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THE ADJACENT SIDES OF HYPERBOLIC LAMBERT QUADRILATERALS

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Abstract. We prove sharp bounds for the product and the sum of a pair of hyperbolic adjacent sides of hyperbolic Lambert quadrilaterals in the unit disk.

Keywords. Hyperbolic Lambert quadrilateral, hyperbolic metric, Hölder mean, convexity

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1. Introduction

Given a pair of points in the closure of the unit disk $\mathbb{B}^2$, there exists a unique hyperbolic geodesic line joining these two points. Hyperbolic lines are simply sets of the form $C \cap \mathbb{B}^2$ where C is a circle perpendicular to the unit circle, or a Euclidean diameter of $\mathbb{B}^2$. For a quadruple of four points $\{a, b, c, d\}$ in the closure of the unit disk we can draw these hyperbolic lines through each of the four pairs of points $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$. If these hyperbolic lines bound a domain $D \subset \mathbb{B}^2$ such that the points $\{a, b, c, d\}$ are in the positive order on the boundary of the domain D, then we say that the quadruple of points $\{a, b, c, d\}$ determines a hyperbolic quadrilateral $Q(a, b, c, d)$ and that the points a, b, c, d are its vertices. A hyperbolic quadrilateral with angles equal to $\pi/2, \pi/2, \pi/2, \phi\,(0 \leq \phi < \pi/2)$, is called a hyperbolic *Lambert* quadrilateral [Be, p. 156], see Figure 1. Observe that one of the vertices of a Lambert quadrilateral may be on the unit circle, in which case the angle at that vertex is $\phi = 0$.

In [VW, Theorems 1.1 and 1.2], the authors gave the sharp bounds of the product and the sum of two hyperbolic distances between the opposite sides of hyperbolic Lambert quadrilaterals in the unit disk. By [VW, Proposition 3.3], we know that the above two hyperbolic distances are also a pair of the lengths of the adjacent sides of hyperbolic Lambert quadrilaterals (see d_1, d_2 in Figure 1). Therefore, it is natural to raise the problem: how does the left pair of hyperbolic adjacent sides behave (see d_3, d_4 in Figure 1)? This is the motivation of this paper, and we will find the sharp bounds of the product and the sum of another pair of hyperbolic adjacent sides of hyperbolic Lambert quadrilaterals in the unit disk. The main results of this paper are formulated as follows.

Theorem 1.1. *Let $Q(v_a, v_b, v_c, v_d)$ be a hyperbolic Lambert quadrilateral in $\mathbb{B}^2$ and let the quadruple of interior angles $(\frac{\pi}{2}, \frac{\pi}{2}, \phi, \frac{\pi}{2})$, $\phi \in [0, \pi/2)$, corresponds to the quadruple (v_a, v_b, v_c, v_d) of vertices. Let $d_3 = \rho(v_c, v_b)$, $d_4 = \rho(v_c, v_d)$ (see Figure 1), and let $s = \mathrm{th}\rho(v_a, v_c) \in (0, 1)$, where ρ is the hyperbolic metric in the unit disk. Then*

$$d_3 d_4 \leq \left(\log \sqrt{\frac{1 + s\sqrt{2 - s^2}}{1 - s^2}} \right)^2$$

Here equality holds if and only if v_c is on the bisector of the interior angle at v_a.

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Theorem 1.2. *Let $Q(v_a\,, v_b\,, v_c\,, v_d)$, d_3, d_4 and s be as in Theorem 1.1.*

$$\log\sqrt{\frac{1+s}{1-s}} < d_3 + d_4 \leq 2\log\frac{1+s\sqrt{2-s^2}}{1-s^2}.$$

Equality holds in the right-hand side if and only if v_c is on the bisector of the interior angle at v_a.

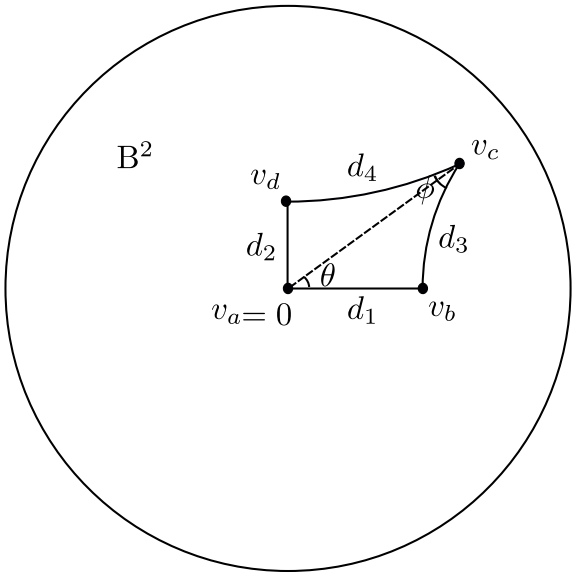

FIGURE 1. A hyperbolic Lambert quadrilateral in $\mathbb{B}^2$

We denote the other two sides of $Q(v_a\,, v_b\,, v_c\,, v_d)$ by

$$d_1 = \rho(v_a\,, v_b) \quad \text{and} \quad d_2 = \rho(v_a\,, v_d).$$

In a Lambert quadrilateral, the angle ϕ is related to the lengths d_1, d_2 of the sides "opposite" to it as follows [Be, Theorem 7.17.1]:

$$\mathrm{sh}\, d_1 \mathrm{sh}\, d_2 = \cos\phi.$$

See also the recent paper of A. F. Beardon and D. Minda [BM, Lemma 5].

