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ON THE NILPOTENT COMMUTATOR OF A NILPOTENT MATRIX

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Abstract. We study the structure of the nilpotent commutator $\mathcal{N}_B$ of a nilpotent matrix B. We show that $\mathcal{N}_B$ intersects all nilpotent orbits for conjugation if and only if B is a square–zero matrix. We describe nonempty intersections of $\mathcal{N}_B$ with nilpotent orbits in the case the $n \times n$ matrix B has rank $n-2$. Moreover, we give some results on the maximal nilpotent orbit that $\mathcal{N}_B$ intersects nontrivially.

1. Introduction

We denote by $M_n(\mathbb{F})$ the algebra of all $n \times n$ matrices over an algebraically closed field $\mathbb{F}$ of characteristic 0 and by $\mathcal{N}_n(\mathbb{F})$ the variety of all nilpotent matrices in $M_n(\mathbb{F})$. Let $B \in \mathcal{N}_n(\mathbb{F})$ and suppose that its Jordan canonical form is given by a partition $\underline{\lambda} \in \mathcal{P}(n)$. We denote by $\mathcal{N}_B$ the nilpotent commutator of B, which is the set of all nilpotent matrices A such that $AB = BA$. Moreover, let us denote by $\mathcal{O}_B = \mathcal{O}_{\underline{\lambda}}$ the orbit of B under the conjugated action of $GL_n(\mathbb{F})$ on $\mathcal{N}_n(\mathbb{F})$, i.e. the set of all nilpotent matrices with their Jordan canonical form given by partition $\underline{\lambda}$.

Recently, the structure of the variety of commuting nilpotent matrices has been widely studied (see e.g. [1, 2, 3, 5, 8]). In this paper we investigate further which intersections $\mathcal{N}_B \cap \mathcal{O}_{\underline{\mu}}$ are nonempty. The answer to this question could be considered as a generalization of the Gerstenhaber–Hesselink Theorem on the partial order of nilpotent orbits [7]. In the first part of the paper, we give anwers for matrices B with extremal kernel, and in the second part we give some results on the maximal partition $\underline{\mu}$, such that the intersection $\mathcal{N}_B \cap \mathcal{O}_{\underline{\mu}}$ is nonempty for a given B.

In the first part of this paper (Sections 2 and 3) we are interested in describing pairs of partitions that are the Jordan canonical forms of two commuting nilpotent matrices. In Section 2, Theorem 2.4, we prove that the nilpotent commutator $\mathcal{N}_B$ intersects every nilpotent orbit $\mathcal{O}_{\underline{\lambda}}$ if and only if B is a square zero-matrix. In Section 3 we investigate the nilpotent commutator of a nilpotent matrix having the dimension of its kernel equal to 2. In Theorem 3.1, we prove that the only pairs of distinct Jordan canonical forms of two commuting nilpotent $n \times n$ matrices, both having exactly 2 parts, are of the form $\left(\left(\frac{n}{2}, \frac{n}{2}\right), \left(\frac{n}{2}+1, \frac{n}{2}-1\right)\right)$ where n is even. Next, we give some additional sufficient and some necessary conditions for partitions to be Jordan canonical forms of matrices in the nilpotent commutator of a nilpotent matrix, having the dimension of its kernel equal to 2. (See Theorems 3.6, 3.9 and Propositions 3.8 and also 2.6.)

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Some of the results in Sections 2 and 3 were already proved in [10]. Note that recently, Britnell and Wildon [6] proved similar results for matrices over finite fields.

Let us recall some definitions and notations we use in the paper.

A nonincreasing sequence of positive integers $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_s)$, such that their sum is equal to n, is called a *partition of an integer* n. It is sometimes convenient to write the partition $\underline{\mu}$ also as $(\mu_1, \mu_2, \ldots, \mu_s) = (m_1^{r_1}, m_2^{r_2}, \ldots, m_l^{r_l})$, where $\sum_{i=1}^{l} r_i = s$, $m_i > m_{i+1}$ and $r_i \neq 0$ for all i. By $\mathcal{P}(n)$, we denote the set of all partitions of n. The *conjugated partition* of a partition $\underline{\mu}$ is the partition $\underline{\mu}^T = (\mu_1^T, \mu_2^T, \ldots, \mu_{\mu_1}^T)$, where $\mu_i^T = |\{j;\ \mu_j \geq i\}|$. It is easy to see that for each $t = 1, 2, \ldots, n$ there exists a uniquely defined partition $r(n,t) := (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathcal{P}(n)$, such that $\lambda_1 - \lambda_t \leq 1$. It can be verified that $r(n,t) = \left(\left\lceil \frac{n}{t} \right\rceil^r, \left\lfloor \frac{n}{t} \right\rfloor^{t-r}\right)$. By the shape of its Ferrer diagram, we call the partition $r(n,t)$ an *almost rectangular partition* of n. Moreover, we define the partial order on $\mathcal{P}(n)$ with $(\lambda_1, \lambda_2, \ldots, \lambda_t) \leq (\mu_1, \mu_2, \ldots, \mu_s)$ if and only if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for all k.

Note that all eigenvalues of a nilpotent matrix are equal to 0 and thus the Jordan canonical form of a nilpotent matrix can be described by a partition, i.e. by the decreasing sequence of sizes of its Jordan blocks. If a nilpotent matrix A has its Jordan canonical form given by partition $\underline{\mu}$, we write $\mathrm{sh}(A) = \underline{\mu}$ and call it the *shape* of matrix A. For every m, we denote the $m \times m$ nilpotent Jordan block by J_m. By computing the lengths of the Jordan chains of J_m^k, $k = 1, 2, \ldots, m$, we observe that the Jordan canonical form of J_m^k is given by partition $r(m,k)$. By $J_{\underline{\mu}} = J_{(\mu_1, \mu_2, \ldots, \mu_s)} = J_{\mu_1} \oplus J_{\mu_2} \oplus \ldots \oplus J_{\mu_s}$ we denote the uppertriangular matrix in its Jordan canonical form, with blocks of sizes $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_s > 0$.

Since $\mathcal{N}_B$ is an irreducible variety (see Basili [2]), there exists a unique partition $\underline{\mu}$ of n such that $\mathcal{O}_{\underline{\mu}} \cap \mathcal{N}_B$ is dense in $\mathcal{N}_B$. Here, $\underline{\mu}$ is the largest partition, such that the intersection $\mathcal{O}_{\underline{\mu}} \cap \mathcal{N}_B$ is nonempty. Following Basili and Iarrobino [3], and Panyushev [11] we define the map $\mathcal{D}$ on $\mathcal{P}(n)$ by $\mathcal{D}(\underline{\lambda}) = \underline{\mu}$.

It is an interesting question (see Panyushev [11, Problem 1]) to describe $\mathcal{D}(\underline{\lambda})$ in terms of the partition $\underline{\lambda}$. Recently, some partial results to this problem were obtained. Basili [2, Prop. 2.4] showed that the number of parts of $\mathcal{D}(\underline{\lambda})$ is equal to the smallest number r such that $\underline{\lambda}$ is a union of r almost rectangular partitions. It was proved in [9, Thm. 16] that the first part of $\mathcal{D}(\underline{\lambda})$, where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, is equal to

$$\max_{1 \leq i \leq t} \left\{2(i-1) + \lambda_i + \lambda_{i+1} + \ldots + \lambda_{i+r};\ \lambda_i - \lambda_{i+r} \leq 1,\ \lambda_{i-1} \geq 2 \text{ if } i > 1\right\}.$$

(Note that recently Basili and Iarrobino in [4] observed the same result for an algebraically closed field $\mathbb{F}$, while [9, Thm. 16] holds for $\mathbb{F}$ with the characteristic 0.)

