, not just the space. More formally, a choice of orbifold group action is known as an equivariant structure, so one must pick equivariant structures on all fields with gauge symmetries.

In the context of heterotic toroidal orbifolds, the choice of orbifold group action on the gauge fields is often called "orbifold Wilson lines." Similarly, we shall outline how the choice of orbifold group action on the $B$ field is what is known as discrete torsion.

## 3.1 Orbifold group actions on principal $U(1)$ bundles with connection

The choice of orbifold group action on $U(1)$ gauge fields (*i.e.*, principal $U(1)$ bundles with connection) forms a precise prototype of discrete torsion, and is well-understood within the mathematics community (see [22] for an early reference). For example, we shall review below how such orbifold group actions are classified by $H^1(G, U(1))$, whereas discrete torsion is classified by $H^2(G, U(1))$. As the technical details of equivariant structures (orbifold group actions) on $U(1)$ gauge fields are both closely analogous to and much simpler than those for discrete torsion, we shall review this analogue before proceeding to discrete torsion.

How can one see the $H^1(G, U(1))$ advertised? First, let us review equivariant structures on principal bundles with connection. The elegant way to proceed is as follows. Let $L$ denote a principal $U(1)$ bundle over a space $X$, then a choice of equivariant structure (orbifold group action) on $L$ is a lift of the action of $G$ to $L$, *i.e.*, for each $g \in G$, one defines a map $g' : L \to L$ making the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{g'} & L \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \end{array} \qquad (8)$$

and obeying the group law, *i.e.*, $(g_1 g_2)' = g_1' g_2'$. An equivariant structure on a principal $U(1)$ bundle with connection is defined with the added constraint that each lift $g'$ must preserve the connection. In general, it is well-known that such equivariant structures need not exist, and even when they do exist, they are not unique. We shall only be concerned with non-uniqueness here.

Now, it is straightforward to see that the set of equivariant structures on a principal $U(1)$ bundle with connection is a torsor under $H^1(G, U(1))$. Given two lifts $g'$, $g''$ of a fixed $g \in G$, $\phi_g \equiv g' \circ (g'')^{-1}$ is a base-preserving bundle automorphism, *i.e.*, a gauge transformation, and the constraint that each lift preserve the connection becomes the constraint that the gauge transformation $\phi_g$ preserve the connection – so if $X$ is connected, $\phi_g$ is a constant map into

$U(1)$. Finally, the constraint that the lifts respect the group law becomes the statement that $\phi_g$ respects[9] the group law. Hence, the $\phi_g$ define an element of $\text{Hom}(G, U(1)) = H^1(G, U(1))$.

Thus, as advertised we see that the difference between any two equivariant structures on a principal $U(1)$ bundle with connection is given by an element of $H^1(G, U(1))$, and in fact it is easy to check that the set of equivariant structures is a torsor under $H^1(G, U(1))$. An analogous analysis of $B$ fields will yield $H^2(G, U(1))$ (together with some other degrees of freedom related to the $B$ field).

For later use in describing $B$ fields, we can also repeat this analysis more mechanically, in terms of data assigned to an open cover of $X$. Let $\{U_\alpha\}$ denote a good open cover of our manifold $X$, i.e., each $U_\alpha$ looks like a (contractible) open ball inside $X$. Then, to each $U_\alpha$ we associate a one-form $A^\alpha$. The one-forms on overlapping $U_\alpha$'s must be related by a gauge transformation, i.e.,

$$A^\alpha \; - \; A^\beta \; = \; d \log g_{\alpha\beta}$$

for some gauge transformations $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$, which close on triple overlaps as

$$g_{\alpha\beta} \, g_{\beta\gamma} \, g_{\gamma\alpha} \; = \; 1$$

Next, we need to define the action of $G$ on such a $U(1)$ gauge field. More precisely, we need to relate[10] $g^* A^\alpha$ to $A^\alpha$ and $g^* g_{\alpha\beta}$ to $g_{\alpha\beta}$. The form of such orbifold group actions is a standard result:

$$\begin{aligned} g^* A^\alpha &= A^\alpha \; + \; d \log h^g_\alpha \\ g^* g_{\alpha\beta} &= (g_{\alpha\beta}) \, (h^g_\alpha) \left( h^g_\beta \right)^{-1} \\ h^{g_1 g_2}_\alpha &= (g_2^* h^{g_1}_\alpha) \, (h^{g_2}_\alpha) \end{aligned}$$

for each $g \in G$, for some collection of maps $h^g_\alpha : U_\alpha \to U(1)$, which define the specific $G$-action. Note, for example, that the first line is merely the statement that $g^* A^\alpha$ and $A^\alpha$ need only agree up to a gauge transformation; in order to specify the $G$-action, one must specify those gauge transformations.

Finally, in order to see $H^1(G, U(1))$ explicitly, we need to study the differences between distinct $G$-actions. Let $(h^g_\alpha)$ define one $G$-action, and $\left( \overline{h}^g_\alpha \right)$ another, on the same $U(1)$ gauge field. Define

$$\phi^g_\alpha \; = \; \frac{h^g_\alpha}{\overline{h}^g_\alpha}$$

so that the $\phi^g_\alpha$ literally define the difference between $G$-actions. It is straightforward to check that the $\phi^g_\alpha$ determine a group homomorphism $G \to U(1)$, as we shall outline below:

[9] Note that we have used the fact that the structure group of the principal bundles is abelian in this step – the analogous statement is not true in general of, say, principal $SU(2)$ bundles.

[10] Technically, for simplicity of presentation we are assuming the elements of cover $\{U_\alpha\}$ are invariant under the action of $G$. So, $\{U_\alpha\}$ is not a good cover, but we can assume each $U_\alpha$ is a disjoint union of contractible open sets, which amounts to the next best thing.

1. First use the fact that

$$\begin{aligned} g^* g_{\alpha\beta} &= (g_{\alpha\beta}) \left(h^g_\alpha\right) \left(h^g_\beta\right)^{-1} \\ \text{also} &= (g_{\alpha\beta}) \left(\overline{h}^g_\alpha\right) \left(\overline{h}^g_\beta\right)^{-1} \end{aligned}$$

   By dividing these two lines, we see that $\phi^g_\alpha = \phi^g_\beta$ on $U_\alpha \cap U_\beta$, hence for any fixed $g$ the $\phi^g_\alpha$ define a global function we shall call $\phi^g$.

2. Next, use

$$\begin{aligned} g^* A^\alpha &= A^\alpha + d \log h^g_\alpha \\ \text{also} &= A^\alpha + d \log \overline{h}^g_\alpha \end{aligned}$$

   to see that $\phi^g$ must be a constant function.

3. Finally, use

$$\begin{aligned} h^{g_1 g_2}_\alpha &= \left(g_2^* h^{g_1}_\alpha\right) \left(h^{g_2}_\alpha\right) \\ \overline{h}^{g_1 g_2}_\alpha &= \left(g_2^* \overline{h}^{g_1}_\alpha\right) \left(\overline{h}^{g_2}_\alpha\right) \end{aligned}$$

   to see that $\phi^{g_1 g_2} = \phi^{g_1} \phi^{g_2}$.

Thus, the $\phi^g$ define a group homomorphism $G \rightarrow U(1)$, i.e., $\phi^g \in \mathrm{Hom}(G, U(1))$. However, it is a true fact that

$$H^1(G, U(1)) = \mathrm{Hom}(G, U(1))$$

so again we see that the difference between any two orbifold group actions on a $U(1)$ gauge field is defined by an element of $H^1(G, U(1))$.

In passing, we should mention that, if one is only interested in finding group cohomology, there is a faster way to get it, by using the Cartan-Leray spectral sequence description of $G$-equivariant cohomology $H^2_G(\mathbf{Z})$. Unfortunately, this cohomology class is not precisely directly physically relevant – this gives information concerning all equivariantizable bundles with all possible equivariant structures, yet we are concerned with the set of equivariant structures on a *fixed* equivariantizable bundle *with connection.* Also, using the Cartan-Leray spectral sequence in this context obscures some important information; for example, the fact that not all bundles are equivariantizable is hidden. Partly the issue here is one of mere style; however, considering that later we shall want to find more than just group cohomology, we shall not quote the Cartan-Leray spectral sequence when discussing discrete torsion.