In [VW, Corollary 1.3], M. Vuorinen and G.-D. Wang provided a connection between d_1, d_2 and $s = \mathrm{th}\rho(v_a, v_c)$ as follows

$$\mathrm{th}^2\, d_1 + \mathrm{th}^2\, d_2 = s^2. \tag{1.3}$$

Proposition 2.10 (in section 2) yields the following corollary, which provides a connection between d_3, d_4 and $s = \mathrm{th}\rho(v_a, v_c)$.

Corollary 1.4. *Let s, d_3 and d_4 be as in Theorem 1.1. Then*

$$\mathrm{sh}^2\, d_3 + \mathrm{sh}^2\, d_4 = \frac{s^2}{1-s^2}. \tag{1.5}$$

By (1.3) and (1.5), we get the following equality

$$\frac{1}{\mathrm{th}^2\, d_1 + \mathrm{th}^2\, d_2} - \frac{1}{\mathrm{sh}^2\, d_3 + \mathrm{sh}^2\, d_4} = 1,$$

which shows the relation between the four sides of the hyperbolic Lambert quadrilateral $Q(v_a\,, v_b\,, v_c\,, v_d)$.

This paper is organized as follows. In Section 2, the notation and facts on the hyperbolic metric are stated, and some lemmas on the inverse hyperbolic trigonometric functions are proved. Section 3 is devoted to the Hölder convexity of the inverse hyperbolic sine function. The main results are proved in Section 4.

2. Preliminaries

It is assumed that the reader is familiar with basic definitions of geometric function theory, see e.g. [Be, V]. We recall here some basic information on hyperbolic geometry [Be].

The chordal distance is defined by

$$\begin{cases} q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, & x\,, y \neq \infty \\ q(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}, & x \neq \infty, \end{cases} \tag{2.1}$$

for $x, y \in \overline{\mathbb{R}^2}$.

For an ordered quadruple a, b, c, d of distinct points in $\overline{\mathbb{R}^2}$ we define the absolute ratio by

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}.$$

It follows from (2.1) that for distinct points $a, b, c, d \in \mathbb{R}^2$

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}. \tag{2.2}$$

The most important property of the absolute ratio is Möbius invariance, see [Be, Theorem 3.2.7], i.e., if f is a Möbius transformation, then

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|,$$

for all distinct a, b, c, d in $\overline{\mathbb{R}^2}$.

For a domain $G \subsetneq \mathbb{R}^2$ and a continuous weight function $w : G \to (0, \infty)$, we define the weighted length of a rectifiable curve $\gamma \subset G$ to be

$$l_w(\gamma) = \int_\gamma w(z)|dz|$$

and the weighted distance between two points $x, y \in G$ by

$$d_w(x,y) = \inf_\gamma l_w(\gamma),$$

where the infimum is taken over all rectifiable curves in G joining x and y ($x = (x_1, x_2)$, $y = (y_1, y_2)$). It is easy to see that d_w defines a metric on G and (G, d_w) is a metric space. We say that a curve $\gamma : [0, 1] \to G$ is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if for all $t \in (0, 1)$, we have

$$d_w(\gamma(0), \gamma(1)) = d_w(\gamma(0), \gamma(t)) + d_w(\gamma(t), \gamma(1)).$$

The hyperbolic distance in $\mathbb{H}^2$ and $\mathbb{B}^2$ is defined in terms of the weight functions $w_{\mathbb{H}^2}(x) = 1/x_2$ and $w_{\mathbb{B}^2}(x) = 2/(1 - |x|^2)$, respectively. We also have the corresponding explicit formulas

$$\cosh \rho_{\mathbb{H}^2}(x,y) = 1 + \frac{|x-y|^2}{2x_2y_2} \tag{2.3}$$

for all $x, y \in \mathbb{H}^2$ [Be, p.35], and

$$\rho_{\mathbb{B}^2}(x,y) = 2\text{arsh}\,\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}} \tag{2.4}$$

for all $x, y \in \mathbb{B}^2$ [Be, p.40]. In particular, for $t \in (0,1)$,

$$\rho_{\mathbb{B}^2}(0, te_1) = \log\frac{1+t}{1-t} = 2\text{arth}t. \tag{2.5}$$

There is a third equivalent way to express the hyperbolic distances. Let $G \in \{\mathbb{H}^2, \mathbb{B}^2\}$, $x, y \in G$ and let L be an arc of a circle perpendicular to ∂G with $x, y \in L$ and let $\{x_*, y_*\} = L \cap \partial G$, the points being labelled so that x_*, x, y, y_* occur in this order on L. Then by [Be, (7.26)]

$$\rho_G(x,y) = \sup\{\log|a,x,y,b| : a, b \in \partial G\} = \log|x_*, x, y, y_*|. \tag{2.6}$$

The hyperbolic distance remains invariant under Möbius transformations of G onto G' for G, $G' \in \{\mathbb{H}^2, \mathbb{B}^2\}$.