We say that a partition $\underline{\lambda}$ is *stable* if $\mathcal{D}(\underline{\lambda}) = \underline{\lambda}$. Basili and Iarrobino [3, Thm 1.12] showed that $\underline{\lambda}$ is stable if and only if its parts differ pairwise by at least 2. We proved in [8, Thm. 6] that $\mathcal{D}^2 = \mathcal{D}$. From these results, we easily obtain $\mathcal{D}(\underline{\lambda})$ if it has at most two parts (see [8, Thm. 7]). Until now, not much is known about $\mathcal{D}(\underline{\lambda})$ if it has more than two parts. In this paper, Theorem 4.1 characterizes partitions $\underline{\lambda}$, such that $\mathcal{D}(\underline{\lambda})$ has parts that differ exactly by two. In the remainder of Section 4, we examine partitions in $\mathcal{D}^{-1}(\underline{\mu})$ for certain families of partitions $\underline{\mu}$.

2. Nilpotent commutator of a square-zero matrix

We say that B is a *square–zero matrix* if $B^2 = 0$. The Jordan canonical form of a square–zero matrix is given by a partition, such that all its parts are at most 2. By [2, Prop. 2.4] or [9, Thm. 16], we have that for such partition, $\mathcal{D}((2^a, 1^b)) = (2a+b)$. In the main result of this section, Theorem 2.4, we show even more: for an $n \times n$ square–zero matrix B, its nilpotent commutator $\mathcal{N}_B$ intersects every nilpotent orbit, i.e. for every partition $\underline{\mu} \in \mathcal{P}(n)$ there exists a nilpotent matrix A, commuting with B, such that $\mathrm{sh}(A) = \underline{\mu}$.

By $\mathcal{P}(\mathcal{N}_B)$ we denote the set of all partitions that are Jordan canonical forms of matrices in $\mathcal{N}_B$. Thus, in Theorem 2.4 we show that $\mathcal{P}(\mathcal{N}_B) = \mathcal{P}(n)$ for every $n \times n$ square–zero matrix B, and moreover, we show that $\mathcal{P}(\mathcal{N}_B) \subsetneqq \mathcal{P}(n)$ for all $n \times n$ matrices, such that $B^2 \neq 0$ and $n \geq 4$.

First, we state next Proposition, that is easy to prove, and then prove two technical lemmas that will simplify the proof of Theorem 2.4.

Proposition 2.1. [9, Prop. 1] A pair of partitions $((n), \underline{\mu})$ is a pair of Jordan canonical forms of two commuting $n \times n$ nilpotent matrices if and only if $\underline{\mu}$ is an almost rectangular partition of n. ■

Lemma 2.2. If B is an $n \times n$ matrix, $n \geq 4$, such that $B^2 \neq 0$, then $\mathcal{P}(\mathcal{N}_B) \subsetneqq \mathcal{P}(n)$.

Proof. We will show that for an arbitrary partition $\underline{\lambda} \in \mathcal{P}(n)$, $\mathrm{sh}(B) = \underline{\lambda}$, there exists $\underline{\mu} \in \mathcal{P}(n)$, such that for every $n \times n$ matrix A with $\mathrm{sh}(A) = \underline{\mu}$, matrices A and B do not commute.

By Proposition 2.1, we have that if $\underline{\lambda}$ is not an almost rectangular partition, then $(n) \notin \mathcal{P}(\mathcal{N}_B)$. Suppose now $\underline{\lambda}$ is almost rectangular and $B^2 \neq 0$. We assume that there exists A, such that $\mathrm{sh}(A) = (n-1, 1)$ and that A commutes with B. We may take that A is in its Jordan canonical form (otherwise, substitute A with PAP^{-1} and B with PBP^{-1} for a suitable invertible matrix P). Then, B is of the form

$$\begin{bmatrix} T & b \\ c^T & 0 \end{bmatrix},$$

where T is an $(n-1) \times (n-1)$ upper triangular Toeplitz matrix and b, c column vectors, such that $b^T = (b_1, 0, \ldots, 0)$ and $c^T = (0, \ldots, 0, c_1)$. Let us define the matrix $B' = T \oplus 0$ and note that $B^k = B'^k$ for all $k \geq 3$. Since B is not a square-zero matrix, but $\underline{\lambda}$ is almost rectangular, it follows that $\mathrm{rk}\, B > 2$. In this case, $\mathrm{rk}\, B = \mathrm{rk}\, B'$ and therefore $\underline{\lambda} = \mathrm{sh}(B) = \mathrm{sh}(B') = (r(n-1, t), 1)$. Since $\underline{\lambda}$ is almost rectangular, it follows that $r(n-1, t) = (2^a, 1^b)$ and therefore is B a square-zero matrix. This contradicts the assumption and finishes the proof that $\mathcal{P}(\mathcal{N}_B) \subsetneqq \mathcal{P}(n)$. ■

Lemma 2.3. If $\mathrm{sh}(B) = (\lambda_1, \lambda_2) \in \mathcal{P}(n)$, then $(2^a, 1^{n-2a}) \in \mathcal{P}(\mathcal{N}_B)$ for all $0 \leq a \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Suppose first that n is odd. We treat the cases where λ_1 is even or λ_1 is odd, separately. Firstly, let λ_1 be even. If $a \leq \frac{\lambda_1}{2}$, then $\mathrm{sh}(J_{\lambda_1}^{\lambda_1-a}) = r(\lambda_1, \lambda_1 - a) = (2^a, 1^{\lambda_1-2a})$ and thus $\mathrm{sh}(J_{\lambda_1}^{\lambda_1-a} \oplus J_{\lambda_2}^{\lambda_2}) = (2^a, 1^{n-2a})$. Obviously, $J_{\lambda_1}^{k_1} \oplus J_{\lambda_2}^{k_2}$ commutes with $J_{\lambda_1} \oplus J_{\lambda_2}$ for any positive integers k_1 and k_2. Otherwise, if $\frac{\lambda_1}{2} < a \leq \frac{n-1}{2}$, then $\mathrm{sh}(J_{\lambda_2}^{n-\frac{\lambda_1}{2}-a}) = r(\lambda_2, n - \frac{\lambda_1}{2} - a) = (2^{a-\frac{\lambda_1}{2}}, 1^{n-2a})$ and thus $\mathrm{sh}(J_{\lambda_1}^{\frac{\lambda_1}{2}} \oplus J_{\lambda_2}^{n-\frac{\lambda_1}{2}-a}) = (2^a, 1^{n-2a})$. Hence, $(2^a, 1^{n-2a}) \in \mathcal{P}(\mathcal{N}_B)$ for all $0 \leq a \leq \frac{n-1}{2}$. Similarly, we prove the theorem in the case λ_2 being even.

If n is even, we treat the case λ_1 and λ_2 being even similarly as before. In the case when λ_1 and λ_2 are both odd, we must consider several cases:

