arXiv:1209.3681v1 [math.RA] 17 Sep 2012

DUALITY FOR GROUPOID (CO)ACTIONS

DAIANA FLÔRES AND ANTONIO PAQUES

Abstract. In this paper we present Cohen-Montgomery-type duality theorems for groupoid (co)actions.

Key words and phrases: groupoid action, groupoid coaction, Cohen-Montgomery duality.

Mathematics Subject Classification: Primary 16S40, 16W50, 20L05, 22D35.

1. Introduction

Groupoids are usually presented as small categories whose morphisms are invertible. This notion is a natural extension of the notion of a group. Notice that a group can be seen as a category with a unique objet.

The notion of a groupoid action that we use in this paper arose from the notion of a partial groupoid action as introduced in [2], which is a natural extension of the notion of a partial group action [10]. First, partial ordered groupoid actions on sets were introduced in the literature, as ordered premorphisms, by N. Gilbert [11]. After, partial ordered groupoid actions on rings were considered by D. Bagio and the authors [1] as a generalization of partial group actions, as introduced by M. Dokuchaev and R. Exel in [10]. And in [2] this notion was extended to the general context of groupoids.

Our purpose is to present a generalization of Cohen-Montgomery duality Theorem for group actions [9, Theorem 3.2] (resp., group coactions or, equivalently, group grading [9, Theorem 3.5]) to the setting of groupoid actions (resp., groupoid coactions or groupoid grading) (see Theorems 3.7 and 4.5).

This paper is organized as follows. In the next section we give preliminaries about groupoids, groupoid actions, skew groupoid rings, weak bialgebras, groupoid gradings, groupoid coactions and weak smash products, these later as introduced by S. Caenepeel and E. De Groot in [7]. In that section we will be concerned only with the results strictly necessary to construct the appropriate environment to prove our main theorems, whose proofs we set in the sections 3 (actions) and 4 (coactions).

We deal with the groupoid ring KG and its dual KG^*, K being a commutative ring and G a finite groupoid. The K-algebras KG and KG^* are perhaps the first examples of weak bialgebras that are not bialgebras.

Accordingly [2], an action of a groupoid G on a K-algebra R is a pair $\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G})$, where for each $g \in G$, $E_g = E_{gg^{-1}}$ is an ideal of R and $\beta_g : E_{g^{-1}} \to E_g$ is an isomorphism of rings satisfying some appropriate conditions (see the subsection 2.3).

If the set G_0 of all identities of G is finite then there exists a one to one correspondence between the structures of left KG-module algebra on R and the actions β of G on R such that each ideal E_e $(e \in G_0)$ is unital and $R = \bigoplus_{e\in G_0} E_e$. In particular, the notion of groupoid action introduced by Caenepeel and De Groot in [8] is equivalent to this previous one (see Proposition 2.2).

Given an action β of a finite groupoid G on a K-algebra R we can consider the skew groupoid ring $R \star_\beta G$ [1], which is a G-graded algebra or, equivalently, a left KG^*-module algebra. The corresponding weak smash product [7] $(R\star_\beta G)\#KG^*$ is a nonunital K-algebra (see the subsection 2.5).

Given any unital K-algebra A graded by a finite groupoid G or, equivalently, a left KG^*-module algebra, there exists an action β of G on the weak smash product $A\#KG^*$ (see Proposition 2.5) and we can consider the corresponding skew group ring $(A\#KG^*)\star_\beta G$, which also is a nonunital K-algebra.

We show in the section 3 (resp., section 4) that $(R\star_\beta G)\#KG^*$ (resp., $(A\#KG^*)\star_\beta G$) contains a unital K-subalgebra that is isomorphic to a finite direct sum of matrix K-algebras. In particular, if G is a group we recover [9, Theorems 3.2 and 3.5].

In [15] D. Nikshych presented a Blattner-Montgomery-type duality for weak Hopf algebras, generalizing the well known result for Hopf algebras obtained in [4] and [3]. There exists a natural relation between $(R\star_\beta G)\#KG^*$ and the double weak smash product $(R\bigotimes_{KG_t}KG)\bigotimes_{KG_t^*}KG^*$ as constructed by Nikshych (in the case that this makes sense), and it will be explicitly given in the section 5.

Throughout, by ring (or algebra) we mean an associative, not necessarily commutative and not necessarily unital ring (or algebra).

2. Preliminary results

2.1. **Groupoids.** The axiomatic version of groupoid that we adopt in this paper was taken from [12]. A *groupoid* is a non-empty set G equipped with a partially defined binary operation, that we will denote by concatenation, for which the usual axioms of a group hold whenever they make sense, that is:

(i) For every $g,h,l\in G$, $g(hl)$ exists if and only if $(gh)l$ exists and in this case they are equal.
(ii) For every $g,h,l\in G$, $g(hl)$ exists if and only if gh and hl exist.
(iii) For each $g\in G$ there exist (unique) elements $d(g),r(g)\in G$ such that $gd(g)$ and $r(g)g$ exist and $gd(g)=g=r(g)g$.
(iv) For each $g\in G$ there exists an element $g^{-1}\in G$ such that $d(g)=g^{-1}g$ and $r(g)=gg^{-1}$.

We will denote by G^2 the subset of the pairs $(g,h)\in G\times G$ such that the element gh exists.

An element $e\in G$ is called an *identity* of G if $e=d(g)=r(g^{-1})$, for some $g\in G$. In this case e is called the *domain identity* of g and the *range identity* of g^{-1}. We will denote by G_0 the set of all identities of G and we will denote by G_e the set of all $g\in G$ such that $d(g)=r(g)=e$. Clearly, G_e is a group, called the *isotropy (or principal) group associated to e*.

The assertions listed in the following lemma are straightforward from the above definition. Such assertions will be freely used along this paper.

Lemma 2.1. *Let G be a groupoid. Then,*

(i) *for every $g\in G$, the element g^{-1} is unique satisfying $g^{-1}g=d(g)$ and $gg^{-1}=r(g)$,*
(ii) *for every $g\in G$, $d(g^{-1})=r(g)$ and $r(g^{-1})=d(g)$,*
(iii) *for every $g\in G$, $(g^{-1})^{-1}=g$,*
(iv) *for every $g,h\in G$, $(g,h)\in G^2$ if and only if $d(g)=r(h)$,*
(v) *for every $g,h\in G$, $(h^{-1},g^{-1})\in G^2$ if and only if $(g,h)\in G^2$ and, in this case, $(gh)^{-1}=h^{-1}g^{-1}$,*
(vi) *for every $(g,h)\in G^2$, $d(gh)=d(h)$ and $r(gh)=r(g)$,*
(vii) *for every $e\in G_0$, $d(e)=r(e)=e$ and $e^{-1}=e$,*
(viii) *for every $(g,h)\in G^2$, $gh\in G_0$ if and only if $g=h^{-1}$,*
(ix) *for every $g,h\in G$, there exists $l\in G$ such that $g=hl$ if and only if $r(g)=r(h)$,*
(x) *for every $g,h\in G$, there exists $l\in G$ such that $g=lh$ if and only if $d(g)=d(h)$.*

2.2. **Weak bialgebras: the finite groupoid algebra and its dual.** Hereafter K will denote a unital commutative ring and unadorned $\otimes$ will mean $\otimes_K$. Following [6], a weak K-bialgebra H is a unital K-algebra, with a K-coalgebra structure (Δ,ϵ) such that

(i) $\Delta(xy)=\Delta(x)\Delta(y)$

(ii) $\Delta^2(1_H) = (\Delta(1_H) \otimes 1_H)(1_H \otimes \Delta(1_H)) = (1_H \otimes \Delta(1_H))(\Delta(1_H) \otimes 1_H)$

(iii) $\varepsilon(xyz) = \sum \varepsilon(xy_1)\varepsilon(y_2 z) = \sum \varepsilon(xy_2)\varepsilon(y_1 z)$,

for all $x, y, z \in H$, where $\Delta^2 = (\Delta \otimes I_H) \circ \Delta = (I_H \otimes \Delta) \circ \Delta$ and I_H denotes the identity map of H. We use the Sweedler-Heyneman notation for the comultiplication, namely $\Delta(x) = \sum x_1 \otimes x_2$, for all $x \in H$.

If H is a bialgebra, that is, the maps Δ and ε are homomorphisms of algebras, then the above axioms (i)-(iii) are trivially satisfied. Here we are concern with the algebra KG of a groupoid G and its dual in the case that G is finite. Both are weak bialgebras but not bialgebras.

Given a groupoid G, the groupoid algebra KG is free as a K-module with basis $\{u_g \mid g \in G\}$, its multiplication is given by the rule

$$u_g u_h = \begin{cases} u_{gh} & \text{if } \ d(g) = r(h) \\ 0 & \text{otherwise,} \end{cases}$$

for all $g, h \in G$, its identity element exists and is $1_{KG} = \sum_{e \in G_0} u_e$ if and only if G_0 is finite [14], and its K-coalgebra structure is given by

$$\Delta(u_g) = u_g \otimes u_g \quad \text{and} \quad \varepsilon(u_g) = 1_K,$$

for all $g \in G$.