Hyperbolic geodesics are arcs of circles which are orthogonal to the boundary of the domain. More precisely, for $a, b \in \mathbb{B}^2$ (or $\mathbb{H}^2$), the hyperbolic geodesic segment joining a to b is an arc of a circle orthogonal to S^1 (or $\partial\mathbb{H}^2$). In a limiting case the points a and b are located on a Euclidean line through 0 (or located on a normal of $\partial\mathbb{H}^2$), see [Be]. Therefore, the points x_* and y_* are the end points of the hyperbolic geodesic. For any two distinct points the hyperbolic geodesic segment is unique (see Figure 2 and 3). For basic facts about the hyperbolic geometry we refer the interested reader to [A], [Be] and [KL].

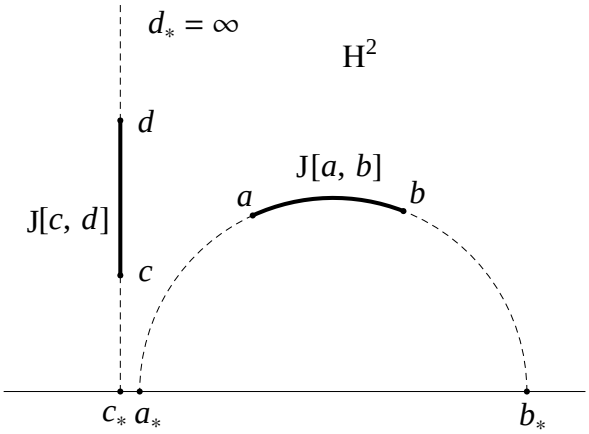

FIGURE 2. Hyperbolic geodesic segments in $\mathbb{H}^2$

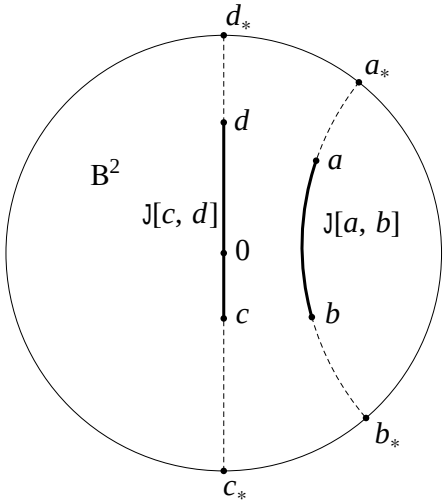

FIGURE 3. Hyperbolic geodesic segments in $\mathbb{B}^2$

By [K, Exercise 1.1.27] and [KV, Lemma 2.2], for $x\,, y\, \in \mathbb{R}^2 \setminus \{0\}$ such that $0, x, y$ are noncollinear, the circle $S^1(a, r_a)$ containing x, y is orthogonal to the unit circle, where

$$a = i\frac{y(1+|x|^2) - x(1+|y|^2)}{2(x_2y_1 - x_1y_2)} \quad \textit{and} \quad r_a = \frac{|x-y|\big||x|y|^2 - y\big|}{2|y||x_1y_2 - x_2y_1|}\,. \tag{2.7}$$

The following *monotone form of l'Hôpital's rule* is useful in deriving monotonicity properties and obtaining inequalities. See the extensive bibliography of [AVZ].

Lemma 2.8. [AVV1, Theorem 1.25]. *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b), and let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing(deceasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

From now on we let $r' = \sqrt{1-r^2}$ for $0 < r < 1$.

Lemma 2.9. *Let $s \in (0,1)$, $m = s/\sqrt{1-s^2}$ and $r \in (0,1)$.*

(1) The function $f_1(r) \equiv \mathrm{arsh}(m\,r)/\mathrm{arth}(s\,r)$ is strictly decreasing with range $(1, 1/\sqrt{1-s^2})$.

(2) The function $f_2(r) \equiv \mathrm{arsh}(m\,r)\mathrm{arsh}(m\,r')$ is strictly increasing on $(0, \frac{\sqrt{2}}{2}]$ and strictly decreasing on $[\frac{\sqrt{2}}{2}, 1)$ with maximum value $(\mathrm{arsh}(\frac{\sqrt{2}}{2}m))^2$.

(3) The function $f_3(r) \equiv \mathrm{arsh}(m\,r) + \mathrm{arsh}(m\,r')$ is strictly increasing on $(0, \frac{\sqrt{2}}{2}]$ and strictly decreasing on $[\frac{\sqrt{2}}{2}, 1)$ with range $(\mathrm{arsh}\, m,\ 2\mathrm{arsh}\,(\frac{\sqrt{2}}{2}m))$.

Proof. (1) Let $f_{11}(r) = \mathrm{arsh}(m\,r)$ and $f_{12}(r) = \mathrm{arth}(s\,r)$. Then $f_{11}(0^+) = f_{12}(0^+) = 0$. By differentiation,

$$\frac{f_{11}'(r)}{f_{12}'(r)} = \frac{1-s^2r^2}{\sqrt{1-s^2r'^2}}$$

which is strictly decreasing. Hence by Lemma 2.8, f_1 is strictly decreasing with $f_1(0^+) = 1/\sqrt{1-s^2}$ and $f_1(1^-) = 1$ because

$$\mathrm{arsh}\, m = \log\sqrt{\frac{1+s}{1-s}} = \mathrm{arth}\, s.$$

(2) By differentiation,

$$f_2'(r) = \frac{m}{r'\sqrt{1+m^2r'^2}\sqrt{1+m^2r^2}}(\phi_2(r') - \phi_2(r)),$$

where $\phi_2(r) = r\sqrt{1+m^2r^2}\mathrm{arsh}(m\,r)$. It is clear that ϕ_2 is strictly increasing. Therefore, we get the result.