- If $0 \leq a \leq \frac{\lambda_1-1}{2}$, then it is easy to see that $\mathrm{sh}(J_{\lambda_1}^{\lambda_1-a} \oplus J_{\lambda_2}^{\lambda_2}) = (2^a, 1^{n-2a})$ and if $\frac{\lambda_1-1}{2} < a < \frac{n}{2}$, then $\mathrm{sh}(J_{\lambda_1}^{\frac{\lambda_1+1}{2}} \oplus J_{\lambda_2}^{\frac{\lambda_1-1}{2}+\lambda_2-a}) = (2^a, 1^{n-2a})$.
- If λ_1 and λ_2 are both odd and $\lambda_1 = \lambda_2 = a = \frac{n}{2}$, then write $A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathcal{N}_B$, where $I \in \mathcal{M}_a(\mathbb{F})$. Since $\mathrm{rk}\, A = a$ and $A^2 = 0$, it follows that $\mathrm{sh}(A) = (2^a, 1^{n-2a}) = (2^{\frac{n}{2}})$.
- Suppose now that λ_1 and λ_2 are both odd, $\lambda_1 > \lambda_2$, and $a = \frac{n}{2}$. We write $\lambda_i = 2k_i+1$, for $i = 1, 2$, and define matrices $A_{12} = \begin{bmatrix} J_{\lambda_2}^{k_2} \\ 0 \end{bmatrix} \in \mathcal{M}_{\lambda_1 \times \lambda_2}(\mathbb{F})$, where $0 \in \mathcal{M}_{(\lambda_1-\lambda_2)\times\lambda_2}(\mathbb{F})$ and $A_{21} = \begin{bmatrix} 0 & -J_{\lambda_2}^{k_2} \end{bmatrix} \in \mathcal{M}_{\lambda_2 \times \lambda_1}(\mathbb{F})$, where $0 \in \mathcal{M}_{\lambda_2\times(\lambda_1-\lambda_2)}(\mathbb{F})$. Here, we define $J_{\lambda_2}^0 = I$. Let $A = \begin{bmatrix} J_{\lambda_1}^{k_1} & A_{12} \\ A_{21} & J_{\lambda_2}^{k_2+1} \end{bmatrix}$. It can be easily seen that $A \in \mathcal{N}_B$, $A^2 = 0$ and $\mathrm{rk}\, A = \frac{n}{2}$. Thus $\mathrm{sh}(A) = (2^{\frac{n}{2}})$. ■

Theorem 2.4. Let B be an $n \times n$ matrix.

- If $n \leq 3$, then $\mathcal{P}(\mathcal{N}_B) = \mathcal{P}(n)$.
- If $n \geq 4$, then $\mathcal{P}(\mathcal{N}_B) = \mathcal{P}(n)$ if and only if B is a square–zero matrix.

Proof. The case $n \leq 3$ is clear, and for $n \geq 4$ the necessity follows by Lemma 2.2.

To prove the sufficiency of the second claim, take an arbitrary $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathcal{P}(n)$ and let $B = J_{\underline{\lambda}}$. The matrix B can be written as a direct sum $B_1 \oplus B_2 \oplus \ldots \oplus B_r$, where either

(a) r is odd and $B_j = J_{\lambda_j}$, where all λ_j are odd (i.e. $\underline{\lambda}$ has an odd number of odd parts), or

(b) each B_i has one of the following forms:
  (i) $B_i = J_{\lambda_j}$, for an even λ_j,
  (ii) $B_i = J_{\lambda_{i_1}} \oplus J_{\lambda_{i_2}}$, where $\lambda_{i_1} + \lambda_{i_2}$ is even,
  (iii) $B_i = J_{\lambda_{i_1}} \oplus J_{\lambda_{i_2}}$, where $\lambda_{i_1} + \lambda_{i_2}$ is odd,
  and at most one B_i is of the form (iii). (Namely, if $\underline{\lambda}$ has an even number of odd parts, then all B_i are of the forms (i) and (ii), otherwise there exists exactly one B_i of the form (iii).)

It is clear that for an odd λ_j and an arbitrary a, $0 \le a \le \frac{\lambda_j - 1}{2}$, the set $\mathcal{P}(\mathcal{N}_{J_{\lambda_j}})$ includes all partitions of the form $r(\lambda_j, \lambda_j - a) = (2^a, 1^{\lambda_j - 2a})$. In the case (a), $\lambda_{2i} + \lambda_{2i+1}$ is even, thus we use Lemma 2.3 to see that $\mathcal{P}(\mathcal{N}_{B_{2i} \oplus B_{2i+1}}) = \mathcal{P}(\lambda_{2i} + \lambda_{2i+1})$. Therefore, $(2^a, 1^{n-2a}) \in \mathcal{P}(\mathcal{N}_B)$ for all $a = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

In the case (b), note that for an even λ_j and an arbitrary a, $0 \le a \le \frac{\lambda_j}{2}$, the set $\mathcal{P}(\mathcal{N}_{J_{\lambda_j}})$ again includes all partitions of the form $(2^a, 1^{\lambda_j - 2a})$. Thus, by Lemma 2.3, it follows that $(2^a, 1^{n-2a}) \in \mathcal{P}(\mathcal{N}_B)$ for all $a = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. ■

Corollary 2.5. For every nilpotent $n \times n$ matrix A and integer $k \le \frac{n}{2}$ there exists a matrix B, such that $B^2 = 0$ and $\operatorname{rk} B = k$.

Moreover, if A and B are $n \times n$ nilpotent matrices, then for each integer $k \le \frac{n}{2}$, there exist a square–zero matrix C, such that $\operatorname{rk} C = k$, and $P \in GL_n(\mathbb{F})$, such that C commutes with A and PCP^{-1} commutes with B. ■

In Theorem 2.4 we proved that if B is not a square-zero matrix, there always exists a partition $\underline{\mu}$, such that the nilpotent orbit $\mathcal{O}_{\underline{\mu}}$ does not intersect the nilpotent commutator of matrix B. Moreover, for a suitable $\underline{\lambda} = \operatorname{sh}(B)$ there exist large families of such $\underline{\mu}$. Let us mention the following obstruction. (See also Propositions 3.8 and 3.9.)

Proposition 2.6. Let $(\underline{\lambda}, \underline{\mu})$ be a pair of Jordan canonical forms of two commuting nilpotent matrices, where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_s)$. If $s \ge n - \frac{\lambda_t}{2}$, then $\mu_1 \le 2$.

Proof. Let $A \in \mathcal{N}_B$, where $\operatorname{sh}(B) = \underline{\lambda}$ and $\operatorname{sh}(A) = \underline{\mu}$ as in the statement. Then, A can be partitioned into blocks $A_{ij} \in \mathcal{M}_{\lambda_i \times \lambda_j}(\mathbb{F})$, all upper triangular and constant along diagonals. Since $s \ge n - \frac{\lambda_t}{2}$, we have $\operatorname{rk} A = n - s \le \frac{\lambda_t}{2}$ and thus for all i, j, $\operatorname{rk} A_{i,j} \le \frac{\lambda_t}{2}$. It follows that $A^2 = 0$ and thus $\mu_1 \le 2$. ■

Example 2.7. Note that Baranovsky proved in [1, Lemma 3] that $(\underline{\lambda}, \underline{\lambda}^T)$ is a pair of Jordan canonical forms of two commuting nilpotent matrices.

In the case $\underline{\lambda} = (\lambda_1, \lambda_2)$, $\underline{\lambda}^T$ has all parts equal to at most two and thus by Theorem 2.4, $(\underline{\lambda}, \underline{\mu})$ is a pair of Jordan canonical forms of two commuting nilpotent matrices for all $\underline{\mu} \le \underline{\lambda}^T$. However, this is not true in general.

Suppose $\operatorname{sh}(B) = \underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, where $t \ge 3$ and $\lambda_t \ge 4$. Then, $\big(\underline{\lambda}, (3, 1^{n-3})\big)$ is **not** a pair of Jordan canonical forms of two commuting nilpotent matrices (see Proposition 2.6) and $(3, 1^{n-3}) \le \underline{\lambda}^T$. □

3. Partitions with 2 parts

Besides the square–zero matrices that have rather large dimension of its kernel, we are also interested in matrices, having its kernel of dimension at most two, i.e. matrices that have at most two Jordan blocks. Jordan canonical forms of matrices in the nilpotent commutator of the matrix with one Jordan block are characterized in Proposition 2.1.

In this section, we give a characterization of pairs of Jordan canonical forms of two commuting nilpotent matrices, each having exactly two Jordan blocks. Namely, we will prove the following.

Theorem 3.1. A pair $((\lambda_1, \lambda_2), (\mu_1, \mu_2))$ of *distinct* partitions of n is a pair of Jordan canonical forms of two nilpotent commuting matrices if and only if n is even and one of them is equal to $\left(\frac{n}{2}, \frac{n}{2}\right)$ and the other one is equal to $\left(\frac{n}{2}+1, \frac{n}{2}-1\right)$.