## 3.2 $B$ fields and $H^2(G, U(1))$

When describing orbifold group actions on $B$ fields, one finds $H^2(G, U(1))$, just as we found $H^1(G, U(1))$ for $U(1)$ gauge fields, together with some related degrees of freedom.

As for bundles with connection, there is an elegant formal description of equivariant structures on gerbes with connection (*i.e.*, $B$ fields). Let $P$ denote a gerbe over a space $X$. A lift of $g \in G$ from $X$ to $P$ is given by a map $g' : P \to P$ making the following diagram commute:

$$\begin{array}{ccc} P & \xrightarrow{g'} & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \end{array} \tag{9}$$

As before, the lifts $g'$ must obey the group law, meaning that

$$P \xrightarrow{g_1'} P \xrightarrow{g_2'} P, \qquad P \xrightarrow{(g_1 g_2)'} P \tag{10}$$

Now, there is an additional layer of subtlety. If we realize the gerbe $P$ as a stack, then the diagram above is a commutative diagram of (sheaves of) functors. For a diagram of functors to commute does not mean that the compositions must be equal, but merely isomorphic. Hence, we must also specify isomorphisms $\omega(g_1, g_2) : g_2' \circ g_1' \Longrightarrow (g_1 g_2)'$, and to be consistent on triples, we must demand

$$\omega(g_1 g_2, g_3) \circ \omega(g_1, g_2) \; = \; \omega(g_1, g_2 g_3) \circ \omega(g_2, g_3)$$

As before, an equivariant structure on a gerbe with connection is defined with added constraints that the connection must be preserved by these maps. As for bundles with connection, $H^2(G, U(1))$ (and related degrees of freedom) emerge when considering differences between orbifold group actions,

As the details of this more elegant description are unfortunately rather lengthy to work out, we shall instead proceed immediately to a description of the $B$ field in terms of data assigned to elements of an open cover. Let $\{U_\alpha\}$ be a cover of our manifold $X$ as before, then globally the $B$ field is described by [21] a collection of two-forms $B^\alpha$ assigned to patches $U_\alpha$, one-forms $A^{\alpha\beta}$ on double overlaps, and $U(1)$-valued functions $h_{\alpha\beta\gamma}$ on triple overlaps, satisfying

$$\begin{array}{rcl} B^\alpha \; - \; B^\beta & = & dA^{\alpha\beta} \\ A^{\alpha\beta} \; + \; A^{\beta\gamma} \; + \; A^{\gamma\alpha} & = & d \log h_{\alpha\beta\gamma} \\ (h_{\alpha\beta\gamma}) \; (h_{\alpha\beta\delta})^{-1} \; (h_{\alpha\gamma\delta}) \; (h_{\beta\gamma\delta})^{-1} & = & 1 \end{array}$$

For physicists reading this discussion, we should stress that the data $A^{\alpha\beta}$ and $h_{\alpha\beta\gamma}$ are *not* some new fields in the theory, just as bundle transition functions are not new fields in a gauge theory; rather, we are merely making gauge transformations on overlaps explicit.

Next, we need to define the $G$-action completely, which is to say, describe how it acts not only on the $B^\alpha$ but also on the overlap data $A^{\alpha\beta}$ and $h_{\alpha\beta\gamma}$. The result can be derived from self-consistency, and also exists in the literature (see for example [23]):

$$\begin{aligned}
g^* B^\alpha &= B^\alpha \,+\, d\chi(g)^\alpha \\
g^* A^{\alpha\beta} &= A^{\alpha\beta} \,+\, d\log \nu^g_{\alpha\beta} \,+\, \chi(g)^\alpha \,-\, \chi(g)^\beta \\
g^* h_{\alpha\beta\gamma} &= (h_{\alpha\beta\gamma}) \left(\nu^g_{\alpha\beta}\right) \left(\nu^g_{\beta\gamma}\right) \left(\nu^g_{\gamma\alpha}\right) \\
\chi(g_1 g_2)^\alpha &= \chi(g_2)^\alpha \,+\, g_2^* \chi(g_1)^\alpha \,-\, d\log h^{g_1,g_2}_\alpha \\
\nu^{g_1 g_2}_{\alpha\beta} &= \left(\nu^{g_2}_{\alpha\beta}\right) \left(g_2^* \nu^{g_1}_{\alpha\beta}\right) (h^{g_1,g_2}_\alpha) \left(g^{g_1,g_2}_\beta\right)^{-1} \\
(h^{g_1,g_2 g_3}_\alpha)\,(h^{g_2,g_3}_\alpha) &= (g_3^* h^{g_1,g_2}_\alpha)\,(h^{g_1 g_2, g_3}_\alpha)
\end{aligned}$$

where $\chi(g)^\alpha$ (local one-forms), $\nu^g_{\alpha\beta}$ (maps $U_\alpha \cap U_\beta \rightarrow U(1)$), and $h^{g_1,g_2}_\alpha$ (maps $U_\alpha \cap U_\beta \cap U_\gamma \rightarrow U(1)$) define the $G$ action on the $B$ field.

This looks more complicated than the description of $G$-actions on $U(1)$ gauge fields, but the basic idea is identical. For example, note from the first line that $g^* B^\alpha$ and $B^\alpha$ differ by a gauge transformation (defined by $\chi(g)^\alpha$), for any $g \in G$. Since the overlap data is more complicated, one has to work harder to express the complete $G$-action, but the basic principle is the same.

Now that we have defined $G$-actions on $B$ fields, we can discuss the differences between $G$-actions on a fixed $B$ field. When we did this for $U(1)$ gauge fields, we found that possible $G$-actions are classified by $H^1(G, U(1))$. Here, we shall find $H^2(G, U(1))$ (among other things).

We shall only outline how this section of the analysis proceeds. Define, for example,

$$T^g_{\alpha\beta} \,\equiv\, \frac{\nu^g_{\alpha\beta}}{\overline{\nu}^g_{\alpha\beta}}$$

Then using the relations

$$\begin{aligned}
g^* h_{\alpha\beta\gamma} &= (h_{\alpha\beta\gamma}) \left(\nu^g_{\alpha\beta}\right) \left(\nu^g_{\beta\gamma}\right) \left(\nu^g_{\gamma\alpha}\right) \\
\text{also} &= (h_{\alpha\beta\gamma}) \left(\overline{\nu}^g_{\alpha\beta}\right) \left(\overline{\nu}^g_{\beta\gamma}\right) \left(\overline{\nu}^g_{\gamma\alpha}\right)
\end{aligned}$$

we see that

$$T^g_{\alpha\beta}\, T^g_{\beta\gamma}\, T^g_{\gamma\alpha} \,=\, 1$$

so the $T^g_{\alpha\beta}$ are transition functions for a bundle, call it $T^g$, for each $g \in G$. Similarly, $\chi(g)^\alpha - \overline{\chi}(g)^\alpha$ defines a flat connection (a flat $U(1)$ gauge field) on $T^g$, and $h^{g_1,g_2}_\alpha / \overline{h}^{g_1,g_2}_\alpha$ defines an isomorphism $T^{g_2} \otimes g_2^* T^{g_1} \rightarrow T^{g_1 g_2}$, satisfying a consistency condition we shall describe momentarily.

A moment's reflection reveals that this story is closely analogous to the case of $G$-actions on $U(1)$ gauge fields. There, recall the difference between any two $G$-actions was defined by

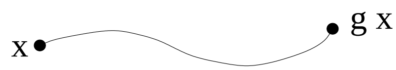


Figure 3: Lift of closed loop in $X/G$ to covering space $X$.

a set of gauge transformations $\phi^g$, one for each $g \in G$, respecting the group law in $G$. Here we see the same thing. After all, globally a gauge transformation of a $B$ field is defined by a $U(1)$ gauge field, so again we see the difference between two $G$-actions is given by a set of gauge transformations, here determined by the $T^g$. The isomorphisms $T^{g_2} \otimes g_2^* T^{g_1} \xrightarrow{\sim} T^{g_1 g_2}$ simply enforce the group law on these gauge transformations.