The dual KG^* of KG, as a K-module, is free with dual basis $\{v_g \mid g \in G\}$, that is, $v_g(u_h) = \delta_{g,h}1_K$, for all $g, h \in G$. If G is finite, its K-algebra structure is given by

$$v_g v_h = \delta_{g,h} v_g \quad \text{and} \quad \sum_{g \in G} v_g = 1_{KG^*},$$

and its K-coalgebra structure is given by

$$\Delta(v_g) = \sum_{hl=g} v_h \otimes v_l = \sum_{d(h)=d(g)} v_{gh^{-1}} \otimes v_h \quad \text{and} \quad \varepsilon(\sum_{g \in G} a_g v_g) = \sum_{e \in G_0} a_e.$$

2.3. **Groupoid actions.** Let G be a groupoid and R a not necessarily unital ring. Following [2], an action of G on R is a pair

$$\beta = (\{E_g\}_{g \in G}, \{\beta_g\}_{g \in G}),$$

where for each $g \in G$, $E_g = E_{r(g)}$ is an ideal of R, $\beta_g : E_{g^{-1}} \to E_g$ is an isomorphism of rings, and the following conditions hold:

(i) β_e is the identity map I_{E_e} of E_e, for all $e \in G_0$,

(ii) $\beta_g \beta_h(r) = \beta_{gh}(r)$, for all $(g, h) \in G^2$ and $r \in E_{h^{-1}} = E_{(gh)^{-1}}$.

In particular, β induces an action of the group G_e on E_e, for every $e \in G_0$.

In [8] Caenepeel and De Groot developed a Galois theory for weak bialgebra actions on algebras. In particular, they considered the situation where the weak bialgebra is a finite groupoid algebra and a notion of groupoid action was introduced. Actually, this later notion and the one above defined, under some additional conditions, are equivalent, as we will see in the next proposition.

Following [8, section 4], a KG-module algebra is a unital K-algebra R, with a left KG-module structure given by $\cdot : KG \otimes R \to R$, $u_g \otimes x \mapsto u_g \cdot x$, such that:

$$u_g \cdot (xy) = (u_g \cdot x)(u_g \cdot y) \quad \text{and} \quad u_g \cdot 1_R = u_{r(g)} \cdot 1_R,$$

for all $x, y \in R$ and $g \in G$.

Proposition 2.2. *Let G be a groupoid such that G_0 is finite, and R be a unital K-algebra. Then the following statements are equivalent:*

(i) *There exists an action $\beta = (\{E_g\}_{g \in G}, \{\beta_g\}_{g \in G})$ of G on R such that every E_e, $e \in G_0$, is unital and $R = \bigoplus_{e \in G_0} E_e$.*

(ii) *R has an structure of KG-module algebra.*

Proof. (i)⇒(ii) Let 1_g denote the identity element of E_g, for every $g \in G$. It easily follows from the assumptions that each 1_g is a central idempotent of R, $1_R = \sum_{e\in G_0} 1_e$ and $E_e \bigcap E_f = 0 = E_e E_f$, for all $e \neq f$.

Consider now the action of KG over R given by

$$u_g \cdot r = \beta_g(r1_{g^{-1}}),$$

for every $g \in G$ and $r \in R$. Such an action induces on R an structure of KG-module. Indeed,

$$1_{KG} \cdot r = \sum_{e\in G_0} u_e \cdot r = \sum_{e\in G_0} \beta_e(r1_e) = \sum_{e\in G_0} r1_e = r \sum_{e\in G_0} 1_e = r1_R = r,$$

for all $r \in R$. Furthermore, it follows from the items (ii),(iv) and (vi) of Lemma 2.1 that $E_{(gh)^{-1}} = E_{h^{-1}}$, $E_h = E_{g^{-1}}$, $E_g = E_{gh}$, for all $(g, h) \in G^2$. Hence,

$$\begin{aligned} u_g \cdot (u_h \cdot r) &= \beta_g(\beta_h(r1_{h^{-1}})1_{g^{-1}}) = \beta_g(\beta_h(r1_{h^{-1}})) \\ &= \beta_{gh}(r1_{h^{-1}}) = \beta_{gh}(r1_{(gh)^{-1}}) \\ &= u_{gh} \cdot r = u_g u_h \cdot r, \end{aligned}$$

for all $r \in R$ and $(g, h) \in G^2$. For $(g, h) \notin G^2$ it is trivial to check that $u_g \cdot (u_h \cdot r) = 0 = u_g u_h \cdot r$. So, R is a left KG-module.

Now, since

$$u_g \cdot (rs) = \beta_g(rs1_{g^{-1}}) = \beta_g(r1_{g^{-1}})\beta_g(s1_{g^{-1}}) = (u_g \cdot r)(u_g \cdot s)$$

and

$$u_g \cdot 1_R = \beta_g(1_R 1_{g^{-1}}) = 1_g = 1_{r(g)} = \beta_{r(g)}(1_R 1_{r(g)^{-1}}) = u_{r(g)} \cdot 1_R,$$

for all $g \in G$ and $r, s \in R$, the required follows.

(ii)⇒(i) Put $1_g = u_g \cdot 1_R$ and $E_g = R1_g$, for every $g \in G$. So, $1_g = 1_{r(g)}$ and by [8, Proposition 4.1] these elements are central orthogonal idempotents in R, and $R = \bigoplus_{e\in G_0} E_e$. Clearly, each E_g, $g \in G$, is an ideal of R and a unital ring. Let $\beta_g : E_{g^{-1}} \to E_g$ given by $\beta_g(r) = u_g \cdot r$, for every $r \in E_{g^{-1}}$ and $g \in G$. It is immediate from the assumptions that β_g is a well defined isomorphism of rings with $\beta_g^{-1} = \beta_{g^{-1}}$. Furthermore,

$$\beta_e(r) = \sum_{e'\in G_0} \beta_{e'}(r1_{e'}) = \sum_{e'\in G_0} u_{e'} \cdot r = 1_{KG} \cdot r = r,$$

for every $r \in E_{e^{-1}} = E_e$, and

$$\beta_g(\beta_h(r)) = u_g \cdot (u_h \cdot r) = u_{gh} \cdot (r) = \beta_{gh}(r),$$

for every $(g, h) \in G^2$ and $r \in E_{h^{-1}} = E_{(gh)^{-1}}$. The proof is complete. □

2.4. **The skew groupoid ring.** Let R, G and $\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ be as in the previous subsection. Accordingly [1, Section 3], the skew groupoid ring $R \star_\beta G$ corresponding to β is defined as the direct sum

$$R \star_\beta G = \bigoplus_{g\in G} E_g \delta_g$$

in which the δ_g's are symbols, with the usual addition, and multiplication determined by the rule

$$(x\delta_g)(y\delta_h) = \begin{cases} x\beta_g(y)\delta_{gh} & \text{if } (g, h) \in G^2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $g, h \in G$, $x \in E_g$ and $y \in E_h$.

This multiplication is well defined. Indeed, if $(g, h) \in G^2$ then $d(g) = r(h)$ (see Lemma 2.1(iv)). So, $E_{g^{-1}} = E_{r(g^{-1})} = E_{d(g)} = E_{r(h)} = E_h$, $\beta_g(y)$ makes sense, and $x\beta_g(y) \in E_g = E_{r(g)} \overset{(\star)}{=} E_{r(gh)} = E_{gh}$, where the equality $(\star)$ is ensured by Lemma 2.1(vi).

By a routine calculation one easily sees that $A = R \star_\beta G$ is associative, and by [1, Proposition 3.3] it is unital if G_0 is finite and E_e is unital for all $e \in G_0$. In this case the identity element of A is $1_A = \sum_{e\in G_0} 1_e\delta_e$, where 1_e denotes the identity element of E_e, for all $e \in G_0$.

Remark 2.3. *Proposition 3.3 in* [1] *asserts that $A = R \star_\beta G$ is unital if and only if G_0 finite. Unfortunately, the existence of the identity element 1_A does not necessarily imply that G_0 is finite, as we will see in the next subsection (note that, in particular, A is a G-graded algebra by construction).*

2.5. **Groupoid gradings, coations and weak smash products.** Let G be a groupoid and A a not necessarily unital K-algebra. We say that A is a G-graded algebra if there exists a family $\{A_g\}_{g\in G}$ of K-submodules of A such that

$$A = \bigoplus_{g\in G} A_g$$

and

$$A_g A_h \begin{cases} \subseteq A_{gh} & \text{if } (g,h) \in G^2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $g, h \in G$. It easily follows from this definition that each A_e, $e \in G_0$, is a subalgebra of A.