Define matrices $M = J_{\lambda_1} \oplus 0$, where $0 \in \mathcal{M}_{\lambda_2}(\mathbb{F})$, and $N = 0 \oplus J_{\lambda_2}$, where $0 \in \mathcal{M}_{\lambda_1}(\mathbb{F})$. Write also $M_0 = I_{\lambda_1 \times \lambda_1} \oplus 0_{\lambda_2 \times \lambda_2}$ and $N_0 = 0_{\lambda_1 \times \lambda_1} \oplus I_{\lambda_2 \times \lambda_2}$. Let us write $M_i = M^i$ and $N_i = N^i$ for $i = 0, 1, \ldots$.

For $k = 0, 1, \ldots, \lambda_2 - 1$ let K_k be an $n \times n$ matrix such that its only nonzero entries are in the positions $(i, \lambda_1 + k + i)$, where $i = 1, 2, \ldots, \lambda_2 - k$, and are all equal to 1. Similarly, let us define matrices L_l for $l = 0, 1, \ldots, \lambda_2 - 1$ such that its only nonzero entries (which are equal to 1) are in the positions $(\lambda_1 + j, \lambda_1 - \lambda_2 + l + j)$, where $j = 1, 2, \ldots, \lambda_2 - l$.

It is easy to see that the only nonzero products of these matrices are:

$$\begin{aligned} M_i \cdot M_j &= M_{i+j} & M_i \cdot K_j &= K_{i+j} \\ K_i \cdot L_j &= M_{\lambda_1 - \lambda_2 + i + j} & K_i \cdot N_j &= K_{i+j} \\ L_i \cdot M_j &= L_{i+j} & L_i \cdot K_j &= N_{\lambda_1 - \lambda_2 + i + j} \\ N_i \cdot L_j &= L_{i+j} & N_i \cdot N_j &= N_{i+j} \end{aligned} \tag{1}$$

where by the definition $M_j = 0$ for all $j \geq \lambda_1$ and $K_i = L_i = N_i = 0$ for $i \geq \lambda_2$.

From now on, let $B = J_{\lambda_1} \oplus J_{\lambda_2}$, where $\lambda_1 \geq \lambda_2 > 0$.

It is well known that nilpotent matrix A, commuting with B, is of the form

$$A = \sum_{i=1}^{\lambda_1 - 1} a_i M_i + \sum_{i=0}^{\lambda_2 - 1} b_i K_i + \sum_{i=0}^{\lambda_2 - 1} c_i L_i + \sum_{i=1}^{\lambda_2 - 1} d_i N_i \tag{2}$$

where $b_0 c_0 = 0$ if $\lambda_1 = \lambda_2$. Equivalently,

$$A = \sum_{i=\alpha}^{\lambda_1 - 1} a_i M_i + \sum_{i=\beta}^{\lambda_2 - 1} b_i K_i + \sum_{i=\gamma}^{\lambda_2 - 1} c_i L_i + \sum_{i=\delta}^{\lambda_2 - 1} d_i N_i \,,$$

where $a_\alpha b_\beta c_\gamma d_\delta \neq 0$ and if $\lambda_1 = \lambda_2$, also $\beta + \gamma \geq 1$. We define $\alpha = \lambda_1$ (resp. $\beta = \lambda_2$, $\gamma = \lambda_2$, $\delta = \lambda_2$) if $a_i = 0$ for $i = 1, 2, \ldots, \lambda_1 - 1$ (resp. $b_i = 0$ for $i = 0, 1, \ldots, \lambda_2 - 1$, $c_i = 0$ for $i = 0, 1, \ldots, \lambda_2 - 1$, $d_i = 0$ for $i = 1, 2 \ldots, \lambda_2 - 1$).

In what follows, we will prove some lemmas that will give the proof of Theorem 3.1. Lemma 3.2 is well known, but we give here full proof for the sake of completeness.

Lemma 3.2. If A is a nilpotent matrix, such that $\mathrm{sh}(A) = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, then the kernel of matrix A^j has dimension equal to $\sum_{i=1}^{j} \lambda_i^T$.

Proof. For a nilpotent matrix A with $\mathrm{sh}(A) = \underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ let $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \ldots \oplus V_{\lambda_t}$ be a decomposition of $\mathbb{F}^n$ corresponding to the Jordan canonical form $J_{\underline{\lambda}} = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \ldots \oplus J_{\lambda_t}$ of A. For matrix $J_{\underline{\lambda}}$ it is clear that

$$\mathrm{codim}_{\ker J^j_{\underline{\lambda}}|_{V_{\lambda_i}}} (\ker J^{j-1}_{\underline{\lambda}}|_{V_{\lambda_i}}) = \begin{cases} 1, & \text{if } \lambda_i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $\mathrm{codim}_{\ker J^j_{\underline{\lambda}}}(\ker J^{j-1}_{\underline{\lambda}}) = |\{\lambda_i;\ \lambda_i \geq j\}| = \lambda_j^T$ and thus $\dim \ker A^j = \dim \ker J^j_{\underline{\lambda}} = \sum_{i=1}^{j} \mathrm{codim}_{\ker J^i_{\underline{\lambda}}}(\ker J^{i-1}_{\underline{\lambda}}) = \sum_{i=1}^{j} \lambda_i^T$. ■

Lemma 3.3. Suppose that either $\lambda_1 - \lambda_2 = 1$ or $\lambda_1 - \lambda_2 \geq 3$. The pair $((\lambda_1, \lambda_2), (\mu_1, \mu_2))$ is a pair of Jordan canonical forms of two commuting nilpotent matrices if and only if $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$.

Proof. Let $B = J_{\lambda_1} \oplus J_{\lambda_2}$ and suppose there exists $A \in \mathcal{N}_B$, such that $\mathrm{sh}(A) = \underline{\mu} = (\mu_1, \mu_2) \neq (\lambda_1, \lambda_2)$.

Suppose first that $\lambda_1 - \lambda_2 \geq 3$. Since (λ_1, λ_2) is stable, we have that $\mu_1 < \lambda_1$ and $\mu_2 \geq \lambda_2 + 1$. By Lemma 3.2, $\mathrm{rk}\, A^{\lambda_2+1} = n - 2(\lambda_2 + 1) = \lambda_1 - \lambda_2 - 2$. On the other hand, since $A \in \mathcal{N}_B$ is of the form (2) and $\lambda_1 - \lambda_2 \geq 2$, it follows that $a_1 d_1 \neq 0$. Since $\lambda_1 - \lambda_2 \geq 3$, it follows that $A^{\lambda_2+1} = \sum_{i=\lambda_2+1}^{\lambda_1-1} a'_i M_i$, where $a'_{\lambda_2+1} \neq 0$. Thus, $\mathrm{rk}\, A^{\lambda_2+1} = \lambda_1 - \lambda_2 - 1$, which is a contradiction.

If $\lambda_1 - \lambda_2 = 1$, we have that $\mu_1 - \mu_2 \geq 3$. It follows from the previous paragraph that $((\mu_1, \mu_2), (\lambda_1, \lambda_2))$ is not a pair of Jordan canonical forms of two commuting nilpotent matrices. ■

Lemma 3.4. If $\lambda_1 - \lambda_2 = 2$ and $((\lambda_1, \lambda_2), (\mu_1, \mu_2))$ is a pair of Jordan canonical forms of two commuting nilpotent matrices, then $\mu_1 = \lambda_1$ or $\mu_1 = \lambda_1 - 1$.

Proof. To prove the lemma, it suffices to prove that $(\lambda_1 - 1, \lambda_1 - 1) \in \mathcal{P}(\mathcal{N}_B)$ for $\mathrm{sh}(B) = (\lambda_1, \lambda_1 - 2)$. Equivalently, we have to prove that $(\lambda + 1, \lambda - 1) \in \mathcal{P}(\mathcal{N}_C)$, where $\mathrm{sh}(C) = (\lambda, \lambda)$.