To summarize our results so far, the difference between two $G$-actions on a $B$ field is defined by

1. Bundles $T^g$ with flat connection (i.e., flat $U(1)$ gauge fields)
2. Maps $\omega^{g,h} : T^h \otimes h^* T^g \xrightarrow{\sim} T^{gh}$ such that the following diagram commutes:

$$\begin{array}{ccc} T^{g_3} \otimes g_3^* \left( T^{g_2} \otimes g_2^* T^{g_1} \right) & \xrightarrow{\omega^{g_1,g_2}} & T^{g_3} \otimes g_3^* T^{g_1 g_2} \\ \downarrow{\scriptstyle \omega^{g_2,g_3}} & & \downarrow{\scriptstyle \omega^{g_1 g_2, g_3}} \\ T^{g_2 g_3} \otimes (g_2 g_3)^* T^{g_1} & \xrightarrow{\omega^{g_1, g_2 g_3}} & T^{g_1 g_2 g_3} \end{array} \qquad (11)$$

Now we can finally see $H^2(G, U(1))$. To make this explicit, take all of the bundles $T^g$ to be canonically trivial with zero connection (i.e., set all of the $U(1)$ gauge fields to zero), then the maps $\omega^{g,h}$ are forced to be constant gauge transformations. Commutivity of diagram (11) becomes precisely the group 2-cocycle condition (7). There are also residual gauge transformations, namely constant gauge transformations of the bundles $T^g$, and these act merely to change the $\omega^{g,h}$ by coboundaries.

Of course, not all flat bundles $T^g$ are trivializable, and not all flat connections on trivializable bundles are gauge trivial. So, clearly there are extra degrees of freedom present besides merely $H^2(G, U(1))$, and a more careful analysis reveals these are precisely momentum-dependent phase factors, a more obscure aspect of string orbifolds also associated with $B$ fields.

### 3.3 Vafa's phases

So far we have derived the classifying group $H^2(G, U(1))$ of discrete torsion from orbifold group actions on $B$ fields. In this section we will explain how one can see the orbifold partition function phases that we originally used to motivate discrete torsion.

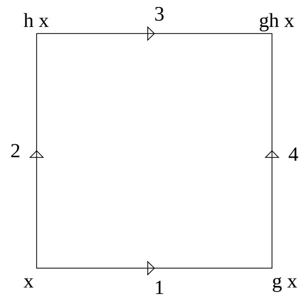


Figure 4: A twisted sector contribution to the one-loop partition function.

These phases are closely analogous to orbifold Wilson lines, so again let us briefly review relevant aspects of orbifold Wilson lines. Consider a path on the covering space $X$, whose ends are related by the action of $G$, as illustrated in figure 3. In other words, consider a point-particle one-loop twisted sector, something which, on the quotient space $X/G$, becomes a closed loop.

If one computes the Wilson loop about that closed loop, but upstairs on $X$, one finds it has the form

$$\varphi \exp\left( \int_x^{g\cdot x} A \right)$$

Mostly this is the holonomy along the path in figure 3, but there is a correcting factor $\varphi$ that implicitly describes the $G$-action on $A$, i.e., $\varphi$ relates $A_x$ to $A_{g\cdot x}$.

Similar considerations for the $B$ field holonomy $\exp(B)$ on polygons with sides identified by $G$ yield Vafa's twisted sector phases.

Consider a one-loop twisted sector (i.e., from a string orbifold on $T^2$) as shown in figure 4. Naively if we calculate the holonomy of the $B$ field about the $T^2 \subset X/G$ one might expect

$$\exp\left( \int_S B \right) \tag{12}$$

corresponding to the holonomy over the interior $S$ of the polygon. However, this omits the contribution from gauge transformations at the boundaries.

How can we take into account such gauge transformations? Recall that a $G$ action on the $B$ field is specified by

- principal $U(1)$ bundles $T^g$ with flat connection $\Lambda(g)$, for all $g \in G$
- connection-preserving bundle maps $\omega^{g,h} : T^h \otimes h^* T^g \xrightarrow{\sim} T^{gh}$ enforcing the group law

Since we have gauge fields and the boundary of the square in figure 4 has one-dimensional components, the first thing to try is to add a factor corresponding to the Wilson lines of the

$U(1)$ gauge fields along the boundaries, as

$$\exp\left( \int_x^{h\cdot x} \Lambda(g) \; - \; \int_x^{g\cdot x} \Lambda(h) \right) \exp\left( \int_S B \right) \tag{13}$$

(Signs are determined by relative orientations.)

However, expression (13) is still not right, for technical reasons (such as the fact that the result is not invariant under gauge transformations of the $U(1)$ gauge fields $\Lambda$). As described in much more detail in [1], to fix this expression we must also take into account the corners, which yields the correct general expression

$$\left( \omega_x^{g,h} \right) \left( \omega_x^{h,g} \right)^{-1} \exp\left( \int_x^{h\cdot x} \Lambda(g) \; - \; \int_x^{g\cdot x} \Lambda(h) \right) \exp\left( \int_S B \right) \tag{14}$$

(Overall normalizations are fixed by comparison to $B$ field holonomies around nontrivial cycles on the quotient space.)

Now, for those degrees of freedom measured by $H^2(G, U(1))$, recall we can assume $\Lambda(g) \equiv 0$ for all $g \in G$ and that $\omega^{g,h}$ is constant, hence for these degrees of freedom the expression (14) reduces to

$$\left( \omega^{g,h} \right) \left( \omega^{h,g} \right)^{-1} \exp\left( \int_S B \right) \tag{15}$$

Note that the resulting phase factor completely agrees with Vafa's one-loop phase factor as stated in equation (6).

Since all contributions to the path integral (in this twisted sector) are weighted with this same phase, the effect is to multiply the partition function for this twisted sector by Vafa's phase. Hence, we find complete agreement with Vafa's original description [17].

Other checks of this description, such as multiloop factorization, also work out nicely and are described in [1].

## 3.4 Douglas's projectivization for D-branes

An action of $G$ on a $U(N)$ gauge field is described by

$$\begin{aligned} g^* A^\alpha &= (\gamma_\alpha^g) \, A^\alpha \, (\gamma_\alpha^g)^{-1} \; + \; (\gamma_\alpha^g) \, d \, (\gamma_\alpha^g)^{-1} & (16) \\ g^* g_{\alpha\beta} &= (\gamma_\alpha^g) \, (g_{\alpha\beta}) \left( \gamma_\beta^g \right)^{-1} & (17) \\ \gamma_\alpha^{gh} &= (h^* \gamma_\alpha^g) \left( \gamma_\alpha^h \right) & (18) \end{aligned}$$

for some $\gamma_\alpha^g : U_\alpha \to U(N)$ defining the $G$-action.

M. Douglas conjectured [19, 20] that when discrete torsion is turned on, the $G$-action on a D-brane gauge field is twisted, meaning that equation (18) is replaced with

$$\left(\omega^{g,h}\right) \left(\gamma^{gh}_{\alpha}\right) \; = \; \left(h^* \gamma^g_{\alpha}\right) \left(\gamma^h_{\alpha}\right) \tag{19}$$

so that instead of an honest representation of the orbifold group $G$, one only has a projective representation. Phrased another fashion, instead of working with equivariant K-theory, one works with projectivized equivariant K-theory.

This projectivized representation comes from the fact that in the presence of a nontrivial $B$ field, the "bundle" on the worldvolume of a D-brane is twisted[11]:

$$\begin{aligned} A^{\alpha} \; - \; g_{\alpha\beta} A^{\beta} g_{\alpha\beta}^{-1} \; - \; d \log g_{\alpha\beta}^{-1} \; &= \; A^{\alpha\beta} I \\ g_{\alpha\beta} \, g_{\beta\gamma} \, g_{\gamma\alpha} \; &= \; h_{\alpha\beta\gamma} I \end{aligned}$$

where $A^{\alpha}$ is a local $U(N)$ gauge field and $\left(B^{\alpha}, A^{\alpha\beta}, h_{\alpha\beta\gamma}\right)$ define the $B$ field (as discussed earlier). This twisting is a consequence of the fact that under $B \mapsto B + d\Lambda$, the Chan-Paton gauge field $A \mapsto A - \Lambda I$, as described in for example [25].

As a direct result of this intermingling between the gauge field and the $B$ field, any $G$-action on the "bundle" must be intertwined with the $G$-action on the $B$ field.