Remark 2.4. *If A is assumed to be unital, then it follows from* [13, Propositions 2.2 and 2.3, and Remark 2.4] *that:*

(i) *the set $J_0 = \{e \in G_0 \mid A_e \neq 0\}$ is finite,*
(ii) *$A_g = 0$ for all $g \in G$ such that either $d(g)$ or $r(g)$ does not belong to J_0,*
(iii) *every A_e, with $e \in J_0$, is unital,*
(iv) *$1_A = \sum_{e\in J_0} 1_e$, with 1_e denoting the identity element of A_e, for all $e \in J_0$.*
(v) *for every $e \in J_0$ and $g \in G$ such that $r(g) = e$ (resp., $d(g) = e$), $1_e a_g = a_g$ (resp., $a_g 1_e = a_g$), for all $a_g \in A_g$.*

Recall from [5] and [7] that a right H-comodule algebra A (H being a weak bialgebra) is a unital K-algebra with a right H-comodule structure given by the coaction $\rho : A \to A \otimes H$ such that $\rho(ab) = \rho(a)\rho(b)$, for all $a, b \in A$, and $(\rho \otimes I_H) \circ \rho(1_A) = (\rho(1_A) \otimes 1_H)(1_A \otimes \Delta(1_H))$

In all what follows we will assume that A is a unital K-algebra and G is finite.

The existence of a G-grading on A is equivalent to say that A has an structure of a right KG-comodule algebra (see [8, Proposition 3.1]), with coation given by $\rho(a) = \sum_{g\in G} a_g \otimes u_g$, for all $a = \sum_{g\in G} a_g \in A$. This is also equivalent to say that A is a KG^*-module algebra, with the action given by $v_h \cdot a = a_h$, for all $a \in A$ and $h \in G$.

Hence, we can consider the weak smash product $A\#KG^*$ of A by KG^* (see [7, section 3]), which is equal to $A \otimes KG^*$ as K-modules and the multiplication is given by the following rule:

$$(a\#v_g)(b\#v_h) = \begin{cases} a(v_{gh^{-1}} \cdot b)\#v_h & \text{if } d(h) = d(g) \\ 0 & \text{otherwise.} \end{cases}$$

A routine calculation easily shows that such a multiplication is associative. Also, $A\#KG^*$ is not unital. To see this it is enough to verify that any element of the type $x = b\#v_h$, with $b \in A_k$ and $d(k) \neq r(h)$, is a right annhilator of $A\#KG^*$. Indeed, in this case $gh^{-1} \neq k$ by Lemma 2.1(x) and therefore $v_{gh^{-1}} \cdot b = 0$ which implies $(a\#v_g)x = 0$ for all $a \in A$ and $g \in G$.

The element $u = 1_A\#1_{KG^*}$ is a preunit of $A\#KG^*$, that is, $ux = xu = xu^2$, for all $x \in A\#KG^*$ (see [7, section 3]).

In the sequel we will show that there also exists an action β of the groupoid G on $A\#KG^*$, which allows us to construct the skew groupoid ring $(A\#KG^*) \star_\beta G$.

Put $B = A\#KG^*$ and, for each $g \in G$, let

$$E_g = \bigoplus_{\substack{l,k\in G \\ d(k)=r(g)}} A_l\#v_k$$

and

$$\beta_g : E_{g^{-1}} \to E_g \quad \text{given by} \quad \beta_g(a_l\#v_k) = a_l\#v_{kg^{-1}}.$$

Proposition 2.5. *The pair* $\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ *is an action of* G *on* B*, and* $B = \bigoplus_{e\in G_0} E_e$.

Proof. First, it is clear that $E_g = E_{r(g)}$, for all $g \in G$. Now, taking $x = a_l\#v_k \in E_g$ and $y = b_s\#v_t \in B$, we have by definition

$$xy = (a_l\#v_k)(b_s\#v_t) = \begin{cases} a_l(v_{kt^{-1}} \cdot b_s)\#v_t & \text{if } d(k) = d(t) \\ 0 & \text{otherwise,} \end{cases}$$

and $v_{kt^{-1}} \cdot b_s \neq 0$ if and only if $kt^{-1} = s$ or, equivalently, $t = s^{-1}k$. Thus,

$$xy = \begin{cases} a_l b_s\#v_t & \text{if } t = s^{-1}k \\ 0 & \text{otherwise,} \end{cases}$$

Since $d(t) = d(s^{-1}k) = d(k) = r(g)$, it follows that $xy \in E_g$. Similarly, we also have $yx \in E_g$. Hence, E_g is an ideal of B.

It is immediate to check that $\beta_g : E_{g^{-1}} \to E_g$ is a well defined additive map, $\beta_{g^{-1}} = \beta_g^{-1}$ for all $g \in G$, and $\beta_e = I_{E_e}$ for all $e \in G_0$. From the above we also have

$$\beta_g(xy) = \begin{cases} a_l b_s\#v_{tg^{-1}} & \text{if } t = s^{-1}k \\ 0 & \text{otherwise,} \end{cases}$$

for all $x = a_l\#v_k$ and $y = b_s\#v_t$ in $E_{g^{-1}}$. On the other hand, since $d(kg^{-1}) = d(g^{-1}) = d(tg^{-1})$ (see Lemma 2.1(vi)), and $d(k) = r(g^{-1}) = d(g)$ we have

$$\begin{aligned} \beta_g(x)\beta_g(y) &= a_l(v_{kg^{-1}(tg^{-1})^{-1}} \cdot b_s)\#v_{tg^{-1}} \\ &= a_l(v_{kg^{-1}gt^{-1}} \cdot b_s)\#v_{tg^{-1}} \\ &= a_l(v_{(k(d(g))t^{-1}} \cdot b_s)\#v_{tg^{-1}} \\ &= a_l(v_{kt^{-1}} \cdot b_s)\#v_{tg^{-1}} \\ &= a_l b_s\#v_{tg^{-1}} \end{aligned}$$

if $t = s^{-1}k$ and 0 otherwise. So, β is multiplicative.

Finally, notice that, for all $(g,h) \in G^2$, $\mathrm{dom}(\beta_g\beta_h) = \beta_{h^{-1}}(E_h \bigcap E_{g^{-1}}) = \beta_{h^{-1}}(E_h) = E_{h^{-1}} = E_{d(h)} = E_{d(gh)} = E_{(gh)^{-1}} = \mathrm{dom}(\beta_{gh})$, and

$$\begin{aligned} \beta_g\beta_h(x) &= \beta_g(\beta_h(a_l\#v_k)) = \beta_g(a_l\#v_{kh^{-1}}) \\ &= a_l\#v_{k(gh)^{-1}} = \beta_{gh}(a_l\#v_k) = \beta_{gh}(x), \end{aligned}$$

for all $x = a_l\#v_k \in E_{h^{-1}}$. Hence, β is an action of G on B.

The last assertion is immediate, since $B = \sum_{g\in G} E_{r(g)}$ and $E_e \bigcap \sum_{\substack{e'\in G_0 \\ e'\neq e}} E_{e'} = 0$. □

The skew groupoid ring $B \star_\beta G$ is clearly associative (see the previous subsection). However, it is not unital, because any element of the type $x = (a_j\#v_k)\delta_s$, with $d(k) = r(s)$ and $r(k) \neq d(j)$, is a left annhilator of $B \star_\beta G$. Indeed, it enough to verify that $xE_g = 0$, for all $g \in G$. Let $y = a_l\#v_h \in E_g$, that is, $d(h) = r(g)$. Clearly $xy = 0$ if $d(s) \neq r(g)$, and, otherwise, we have

$$\begin{aligned} xy &= (a_j\#v_k)\beta_s(a_l\#v_h)\delta_{sg} \\ &= (a_j\#v_k)(a_l\#v_{hs^{-1}})\delta_{sg} \\ &= (a_j(v_{ksh^{-1}} \cdot a_l)\#v_{hs^{-1}})\delta_{sg} \end{aligned}$$

But, $v_{ksh^{-1}} \cdot a_l \neq 0$ if and only if $ksh^{-1} = l$, which implies $r(l) = r(k)$ and $a_j(v_{ksh^{-1}} \cdot a_l) = a_j a_l = 0$ because $d(j) \neq r(l)$.

3. Duality for groupoid actions

In this section R will denote a not necessarily unital K-algebra, G a finite groupoid and $\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ an action of G on R such that E_e is unital, for all $e \in G_0$.

We are concerned with a Cohen-Montgomery-type duality theorem [9] for groupoid actions. More precisely, we show that the weak smash product $(R \star_\beta G)\#KG^*$ (which is not unital) contains a unital subalgebra isomorphic to a finite direct sum of matrix K-algebras. In particular, if G is a group we recover Theorem 3.2 of [9].

Let $B = (R \star_\beta G)\#KG^*$. The subalgebra that we are looking for is

$$B_0 = \bigoplus_{d(g)=r(g)=r(h)} E_g\delta_g\#v_h,$$

and, in order to get our main result, the strategy is to obtain a decomposition of B_0 into a direct sum of suitable unital subalgebras satisfying the conditions of the following lemma due to D. S. Passman (see [16, Lemma 1.6, p. 228]).

Lemma 3.1. *Let S be a unital ring, and $1_S = e_1 + \cdots + e_n$ be a decomposition of 1_S into a sum of orthogonal idempotents. Let U be a subgroup of the group of the units of S, and assume that U permutes the set $\{e_1, \dots, e_n\}$ transitively by conjugation. Then $S \simeq M_n(e_1Se_1)$.*

The next lemmas are the necessary preparation to get our purpose. For each $e \in G_0$, put $S_e = \{g \in G \mid \ d(g) = e\}$ and $T_e = \{g \in G \mid r(g) = e\}$. Notice that $G_e = S_e \bigcap T_e$, for all $e \in G_0$.