(3) By differentiation,

$$f_3'(r) = \frac{m}{r'\sqrt{1+m^2r'^2}\sqrt{1+m^2r^2}}(\phi_3(r') - \phi_3(r)),$$

where $\phi_3(r) = r\sqrt{1+m^2r^2}$. It is clear that ϕ_3 is strictly increasing and hence the result follows immediately. □

Proposition 2.10. *Let $Q(v_a\,, v_b\,, v_c\,, v_d)$ be a hyperbolic Lambert quadrilateral in $\mathbb{B}^2$ and let the quadruple of interior angles $(\frac{\pi}{2}\,, \frac{\pi}{2}\,, \phi\,, \frac{\pi}{2})$, $\phi \in [0, \pi/2)$, corresponds to the quadruple $(v_a\,, v_b\,, v_c\,, v_d)$ of vertices. Let $d_1 = \rho(v_a\,, v_b)\,$, $d_2 = \rho(v_a\,, v_d)$ $d_3 = \rho(v_c\,, v_b)\,$, $d_4 = \rho(v_c\,, v_d)$, and let $s = \mathrm{th}\rho(v_a, v_c) \in (0,1)$ and $m = \frac{s}{\sqrt{1-s^2}}$. Then*

$$d_1 = \mathrm{arth}(s\,r), \quad d_2 = \mathrm{arth}(s\,r'), \quad d_3 = \mathrm{arsh}(m\,r'), \quad d_4 = \mathrm{arsh}(m\,r),$$

where $r \in (0,1)$ is a constant.

Proof. Since the hyperbolic distance is Möbius invariant, we may assume that $v_a = 0$, v_b is on the real axis X, v_d is on the imaginary axis Y and $v_c = te^{i\theta}$, $0 < t < 1$ and

$0 < \theta < \frac{\pi}{2}$ (see Figure 1). Let $r = \cos\theta$. Then by (2.7) the circle $S^1(b, r_b)$ through v_c and $\overline{v_c}$ is orthogonal to $\partial\mathbb{B}^2$, where

$$b = \frac{1+t^2}{2t\cos\theta} \quad \text{and} \quad r_b = \frac{\sqrt{(1+t^2)^2 - 4t^2\cos^2\theta}}{2t\cos\theta}.$$

d_1 and d_2 are from the proof of Theorem 1.1 in [VW] or directly derived by (2.5).

By (2.4), we get

$$d_3 = \rho(v_c, v_b) = 2\text{arsh}\, f_s(r),$$

where

$$f_s(r) = \sqrt{\frac{g_s^2(r) + g_s^2(1) - 2r g_s(r) g_s(1)}{(1-g_s^2(r))(1-g_s^2(1))}}$$

and

$$g_s(r) = \frac{1-\sqrt{1-s^2r^2}}{s\,r}.$$

Similarly, we get

$$d_4 = \rho(v_c, v_d) = 2\text{arsh}\, f_s(r').$$

By calculation, we get the equality

$$f_s(r)\sqrt{1+f_s^2(r)} = \frac{mr'}{2},$$

which implies $\text{sh}d_3 = mr'$ and $\text{sh}d_4 = mr$. Therefore,

$$d_3 = \text{arsh}(m\,r') \quad \text{and} \quad d_4 = \text{arsh}(m\,r).$$

□

By Proposition 2.10 and Lemma 2.9(1), we have the following theorem.

Theorem 2.11. *Let $Q(v_a, v_b, v_c, v_d)$, d_1, d_2, d_3, d_4 and s be as in Proposition 2.10. Then*

$$d_2 < d_3 < \frac{1}{\sqrt{1-s^2}} d_2 \quad \text{and} \quad d_1 < d_4 < \frac{1}{\sqrt{1-s^2}} d_1.$$

Remark 2.12. *M. Vuorinen and G.-D. Wang gave the bounds for the product and sum of d_1 and d_2* [VW, Theorem 1.1 and Theorem 1.2] *by which and Theorem 2.11 we can get the bounds for the product and sum of d_3 and d_4. But the results are weaker than that of Theorem 1.1 and Theorem 1.2 in this paper.*

3. The Hölder convexity for the inverse hyperbolic sine function

The inverse hyperbolic sine and tangent functions play important roles in the study of the hyperbolic metric. In [VW, Theorem 2.21], the authors showed the Hölder convexity of the inverse hyperbolic tangent function. We will study the similar property of the inverse hyperbolic sine function in this section.