Using self-consistency arguments, we find that in general, the $G$-action on the D-brane "bundle" is actually of the form

$$g^* A^{\alpha} \; = \; \left(\gamma^g_{\alpha}\right) A^{\alpha} \left(\gamma^g_{\alpha}\right)^{-1} \, + \, \left(\gamma^g_{\alpha}\right) d \left(\gamma^g_{\alpha}\right)^{-1} \, + \, \chi(g)^{\alpha} I \tag{20}$$

$$g^* g_{\alpha\beta} \; = \; \left(\nu^g_{\alpha\beta}\right) \left[ \left(\gamma^g_{\alpha}\right) \left(g_{\alpha\beta}\right) \left(\gamma^g_{\beta}\right)^{-1} \right] \tag{21}$$

$$\left(\omega^{g,h}_{\alpha}\right) \left(\gamma^{gh}_{\alpha}\right) \; = \; \left(h^* \gamma^g_{\alpha}\right) \left(\gamma^h_{\alpha}\right) \tag{22}$$

where $\chi(g)^{\alpha}$, $\nu^g_{\alpha\beta}$, and $\omega^{g,h}_{\alpha}$ are data defining the $G$-action on the $B$ field.

To be very brief, from equation (22) we see Douglas's projectivization (suitably generalized).

### 3.5 Discrete torsion for 3-form potentials

So far we have only discussed $B$ fields. However, string theory and $M$ theory have other tensor field potentials. These other tensor field potentials have precise analogues of discrete torsion [2].

[11] Technically, we should mention that the equivariant structure described below is almost but not quite uniquely fixed by self-consistency; rather, this reflects some minor minimal choices, and is the analogue of a 'true' equivariant structure as opposed, for example, to a projective equivariant structure.

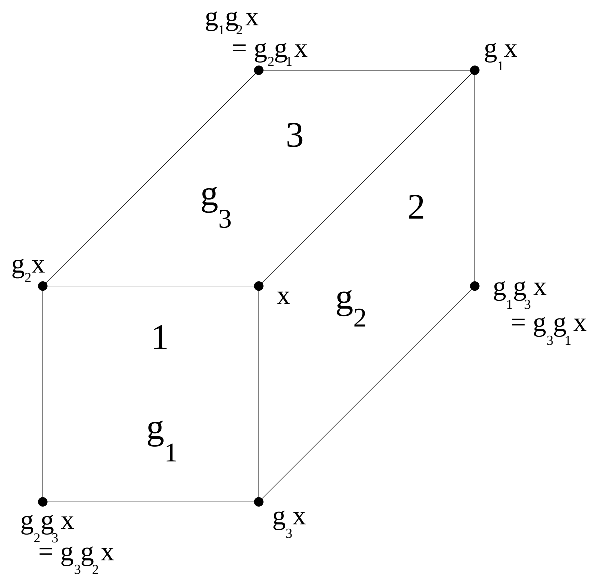


Figure 5: Three-torus seen as open box on covering space.

For example, consider a three-form potential $C_{\mu\nu\rho}$. It can be shown [2] that just as $G$-actions on $U(1)$ gauge fields are classified by $H^1(G, U(1))$, and $G$-actions on $B$ fields are (partially) classified by $H^2(G, U(1))$, possible $G$-actions on $C$ fields are (partially) classified by $H^3(G, U(1))$.

One can also get phase factors for membranes, closely analogous to Vafa's phase factors. For example, recall that for a $T^2$ twisted sector (as illustrated in figure 4), from $\exp\left(\int B\right)$ one has a phase factor

$$\omega(g, h) \,-\, \omega(h, g) \tag{23}$$

(where we have chosen here to write the abelian product additively instead of multiplicatively). In the present case, for membranes on $T^3$ twisted sectors as illustrated in figure 5, from $\exp\left(\int C\right)$ one has a phase factor

$$\omega(g_1, g_2, g_3) - \omega(g_2, g_1, g_3) - \omega(g_3, g_2, g_1) + \omega(g_3, g_1, g_2) + \omega(g_2, g_3, g_1) - \omega(g_1, g_3, g_2) \tag{24}$$

Just as Vafa's original phase factor (23) for $T^2$ was $SL(2, \mathbf{Z})$ invariant, the phase factor (24) for $T^3$ is $SL(3, \mathbf{Z})$ invariant (reflecting the fact that it is well-defined on $T^3$).

## 3.6 Vafa-Witten and supersymmetric moduli

In the paper [18] it was pointed out that in string orbifolds with discrete torsion 'turned on,' supersymmetric moduli are often obstructed. It should be mentioned that because in many simple cases, supersymmetric moduli can be identified with Calabi-Yau moduli, the paper

[18] has sometimes been misinterpreted to mean that Calabi-Yau are somehow obstructed, that the geometry of the Calabi-Yau somehow changes, but in fact the authors of [18] argued only a much weaker statement.

One of their proposed explanations for this behavior now seems extremely reasonable (for reasons explained in much greater detail in [1]). Namely, the authors of [18] speculated that, after 'turning on' discrete torsion, attempts to resolve or deform singularities will often result in nonzero $B$ field curvature (known to physicists as 'torsion,' and denoted by $H$). (Such behavior is closely analogous to one description of the McKay correspondence [30], for example.) Nonzero $B$ field curvature breaks supersymmetry, and so such deformations would be obstructed.

Although this sounds extremely plausible in the present context, it has not yet been verified to date, and is essentially the only remaining aspect of classical discrete torsion that is not yet completely understood. (See however [24] for a recent attempt to understand this effect on D-branes using noncommutative geometry.)

# 4 Quotient stacks and string orbifolds

In this first half of this paper we gave a largely complete understanding of discrete torsion, as the choice of equivariant structure on the $B$ field (a gerbe with connection). In the second half, we shall shift gears and describe the relation between string orbifolds and quotient stacks. Although in the first half of this paper we were able to give a fairly complete understanding of discrete torsion, in the second half we shall only set up the basics required to relate string orbifolds to quotient stacks, and emphasize that much work remains to be done.

As mentioned in the introduction, historically physicists have assumed that string orbifolds describe strings on quotient spaces with some sort of 'stringy' behavior at singularities. Of course, string orbifolds are not sigma models on quotient spaces; rather, they describe group actions on covers. However, physicists have assumed that this was merely scaffolding. In fact, in practice, although physicists often speak of 'stringy' behavior at singularities, when relating string orbifolds to the rest of string theory, we often implicitly identify string orbifolds with quotient spaces. For example, descriptions of string moduli spaces, fundamental to topics ranging from mirror symmetry to string/string duality, all implicitly assume that the deformation theory of a string orbifold is the same as that of a quotient space.

Unfortunately, thinking about string orbifolds in terms of quotient spaces with 'stringy' behavior at singularities is not wholly satisfactory – various physical features of string orbifold CFT's, such as well-behavedness of the CFT, have only clumsy explanations within the present physics lore. For such reasons, a more subtle alternative was proposed in [5], namely that string orbifolds do not describe strings on quotient spaces, with or without 'stringy' behavior, but rather describe strings compactified on quotient *stacks*.

To most physicists, the idea that string orbifold CFT's coincide with CFT's for strings compactified on quotient stacks, a formal geometric structure assigned to the group-actions-on-covers scaffolding, is somewhat radical. Certainly nothing of the sort has previously been believed within the physics community. Reference [5] concentrated on trying to make the notion palatable to physicists. In a nutshell, such a description has nontrivial physical implications. For example, by thinking about string orbifolds as sigma models on stacks, one can immediately resolve a number of puzzling issues about the physics of string orbifolds, which were not satisfactorily resolved within the physics lore. Features often ascribed to 'stringy' effects can be seen to be easy consequences of geometrical features of stacks. This also has nontrivial consequences for the understanding of string moduli spaces.

To mathematicians familiar with stacks, on the other hand, the idea that string orbifold CFT's coincide with CFT's for strings compactified on quotient stacks is much more natural. After all, one way of thinking about quotient stacks is as an overcomplicated way to describe group actions on covers, which is certainly the language that string orbifolds are phrased in. Certainly one can efficiently manipulate group actions on covers by working with quotient stacks. Moreover, quotient stacks are more than just an overcomplicated way of describing group actions on covers – they also can be interpreted as 'generalized spaces.' For example, one sometimes hears[12] that it is possible to do differential geometry on quotient stacks. So, someone who was not acquainted with the physics literature might be led to (very naively) believe that the notion of string compactification on a stack is sensible, and furthermore that a string orbifold coincides with a string on a quotient stack.

Unfortunately, the notion that that extra structure of a 'generalized space' has physical content, in the fashion above, has not been justified. Any competent physicist would point out that, not only does this appear to contradict the standard physics lore, but also a tremendous amount of work must be done to even check whether the idea of string compactification on a stack is sensible, much less reconcile it with the role string orbifolds play in the rest of string theory, or resolve the basic physical contradictions that appear.