Lemma 3.2. *The following statements hold:*

(i) *B_0 is a unital subalgebra of B, with identity element $w = \sum\limits_{\substack{e\in G_0\\ g\in T_e}} 1_e\delta_e\#v_g$.*

(ii) *$W = \{w_e := 1_e\delta_e\# \sum\limits_{g\in T_e} v_g \mid e \in G_0\}$ is a set of central orthogonal idempotents of B_0.*

(iii) *$B_0 = \bigoplus_{e\in G_0} B_e$, where $B_e = B_0w_e = \sum\limits_{\substack{g\in G_e\\ h\in T_e}} E_g\delta_g\#v_h$ is a unital ideal (so, a subalgebra) of B_0 with identity element w_e.*

(iv) *For each $e \in G_0$, $W_e = \{w_{e,g} := 1_e\delta_e\#v_g \mid g \in T_e\}$ is a set of noncentral orthogonal idempotents of B_e whose sum is w_e.*

Proof. (i) It is clear that B_0 is a K-submodule of B. Given $x = a_g\delta_g\#v_h$ and $y = b_l\delta_l\#v_k$ in B_0 we have

$$xy = \begin{cases} (a_g\delta_g)(v_{hk^{-1}} \cdot b_l\delta_l)\#v_k & \text{if } d(k) = d(h)\\ 0 & \text{otherwise,}\end{cases}$$

and, consequently,

$$xy = \begin{cases} (a_g\delta_g)(b_l\delta_l)\#v_k & \text{if } k = l^{-1}h\\ 0 & \text{otherwise,}\end{cases}$$

Since $d(g) = r(h) = d(l^{-1}) = r(l)$, it follows that

$$(a_g\delta_g)(b_l\delta_l) = a_g\beta_g(b_l)\delta_{gl} \in E_{gl}\delta_{gl}.$$

So, $xy \in B_0$ because $r(gl) = r(g) = r(h) = r(l) = r(k)$ and $d(gl) = d(l) = r(k)$.

Also,

$$\begin{aligned} xw &= \sum_{e\in G_0}(a_g\delta_g\#v_h)(1_e\delta_e\#\sum_{l\in T_e}v_l)\\ &= \sum_{e\in G_0}\sum_{\substack{l\in T_e\\ d(l)=d(h)}} a_g\delta_g(v_{hl^{-1}}\cdot 1_e\delta_e)\#v_l.\\ &= (a_g\delta_g)(1_{r(h)}\delta_{r(h)})\#v_h\\ &= (a_g\delta_g)(1_{d(g)}\delta_{d(g)})\#v_h\\ &= a_g\beta_g(1_{d(g)})\delta_{gd(g)}\#v_h\\ &= a_g\delta_g\#v_h\\ &= x, \end{aligned}$$

and

$$\begin{aligned} wx &= \sum_{e\in G_0}\sum_{l\in T_e}(1_e\delta_e\#v_l)(a_g\delta_g\#v_h)\\ &= \sum_{e\in G_0}\sum_{\substack{l\in T_e\\ d(l)=d(h)}} 1_e\delta_e(v_{lh^{-1}}\cdot a_g\delta_g)\#v_h\\ &= (1_{r(g)}\delta_{r(g)})(a_g\delta_g)\#v_h\\ &= \beta_{r(g)}(a_g)\delta_{r(g)g}\#v_h\\ &= a_g\delta_g\#v_h\\ &= x. \end{aligned}$$

(ii) Let $e,f\in G_0$, $w_e=1_e\delta_e\#\sum_{g\in T_e}v_g$ and $w_f=1_f\delta_f\#\sum_{h\in T_f}v_h$. Then,

$$w_ew_f=\sum_{g\in T_e}\sum_{d(k)=d(g)}1_e\delta_e(v_{gk^{-1}}\cdot 1_f\delta_f)\#v_k(\sum_{h\in T_f}v_h),$$

and so

$$w_ew_f=\begin{cases}\sum\limits_{g\in T_e}\sum\limits_{\substack{h\in T_f\\ d(h)=d(g)}}1_e\delta_e(v_{gh^{-1}}\cdot 1_f\delta_f)\#v_h & \\ 0 & \text{if } d(h)\neq d(g),\ \forall g\in T_e, h\in T_f.\end{cases}$$

Noticing that $v_{gh^{-1}}.1_f\delta_f\neq 0$ if and only if $h=g$ and $e=r(g)=f$, we have $w_ew_e=\sum_{g\in T_e}(1_e\delta_e)(1_e\delta_e)\#v_g=1_e\delta_e\#\sum_{g\in T_e}v_g=w_e$. Therefore,

$$w_ew_f=\begin{cases}w_e & \text{if } e=f\\ 0 & \text{otherwise.}\end{cases}$$

It remains to show that w_e is central in B_0. Take $x=a_g\delta_g\#v_h\in B_0$. Then,

$$xw_e=(a_g\delta_g\#v_h)(1_e\delta_e\#\sum_{l\in T_e}v_l)=\sum_{d(t)=d(h)}a_g\delta_g(v_{ht^{-1}}\cdot 1_e\delta_e)\#v_t(\sum_{l\in T_e}v_l).$$

Observe that if $r(t)\neq e$, for all $t\in G$ such that $d(t)=d(h)$, then $v_t(\sum_{l\in T_e}v_l)=0$. Hence,

$$xw_e=\begin{cases}\sum\limits_{\substack{l\in T_e\\ d(l)=d(h)}}a_g\delta_g(v_{hl^{-1}}.1_e\delta_e)\#v_l & \\ 0 & \text{if } d(h)\neq d(l),\ \forall l\in T_e.\end{cases}$$

Since $v_{hl^{-1}}\cdot 1_e\delta_e\neq 0$ if and only if $l=h$, and $e=r(h)=d(g)$, it follows that

$$\sum_{\substack{l\in T_e\\ d(l)=d(h)}}a_g\delta_g(v_{hl^{-1}}\cdot 1_e\delta_e)\#v_l=(a_g\delta_g)(1_e\delta_e)\#v_h=a_g\beta_g(1_e)\delta_{ge}\#v_h=a_g\delta_g\#v_h=x.$$

Hence,

$$xw_e=\begin{cases}x & \text{if } h\in T_e\\ 0 & \text{otherwise.}\end{cases}$$

By a similar calculation one obtains $w_e x = x w_e$.

(iii) It easily follows from (ii).

(iv) Take $w_{e,g} = 1_e\delta_e\#v_g$ and $w_{e,h} = 1_e\delta_e\#v_h$ in W_e. Then,

$$w_{e,g}w_{e,h} = \begin{cases} 1_e\delta_e(v_{gh^{-1}} \cdot 1_e\delta_e)\#v_h & \text{if } d(h) = d(g) \\ 0 & \text{otherwise.} \end{cases}$$

Since $v_{gh^{-1}} \cdot 1_e\delta_e \neq 0$ if and only if $h = g$, and $r(g) = e$ by assumption, it follows that

$$w_{e,g}w_{e,h} = \begin{cases} w_{e,g} & \text{if } h = g \\ 0 & \text{otherwise.} \end{cases}$$

To see that each $w_{e,l} = 1_e\delta_e\#v_l$ is not central in B_e, take $x = a_g\delta_g\#v_h \in B_e$. Then, $r(g) = d(g) = r(h) = e$ and

$$xw_{e,l} = (a_g\delta_g\#v_h)(1_e\delta_e\#v_l) = \begin{cases} a_g\delta_g(v_{hl^{-1}} \cdot 1_e\delta_e)\#v_l & \text{if } d(l) = d(h) \\ 0 & \text{otherwise,} \end{cases}$$

that easily implies

$$xw_{e,l} = \begin{cases} x & \text{if } l = h \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$w_{e,l}x = (1_e\delta_e\#v_l)(a_g\delta_g\#v_h) = \begin{cases} 1_e\delta_e(v_{lh^{-1}} \cdot a_g\delta_g)\#v_h & \text{if } d(l) = d(h) \\ 0 & \text{otherwise} \end{cases}$$

and, consequently,

$$w_{e,l}x = \begin{cases} x & \text{if } l = gh \\ 0 & \text{otherwise.} \end{cases}$$

The proof is complete. □

Lemma 3.3. *For each $e \in G_0$,*

(i) $U_e = \{u_{g,e} := 1_g\delta_g\#\sum_{h\in T_e} v_h \mid g \in G_e\}$ *is a subgroup of the group of the units of B_e,*

(ii) U_e *acts on W_e by conjugation,*

(iii) $w_{e,h}^{U_e} = \{w_{e,gh} \mid g \in G_e\} = \{w_{e,l} \mid l \in S_{d(h)}\}$ *is the orbit of the element $w_{e,h} \in W_e$, under the action of U_e on W_e.*

Proof. (i) Let $u_{g,e} = 1_g\delta_g\#\sum_{h\in T_e} v_h$ and $u_{l,e} = 1_l\delta_l\#\sum_{k\in T_e} v_k$ be elements of U_e. Then,

$$\begin{aligned} u_{g,e}u_{l,e} &= \sum_{h\in T_e}(1_g\delta_g\#v_h)(1_l\delta_l\#\sum_{k\in T_e} v_k) \\ &= \sum_{h\in T_e}\sum_{\substack{k\in T_e \\ d(k)=d(h)}} 1_g\delta_g(v_{hk^{-1}} \cdot 1_l\delta_l)\#v_k \\ &= \sum_{h\in T_e} 1_g\delta_g 1_l\delta_l\#v_{l^{-1}h} \\ &= 1_{gl}\delta_{gl}\#\sum_{h\in T_e} v_{l^{-1}h} \end{aligned}$$

The last equality follows from the fact that $1_g = 1_{r(g)} = 1_e = 1_{r(l)} = 1_l$ and $1_{gl} = 1_{r(gl)} = 1_{r(g)} = 1_g$. Since $r(l^{-1}h) = r(l^{-1}) = d(l) = e$ and $gl \in G_e$, it follows that $u_{g,e}u_{l,e} \in U_e$.