For $r, s \in (0, +\infty)$, the *Hölder mean of order p* is defined by

$$H_p(r,s) = \left(\frac{r^p + s^p}{2}\right)^{1/p} \quad \text{for} \quad p \neq 0, \quad H_0(r,s) = \sqrt{r\,s}.$$

For $p = 1$, we get the arithmetic mean $A = H_1$; for $p = 0$, the geometric mean $G = H_0$; and for $p = -1$, the harmonic mean $H = H_{-1}$. It is well-known that $H_p(r,s)$ is continuous and increasing with respect to p. Many interesting properties of Hölder means are given in [Bu] and [HLP].

A function $f : I \to J$ is called *$H_{p,q}$-convex(concave)* if it satisfies

$$f(H_p(r,s)) \leq (\geq) H_q(f(r), f(s))$$

for all $r, s \in I$, and *strictly $H_{p,q}$-convex(concave)* if the inequality is strict except for $r = s$. For $H_{p,q}$-convexity of some special functions the reader is referred to [AVV2, AVZ, Ba1, Ba2, CWZQ, ZWC].

Lemma 3.1. *Let $r \in (0, +\infty)$.*

(1) The function $f_1(r) \equiv \frac{\operatorname{arsh} r}{r}$ is strictly decreasing with range $(0, 1)$.

(2) The function $f_2(r) \equiv \frac{r(1+r^2)-\sqrt{1+r^2}\operatorname{arsh} r}{r^3}$ is strictly increasing with range $(2/3, 1)$.

Proof. (1) $f_{11}(r) = \operatorname{arsh} r$ and $f_{12}(r) = r$, it is easy to see that $f_{11}(0^+) = f_{12}(0^+) = 0$, then

$$\frac{f'_{11}(r)}{f'_{12}(r)} = \frac{1}{\sqrt{1+r^2}}$$

which is strictly decreasing. Hence by Lemma 2.8, f_1 is strictly decreasing with $f_1(0^+) = 1$ and $f_1(+\infty) = \lim\limits_{r\to+\infty} f_1(r) = \lim\limits_{r\to+\infty} \frac{f'_{11}(r)}{f'_{12}(r)} = 0$.

(2) Let $f_{21}(r) = r(1+r^2) - \sqrt{1+r^2}\operatorname{arsh} r$ and $f_{22}(r) = r^3$, then $f_{21}(0^+) = f_{22}(0^+) = 0$. By differentiation, we have

$$\frac{f'_{21}(r)}{f'_{22}(r)} = 1 - \frac{1}{3}\frac{1}{\sqrt{1+r^2}}\frac{\operatorname{arsh} r}{r},$$

which is strictly increasing by (1). Hence by Lemma 2.8, f_2 is strictly increasing with $f_2(0^+) = 2/3$ and $f_2(+\infty) = \lim\limits_{r\to+\infty} f(r) = 1$. □

Lemma 3.2. *For $p \in \mathbb{R}$ and $r \in (0, +\infty)$ define*

$$h_p(r) \equiv 1 + p\sqrt{1+r^2}\frac{\operatorname{arsh} r}{r} - \frac{1}{\sqrt{1+r^2}}\frac{\operatorname{arsh} r}{r}.$$

(1) If $p \leq -2$, then h_p is strictly decreasing with range $(-\infty, p)$.

(2) If $p > 0$, then h_p is strictly increasing with range $(p, +\infty)$.

(3) If $p = 0$, then h_p is strictly increasing with range $(0, 1)$.

(4) If $-2 < p < 0$, then the range of h_p is $(-\infty, C(p)]$, where $C(p) \equiv \sup\limits_{0<r<+\infty} h_p(r) \in (p\,, 1)$. Moreover, $\lim\limits_{p\to-2} C(p) = -2$ and $\lim\limits_{p\to0} C(p) = 1$.

Proof. By l'Hôpital's Rule, we get

$$\lim_{r\to+\infty} \frac{\sqrt{1+r^2}\operatorname{arsh} r}{r} = \lim_{r\to+\infty} \left(1 + \frac{r\operatorname{arsh} r}{\sqrt{1+r^2}}\right) = +\infty.$$

Together with Lemma 3.1 (1), we have $h_p(0^+) = p$ and

$$h_p(+\infty) = \lim_{r\to+\infty} h_p(r) = \begin{cases} -\infty & p < 0, \\ 1 & p = 0, \\ +\infty & p > 0. \end{cases} \tag{3.3}$$

Next by differentiation, we have

$$h'_p(r) = \frac{1}{r}\left(1 - \frac{1}{\sqrt{1+r^2}}\frac{\operatorname{arsh} r}{r}\right)[p - f(r)],$$

where $f(r) = 2 - \frac{1}{1+r^2} - \frac{2}{f_2(r)}$ and $f_2(r)$ is as in Lemma 3.1 (2). Therefore, f is strictly increasing from $(0, +\infty)$ onto $(-2, 0)$. Hence we get (1)-(3).