For example, before one can make sense out of the statement that string orbifolds are the same thing as strings compactified on *quotient* stacks, one must first describe string compactification on stacks in general, and check whether this is even a sensible notion. The fact that one can do differential geometry on stacks is a necessary condition, but by no means is it sufficient. One way to do this (which we shall outline the basics of) is to write down the classical action for a sigma model on a stack, and then try to check whether it can be quantized (which involves studying global issues). As the second half of such requirements is rather technical, one uses consistency checks to gain insight into whether or not this is

[12] Though good references are unfortunately very difficult to find. For this reason, reference [5] includes a lengthy discussion of differential geometry on stacks in general, and quotient stacks in particular.

reasonable. At the end of the day, in order to claim that one completely understands these matters, one must be able to answer questions ranging from basics such as "what is a string on another stack, *e.g.* a gerbe" to more difficult ones such as "how does one make sense of sigma model anomalies in this context" (*i.e.*, "what about global issues") and "sigma models with stack targets are one thing, but can do one quantum field theory directly on a stack, now viewed as spacetime itself?" "What is the propagator for a scalar field on $[R^4/Z_2]$?" and so forth.

Even after one attempts to make sense of the notion of a string on a stack, and checks whether a string orbifold really is a sigma model on a quotient stack, one still must reconcile quotient stacks with the role string orbifolds play in the rest of string theory. The statement that string orbifold CFT's coincide with CFT's for strings compactified on quotient stacks has nontrivial physical implications, which cannot be ignored. For example, as mentioned above, in constructing moduli spaces (important to justify everything from mirror symmetry to string/string duality), physicists have assumed that the deformation theory of string orbifolds was that of a quotient space, as indeed we assumed string orbifolds described strings on quotient spaces. If string orbifolds are indeed sigma models on quotient stacks, then one must explain how the deformation theory agrees, despite the fact that deformation theory of a quotient stack is not the same as that of a quotient space [26].

Perhaps the best question one can ask is simply: why bother? If, at the end of the day, describing string orbifolds in terms of quotient stacks accomplishes nothing more than providing an overcomplicated description of group actions on covers, then there is hardly a point. However, we shall argue that such a statement has nontrivial physical implications. First, by thinking about string orbifolds as sigma models, one gains a much clearer understanding of certain physical features of string orbifolds. Second, as mentioned above, such a statement has nontrivial implications for our understanding of string moduli spaces – if a string orbifold really is a sigma model on a quotient stack (assuming that is a sensible notion), then any deformations of the CFT must be understood in terms of deformations of the quotient stack, not the quotient space.

In the next few subsections we shall begin by describing classical sigma models on stacks, verify that string orbifolds are indeed sigma models on quotient stacks, and describe strings on some other stacks. We shall discuss the massless spectrum of sigma models on stacks, and in particular resolve the apparent contradiction that the massless spectrum is not given by the cohomology of the target. We shall also outline how one can understand the well-behavedness of string orbifold CFT's in terms of the geometry of stacks. Finally we shall conclude with some comments on and questions about deformation theory.

More generally there seems to be a considerable gap between the mathematics lore and the physics lore on string orbifolds. We shall address these distinctions as they arise, in an attempt to help bridge certain gaps.

Physicists who are not comfortable with quotient stacks are encouraged to read [5], where we have spoken to their concerns in print. As a description for physicists already exists, the rest of this lecture shall instead be oriented towards mathematicians, and in particular, that aspect of the mathematics community which feels that they already know that string orbifolds are the same as strings compactified on quotient stacks.

We should emphasize that we do not wish to claim that this matter is completely resolved – there is still a tremendous amount of work that must be done to verify that string orbifolds really can be consistently interpreted as sigma models on quotient stacks, to verify that this is not only itself reasonable but consistent with the role string orbifolds play in string theory as a whole. Although we shall outline some of the basic work required to make sense of such notions, and resolve some of the basic paradoxes that crop up, there are still some strong arguments that ultimately this program must fail, which we have not yet been able to resolve. In other words, at present, despite what some might like to believe, there is still no guarantee that string orbifolds really do describe strings on quotient stacks.

## 4.1 Sigma models on stacks

As described above, before one can say that string orbifolds are the same as strings compactified on quotient stacks, one must first describe string compactification on general stacks. A necessary condition for this is the ability to do differential geometry on stacks – something one sometimes hears mentioned. Reference [5] contains a lengthy discussion of this matter, something that seems to be largely missing from the literature.

Now, being able to do differential geometry on stacks is not nearly sufficient to enable one to speak of describing strings on stacks. One must be able to make sense out of the notion of a sigma model on a stack, and answer a long list of related questions. We shall outline classical actions for sigma models on stacks (see [5] for more details), answer some of the basic questions one can ask [5], and list more questions that must be answered before compactification on stacks can be described as well understood.

Let $\mathcal{F}$ be a stack, with atlas $X$. (We shall only attempt to describe sigma models on stacks with atlases.) For readers not well-acquainted with stacks, for $X$ to be an atlas for $\mathcal{F}$ implies that

- implicitly there is also a fixed map $X \rightarrow \mathcal{F}$ (which is required to be a surjective local homeomorphism)
- for any space $Y$ and map $Y \rightarrow \mathcal{F}$, the fibered product $Y \times_{\mathcal{F}} X$ is an honest space, not a stack.

For example, if $\mathcal{F}$ is a space, not just a stack (spaces are special cases of stacks), then $\mathcal{F}$

is its own atlas, and $Y \times_{\mathcal{F}} X = Y$ for any $Y$. For another example, suppose $\mathcal{F}$ is a quotient stack $[X/G]$, with $G$ discrete and acting by diffeomorphisms on a smooth space $X$. In such a case, $X$ is an atlas for $[X/G]$. In this case, $Y \times_{[X/G]} X$ is a principal $G$-bundle over $Y$, partially specifying the map $Y \rightarrow [X/G]$, and the projection map $Y \times_{[X/G]} X \longrightarrow X$ is the $G$-equivariant map from the total space of the bundle to $X$, specifying the rest of the map $Y \rightarrow [X/G]$.

Now, the natural description of a sigma model with target $\mathcal{F}$, formulated on (base) space $Y$, is a sum over equivalence classes[13] of maps $Y \rightarrow \mathcal{F}$, weighted by $\exp(iS)$, where the classical action $S$ is formulated as follows. Fix a map $\phi : Y \rightarrow \mathcal{F}$. If we let[14] $\Phi : Y \times_{\mathcal{F}} X \rightarrow X$ denote the second projection map (implicitly encoding part of the map $\phi : Y \rightarrow \mathcal{F}$), then the natural proposal for the bosonic part of the classical action for a sigma model on $\mathcal{F}$ is given by [5]

$$\int d^2\sigma \ (\pi_1^* \phi^* G_{\mu\nu}) \, \pi_1^* h^{\alpha\beta} \left( \frac{\partial \Phi^\mu}{\partial \sigma^\alpha} \right) \left( \frac{\partial \Phi^\nu}{\partial \sigma^\beta} \right) \tag{25}$$

where $h^{\alpha\beta}$ is the worldsheet metric, $\phi^* G$ denotes the pullback of the metric on $\mathcal{F}$ to $Y$ (metrics on $\mathcal{F}$ are described in terms of their pullbacks), $\pi_1 : Y \times_{\mathcal{F}} X \rightarrow Y$ is the projection map, and this action is integrated over a lift[15] of $Y$ to $Y \times_{\mathcal{F}} X$.

A few examples should help clarify this description:

1. Suppose $\mathcal{F}$ is an ordinary space. Then the path integral is a sum over maps into that space, and as $Y \times_{\mathcal{F}} X = Y$ (taking the atlas $X$ to be $\mathcal{F}$ itself), we see that $\Phi = \phi$, and so in this case the classical action proposed above duplicates the usual classical action, as described in equation (2), as well as the path integral sum. Thus, the description above duplicates sigma models on ordinary spaces.