Finally, taking $u_{g^{-1},e} = 1_{g^{-1}}\delta_{g^{-1}}\#\sum_{l\in T_e} v_l \in U_e$ we have

$$\begin{aligned}
u_{g,e}u_{g^{-1},e} &= (1_g\delta_g\# \sum_{h\in T_e} v_h)(1_{g^{-1}}\delta_{g^{-1}}\# \sum_{l\in T_e} v_l) \\
&= \sum_{h\in T_e} \sum_{\substack{l\in T_e \\ d(l)=d(h)}} 1_g\delta_g(v_{hl^{-1}}\cdot 1_{g^{-1}}\delta_{g^{-1}})\# v_l \\
&= \sum_{h\in T_e} 1_g\delta_g 1_{g^{-1}}\delta_{g^{-1}}\# v_h \quad (\text{since} \quad 1_{r(h)} = 1_e = 1_{d(g)} = 1_{g^{-1}}) \\
&= \sum_{h\in T_e} 1_{r(g)}\delta_{r(g)}\# v_h \\
&= 1_e\delta_e\# \sum_{h\in T_e} v_h \\
&= 1_{B_e}.
\end{aligned}$$

By a similar calculation one also gets $u_{g^{-1},e}u_{g,e} = 1_{B_e}$.

(ii) Taking $w_{e,l} = 1_e\delta_e\# v_l \in W_e$ and $u_{g,e} = 1_g\delta_g\# \sum_{h\in T_e} v_h \in U_e$ we have

$$\begin{aligned}
u_{g,e}w_{e,l} &= (1_g\delta_g\# \sum_{h\in T_e} v_h)(1_e\delta_e\# v_l) \\
&= \sum_{h\in T_e} \sum_{d(t)=d(h)} 1_g\delta_g(v_{ht^{-1}}\cdot 1_e\delta_e)\# v_t v_l \\
&= \sum_{\substack{h\in T_e \\ d(h)=d(l)}} 1_g\delta_g(v_{hl^{-1}}.1_e\delta_e)\# v_l \\
&= 1_g\delta_g 1_e\delta_e\# v_l \\
&= 1_g\delta_g\# v_l.
\end{aligned}$$

Hence,

$$\begin{aligned}
u_{g,e}w_{e,l}u_{g^{-1},e} &= (1_g\delta_g\# v_l)(1_{g^{-1}}\delta_{g^{-1}}\# \sum_{k\in T_e} v_k) \\
&= \sum_{d(t)=d(l)} 1_g\delta_g(v_{lt^{-1}}\cdot 1_{g^{-1}}\delta_{g^{-1}})\# v_t(\sum_{k\in T_e} v_k) \\
&= \sum_{\substack{k\in T_e \\ d(k)=d(l)}} 1_g\delta_g(v_{lk^{-1}}\cdot 1_{g^{-1}}\delta_{g^{-1}})\# v_k \\
&= 1_g\delta_g 1_{g^{-1}}\delta_{g^{-1}}\# v_{gl} \\
&= 1_{r(g)}\delta_{r(g)}\# v_{gl} \\
&= 1_e\delta_e\# v_{gl},
\end{aligned}$$

which belongs to W_e because $r(gl) = r(g) = e$.

(iii) It easily follows from the proof of (ii). □

It follows from the above that any two elements of W_e, say $w_{e,g}$ and $w_{e,h}$, are in the same orbit by the action of U_e if and only if $d(g) = d(h)$. Hence, the action of U_e on W_e in general is not transitive. Nevertheless, W_e contains subsets $\Omega_{e,h_1}, \ldots, \Omega_{e,h_{n_e}}$ and U_e contains subgroups $U_{e,h_1}, \ldots, U_{e,h_{n_e}}$ such that each U_{e,h_i} acts transitively on Ω_{e,h_i} by conjugation, as we shall see in the sequel.

Lemma 3.4. *For each $e \in G_0$, let $w_{e,h_1}^{U_e}, \ldots, w_{e,h_{n_e}}^{U_e}$ be the distinct orbits of U_e in W_e, and ω_{e,h_i} denote the sum of all elements of the orbit $w_{e,h_i}^{U_e}$, for each $1 \le i \le n_e$. Then,*

(i) *$d(h_i) \neq d(h_j)$, for all $i \neq j$,*

(ii) *$\omega_{e,h_i} = 1_e\delta_e\# \sum_{l\in T_e\cap S_{d(h_i)}} v_l$ and $w_e = \sum_{1\le i\le n_e} \omega_{e,h_i}$,*

(iii) *the elements* ω_{e,h_i} *are central orthogonal idempotents of* B_e,

(iv) *for each* $1 \leq i \leq n_e$, $B_{e,h_i} := B_e\omega_{e,h_i} = \bigoplus_{\substack{g\in G_e \\ l\in T_e\bigcap S_{d(h_i)}}} E_g\delta_g\#v_l$ *is a unital ideal (hence, a subalgebra) of* B_e *with identity element* ω_{e,h_i},

(v) $B_e = \bigoplus_{1\leq i\leq n_e} B_{e,h_i}$.

Proof. (i) It follows from Lemma 3.3(iii).

(ii) Immediate.

(iii) Notice that each ω_{e,h_i} is a sum of orthogonal idempotents of W_e, which are all orthogonal. So, it is immediate to check that all the ω_{e,h_i}, $1 \leq i \leq n_e$, are orthogonal.

It remains to show that such idempotents are central in B_e.

Take $x = a_l\delta_l\#v_k \in B_e$. Then, $l \in G_e$, $k \in T_e$ and

$$\begin{aligned}
\omega_{e,h_i}x &= (1_e\delta_e\#\sum_{t\in T_e\bigcap S_{d(h_i)}} v_t)(a_l\delta_l\#v_k)\\
&= \sum_{t\in T_e\bigcap S_{d(h_i)}} (1_e\delta_e\#v_t)(a_l\delta_l\#v_k)\\
&= \sum_{t\in T_e\bigcap S_{d(h_i)}}\ \sum_{d(s)=d(t)} 1_e\delta_e(v_{ts^{-1}}\cdot a_l\delta_l)\#v_sv_k\\
&= \begin{cases} \sum\limits_{t\in T_e\bigcap S_{d(h_i)}} 1_e\delta_e(v_{tk^{-1}}\cdot a_l\delta_l)\#v_k & \text{if } d(k)=d(h_i)\\ 0 & \text{otherwise}\end{cases}\\
&= \begin{cases} a_l\delta_l\#v_k & \text{if } d(k)=d(h_i)\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

since $v_{tk^{-1}}\cdot a_l\delta_l \neq 0$ if and only if $t = lk$ and, in this case, $d(t) = d(lk) = d(k)$ and $r(t) = r(lk) = r(l) = e$.

By a similar calculation one obtains $x\omega_{e,h_i} = \omega_{e,h_i}x$.

(iv) It follows from (iii).

(v) It follows from (ii) and (iii). □

Lemma 3.5. *For each* $e \in G_0$ *and* $1 \leq i \leq n_e$,

(i) $\Omega_{e,h_i} = \{\omega_{e,h_i,l} := 1_e\delta_e\#v_l \mid l \in T_e \cap S_{d(h_i)}\}$ *is a set of noncentral orthogonal idempotents of* B_{e,h_i} *whose sum is* $1_{B_{e,h_i}} = \omega_{e,h_i}$,

(ii) *the set* $U_{e,h_i} = \{u_{g,e,h_i} := 1_g\delta_g\#\sum_{l\in T_e\bigcap S_{d(h_i)}} v_l \mid g \in G_e\}$ *is a subgroup of* U_e,

(iii) U_{e,h_i} *acts transitively on* Ω_{e,h_i} *by conjugation.*

Proof. (i) It follows from Lemma 3.2(iv) and Lemma 3.4(ii)(iv).