(4) If $-2 < p < 0$, since the range of f is $(-2, 0)$, we see that there exists exactly one point $r_0 \in (0, +\infty)$ such that $p = f(r_0)$. Then h_p is increasing on $(0, r_0)$ and decreasing on $(r_0, +\infty)$. Since

$$h_p(r) = 1 - \left(-p\sqrt{1+r^2}\frac{\operatorname{arsh} r}{r} + \frac{1}{\sqrt{1+r^2}}\frac{\operatorname{arsh} r}{r} \right) < 1,$$

by the continuity of h_p, there is a continuous function

$$C(p) \equiv \sup_{0<r<+\infty} h_p(r)$$

with $p < C(p) < 1$. Moreover, $\lim_{p\to -2} C(p) = -2$ and $\lim_{p\to 0} C(p) = 1$. □

Lemma 3.4. *Let p, q be real numbers and $r \in (0, +\infty)$. Let*

$$g_{p,q}(r) \equiv \frac{\operatorname{arsh}^{q-1} r}{r^{p-1}\sqrt{1+r^2}}.$$

(1) If $p \le -2$, then $g_{p,q}$ is strictly increasing for each $q \ge p$, and $g_{p,q}$ is not monotone for any $q < p$.

(2) If $p > 0$, then $g_{p,q}$ is strictly decreasing for each $q \le p$, and $g_{p,q}$ is not monotone for any $q > p$.

(3) If $p = 0$, then $g_{p,q}$ is strictly increasing for each $q \ge 1$, $g_{p,q}$ is strictly decreasing for each $q \le 0$, and $g_{p,q}$ is not monotone for any $0 < q < 1$.

(4) If $-2 < p < 0$, then $g_{p,q}$ is strictly increasing for each $q \ge C(p)$, and $g_{p,q}$ is not monotone for any $q < C(p)$.

Here $C(p)$ is the same as in Lemma 3.2.

Proof. By logarithmic differentiation in r,

$$\frac{g'_{p,q}(r)}{g_{p,q}(r)} = \frac{1}{\sqrt{1+r^2}\operatorname{arsh} r}[q - h_p(r)],$$

where $h_p(r)$ is the same as Lemma 3.2. Hence the results follow from Lemma 3.2. □

The following theorem studies the $H_{p,q}$-convexity of arsh.

Theorem 3.5. *The inverse hyperbolic sine function* arsh *is strictly $H_{p,q}$-convex on $(0, \infty)$ if and only if $(p, q) \in D_1 \cup D_2$, while* arsh *is strictly $H_{p,q}$-concave on $(0, \infty)$ if and only if $(p, q) \in D_3$, where*

$$D_1 = \{(p, q) | -\infty < p < -2,\ p \le q < +\infty\},$$

$$D_2 = \{(p, q) | -2 \le p \le 0,\ C(p) \le q < +\infty\},$$

$$D_3 = \{(p, q) | 0 \le p < +\infty,\ -\infty < q \le p\},$$

and $C(p)$ is a continuous function with $C(-2) = -2$, $C(0) = 1$ and $p < C(p) < 1$ for $-2 < p < 0$.

Proof. The proof is divided into the following four cases.

Case 1. $p \ne 0$ and $q \ne 0$.

We may suppose that $0 < x \le y < 1$. Define

$$F(x, y) = \operatorname{arsh}^q (H_p(x, y)) - \frac{\operatorname{arsh}^q x + \operatorname{arsh}^q y}{2}.$$

Let $t = H_p(x, y)$, then $\frac{\partial t}{\partial x} = \frac{1}{2}(\frac{x}{t})^{p-1}$. If $x < y$, we see that $t > x$. By differentiation, we have

$$\frac{\partial F}{\partial x} = \frac{q}{2}x^{p-1}\left(\frac{\operatorname{arsh}^{q-1}t}{t^{p-1}\sqrt{1+t^2}} - \frac{\operatorname{arsh}^{q-1}x}{x^{p-1}\sqrt{1+x^2}}\right).$$

Case 1.1. $p \le -2$, $q \ge p$ and $q \ne 0$.

By Lemma 3.4(1), $\frac{\partial F}{\partial x} < 0$ if $0 > q \ge p$, and $\frac{\partial F}{\partial x} > 0$ if $q > 0 (\ge p)$. Then $F(x, y)$ is strictly decreasing and $F(x, y) \ge F(y, y) = 0$ if $0 > q \ge p$, and $F(x, y)$ is strictly increasing and $F(x, y) \le F(y, y) = 0$ if $q > 0$. Hence we have

$$\operatorname{arth}(H_p(x, y)) \le H_q(\operatorname{arth}x, \operatorname{arth}y)$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-convex on $(0, +\infty)$ for $(p, q) \in \{(p, q)|p \le -2,\ p \le q < 0\} \cup \{(p, q)|p \le -2,\ q > 0\}$.

Case 1.2. $p \le -2$, $q < p$.

By Lemma 3.4(1), with an argument similar to Case 1.1, it is easy to see that arsh is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(0, +\infty)$.

Case 1.3. $p > 0$, $q \le p$ and $q \ne 0$.

By Lemma 3.4(2), $\frac{\partial F}{\partial x} > 0$ if $q < 0 (\le p)$, and $\frac{\partial F}{\partial x} < 0$ if $0 < q \le p$. Then $F(x, y)$ is strictly increasing and $F(x, y) \le F(y, y) = 0$ if $q < 0$, and $F(x, y)$ is strictly decreasing and $F(x, y) \ge F(y, y) = 0$ if $0 < q \le p$. Hence we have

$$\operatorname{arth}(H_p(x, y)) \ge H_q(\operatorname{arth}x, \operatorname{arth}y)$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-concave on $(0, +\infty)$ for $(p, q) \in \{(p, q)|p > 0,\ q < 0\} \cup \{(p, q)|p > 0,\ 0 < q \le p\}$.