2. Suppose $\mathcal{F} = [X/G]$, where $X$ is smooth and $G$ is a nontrivial action of a discrete group by diffeomorphisms. Then the path integral is a sum over equivalence classes of maps $Y \rightarrow [X/G]$, which is to say, equivalence classes of principal $G$-bundles on $Y$ together with $G$-equivariant maps from the total space of the bundle into $X$. It is easy to check that the proposed classical action above duplicates the usual classical action for a string orbifold. Also, by summing over (equivalence classes of) maps $Y \rightarrow \mathcal{F}$, note we are summing over both twisted sectors as well as maps within any given twisted sector.

   Now, for each such map $\phi : Y \rightarrow \mathcal{F}$, there are $|G|$ lifts of $Y$ to the total space of the bundle (which is $Y \times_{[X/G]} X$), *i.e.*, $|G|$ twisted sector maps, as they usually appear in physics.

[13]A sigma model path integral is a sum over maps, after all, hence one must take equivalence classes in order to make sense out of such a sum.

[14] Note that since both $Y \times_{\mathcal{F}} X$ and $X$ are ordinary spaces, $\Phi$ is a map in the ordinary sense of the term.

[15]Sensible essentially because the (projection) map $\pi_1 : Y \times_{\mathcal{F}} X \rightarrow Y$ is a surjective local homeomorphism.

Note we are only summing over equivalence classes of bundles, not all possible twisted sector maps. However, we can trivially sum over all possible twisted sector maps, at the cost of overcounting by $|G|$. Hence, we can equivalently describe this in terms of a sum over twisted sector maps, but weighted by $|G|^{-1}$. Hence we recover both the path integral sum and the overall multiplicative factor of $|G|^{-1}$ appearing in string orbifold partition functions (*e.g.* equation (3)).

Thus, we see that the natural definition of a sigma model on a stack duplicates not only sigma models on ordinary spaces, but also string orbifolds when the target is a quotient stack, even down to the $|G|^{-1}$ factor appearing in partition functions. Also, note that the fact that the path integral sum duplicates both the twisted sector sum and the functional integral within each twisted sector is independent of our proposal for a classical action – even if our proposed classical action is wrong, agreement between path integral sums still holds true, and is a 'smoking gun' for some sort of interpretation as a sigma model, as emphasized in [5].

We should take this opportunity to also note that this description of sigma models does not make any assumptions concerning the dimension of the base space $Y$ – classically there are analogues of 'string' orbifolds in every dimension, all obtained precisely by gauging the action of a discrete group on the target space of a sigma model.

So far we have only recovered known results; let us now try something new. Suppose the target $\mathcal{F}$ is a gerbe. For simplicity, we shall assume that $\mathcal{F}$ is the canonical trivial $G$-gerbe on a space $X$. Such a gerbe is described by the quotient stack $[X/G]$, where the action of $G$ on $X$ is trivial. Using the notion of sigma model on a stack as above, one quickly finds that the path integral for this target space is the same as the path integral for a sigma model on $X$, up to an overall multiplicative factor (equal to the number of equivalence classes of principal $G$-bundles on $Y$). As overall factors are irrelevant in path integrals, the result appears to be that a string on the canonical trivial gerbe is the same as a string on the underlying space. More generally, it is natural to conjecture that strings on flat gerbes are equivalent to strings on underlying spaces, but with flat $B$ fields. In particular, such a result would nicely dovetail with the well-known fact that a coherent sheaf on a flat gerbe is equivalent to a 'twisted' sheaf on the underlying space, the same twisting that occurs in the presence of a $B$ field. (For physicists, this is an alternative to the description in terms of modules over Azumaya algebras that has recently been popularized [27].)

So far we have only discussed classical actions for sigma models on stacks, but there is much more that must be done before one can verify that the notion of a sigma model on a stack is necessarily sensible. In effect, we have only considered local behavior, but in order to be sure this notion is sensible after quantization, one also needs to consider global phenomena. Such considerations were the source of much hand-wringing when nonlinear sigma models on ordinary spaces were first introduced (see for example [28]), and must be repeated for stacks. We have not performed such global analyses, but instead shall perform several consistency checks. For example, in the next section we will perform such a check by examining the massless spectrum of the purported sigma model, which ordinarily must coincide with some cohomology of the target space. Interestingly enough, we will find that stacks seem to fail this consistency check! Although we will overcome this particular difficulty, we will not have solutions to the puzzles posed by other failed consistency checks we will describe later.

## 4.2 Massless spectrum of a sigma model on a stack

In [5], we argued that thinking about (supersymmetric) sigma models on quotient stacks $[X/G]$ (with $G$ acting effectively) led one to conclude that the massless spectrum should be given by cohomology of the associated inertia group stack $I_{[X/G]}$:

$$I_{[X/\Gamma]} \cong \coprod_{[g]} [\, X^g / C(g) \,] \tag{26}$$

a result that, although obscure to most physicists, is known to some mathematicians. (Note the obvious relation to the Hirzebruch-Höfer [9] description of orbifold Euler characteristics, as in expression (5).)

Note that we seem to immediately find a contradiction. The massless spectrum of a string sigma model is given by some cohomology of the target; yet, in a string orbifold, the twisted sectors of the massless spectrum are not described by cohomology of the (quotient stack) target. On the face of it, this completely contradicts the idea that string orbifolds can consistently be considered to be sigma models on quotient stacks, and indeed, many physicists would take this as strong evidence that string orbifolds *cannot* possibly be the same as strings compactified on stacks.

In fact, a physics subtlety saves the day. As observed in [5], the usual statement that the massless spectrum of a string sigma model is the cohomology of the target is a bit imprecise – it would be better to say, the massless spectrum of a string sigma model is the cohomology of the zero-momentum part of the loop space of the target. When the target is an ordinary space, the zero-momentum part of the loop space is the original space itself. However, when the target is a stack, there is a distinction. Although we were able to resolve this apparent paradox, we shall see more apparent contradictions later, and unfortunately we will not be able to describe how to solve the puzzles they pose.

Another important point to note is that we have been naturally led to a description of twist fields that is very different from the description most physicists assumed would hold true. Indeed, the form of this description of twist fields we have just given is largely unknown within the physics community, although after unraveling definitions, it boils down to a description in terms of group actions on covers that appears more familiar. As described in section 2, in the physics community many assumed that twist fields could be understood in terms of some cohomology of the quotient space, called 'orbifold cohomology,' that would implicitly encode information about the hypothesized 'stringy' behavior at singularities. Put another way, the physics of string orbifolds seemed to predict the existence of such a description of twist fields, not just in terms of group actions on covers, but in terms of some cohomology of a quotient space. However, not only is the description above not a cohomology of quotient spaces, it is not even a cohomology of a quotient stack! By thinking about the physics of string orbifolds in terms of stacks, a more subtle approach than most physicists have previously considered, we have been led to a very different description of twist fields than most physicists have assumed would hold true. Put another way, by thinking in terms of stacks, one is led to conclude that a description of twist fields in terms of group actions on covers is the best one can hope to manage, that previous expectations of some description directly in terms of the underlying quotient space were naive.

Although expression (26) is largely unknown within the physics community (and indeed, substantially deviates from what physicists have traditionally expected), it is not unknown within the mathematics community. For example, it is formally equivalent[16] to the description of twist fields given in [16]. We should emphasize again, however, that the definition of 'orbifold cohomology' presently used in the mathematics community appears to be somewhat different from the definition used by many members of the physics community. A description of twist fields in terms of group actions on covers, although interesting, appears neither to be the 'orbifold cohomology' that many physicists have traditionally desired, nor does it seem to shed insight into the physics questions that physicists have hoped orbifold cohomology would answer. We have argued, from an understanding of string orbifolds as sigma models on quotient stacks, that [16] is nevertheless the correct and 'final' understanding of twist fields. However, if future analysis reveals that we are wrong, then although [16] is very interesting, some physicists would argue that it is not necessarily the final word on twist fields.

## 4.3 Well-behavedness of string orbifold CFT's

So far we have only set up *how* one could describe string orbifolds in terms of stacks. We have yet to explain *why* one would wish to do so. After all, if we are only using quotient stacks as a radically overcomplicated way to describe group actions on covers, then there is hardly a point. However, this description has nontrivial physical consequences, which is the real reasons why physicists should be interested in such a description. We shall describe in this section how this description greatly clarifies the physics of string orbifolds. Viewing string orbifolds as sigma models on stacks sheds new light on the physics of string orbifolds, and

[16] Technically the paper [16] used a description of twist fields (in terms of group actions on covers) equivalent to the one above as a starting point, and concentrated on developing an ansatz for a cup product which, by construction, duplicates the twist field correlation functions studied in the older physics literature. The distinction we are trying to draw here appears to be somewhat more basic – many physicists would like some cohomology theory of a quotient space, from which twist fields emerge naturally, whereas a description of twist fields in terms of group actions on covers that begins with expression (26) does not sound like a final answer to some physicists.

many properties of string orbifolds that were previously attributed to some sort of 'stringy' effect can be seen to be easy consequences of the geometry of stacks. In later sections we shall describe other nontrivial consequences of working with stacks.