(ii) Take $u_{l,e,h_i} = 1_l\delta_l\# \sum_{k\in T_e\bigcap S_{d(h_i)}} v_k$ and $u_{t,e,h_i} = 1_t\delta_t\# \sum_{s\in T_e\bigcap S_{d(h_i)}} v_s$ in U_{e,h_i}. Then,

$$\begin{aligned} u_{l,e,h_i}u_{t,e,h_i} &= (1_l\delta_l\# \sum_{k\in T_e\bigcap S_{d(h_i)}} v_k)(1_t\delta_t\# \sum_{s\in T_e\bigcap S_{d(h_i)}} v_s) \\ &= \sum_{k,s\in T_e\bigcap S_{d(h_i)}} (1_l\delta_l\#v_k)(1_t\delta_t\#v_s) \\ &= \sum_{k,s\in T_e\bigcap S_{d(h_i)}} (1_l\delta_l)(v_{ks^{-1}}\cdot 1_t\delta_t)\#v_s \\ &= \sum_{s\in T_e\bigcap S_{d(h_i)}} (1_l\delta_l)(1_t\delta_t)\#v_s \\ &= 1_{lt}\delta_{lt}\# \sum_{s\in T_e\bigcap S_{d(h_i)}} v_s, \end{aligned}$$

because $l \in G_e$, $t \in T_e$ and $1_l = 1_{r(l)} = 1_{r(lt)} = 1_{lt}$, which implies $(1_l\delta_l)(1_t\delta_t) = \beta_l(1_t)\delta_{lt} = \beta_l(1_{r(t)})\delta_{lt} = \beta_l(1_{d(l)})\delta_{lt} = 1_l\delta_{lt} = 1_{lt}\delta_{lt}$. And such a product belongs to U_{e,h_i} because $r(lt) = r(l) = e = d(t) = d(lt)$.

Finally, it is straightforward to check that $u_{l^{-1},e,h_i} = 1_{l^{-1}}\delta_{l^{-1}}\# \sum_{t\in T_e\bigcap S_{d(h_i)}} v_t$ is the inverse of u_{l,e,h_i}, and clearly it also belongs to U_{e,h_i}.

(iii) Take $u_{g,e,h_i} = 1_g\delta_g\# \sum_{l\in T_e\cap S(d(h_i))} v_l \in U_{e,h_i}$ and $\omega_{e,h_i,k} = 1_e\delta_e\#v_k \in W_{e,h_i}$. By a calculation identical to that done in the proof of the item (ii) of Lemma 3.3 one gets

$$u_{g,e,h_i}\omega_{e,h_i,k}u_{g^{-1},e,h_i} = 1_e\delta_e\#v_{gk},$$

and this action is clearly transitive. □

Proposition 3.6. *$B_{e,h_i} \simeq M_{n_{e,h_i}}(E_e)$ as unital K-algebras, for all $e \in G_0$ and $1 \le i \le n_e$.*

Proof. It follows from Lemmas 3.1 and 3.5 that $B_{e,h_i} \simeq M_{n_{e,h_i}}(C_{e,h_i})$, where

$$C_{e,h_i} = (\omega_{e,h_i,l_1})B_{e,h_i}(\omega_{e,h_i,l_1}),$$

with $l_1 \in T_e\bigcap S_{d(h_i)}$. It remains to prove that C_{e,h_i} is isomorphic to E_e. Recalling from Lemma 3.4 that any element of B_{e,h_i} is of the form $x = \sum_{\substack{g\in G_e\\ k\in T_e\cap S_{d(h_i)}}} a_g\delta_g\#v_k$, we have

$$\begin{aligned} x\omega_{e,h_i,l_1} &= \sum_{\substack{g\in G_e\\ k\in T_e\cap S_{d(h_i)}}} a_g\delta_g(v_{kl_1^{-1}}\cdot 1_e\delta_e)\#v_{l_1} \\ &= \sum_{g\in G_e} (a_g\delta_g)(1_e\delta_e)\#v_{l_1} \\ &= \sum_{g\in G_e} a_g\delta_g\#v_{l_1}. \end{aligned}$$

and

$$\begin{aligned}\omega_{e,h_i,l_1} x \omega_{e,h_i,l_1} &= (1_e\delta_e\#v_{l_1})(\sum_{g\in G_e} a_g\delta_g\#v_{l_1})\\ &= \sum_{g\in G_e}(1_e\delta_e)(v_{l_1 l_1^{-1}}\cdot 1_g\delta_g)\#v_{l_1}\\ &= \sum_{g\in G_e}(1_e\delta_e)(v_e.a_g\delta_g)\#v_{l_1}\\ &= (1_e\delta_e)(a_e\delta_e)\#v_{l_1}\\ &= a_e\delta_e\#v_{l_1}.\end{aligned}$$

Hence, $C_{e,h_i} = E_e\delta_e\#v_{l_1} = E_e(1_A\#v_{l_1})$, which is naturally isomorphic to E_e as K-algebras via the map $a_e \mapsto a_e\delta_e\#v_{l_1} = a_e(1_A\#v_{l_1})$. Observe that this map is surjective by definition and it is straightforward to check that it is a homomorphism of K-algebras. Its injectivity follows from the freeness of $1_A\#v_{l_1}$ over $A = R\star_\beta G$. □

Theorem 3.7.

$$B_0 \simeq \bigoplus_{e\in G_0}(\bigoplus_{i=1}^{n_e} M_{n_{e,h_i}}(E_e))$$

as unital K-algebras.

Proof. It follows from Lemmas 3.2(iii) and 3.4(v), and Proposition 3.6. □

4. Duality for groupoid coactions

In all this section G is a finite groupoid and A is a unital G-graded K-algebra. Recall from the subsection 2.5 that A is a left KG^*-module algebra via the action $v_k\cdot\sum_{g\in G} a_g = a_k$, for all $k\in G$, and there exists an action β of G on the corresponding weak smash product $B = A\#KG^*$. Let $C = B\star_\beta G$ be the corresponding skew groupoid ring. Like in the section 3, also here we obtain a Cohen-Montgomery-type duality theorem, that is, we show that the K-algebra C contains a unital subalgebra isomorphic to a finite direct sum of matrix K-algebras. In particular, if G is a group we recover [9, Theorem 3.5].

The steps to get our purpose are similar to those in the previous section. Recall from Proposition 2.5 that

$$C = \bigoplus_{g\in G}(\bigoplus_{d(k)=r(g)} A_l\#v_k)\delta_g.$$

Let

$$C_0 = \bigoplus_{e\in G_0}(\bigoplus_{g\in G_e}(\bigoplus_{\substack{r(l)=d(l)=r(k)\\ d(k)=e}} A_l\#v_k)\delta_g).$$

Lemma 4.1. *The following statements hold:*

(i) *C_0 is a unital K-algebra with identity element $w = \sum_{e\in G_0}(\sum_{f\in G_0} 1_f\#\sum_{h\in T_f\bigcap S_e} v_h)\delta_e$,*

(ii) *$W = \{w_e := (\sum_{f\in G_0} 1_f\#\sum_{h\in T_f\bigcap S_e} v_h)\delta_e \mid e\in G_0\}$ is a set of central orthogonal idempotents of C_0, whose sum is w,*

(iii) *$C_0 = \bigoplus_{e\in G_0} C_e$, where $C_e = C_0 w_e = \bigoplus_{g\in G_e}(\bigoplus_{\substack{r(l)=d(l)=r(k)\\ d(k)=e}} A_l\#v_k)\delta_g$ is a unital ideal (so, subalgebra) of C_0 with identity element w_e.*

Proof. (i) Clearly, C_0 is a K-submodule of C. Now, take $x = (a_l\#v_k)\delta_g$ and $y = (b_s\#v_t)\delta_h$ in C_0. Then, $r(l) = d(l) = r(k)$, $d(k) = e$, $r(s) = d(s) = r(t)$, $d(t) = f$, $g\in G_e$ and $h\in G_f$, with

$e,f\in G_0$. If $e\neq f$ it is immediate that $xy=0$. Otherwise, we have

$$\begin{array}{rcl} xy & = & (a_l\#v_k)\beta_g(b_s\#v_t)\delta_{gh} \\ & = & (a_l\#v_k)(b_s\#v_{gt^{-1}})\delta_{gh} \\ & = & (a_l(v_{k(tg^{-1})}.b_s)\#v_{tg^{-1}})\delta_{gh} \\ & = & (a_lb_s\#v_{tg^{-1}})\delta_{gh}, \\ & = & (a_lb_s\#v_{s^{-1}k})\delta_{gh} \end{array}$$

if $s=kgt^{-1}$ or, equivalently, $tg^{-1}=s^{-1}k$. Observing that

— $a_lb_s\in A_{ls}$, because $d(l)=r(k)=r(s)$,

— $d(ls)=d(s)$, $r(ls)=r(l)=r(k)=r(s)=d(s)$ and $r(s^{-1}k)=r(s^{-1})=d(s)$,

— $d(s^{-1}k)=d(k)=e=r(g)=r(gh)$, and

— $gh\in G_e$

we conclude that $xy\in C_0$. It remains to prove that $w=\sum\limits_{e\in G_0}(\sum\limits_{f\in G_0}1_f\#\sum\limits_{h\in T_f\bigcap S_e}v_h)\delta_e$ is the identity element of C_0. Indeed, taking $x=(a_l\#v_k)\delta_g$, with $g\in G_{e'}$, for some $e'\in G_0$, $r(l)=d(l)=r(k)$ and $d(k)=e'$, we have

$$\begin{aligned} wx &= \sum_{e\in G_0}(\sum_{f\in G_0}1_f\#\sum_{h\in T_f\bigcap S_e}v_h)\delta_e(a_l\#v_k)\delta_g \\ &= (\sum_{f\in G_0}1_f\#\sum_{h\in T_f\bigcap S_{e'}}v_h)\delta_{e'}(a_l\#v_k)\delta_g \\ &= (\sum_{f\in G_0}1_f\#\sum_{h\in T_f\bigcap S_{e'}}v_h)\beta_{e'}(a_l\#v_k)\delta_g \\ &= (\sum_{f\in G_0}\sum_{h\in T_f\bigcap S_{e'}}(1_f\#v_h)(a_l\#v_k))\delta_g \\ &= (\sum_{f\in G_0}\sum_{h\in T_f\bigcap S_{e'}}1_f(v_{hk^{-1}}\cdot a_l)\#v_k)\delta_g \\ &= (1_{r(l)}a_l\#v_k)\delta_g. \\ &= (a_l\#v_k)\delta_g \quad \text{(by Remark 2.4(v))} \\ &= x \end{aligned}$$

One easily verifies that $xw=x$ by the same way.