Case 1.4. $p > 0$, $q > p$.

By Lemma 3.4(2), with an argument similar to Case 1.3, it is easy to see that arsh is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(0, +\infty)$.

Case 1.5. $-2 < p < 0$, $q \ge C(p)$ and $q \ne 0$.

By Lemma 3.4(4), $\frac{\partial F}{\partial x} < 0$ if $0 > q \ge C(p)$, and $\frac{\partial F}{\partial x} > 0$ if $q \ge C(p)$ and $q > 0$. Then $F(x, y)$ is strictly decreasing and $F(x, y) \ge F(y, y) = 0$ if $0 > q \ge C(p)$, and $F(x, y)$ is strictly increasing and $F(x, y) \le F(y, y) = 0$ if $q \ge C(p)$ and $q > 0$. Hence we have

$$\operatorname{arth}(H_p(x, y)) \le H_q(\operatorname{arth}x, \operatorname{arth}y)$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-convex on $(0, +\infty)$ for $(p, q) \in \{(p, q)| -2 < p < 0,\ 0 > q \ge C(p)\} \cup \{(p, q)| -2 < p < 0,\ q \ge C(p), q > 0\}$.

Case 1.6. $-2 < p < 0$, $q < C(p)$ and $q \ne 0$.

By Lemma 3.2(4), with an argument similar to Case 1.5, it is easy to see that arsh is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(0, +\infty)$.

Case 2. $p \ne 0$ and $q = 0$.

For $0 < x \le y < 1$, let

$$F(x, y) = \frac{\operatorname{arsh}^2(H_p(x, y))}{\operatorname{arsh}x \operatorname{arsh}y},$$

and $t = H_p(x, y)$. If $x < y$, we see that $t > x$. By logarithmic differentiation, we obtain

$$\frac{1}{F}\frac{\partial F}{\partial x} = x^{p-1}\left(\frac{(\mathrm{arsh}t)^{-1}}{t^{p-1}\sqrt{1+t^2}} - \frac{(\mathrm{arsh}x)^{-1}}{x^{p-1}\sqrt{1+x^2}}\right).$$

Case 2.1. $p \leq -2$ and $q = 0 (> p)$.

By Lemma 3.4(1), we have $\frac{\partial F}{\partial x} > 0$ and $F(x, y) \leq F(y, y) = 1$. Hence we have

$$\mathrm{arsh}(H_p(x, y)) \leq \sqrt{\mathrm{arsh}x\,\mathrm{arsh}y}$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-convex on $(0, +\infty)$ for $(p, q) \in \{(p, q)| - p \leq -2, q = 0\}$.

Case 2.2. $p > 0$ and $q = 0 (< p)$.

By Lemma 3.4(2), we have $\frac{\partial F}{\partial x} < 0$ and $F(x, y) \geq F(y, y) = 1$. Hence we have

$$\mathrm{arsh}(H_p(x, y)) \geq \sqrt{\mathrm{arsh}x\,\mathrm{arsh}y}$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-concave on $(0, +\infty)$ for $(p, q) \in \{(p, q)|p > 0, q = 0\}$.

Case 2.3. $-2 < p < 0$ and $q = 0 \geq C(p)$.

By Lemma 3.4(4), we have $\frac{\partial F}{\partial x} > 0$ and $F(x, y) \leq F(y, y) = 1$. Hence we have

$$\mathrm{arsh}(H_p(x, y)) \leq \sqrt{\mathrm{arsh}x\,\mathrm{arsh}y}$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-convex on $(0, +\infty)$ for $(p, q) \in \{(p, q)| - 2 < p < 0, q = 0 \geq C(p)\}$.

Case 2.4. $-2 < p < 0$ and $q = 0 < C(p)$.

By Lemma 3.4(4), with an argument similar to Case 2.3, it is easy to see that arsh is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(0, +\infty)$.

Case 3. $p = 0$ and $q \neq 0$.

For $0 < x \leq y < 1$, let

$$F(x, y) = \mathrm{arsh}^q(\sqrt{xy}) - \frac{\mathrm{arsh}^q x + \mathrm{arsh}^q y}{2},$$

and $t = \sqrt{xy}$. If $x < y$, we have that $t > x$. By differentiation, we obtain

$$\frac{\partial F}{\partial x} = \frac{q}{2x}\left(\frac{\mathrm{arsh}^{q-1}t}{t^{-1}\sqrt{1+t^2}} - \frac{\mathrm{arsh}^{q-1}x}{x^{-1}\sqrt{1+x^2}}\right).$$

Case 3.1. $p = 0$ and $q \geq 1$.

By Lemma 3.4(3), we have $\frac{\partial F}{\partial x} > 0$ and $F(x, y) \leq F(y, y) = 0$. Hence we have

$$\mathrm{arsh}(\sqrt{xy}) \leq H_q(\mathrm{arsh}x, \mathrm{arsh}y)$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-convex on $(0, 1)$ for $(p, q) \in \{(p, q)|p = 0, q \geq 1\}$.

Case 3.2. $p = 0$ and $q < 0$.