For example, historically one of the more interesting and useful features of string orbifold CFT's was their well-behavedness: string orbifolds were constructed in an attempt to describe strings on quotient spaces, yet even when the quotient space is singular, the CFT is well-behaved. Historically physicists have often quoted two general mechanisms in connection with this behavior:

- "String orbifold CFT's are well-behaved because a CFT operator typically associated with holonomy of the $B$ field on exceptional divisors (the 'theta angle') is expected to often have a nonzero vacuum expectation value" [11, 12, 13].

  Certainly the string orbifold point in a moduli space of string vacua is distinct from the point corresponding to a sigma model on the quotient space, and one natural mechanism for this to occur is if the string orbifold point corresponds to a nonzero theta angle. Moreover, from a linear sigma model perspective this mechanism is extremely natural [11].

  Now, one might ask if this phenomenon has a more intrinsic explanation. This nonzero theta angle is described either by studying sigma models on resolutions of the underlying quotient space, and considering rational-curve-counting in limits when the exceptional divisors shrink to zero size, or in terms of linear sigma models, in which the physical theory is described very indirectly in terms of a distinct theory which is believed to flow (in the sense of the renormalization group) to the relevant physical theory. Neither of these descriptions involves the string orbifold CFT directly, but rather only deformations of that physical theory.

  If one wishes to propose a description of string orbifolds as something other than strings on quotient spaces with stringy effects at singularities, then one natural question that will be asked is, can one understand this nonzero theta angle business geometrically? We have described nonzero theta angles in terms of CFT operators, but they might have a more geometric description. Since the operator in question is associated to $B$ field holonomies on exceptional divisors, sometimes people describe this phenomenon as having a "$B$ field at a quotient singularity" – if taken literally, what would such a statement mean? Can this be understood mathematically, or is it an intrinsically-CFT phenomenon? Why, directly in the language of the CFT (as opposed to a massive theory), does this have the effect of making the CFT well-behaved? And perhaps best of all, how can one see such a $B$ field directly in the string orbifold CFT? (The arguments given in [11, 12, 13] and elsewhere do not directly speak to the CFT itself, but either to deformations of it, or massive theories believed to flow to it in the IR.)

- "String orbifold CFT's are well-behaved because they describe strings on ('infinitesimal') resolutions."

> Since string orbifold CFT's are well-behaved (*i.e.*, they behave as though they described strings on smooth spaces), and since they have twist fields which are often associated to exceptional divisors in a minimal resolution of the underlying quotient space, physicists sometimes speak loosely of string orbifold CFT's as describing strings on resolutions. Of course, this cannot be literally correct in general, simply because terminal singularities exist – not all quotient singularities of interest to physics can be resolved (and even those that can, typically do not have unique resolutions). Thus, any attempt to describe properties of string orbifolds in terms of a resolution simply cannot possibly be the general story.
>
> Having said that, given an orbifold that can be resolved, it is true that the full string theory (not just the CFT) will often see that resolution – as fields in the full string theory fluctuate, twist fields corresponding to blowup modes will be excited, and so the full string theory will probe resolutions as it probes other small deformations of the original CFT. So, again, describing string orbifold CFT's in terms of strings on resolutions gives a not unreasonable intuition for some of their properties. However, the fact that the full string theory is well-behaved close to a given point in string moduli space is hardly itself evidence that the CFT at that point is well-behaved – if it were, badly-behaved CFT's would be far more rare, as one can usually deform them to better-behaved CFT's.

Stacks, on the other hand, seem to give a different perspective, which appears to be simpler and cleaner. Quotient stacks, the target spaces of string orbifolds viewed as sigma models, are smooth[17], and, moreover, smooth in precisely the sense relevant for sigma models.

The fact that quotient stacks themselves are smooth is not itself sufficient to explain why string orbifold CFT's are smooth, one must also check whether the sense in which they are smooth is physically relevant. In this particular case, the sense in which a quotient stack is smooth precisely coincides with the notion of smoothness relevant for a sigma model to be well-behaved.

However, the bottom line is that one can now see that string orbifold CFT's are well-behaved because they are sigma models on smooth spaces, namely, quotient stacks.

The business involving the $B$ field also appears to have a new understanding from a stack perspective. To see how it arises, we shall consider D-brane probes, described as coherent sheaves. For readers who might be leery of this usage in a stack context, we should point out two important facts:

- First, a coherent sheaf on a quotient stack $[X/G]$ is precisely a $G$-equivariant coherent sheaf on $X$ (which is not quite the same as a sheaf on the quotient space $X/G$). Recall that the Douglas-Moore construction [29] of D-branes on string orbifolds describes

[17] Technically, $[X/G]$ is smooth if $X$ is smooth and $G$ is a discrete group acting on $X$ by diffeomorphisms.

$G$-equivariant objects on the covering space, so in other words, the Douglas-Moore construction of branes on orbifolds precisely corresponds to coherent sheaves on quotient stacks.

- Second, a coherent sheaf on a flat gerbe is the same thing as a 'twisted' sheaf on the underlying space, *i.e.*, twisted in the sense of 'bundles' on D-branes with $B$ fields. Put another way, sheaves on gerbes are an alternative to modules over Azumaya algebras as popularized in [27].

Given that all stacks look locally like either orbifolds or gerbes, these two cases justify using coherent sheaves to describe D-branes on more general stacks.

Now, in order to see what quotients stacks have to do with $B$ fields, let us consider a naive 'blowup' of the stack $[\mathbf{C}^2/\mathbf{Z}_2]$. In particular, the minimal resolution of the quotient singularity $\mathbf{C}^2/\mathbf{Z}_2$ is the same as the quotient $(\mathrm{Bl}_1\mathbf{C}^2)/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ action has been extended trivially over the exceptional divisor of the blowup. Thus, the quotient stack $[(\mathrm{Bl}_1\mathbf{C}^2)/\mathbf{Z}_2]$ is a naive stacky analogue of the resolution of the quotient space $\mathbf{C}^2/\mathbf{Z}_2$, and among other things, is a stack over the resolution.

Finally, consider D-brane probes of this stack, viewed as coherent sheaves. Away from the exceptional divisor, this stack looks like the corresponding space, so a D-brane away from the exceptional divisor thinks it is propagating on the underlying space. A coherent sheaf over the exceptional divisor, on the other hand, is a sheaf on a gerbe, and so describes a D-brane in the presence of a $B$ field. Thus, we see the advertised $B$ field. In fact, we can read off even more – the gerbe over the exceptional divisor is a $\mathbf{Z}_2$-gerbe, so the corresponding $B$ field holonomy must be either 0 or 1/2. Determining which requires a detailed calculation, but notice that without doing any work, we have quickly arrived in the right ballpark.

Although we have spoken about D-brane probes, the same remarks also hold true for sigma models, using the result that a sigma model on a flat gerbe is equivalent to a sigma model on the underlying space with a nontrivial $B$ field.

Having said all of this, in order to properly understand the old lore concerning $B$ fields at quotient singularities would require a detailed understanding of Kähler moduli of quotient stacks, something that does not presently seem to exist. However, the point is that from the perspective of quotient stacks, the lore concerning $B$ fields is extremely natural – having a "$B$ field at a quotient singularity" is a natural notion forced on one by virtue of working with stacks, whereas understanding such a statement in terms of quotient spaces seems impossibly obscure.

A few more words should be said. We have argued that quotient stacks contain within themselves an intrinsic notion of a "$B$ field at a singularity," and thereby potentially cleared up one confusing issue involving string orbifolds. However, linking this perspective up with traditional computations is a nontrivial matter. Historically, physicists (implicitly assuming that the deformation theory relevant to string orbifolds was that of quotient *spaces*) arrived at the same conclusions about the occurrence of a $B$ field by considering rational curve counting arguments in limits in which exceptional divisors shrink to zero size. To be consistent, once we accept that string orbifolds describe strings on quotient stacks, any Kähler deformation must be a Kähler deformation of the *stack*, not the quotient space, and the resulting stack may well have a different rational curve count. In order for the statements that the physics community have assumed to hold true, we need some theorems regarding the relationship between rational curves in resolutions of quotient spaces and Kähler deformations of quotient stacks, something we will speak about at greater length in the next subsection.