(ii) Let $e,e'\in G_0$. It is clear that $w_ew_{e'}=0$ if $e\neq e'$. Otherwise we have

$$\begin{array}{rcl} w_ew_{e'} & = & [(\sum\limits_{f\in G_0}1_f\#\sum\limits_{h\in T_f\bigcap S_e}v_h)\delta_e][(\sum\limits_{f'\in G_0}1_{f'}\#\sum\limits_{l\in T_{f'}\bigcap S_{e'}}v_l)\delta_e] \\ & = & (\sum\limits_{f\in G_0}1_f\#\sum\limits_{h\in T_f\bigcap S_e}v_h)(\sum\limits_{f'\in G_0}1_{f'}\#\sum\limits_{l\in T_{f'}\bigcap S_e}v_l)\delta_e \\ & = & \sum\limits_{f,f'\in G_0}\sum\limits_{\substack{h\in T_f\cap S_e\\ l\in T_{f'}\cap S_e}}(1_f\#v_h)(1_{f'}\#v_l)\delta_e \\ & = & \sum\limits_{f,f'\in G_0}\sum\limits_{\substack{h\in T_f\cap S_e\\ l\in T_{f'}\cap S_e}}1_f(v_{hl^{-1}}\cdot 1_{f'})\#v_l\delta_e \\ & = & \sum\limits_{f\in G_0}1_f\#\sum\limits_{h\in T_f\bigcap S_e}v_h\delta_e \\ & = & w_e \end{array}$$

because $v_{hl^{-1}}\cdot 1_{f'}\neq 0$ if and only if $h=l$, and, in this case, $f=f'$.

It remains to prove that w_e is central in C_0, for all $e\in G_0$. Let $x=(a_l\#v_k)\delta_g\in C_0$, that is, $g\in G_{e'}$, for some $e'\in G_0$, $r(l)=d(l)=r(k)$ and $d(k)=e'$. Again, it is clear that $xw_e=0$ if

$e \neq e'$. Otherwise,

$$\begin{aligned} xw_e &= \sum_{f\in G_0} \sum_{h\in T_f \bigcap S_e} (a_l \# v_k)\beta_g(1_f \# v_h)\delta_{ge} \\ &= \sum_{f\in G_0} \sum_{h\in T_f \bigcap S_e} (a_l \# v_k)(1_f \# v_{hg^{-1}})\delta_{ge} \\ &= \sum_{f\in G_0} \sum_{h\in T_f \bigcap S_e} (a_l(v_{kgh^{-1}} \cdot 1_f) \# v_{hg^{-1}})\delta_g \\ &= (a_l 1_{r(k)} \# v_k)\delta_g \\ &= x \end{aligned}$$

by Remark 2.4(v). Similarly, one also gets $w_e x = x$.

(iii) It is immediate from the above. □

Lemma 4.2. *The following statements hold:*

(i) *For any* $f \in G_0$, $W_f = \{w_{e,f} := (1_f \# \sum_{h\in T_f \bigcap S_e} v_h)\delta_e \mid e \in G_0\}$ *is a set of central orthogonal idempotents of* C_e*, whose sum is* w_e*,*

(ii) *For any* $e \in G_0$, $C_e = \bigoplus_{f\in G_0} C_{e,f}$ *where* $C_{e,f} = C_e w_{e,f} = \bigoplus_{g\in G_e} (\bigoplus_{\substack{l\in G_f \\ k\in T_f \bigcap S_e}} A_l \# v_k)\delta_g$ *is a unital ideal (so, subalgebra) of* C_e*, with identity element* $w_{e,f}$*.*

Proof. (i) For any $e, f, f' \in G_0$ we have

$$\begin{aligned} w_{e,f} w_{e,f'} &= (1_f \# \sum_{h\in T_f \bigcap S_e} v_h)\delta_e (1_{f'} \# \sum_{l\in T_{f'} \cap S_e} v_l)\delta_e \\ &= (1_f \# \sum_{h\in T_f \bigcap S_e} v_h)\beta_e(1_{f'} \# \sum_{l\in T_{f'} \bigcap S_e} v_l)\delta_e \\ &= \sum_{h\in T_f \bigcap S_e} \sum_{l\in T_{f'} \bigcap S_e} (1_f \# v_h)(1_{f'} \# v_{le})\delta_e \\ &= \sum_{h\in T_f \bigcap S_e} \sum_{l\in T_{f'} \bigcap S_e} (1_f \# v_h)(1_{f'} \# v_l)\delta_e \\ &= \sum_{h\in T_f \bigcap S_e} \sum_{l\in T_{f'} \bigcap S_e} (1_f(v_{hl^{-1}} \cdot 1_{f'}) \# v_l)\delta_e \\ &= \begin{cases} (1_f \# \sum_{h\in T_f \bigcap S_e} v_h)\delta_e = w_{e,f} & \text{if } f' = f \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To show that each $w_{e,f}$ is central in C_e take $x = (a_l \# v_k)\delta_g \in C_e$. So, $g \in G_e$, $r(l) = d(l) = r(k)$, $d(k) = e$, and

$$\begin{aligned} xw_{e,f} &= (a_l \# v_k)\beta_g(1_f \# \sum_{h\in T_f \bigcap S_e} v_h)\delta_g \\ &= (a_l \# v_k)(1_f \# \sum_{h\in T_f \bigcap S_e} v_{hg^{-1}})\delta_g \\ &= \sum_{h\in T_f \bigcap S_e} (a_l \# v_k)(1_f \# v_{hg^{-1}})\delta_g \\ &= \sum_{h\in T_f \bigcap S_e} (a_l(v_{kgh^{-1}} \cdot 1_f) \# v_{hg^{-1}})\delta_g \\ &= \begin{cases} (a_l \# v_k)\delta_g = x & \text{if } f = r(k) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $v_{kgh^{-1}}.1_f \neq 0$ if and only if $h = kg$ if and only if $f = r(h) = r(k)$ (Lemma 2.1(ix)). One also gets $w_{e,f}x = xw_{e,f}$ in a similar way.

(ii) It is clear from the above. □

Lemma 4.3. *For each $e, f \in G_0$,*

(i) $W_{e,f} = \{w_{e,f,h} := (1_f \# v_h)\delta_e \mid h \in T_f \bigcap S_e\}$ *is a set of noncentral orthogonal idempotents, with sum* $w_{e,f} = 1_{C_{e,f}}$,

(ii) $U_{e,f} = \{u_{e,f,g} := (1_f \# \sum\limits_{l \in T_f \bigcap S_e} v_l)\delta_g \mid g \in G_e\}$ *is a subgroup of the group of the units of* $C_{e,f}$,

(iii) $U_{e,f}$ *acts transitively on* $W_{e,f}$ *by conjugation.*

Proof. (i) Let $e, f \in G_0$ and $h, l \in T_f \bigcap S_e$. Then,

$$\begin{aligned} w_{e,f,h}w_{e,f,l} &= (1_f \# v_h)\delta_e(1_f \# v_l)\delta_e \\ &= (1_f \# v_h)\beta_e(1_f \# v_l)\delta_e \\ &= (1_f \# v_h)(1_f \# v_l)\delta_e \\ &= (1_f(v_{hl^{-1}} \cdot 1_f) \# v_l)\delta_e \\ &= \begin{cases} (1_f \# v_h)\delta_e = w_{e,f,h} & \text{if } h = l \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, $\sum\limits_{h \in T_e \bigcap S_e} w_{e,f,h} = w_{e,f} = 1_{C_{e,f}}$. Also, $w_{e,f,h}$ is not central in $C_{e,f}$. Indeed, let $x = (a_l \# v_k)\delta_g \in C_{e,f}$. So, $g \in G_e$, $l \in G_f$, $k \in T_f \cap S_e$, and

$$\begin{aligned} w_{e,f,h}x &= (1_f \# v_h)\delta_e(a_l \# v_k)\delta_g \\ &= (1_f \# v_h)\beta_e(a_l \# v_k)\delta_{eg} \\ &= (1_f \# v_h)(a_l \# v_k)\delta_g \\ &= (1_f(v_{hk^{-1}} \cdot a_l) \# v_k)\delta_g \\ &= \begin{cases} x & \text{if } h = lk \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} xw_{e,f,h} &= (a_l \# v_k)\delta_g(1_f \# v_h)\delta_e \\ &= (a_l \# v_k)\beta_g(1_f \# v_h)\delta_{ge} \\ &= (a_l \# v_k)(1_f \# v_{hg^{-1}})\delta_g \\ &= (a_l(v_{kgh^{-1}} \cdot 1_f) \# v_{hg^{-1}})\delta_g \\ &= \begin{cases} x & \text{if } h = kg \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $kgg^{-1}k^{-1} = kr(g)k^{-1} = kek^{-1} = kk^{-1} = r(k) = f$ and $a_l 1_f = a_l$ by Remark 2.4(v).