By Lemma 3.4(3), we have $\frac{\partial F}{\partial x} > 0$ and $F(x, y) \leq F(y, y) = 0$. Hence we have

$$\mathrm{arsh}(\sqrt{xy}) \geq H_q(\mathrm{arsh}x, \mathrm{arsh}y)$$

with equality if and only if $x = y$.

In conclusion, arsh is strictly $H_{p,q}$-concave on $(0,1)$ for $(p,q) \in \{(p,q)|p=0, q \le 0\}$.

Case 3.3. $p=0$ and $0<q<1$.

By Lemma 3.4(3), with an argument similar to Case 3.1 or Case 3.2, it is easy to see that arsh is neither $H_{p,q}$-concave nor $H_{p,q}$-convex on the whole interval $(0,+\infty)$.

Case 4. $p=q=0$.

By Case 2.2, for all $x\,, y \in (0,+\infty)$, we have

$$\operatorname{arsh}(H_p(x,y)) \ge \sqrt{\operatorname{arsh}x \operatorname{arsh}y}, \quad \text{for} \quad p>0.$$

By the continuity of H_p in p and arsh in x, we have

$$\operatorname{arsh}(H_0(x,y)) \ge H_0(\operatorname{arsh}x, \operatorname{arsh}y).$$

In conclusion, arsh is strictly $H_{0,0}$-concave on $(0,+\infty)$.

This completes the proof of Theorem 3.5. □

Setting $p=1=q$ in Theorem 3.5, we obtain the concavity of arsh easily.

Corollary 3.6. *The inverse hyperbolic sine function* arsh *is strictly concave on* $(0,+\infty)$.

4. Proof of Main Results

Proof of Theorem 1.1. By the proof of Proposition 2.10, we have

$$d_3 d_4 = \operatorname{arsh}(m\,r)\operatorname{arsh}(m\,r')$$

where m, r are the same as in the proof of Proposition 2.10. By Lemma 2.9(2), we have

$$d_3 d_4 \le \left(\operatorname{arsh}\left(\frac{\sqrt{2}}{2}m\right)\right)^2.$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By the proof of Proposition 2.10, we have

$$d_3 + d_4 = \operatorname{arsh}(m\,r) + \operatorname{arsh}(m\,r')$$

where m, r are the same as in the proof of Proposition 2.10. By Lemma 2.9(3), the desired conclusion follows. □

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References

[AVV1] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, a Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1997.

[AVV2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), 1294–1308.

[AVZ] G. D. Anderson, M. Vuorinen, and X.-H. Zhang, *Topics in special functions III*, Analytic Number Theory, Approximation Theory and Special Functions, ed. by K. Alladi, G. Milovanovic, and M. Th. Rassias, Springer-Verlag, arXiv math 1209.1696 [math.CA].

[A] J. W. Anderson, *Hyperbolic Geometry*, 2nd edition, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2005.

[Ba1] Á. Baricz, *Generalized Bessel functions of the first kind*, Lecture Notes in Mathematics **1994**, Springer-Verlag, Berlin Heidelberg, 2010.

[Ba2] Á. Baricz, *Convexity of the zero-balanced Gaussian hypergeometric functions with respect to Hölder means*, JIPAM. J. Inequal. Pure Appl. Math. **8** (2007), Article 40, 9 pp.

[Be] A. F. Beardon, *The Geometry of Discrete Groups*, Graduate Texts in Mathematics **91**, Springer-Verlag, New York, 1983.

[BM] A. F. Beardon and D. Minda, *Conformal images and the angle of parallelism*, Comput. Methods Funct. Theory **12** (2012), 87–100.

[Bu] P. S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and its Applications **560**, Kluwer Academic Publishers Group, Dordrecht, 2003.

[CWZQ] Y.-M. Chu, G.-D. Wang, X.-H. Zhang, and S.-L. Qiu, *Generalized convexity and inequalities involving special functions*, J. Math. Anal. Appl. **336** (2007), 768–776.

[HLP] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed. Cambridge University Press, Cambridge, 1952.

[KL] L. Keen and N. Lakic, *Hyperbolic Geometry from a Local Viewpoint*, London Mathematical Society Student Texts **68**, Cambridge University Press, Cambridge, 2007.

[KV] R. Klén and M. Vuorinen, *Apollonian circles and hyperbolic geometry*, J. Analysis **19** (2011), 41–60, arXiv:1010.2630[math. MG].

[K] J. G. Krzyż, *Problems in Complex Variable Theory*, Translation of the 1962 Polish original, Modern Analytic and Computational Methods in Science and Mathematics, No. 36, American Elsevier Publishing Co., Inc., New York, PWN—Polish Scientific Publishers, Warsaw, 1971.

[V] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics **1319**, Springer-Verlag, Berlin, 1988.

[VW] M. Vuorinen and G.-D. Wang, *Hyperbolic Lambert quadrilaterals and quasiconformal mappings*, Ann. Acad. Sci. Fenn. Math. **38**(2013), 433–453, arXiv:1203. 6494 [math. MG].

[ZWC] X.-H. Zhang, G.-D. Wang and Y.-M. Chu, *Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions*, J. Math. Anal. Appl. **353**(2009), 256–259.

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