Define an upper triangular matrix $A = \sum_{i=1}^{\lambda-1} a_i M_i + \sum_{i=0}^{\lambda-1} b_i K_i + \sum_{i=1}^{\lambda-1} d_i N_i \in \mathcal{N}_C$, where a_1, b_0 and d_1 are algebraically independent over $\mathbb{Q}$. From (1), it easily follows that $A^k = \sum_{i=k}^{\lambda-1} a'_i M_i + \sum_{i=k-1}^{\lambda-1} b'_i K_i + \sum_{i=k}^{\lambda-1} d'_i N_i$. Thus, $\mathrm{rk}\, A^k = n - 2k$ for $k = 1, 2, \ldots, \lambda - 1$, $\mathrm{rk}\, A^\lambda = 1$ and $A^{\lambda+1} = 0$. Therefore, $\mathrm{sh}(A) = (\lambda + 1, \lambda - 1)$ and thus $(\lambda + 1, \lambda - 1) \in \mathcal{P}(\mathcal{N}_C)$. ■

Now, Theorem 3.1 follows from Lemmas 3.3 and 3.4. As a Corollary of Theorem 3.1, we get the partitions of maximal rank in $\mathcal{P}(\mathcal{N}_B)$.

Corollary 3.5. Let $\mathrm{sh}(B) = (\lambda_1, \lambda_2) \in \mathcal{P}(n)$. The set of Jordan canonical forms of matrices $A \in \mathcal{N}_B$ of maximal rank is equal to

(a) $\{(n)\}$ if $\lambda_1 - \lambda_2 \leq 1$,
(b) $\{(\lambda_1, \lambda_2), (\lambda_1 - 1, \lambda_2 + 1)\}$ if $\lambda_1 - \lambda_2 = 2$,
(c) $\{(\lambda_1, \lambda_2)\}$ if $\lambda_1 - \lambda_2 \geq 3$. ■

Next, we add some Jordan canonical forms of matrices in the nilpotent commutator of matrix $J_{\lambda_1} \oplus J_{\lambda_2}$, which are not almost rectangular subpartitions of (λ_1, λ_2).

Theorem 3.6. Let $\mathrm{sh}(B) = (\lambda_1, \lambda_2) \in \mathcal{P}(n)$. Choose integers j, ℓ such that $0 \leq j \leq \ell < \lambda_2$ and write $w = \lambda_1 - \lambda_2 + j + \ell$.

(a) If there exists an integer k, such that $\lambda_2 \leq kw < \lambda_1$, then
$$\left((2k+1)^{\lambda_1 - kw}, (2k)^{w+\lambda_2-\lambda_1}, (2k-1)^{kw-\lambda_2}\right) \in \mathcal{P}(\mathcal{N}_B)\,.$$
(b) If there exists an integer k, such that $\lambda_1 - \ell \leq kw < \lambda_2 - j$, then
$$\left((2k+2)^{\lambda_2-kw-j}, (2k+1)^{w+j-\ell}, (2k)^{kw+\ell-\lambda_2}\right) \in \mathcal{P}(\mathcal{N}_B)\,.$$
(c) Otherwise, $r(n, w) \in \mathcal{P}(\mathcal{N}_B)$.

Proof. Given j, ℓ with the desired properties, let us define the matrix $A = bK_j + cL_\ell$ and $w = \lambda_1 - \lambda_2 + j + \ell$. For such A, using induction on m, it is easy to verify that for all $m \geq 1$:

$$\begin{array}{rlrl} \mathrm{rk}\,(A^{2m})_{11} & = \max\{\lambda_1 - mw, 0\} & (A^{2m-1})_{11} & = 0 \\ \mathrm{rk}\,(A^{2m})_{22} & = \max\{\lambda_2 - mw, 0\} & (A^{2m-1})_{22} & = 0 \\ \mathrm{rk}\,(A^{2m-1})_{12} & = \max\{\lambda_1 + \ell - mw, 0\} & (A^{2m})_{12} & = 0 \\ \mathrm{rk}\,(A^{2m-1})_{21} & = \max\{\lambda_1 + j - mw, 0\} & (A^{2m})_{21} & = 0 \end{array} \tag{3}$$

Note that $\mathrm{rk}\,(A^{2m})_{11} \geq \mathrm{rk}\,(A^{2m})_{22}$ and $\mathrm{rk}\,(A^{2m-1})_{21} \leq \mathrm{rk}\,(A^{2m-1})_{12}$ for all $m \geq 1$, since $j \leq \ell$.

(a) First, suppose there exists an integer k such that $\lambda_1 > kw$ and $\lambda_2 \leq kw$. By (3), we have that $\mathrm{rk}\,(A^{2k})_{11} > 0$ and $\mathrm{rk}\,(A^{2k})_{22} = 0$. Since $\mathrm{rk}\,(A^{2k+1})_{12} = 0$, it follows that $A^{2k+1} = 0$. On the other hand $\mathrm{rk}(A^{2k-1})_{12} \geq \mathrm{rk}\,(A^{2k-1})_{21} > 0$.

If m is even and $m \leq 2k-1$, then by (3) it follows that $\dim\ker A^m = \lambda_1 + \lambda_2 - \mathrm{rk}\,(A^m)_{11} - \mathrm{rk}(A^m)_{22} = mw$. Similarly, if $m \leq 2k-1$ is odd, then $\dim\ker A^m = \lambda_1 + \lambda_2 - \mathrm{rk}\,(A^m)_{12} - \mathrm{rk}(A^m)_{21} = mw$. Since $kw < \lambda_1$, it follows that $\lambda_2 - (k-1)w > \lambda_1 - \lambda_2 + w \geq 0$ and therefore
$$\begin{aligned} \mathrm{sh}(A) &= \left(w^{2k-1}, \lambda_2 - (k-1)w, \lambda_1 - kw\right)^T = \\ &= \left((2k+1)^{\lambda_1-kw}, (2k)^{w+\lambda_2-\lambda_1}, (2k-1)^{kw-\lambda_2}\right)\,. \end{aligned}$$

(b) If there exists an integer k such that $\lambda_1 - \ell \leq kw < \lambda_2 - j$, we proceed similarly as in (a) and observe that $\mathrm{rk}\,(A^{2k+1})_{12} > 0$, $\mathrm{rk}\,(A^{2k+1})_{21} = 0$, and $\mathrm{rk}(A^{2k-1})_{12} \geq \mathrm{rk}\,(A^{2k-1})_{21} > 0$. Since $\mathrm{rk}\,(A^{2k+2})_{11} = 0$ it follows by (3) that $A^{2k+2} = 0$. Again, similarly as in (a), we can compute that $\dim\ker(A^m) = mw$ for all $1 \leq m \leq 2k$ and thus
$$\begin{aligned} \mathrm{sh}(A) &= \left(w^{2k}, \lambda_1 + j - kw, \lambda_2 - j - kw\right)^T = \\ &= \left((2k+2)^{\lambda_2-kw-j}, (2k+1)^{w+j-\ell}, (2k)^{kw+\ell-\lambda_2}\right)\,. \end{aligned}$$

(c) Since there does not exist an integer k such that $\lambda_2 \le kw < \lambda_1$, it follows that $\operatorname{rk}(A^k)_{11} = 0$ if and only if $\operatorname{rk}(A^k)_{22} = 0$. Similarly, since there does not exist an integer k such that $\lambda_1 - \ell \le kw < \lambda_2 - j$, we can conclude that $\operatorname{rk}(A^k)_{12} = 0$ if and only if $\operatorname{rk}(A^k)_{21} = 0$ for all k. Write $n = sw + r$, where $0 \le r < w$. Similarly as in (a), we conclude that $\dim\ker(A^i) = iw$ for all $1 \le i \le s$ and $A^{s+1} = 0$. Therefore $\operatorname{sh}(A) = (w^s, r)^T = r(n, w)$. ■

On the other hand, there exist plenty of partitions, that are not Jordan canonical forms of matrices in the nilpotent commutator $\mathcal{N}_B$ for $\operatorname{sh}(B) = \underline{\lambda}$. (We already proved Proposition 2.6.)