## 4.4 Deformation theory and other unresolved issues

In the last several subsections we have described a number of positive developments in understanding string orbifolds as strings on quotient stacks:

- We have described the notion of a sigma model on a general stack with an atlas, and verified that, indeed, within the context of that definition a string orbifold is literally a sigma model on a quotient stack – in other words, at least formally at a classical level, a string orbifold appears to be the same as a string compactified on a quotient stack.
- We described the massless spectrum of a sigma model on a quotient stack, and in so doing, resolved the apparent contradiction that the massless spectrum of a string orbifold is *not* the cohomology of the target of the purported corresponding sigma model.
- We have used this description of string orbifolds to shed light on some physical properties of string orbifolds that have historically been considered very mysterious.

However, it must be emphasized that there is still a tremendous amount of work that must be done to completely nail down these ideas. Even to completely nail down the notion of a sigma model on a stack, much more work must be done to answer questions ranging from

> How does one describe sigma model anomalies in the context of stack targets?

to

> Sigma models with stack targets are one thing, but can one do quantum field theory directly on a stack, where the stack is now taken to be the underlying

spacetime? Answering this question positively would imply being able to explicitly write down, for example, the propagator for a scalar field on the stack $[\mathbf{R}^4/\mathbf{Z}_2]$. More generally, for sigma models, if the target looks like a space merely locally, one can go a long way, but in order to do quantum field theory on a space, global properties are important from the outset.

One of the larger remaining unresolved issues involves deformation theory. For the last fifteen years, physicists have assumed that the deformation theory of a string orbifold is the same as that of a quotient space. This assumption has figured into topics ranging from mirror symmetry to string/string duality. Indeed, the fact that this assumption has given consistent results is a strong argument to many physicists that string orbifolds describe strings on quotient *spaces*, that stacks have no direct physical relevance.

Clearly, one of the more important questions that must be answered before physicists will agree that quotient stacks are physically relevant, is simply, why have we been able to get away with assuming the relevant deformation theory is that of a quotient *space* ? For example, Kähler deformations of string orbifolds have historically been described by physicists in terms of resolutions of the underlying quotient space $X/G$, because physicists have made naive assumptions that string orbifolds describe strings on quotient spaces (suitably decorated). Once one accepts that string orbifolds describe strings on something other than quotient spaces, these old arguments must be completely reexamined. For example, a Kähler deformation of a string orbifold must be a stacky "resolution" of the quotient stack $[X/G]$, if string orbifolds really do describe strings on quotient stacks, *not* a resolution of the quotient space $X/G$. One cannot consistently speak of string orbifolds in terms of stacks, and also simultaneously talk about deformations and resolutions of quotient spaces.

A complete understanding of the deformation theory of quotient stacks, even if only for a family of simple examples such as $[\mathbf{C}^2/G]$, would be extremely desirable. Also, a complete understanding would probably also shed light on other matters – for example, the author would not be at all surprised if yet another way of thinking about the McKay correspondence emerged from such considerations.

There are other matters related to deformation theory that one could also ask. For example, in string theory, Kähler moduli are paired with "theta angles." It is tempting to speculate that these theta angles might have some sort of understanding in terms of extra data needed to completely specify stack moduli.

Another matter concerns the lore that "string orbifolds have $B$ fields at quotient singularities." Previously we described how one could clearly see, at least in naive calculations, that $B$ fields arise very naturally in stack contexts. However, to actually calculate holonomies on exceptional divisors requires precisely understanding the stack-y analogue of blowups, for example. Again, knowing the deformation theory is very important.

In fact, understanding the $B$-field-at-singularities business properly implies a nontrivial consistency check involving rational curve counting arguments, as we implied in the last subsection. As we described there, one of the attractive features of quotient stacks, from a physics perspective, is that they appear to give an implicit understanding of what it means to have a "$B$ field at a singularity," as is spoken of in the physics literature. This feature was derived in the past by considering rational curve sums in limits in which exceptional divisors shrunk to zero size (and so could be understood as a limit, but was very mysterious from the direct perspective of the string orbifold CFT). Now, to connect the implicit understanding outlined in the previous section with the standard derivation, implies a nontrivial relationship. After all, as we have stressed here, one of the implications of the idea that string orbifolds describe strings on quotient stacks, is that the relevant deformation theory is that of a stack. Physicists have long equated such deformation theory with that of a quotient space because of some slightly naive assumptions concerning string orbifolds, namely that they describe strings on quotient *spaces* (decorated with some quantum effects at singularities, which are typically glossed over in these discussions). In order to agree with the usual calculations physicists perform, rational curve counting in the stacky "resolution" of a stack $[X/G]$, combined with any nontrivial $B$ field holonomy (from any gerbe structure on exceptional divisors), must combine to yield physical results equivalent to those obtained by just counting rational curves in a resolution of the quotient space $X/G$. In other words, on the face of it, in order to be consistent, stacky effects combined with stacky rational curves had better combine to agree with rational curve counting in resolutions of quotient spaces. If this does not happen, then either string orbifolds do not describe strings on quotient stacks, because of some nonobvious subtlety, or the physics analysis of string moduli spaces must be modified, which would have serious ramifications for our understanding of everything from mirror symmetry to string/string duality.

Yet another matter concerns intermediate points in the moduli space. In string theory, one can interpolate between the string orbifold point in moduli space (with 'nonzero $B$ field') and the singular quotient space point (with 'zero $B$ field'). One could ask, how can the intermediate points be interpreted? Is there a family of stacks interpolating between quotient stacks and quotient spaces, *i.e.*, do all of those intermediate points have stack interpretations? If not, perhaps some slight generalization of stacks is required to understand those points, or perhaps those points can only be understood within conformal field theory. The usual physics picture is that walking along those intermediate points corresponds to changing the vacuum expectation value of some operator describing strings on a (decorated) quotient space. To many physicists, that picture sounds much more natural than the idea that an operator which previously had no geometric interpretation, acts to interpolate between distinct 'spaces,' *i.e.*, between quotient spaces and quotient stacks. This also has been used as an indirect argument in some quarters that string orbifolds are not the same as strings on quotient stacks.

# 5 Conclusions

In this lecture we have given an overview of two recent developments tied to string orbifolds in physics.

First, we gave a mostly complete understanding of discrete torsion, a degree of freedom associated with string orbifolds, simply as the choice of orbifold group action on a field in the theory known as the $B$ field. We outlined how to derive the $H^2(G, U(1))$ classification, Vafa's twisted sector phase factors, Douglas's projectivized equivariant K-theory, and analogues for other tensor field potentials. Some work remains to be done to completely understand supersymmetric moduli obstruction.

Second, we described the beginnings of a program to understand the role of quotient stacks in string orbifolds. Although someone not acquainted with the physics lore might claim they 'know' that string orbifolds are the same as strings compactified on quotient stacks, there is a tremendous amount of work that must be done to be able to justify such a statement – not only to make sense out of the notion of a string compactified on *any* stack, but also to reconcile such claims with the role that string orbifolds play in the rest of string theory. We have outlined the basics, such as how one defines a sigma model, and how to resolve some of the basic apparent paradoxes, such as the fact that the massless spectrum of such sigma models is not given by the cohomology of the target space, but a tremendous amount of work remains to be done, In particular, this issue appears to have nontrivial implications:

- If the physics lore is correct, and string orbifolds do not describe strings on quotient stacks (and indeed, we have described several arguments that support such a conclusion), then some physicists would question whether a description of twist fields in terms of group actions on covers can hope to be the final word on the matter.
- If, on the other hand, string orbifolds do describe strings on quotient spaces, then some work must be done to properly understand string moduli spaces. If string orbifolds describe strings on quotient stacks, then one cannot identify moduli of the CFT with deformations or resolutions of the corresponding quotient *space*, but rather moduli of the CFT must be identified with deformations and analogues of resolutions for the quotient stack.

As a description of string orbifolds in terms of sigma models on quotient stacks appears to greatly clarify the physics of string orbifolds, we think it very important that these issues be resolved.

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