The following lemma can be verified straightforwardly and will be used to prove Proposition 3.8.

Lemma 3.7. (1) If $C \in \mathcal{M}_{p\times r}(\mathbb{F})$ and $D \in \mathcal{M}_{r\times q}(\mathbb{F})$ are uppertriangular matrices, constant along diagonals, then their product CD is also of the same form and $\operatorname{rk}(CD) = \max\{\operatorname{rk}(C) + \operatorname{rk}(D) - r, 0\}$.

(2) If $A = \sum_{i=\alpha}^{\lambda_1-1} a_i M_i + \sum_{i=\beta}^{\lambda_2-1} b_i K_i + \sum_{i=\gamma}^{\lambda_2-1} c_i L_i + \sum_{i=\delta}^{\lambda_2-1} d_i N_i$, where $a_\alpha b_\beta c_\gamma d_\delta \neq 0$, then $\operatorname{rk} A \le \max\{n - \alpha - \delta, 2\lambda_2 - \beta - \gamma\}$. ■

Proposition 3.8. Let $((\lambda_1, \lambda_2), (\mu_1, \mu_2, \dots, \mu_s))$ be a pair of Jordan canonical forms of two commuting nilpotent matrices. If $s > \lambda_1$, then $\mu_1 \le \left\lceil \frac{\lambda_2}{s-\lambda_1} \right\rceil$.

Proof. Let $\operatorname{sh}(B) = (\lambda_1, \lambda_2)$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{N}_B$, $A_{ij} \in \mathcal{M}_{\lambda_i\times\lambda_j}(\mathbb{F})$. Note that A_{ij}, $1 \le i, j \le 2$, are upper triangular and constant along diagonals. Let $\operatorname{sh}(A) = (\mu_1, \mu_2, \dots, \mu_s)$ and denote $q = \operatorname{rk} A = n - s$. It follows that $\operatorname{rk} A_{ij} \le q$ for all $1 \le i, j \le 2$. By assumption, $s > \lambda_1$ and thus $q < \lambda_2$.

We first prove that $\operatorname{rk}(A^k)_{ij} \le \max\{kq - (k-1)\lambda_2, 0\}$ for all $k \ge 1$.

The case $k = 1$ is clear and then we proceed by induction. Suppose that $\operatorname{rk}(A^k)_{ij} \le \max\{kq - (k-1)\lambda_2, 0\}$ for all $1 \le i, j \le 2$. If $kq - (k-1)\lambda_2 \ge 0$, then $(k+1)q - k\lambda_2 \ge q - \lambda_2$. Therefore, by Lemma 3.7,

$$\begin{aligned}
\operatorname{rk}(A^{k+1})_{ij} &\le \max\left\{\operatorname{rk}\left((A^k)_{i1}A_{1j}\right), \operatorname{rk}((A^k)_{i2}A_{2j})\right\} \le \\
&\le \max\{kq - (k-1)\lambda_2 + q - \lambda_1, q - \lambda_1, \\
&\qquad kq - (k-1)\lambda_2 + q - \lambda_2, q - \lambda_2\} = \\
&\le \max\{(k+1)q - k\lambda_2, q - \lambda_2\} = \\
&= (k+1)q - k\lambda_2 \le \max\{(k+1)q - k\lambda_2, 0\}
\end{aligned}$$

for all $1 \le i, j \le 2$. If $kq - (k-1)\lambda_2 \le 0$, then $(A^k)_{ij} = 0$ for $1 \le i, j \le 2$ and thus $\operatorname{rk}(A^{k+1})_{ij} = 0$ for all $1 \le i, j \le 2$.

Again, using Lemma 3.7 it follows that

$$\begin{aligned}
\operatorname{rk} A^k &\le \max\{\operatorname{rk} A^k{}_{11} + \operatorname{rk} A^k{}_{22}, \operatorname{rk} A^k{}_{12} + \operatorname{rk} A^k{}_{21}\} \le \\
&\le 2\max\{kq - (k-1)\lambda_2, 0\} = \\
&= \max\{2\lambda_2 - 2k(\lambda_2 - q), 0\}
\end{aligned}$$

for all $k \geq 1$.

Thus for all $k \geq \frac{\lambda_2}{\lambda_2 - q}$ it follows that $\operatorname{rk} A^k = 0$. Therefore, $\mu_1 \leq \left\lceil \frac{\lambda_2}{\lambda_2 - q} \right\rceil = \left\lceil \frac{\lambda_2}{s - \lambda_1} \right\rceil$. ■

Proposition 3.9. Let $\operatorname{sh}(B) = (\lambda, \lambda)$ and $A \in \mathcal{N}_B$. If a partition $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_s) \in \mathcal{P}(\mathcal{N}_B)$, then either $\underline{\mu} = (n)$ or $\mu_1 \leq \lambda + 1$.

Proof. Write $A \in \mathcal{N}_B$ as in (2), $b_0 c_0 = 0$, and assume that $A^{n-1} = 0$. Without loss of generality suppose that $c_0 = 0$.

Since $A^{n-1} = A^{2\lambda - 1} = b_0^\lambda c_1^{\lambda-1} K_{\lambda-1}$, it follows that $b_0 c_1 = 0$. If $b_0 = 0$, then $A = \sum_{i=1}^{\lambda-1} a_i M_i + \sum_{i=1}^{\lambda-1} b_i K_i + \sum_{i=1}^{\lambda-1} c_i L_i + \sum_{i=1}^{\lambda-1} d_i N_i$ and by (1), there are no summands in A^λ having index less than λ. Therefore, $A^\lambda = 0$. In the case $c_1 = 0$, the only summands of $A^{\lambda+1}$ having smaller index that the number of factors, are $(a_1 M)^\lambda b_0 K_0 = 0$ and $b_0 K_0 (d_1 N)^\lambda = 0$. Thus $A^{\lambda+1} = 0$. ■

4. Inverse image of $\mathcal{D}(\underline{\lambda})$

In the case, when $\mathcal{D}(\underline{\lambda})$ has at most 2 parts, the partition $\mathcal{D}(\underline{\lambda})$ can be easily characterized in the terms of $\underline{\lambda}$ (see [8, Thm. 7]). There is not much known about $\mathcal{D}(\underline{\lambda})$ if it has at least 3 parts.

Theorem 4.1. For a partition $\underline{\lambda}$,

$$\mathcal{D}(\underline{\lambda}) = (\mu, \mu - 2, \mu - 4, \ldots, \mu - 2k)$$

if and only if

$$\underline{\lambda} = (\mu, \mu - 2, \mu - 4, \ldots, \mu - 2k + 2, r(\mu - 2k, t)),$$

for $t = 1, 2, \ldots, \mu - 2k$.

Therefore, $\left|\mathcal{D}^{-1}(\mu, \mu - 2, \mu - 4, \ldots, \mu - 2k)\right| = \mu - 2k$.

Proof. Suppose $\mathcal{D}(\underline{\lambda}) = (\mu, \mu - 2, \mu - 4, \ldots, \mu - 2k)$ and obtain that $\underline{\lambda}$ is a partition of $n = (k+1)\mu - k(k+1)$. Since $\mathcal{D}(\underline{\lambda})$ has $k+1$ parts, it follows by [2, Thm. 2.4], that $\underline{\lambda}$ is of the form

$$\underline{\lambda} = (\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,t_1}, \lambda_{2,1}, \lambda_{2,2}, \ldots, \lambda_{2,t_2}, \ldots, \lambda_{k+1,1}, \lambda_{k+1,2}, \ldots, \lambda_{k+1,t_{k+1}}),$$

where $(\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,t_i})$ are almost rectangular partitions and $t_i \geq 1$, $i = 1, 2, \ldots, k+1$. Since the first part of $\mathcal{D}(\underline{\lambda})$ is equal to μ, it follows by [9, Thm. 16] that

$$\begin{aligned}
\lambda_{1,1} + \lambda_{1,2} + \ldots + \lambda_{1,t_1} &\leq \mu, \\
2t_1 + \lambda_{2,1} + \lambda_{2,2} + \ldots + \lambda_{2,t_2} &\leq \mu, \\
2(t_1 + t_2) + \lambda_{3,1} + \lambda_{3,2} + \ldots + \lambda_{3,t_3} &\leq \mu, \\
&\vdots \\
2(t_1 + t_2 + \ldots + t_k) + \lambda_{k+1,1} + \lambda_{k+1,2} + \ldots + \lambda_{k+1,t_{k+1}} &\leq \mu,
\end{aligned} \tag{4}$$

where at least one of the inequalities is actually an equality.

By summing all inequalities, we have $2(kt_1+(k-1)t_2+\ldots+2t_{k-1}+t_k)+n \leq (k+1)\mu$. Since $t_i \geq 1$ for all i, it follows that $k(k+1) \geq 2(kt_1+(k-1)t_2+\ldots+2t_{k-1}+t_k) \geq 2(k+(k-1)+\ldots+2+1) = k(k+1)$, and therefore, $t_i = 1$ for $i = 1,2,\ldots,k$ and all inequalities in (4) are equalities. By the last inequality in (4) it follows that $\lambda_{k+1,1} + \lambda_{k+1,2} + \ldots + \lambda_{k+1,t_{k+1}} = \mu - 2k$.

Now, $\underline{\lambda}$ has the form $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1,1}, \lambda_{k+1,2}, \ldots, \lambda_{k+1,t_{k+1}})$, and since $\mathcal{D}(\underline{\lambda})$ has $k+1$ parts, it follows that $\lambda_{i-1} - \lambda_i \geq 2$ for $i = 1,2,\ldots,k$. Suppose there exists j, $2 \leq j \leq k$, such that $\lambda_j < \lambda_1 - 2(j-1)$ and let j be minimal such. Thus, for $i = 1,2,\ldots,j-1$ we have $\lambda_i = \lambda_1 - 2(i-1)$ and for $i = j, j+1, \ldots, k$, we have $\lambda_i \leq \lambda_1 - 2i + 1$. Using the above equalities, we now obtain

$$\begin{aligned} k\lambda_1 - k(k-1) &\leq k\mu - k(k-1) = n - (\mu - 2k) = \\ &= n - (\lambda_{k+1,1} + \lambda_{k+1,2} + \ldots + \lambda_{k+1,t_{k+1}}) = \\ &= \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{j-1} \lambda_i + \sum_{i=j}^{k} \lambda_i \leq \\ &\leq k\lambda_1 - k(k-1) - (k-j+1) \end{aligned}$$

and therefore $j \geq k+1$, which contradicts the existence of j, $2 \leq j \leq k$, such that $\lambda_j < \lambda_1 - 2(j-1)$. Thus, $\lambda_i = \lambda_1 - 2(i-1)$ for $i = 1,2,\ldots,k$ and thus $(\lambda_{k+1,1}, \lambda_{k+1,2}, \ldots, \lambda_{k+1,t_{k+1}})$ is an almost rectangular partition of $\mu - 2k$. ■

Remark 4.2. Note that Theorem 4.1 does not hold if the parts of $\mathcal{D}(\underline{\lambda})$ differ for at least 3. For example, $\mathcal{D}((3,1,1)) = (4,1)$.

As mentioned in the begining of this section, $\mathcal{D}(\underline{\lambda})$ is known when $\underline{\lambda}$ has at most 2 parts. (See [8, Thm. 7]). Here, we describe the preimage of $\mathcal{D}$ for certain partitions and give a conjecture on the size of $\mathcal{D}^{-1}(\underline{\mu})$ in the case $\underline{\mu}$ has two parts. We will need the following lemma.

Lemma 4.3. If $\mathcal{D}(\underline{\lambda}) = (\mu, \mu - r)$, where $2 \leq r < \mu$, then the partition $\underline{\lambda}$ is of the form $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_t)$, where

- $\lambda_1 - \lambda_s \leq 1$, $\lambda_{s+1} - \lambda_t \leq 1$,
- $\lambda_1 - \lambda_t \geq 2$,
- $s \leq \frac{r}{2}$.

Proof. If $\mathcal{D}(\underline{\lambda}) = (\mu, \mu - r)$, then $\underline{\lambda}$ is of the form $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_t)$, where $\lambda_1 - \lambda_s \leq 1$, $\lambda_{s+1} - \lambda_t \leq 1$ and $\lambda_1 - \lambda_t \geq 2$. (See Basili [2, Prop. 2.4].) Since the first part of $\mathcal{D}(\underline{\lambda})$ is equal to μ, it follows by [9, Thm. 16] that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_s \leq \mu \tag{5}$$

$$2s + \lambda_{s+1} + \lambda_{s+2} + \ldots + \lambda_t \leq \mu\,. \tag{6}$$

Thus, $2\mu - r = \lambda_1 + \lambda_2 + \ldots + \lambda_t \leq 2\mu - 2s$ and therefore $s \leq \frac{r}{2}$. ■

Proposition 4.4. For a partition $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathcal{P}(n)$ and $n \geq 4$, it follows that

$$\mathcal{D}(\underline{\lambda}) = (n-1, 1)$$

if and only if $\lambda_1 - \lambda_t \geq 2$ and either $\underline{\lambda} = (r(n-1, t-1), 1)$ or $\underline{\lambda} = (3, r(n-3, t-1))$.

Proof. It is clear that

- if the last part of $r(n-1, t-1)$ is not equal to 1, then $\mathcal{D}(r(n-1, t-1), 1) = (n-1, 1)$ and
- if the first part of $r(n-3, t-1)$ is at most 2, then $\mathcal{D}(3, r(n-3, t-1)) = (n-1, 1)$.

Suppose now, $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ and $\mathcal{D}(\underline{\lambda}) = (n-1, 1)$. By Lemma 4.3, we have that $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_s, \lambda_{s+1}, \ldots, \lambda_t)$, where $\lambda_1 + \lambda_2 + \ldots + \lambda_s = n-1$ or $2s + \lambda_{s+1} + \ldots + \lambda_t = n-1$. In the first case, clearly, $\underline{\lambda} = (r(n-1, s), 1)$. In the second case we have that $\sum_{i=1}^{s}(\lambda_i - 2) = 1$ and thus, $s = 1$ and $\lambda_1 = 3$. ■

Proposition 4.5. If $2 \leq r \leq 5$ and $\mu - r \geq 1$, then

$$\left|\mathcal{D}^{-1}(\mu, \mu - r)\right| = (r-1)(\mu - r).$$

Moreover,

$$\mathcal{D}^{-1}(\mu, \mu-2) = \{(\mu, \lambda_2, \lambda_3, \ldots, \lambda_t);\ (\lambda_2, \lambda_3, \ldots, \lambda_t) \text{ is almost rectangular, } \mu - \lambda_t \geq 2\},$$

$$\mathcal{D}^{-1}(\mu, \mu-3) = \{(\mu - \varepsilon, \lambda_2, \lambda_3, \ldots, \lambda_t);\ \lambda_2 - \lambda_t \leq 1,\ \mu - \varepsilon - \lambda_t \geq 2,\ \varepsilon \in \{0, 1\}\}$$

and

$$\begin{aligned}\mathcal{D}^{-1}(\mu, \mu-4) \quad = \quad & \{(\mu_1, \mu_2, \lambda_3, \lambda_4, \ldots, \lambda_t);\ \mu_1 - \mu_2 \leq 1,\ \lambda_3 - \lambda_t \leq 1,\ \mu_1 - \lambda_t \geq 2\} \cup \\ & \cup\ \{(\mu - \varepsilon, \lambda_2, \lambda_3, \ldots, \lambda_t);\ \lambda_2 - \lambda_t \leq 1,\ \mu - \varepsilon - \lambda_t \geq 2,\ \varepsilon \in \{0, 2\}\}.\end{aligned}$$

Proof. As a corollary of Theorem 4.1 we obtain the set $\mathcal{D}^{-1}(\mu, \mu-2)$ and $\left|\mathcal{D}^{-1}(\mu, \mu-2)\right| = \mu - 2$.

If $\mathcal{D}(\underline{\lambda}) = (\mu, \mu-3)$, then $\underline{\lambda}$ is of the form of Lemma 4.3, where $s = 1$. By (6), we get $\lambda_1 = 2\mu - 3 - (\lambda_2 + \lambda_3 + \ldots + \lambda_t) \geq 2\mu - 3 - (\mu - 2) = \mu - 1$ and from (5) it follows that $\lambda_1 \leq \mu$. Thus, we have that either $\lambda_1 = \mu$ or $\lambda_1 = \mu - 1$. It can be easily verified that all such partitions $\underline{\lambda}$ satisfy the condition $\mathcal{D}(\underline{\lambda}) = (\mu, \mu-3)$.

If $\lambda_1 = \mu$, then $(\lambda_2, \lambda_3, \ldots, \lambda_t)$ is an arbitrary almost rectangular partition of $\mu - 3$. If $\lambda_1 = \mu - 1$, then $(\lambda_2, \lambda_3, \ldots, \lambda_t)$ is an almost rectangular partition of $\mu - 2$, where $t \geq 3$. (Note, that otherwise $\lambda_1 = \lambda_t + 1$.) Therefore, $\left|\mathcal{D}^{-1}(\mu, \mu-3)\right| = 2(\mu - 3)$.

If $\mathcal{D}(\underline{\lambda}) = (\mu, \mu-4)$, then by Lemma 4.3, it follows that $s \leq 2$.

If $s = 1$, then by (6) we have that $\lambda_1 = 2\mu - 4 - (\lambda_2 + \lambda_3 + \ldots + \lambda_t) \geq 2\mu - 4 - (\mu - 2) = \mu - 2$. Since by (5), $\lambda_1 \leq \mu$ we consider 3 cases.

If $\lambda_1 = \mu$, then $(\lambda_2, \lambda_3, \ldots, \lambda_t)$ is an almost rectangular partition of $\mu - 4$ and by [8, Thm. 7], it follows that $\mathcal{D}(\mu, \lambda_2, \lambda_3, \ldots, \lambda_t) = (\mu, \mu-4)$. If $\lambda_1 = \mu - 1$ and $(\lambda_2, \lambda_3, \ldots, \lambda_t)$ is an almost rectangular partition of $\mu - 3$, then $\mathcal{D}(\mu - 1, \lambda_2, \lambda_3, \ldots, \lambda_t) = (\mu - 1, \mu - 3)$ and thus no such partition is in $\mathcal{D}^{-1}(\mu, \mu-4)$. If $\lambda_1 = \mu - 2$, then $(\lambda_2, \lambda_3, \ldots, \lambda_t)$ is an almost rectangular partition of $\mu - 2$. If, in addition, $t \geq 3$, then $\mathcal{D}((\mu - 2, \lambda_2, \lambda_3, \ldots, \lambda_t)) = (\mu, \mu-4)$.

If $s = 2$, then, by (6), we have that $\lambda_1+\lambda_2 = 2\mu-4-(\lambda_3+\lambda_4+\ldots+\lambda_t) \geq 2\mu-4-(\mu-4) = \mu$. Thus, (λ_1, λ_2) is an almost rectangular partition of μ and $(\lambda_3, \lambda_4, \ldots, \lambda_t)$ is an almost rectangular partition of $\mu - 4$, such that $\mu_1 - \lambda_t \geq 2$. This is true if and only if $t \geq 4$.

Now, it is easy to compute that $\left|\mathcal{D}^{-1}(\mu, \mu-4)\right| = \mu - 4 + \mu - 3 + \mu - 5 = 3(\mu - 4)$. ■

Question. Is it true that

$$\left|\mathcal{D}^{-1}(\mu, \mu-r)\right| = (r-1)(\mu-r)$$

for all $r \geq 5$?

One can also ask a question, what are maximal and minimal partitions in $\mathcal{D}^{-1}(\underline{\mu})$? Clearly, the maximal partition in $\mathcal{D}^{-1}(\underline{\mu})$ is $\underline{\mu}$. However, there is not a unique minimal partition in $\mathcal{D}^{-1}(\underline{\mu})$, as shown in the next example.

Example 4.6. One can easily check that

$$\mathcal{D}^{-1}(6,2) = \{(6,2),\ (6,1^2),\ (4,2^2),\ (4,2,1^2),\ (4,1^4),\ (3^2,1^2)\}$$

and that there are 2 minimal partitions $(3^2, 1^2)$ and $(4, 1^4)$ in $\mathcal{D}^{-1}(6,2)$.

Recall that the *rank* of partition $(\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathcal{P}(n)$ is defined as the number $n - t$. So, the partition with the minimal rank is the partition with the most parts. Now, we can prove the following:

Proposition 4.7. For every $r \geq 2$, the partition $(\mu+2, 1^{\mu+r-2})$ is in $\mathcal{D}^{-1}(\mu+r, \mu)$ and this is the unique partition with the minimal rank in $\mathcal{D}^{-1}(\mu+r, \mu)$.

Proof. Since $r \geq 2$, we have by [8, Thm. 7], that $\mathcal{D}(\mu+2, 1^{\mu+r-2}) = (\mu+r, \mu)$.

Suppose that $\mathcal{D}(\underline{\lambda}) = (\mu+r, \mu)$ and that $\underline{\lambda}$ has a rank at most $\mu+1$, which is the rank of $(\mu+2, 1^{\mu+r-2})$. Then, $\underline{\lambda}$ is of the form $(\lambda_1, \lambda_2, \ldots \lambda_s, \lambda_{s+1}, \ldots, \lambda_t)$, where $t \geq \mu+r-1$, $\lambda_1 - \lambda_s \leq 1$, $\lambda_{s+1} - \lambda_t \leq 1$ and $\lambda_1 - \lambda_t \geq 2$. Now, we have

$$\begin{aligned} \lambda_1 + \lambda_2 + \ldots + \lambda_s &\leq \mu + r \\ 2s + \lambda_{s+1} + \lambda_{s+2} + \ldots + \lambda_t &\leq \mu + r\,. \end{aligned}$$

If $t \geq \mu+r$, it follows that $\mu+r-s \leq t-s \leq \lambda_{s+1}+\lambda_{s+2}+\ldots+\lambda_t \leq \mu+r-2s$, which is a contradiction. Otherwise, if $t = \mu+r-1$, then $\mu+r-1-s = t-s \leq \lambda_{s+1}+\lambda_{s+2}+\ldots+\lambda_t \leq \mu+r-2s$ and thus $s = 1$. Now, we conclude that $\lambda_2 = \lambda_3 = \cdots = \lambda_{\mu+r} = 1$ and $\underline{\lambda} = (\mu+2, 1^{\mu+r-2})$. Thus, $(\mu+2, 1^{\mu+r-2})$ is the unique partition with rank equal to $\mu+r-1$ and no partition in $\mathcal{D}^{-1}(\mu+r, \mu)$ has greater rank. ■

Question. Let $(\mu_1, \mu_2, \ldots, \mu_s)$ be a stable partition. Is it true that the partition with the minimal rank in $\mathcal{D}^{-1}(\mu_1, \mu_2, \ldots, \mu_s)$ is equal to $(\mu_2+2, \mu_3+2, \ldots, \mu_s+2, 1^{\mu_1-2(s-1)})$?

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