(ii) Let $e, f \in G_0$ and $g, h \in G_e$. Then,

$$\begin{aligned}
u_{e,f,g}u_{e,f,h} &= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)\delta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_h \\
&= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)\beta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{gh} \\
&= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)(1_f\# \sum_{k\in T_f\bigcap S_e} v_{kg^{-1}})\delta_{gh} \\
&= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)(1_f\# \sum_{t\in T_f\bigcap S_e} v_t)\delta_{gh} \quad (\text{setting } t = kg^{-1}) \\
&= \sum_{l,t\in T_f\bigcap S_e} (1_f(v_{lt^{-1}}\cdot 1_f)\# v_t)\delta_{gh} \\
&= \sum_{l\in T_f\bigcap S_e} (1_f\# v_l)\delta_{gh},
\end{aligned}$$

and this last term belongs to $U_{e,f}$ because $r(gh) = r(g) = e = d(h) = d(gh)$ (Lemma 2.1(vi)). The equality $u_{e,f,g^{-1}} = u_{e,f,g}^{-1}$ is straightforward.

(iii) Let $e, f \in G_0$, $g \in G_e$ and $h \in T_f\bigcap S_e$. Then,

$$\begin{aligned}
u_{e,f,g}w_{e,f,h}u_{e,f,g^{-1}} &= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)\delta_g(1_f\# v_h)\delta_e(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{g^{-1}} \\
&= (1_f\# \sum_{l\in T_f\bigcap S_e} v_l)\beta_g(1_f\# v_h)\delta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{g^{-1}} \\
&= \sum_{l\in T_f\bigcap S_e} (1_f\# v_l)(1_f\# v_{hg^{-1}})\delta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{g^{-1}} \\
&= \sum_{l\in T_f\bigcap S_e} (1_f(v_{l(hg^{-1})^{-1}}\cdot 1_f)\# v_{hg^{-1}})\delta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{g^{-1}} \\
&= (1_f\# v_{hg^{-1}})\delta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{g^{-1}} \\
&= (1_f\# v_{hg^{-1}})\beta_g(1_f\# \sum_{k\in T_f\bigcap S_e} v_k)\delta_{gg^{-1}} \\
&= (1_f\# v_{hg^{-1}})(1_f\# \sum_{k\in T_f\bigcap S_e} v_{kg^{-1}})\delta_e \\
&= \sum_{k\in T_f\bigcap S_e} (1_f(v_{hg^{-1}(kg^{-1})^{-1}}\cdot 1_f)\# v_{kg^{-1}})\delta_e \\
&= \sum_{k\in T_f\bigcap S_e} (1_f(v_{hk^{-1}}\cdot 1_f)\# v_{kg^{-1}})\delta_e \\
&= (1_f\# v_{hg^{-1}})\delta_e,
\end{aligned}$$

and this last term belongs to $W_{e,f}$ since $r(gh^{-1}) = r(h) = f$ and $d(gh^{-1}) = r(g) = e$. One easily sees from the above that this action of $U_{e,f}$ on $W_{e,f}$ is transitive. □

Proposition 4.4. *Let $e, f \in G_0$ and $h \in T_f\bigcap S_e$. Then,*

$$C_{e,f} \simeq M_{n_{e,f}}(\bigoplus_{g\in G_e} A_g),$$

as unital K-algebras, where $n_{e,f}$ denotes the cardinality of the orbit of $w_{e,f,h}$.

Proof. It follows from Lemmas 4.3 and 3.1 that

$$C_{e,f} \simeq M_{n_{e,f}}(S_{e,f}),$$

where $S_{e,f} = w_{e,f,h}C_{e,f}w_{e,f,h}$, for some $w_{e,f,h} \in W_{e,f}$.

Now, note that $S_{e,f} = \bigoplus_{g\in G_e}(A_{hgh^{-1}}\#v_{hg^{-1}})\delta_g$. Indeed, for any $x = (a_l\#v_k)\delta_g \in C_{e,f}$ we have

$$\begin{aligned} xw_{e,f,h} &= (a_l\#v_k)\delta_g(1_f\#v_h)\delta_e \\ &= (a_l\#v_k)\beta_g(1_f\#v_h)\delta_g \\ &= (a_l\#v_k)(1_f\#v_{hg^{-1}})\delta_g \\ &= (a_l(v_{kgh^{-1}}\cdot 1_f)\#v_{hg^{-1}})\delta_g \\ &= \begin{cases} (a_l\#v_{hg^{-1}})\delta_g & \text{if } kg = h \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $l \in G_f$ and $a_l1_f = a_l$ by Remark 2.4(v). Thus, if $kg = h$

$$\begin{aligned} w_{e,f,h}xw_{e,f,h} &= (1_f\#v_h)\delta_e(a_l\#v_{hg^{-1}})\delta_g \\ &= (1_f\#v_h)\beta_e(a_l\#v_{hg^{-1}})\delta_g \\ &= (1_f\#v_h)(a_l\#v_{hg^{-1}})\delta_g \\ &= (1_f(v_{hgh^{-1}}\cdot a_l)\#v_{hg^{-1}})\delta_g \\ &= (a_{hgh^{-1}}\#v_{hg^{-1}})\delta_g, \end{aligned}$$

because $v_{hgh^{-1}}.a_l \neq 0$ if and only if $hgh^{-1} = l$, and $1_fa_l = a_l$ by Remark 2.4(v).

Since $h \in T_f \bigcap S_e$, it is routine to check that the map $g \mapsto hgh^{-1}$ from G_e to G_f is a bijection and induces the isomorphism of K-algebras $\theta_h : \bigoplus_{g\in G_e} A_g \to \bigoplus_{g\in G_e} A_{hgh^{-1}}$ given by $\theta_h(a_g) = a_{hgh^{-1}}$, for all $a_g \in A_g$.

Finally, the map $\gamma : \bigoplus_{g\in G_e} A_g \to \bigoplus_{g\in G_e}(A_{hgh^{-1}}\#v_{hg^{-1}})\delta_g$ given by $\gamma(a_g) = (a_{hgh^{-1}}\#v_{hg^{-1}})\delta_g$, for all $g \in G_e$ and $a_g \in A_g$, is an isomorphism of K-algebras. Indeed, clearly γ is an isomorphism of K-modules (induced by θ_h), and

$$\begin{aligned} \gamma(a_g)\gamma(b_l) &= (a_{hgh^{-1}}\#v_{hg^{-1}})\delta_g(b_{hlh^{-1}}\#v_{hl^{-1}})\delta_l \\ &= (a_{hgh^{-1}}\#v_{hg^{-1}})\beta_g(b_{hlh^{-1}}\#v_{hl^{-1}})\delta_{gl} \\ &= (a_{hgh^{-1}}\#v_{hg^{-1}})(b_{hlh^{-1}}\#v_{hl^{-1}g^{-1}})\delta_{gl} \\ &= (a_{hgh^{-1}}(v_{hg^{-1}(hl^{-1}g^{-1})^{-1}}\cdot b_{hlh^{-1}})\#v_{h(gl)^{-1}})\delta_{gl} \\ &= (a_{hgh^{-1}}(v_{hgh^{-1}}\cdot b_{hgh^{-1}})\#v_{h(gl)^{-1}})\delta_{gl} \\ &= (a_{hgh^{-1}}b_{hlh^{-1}}\#v_{h(gl)^{-1}})\delta_{gl} \\ &= (\theta(a_g)\theta(b_l)\#v_{h(gl)^{-1}})\delta_{gl} \\ &= (\theta(a_gb_l)\#v_{h(gl)^{-1}})\delta_{gl} \\ &= ((a_gb_l)_{h(gl)h^{-1}}\#v_{h(gl)^{-1}})\delta_{gl} \\ &= \gamma(a_gb_l), \end{aligned}$$

for all $a_g \in A_g$ and $b_l \in A_l$. Therefore, we have from the above that

$$S_{e,f} = \bigoplus_{g\in G_e}(A_{hgh^{-1}}\#v_{hg^{-1}})\delta_g \simeq \bigoplus_{g\in G_e} A_g \quad \text{and} \quad C_{e,f} \simeq M_{n_{e,f}}(\bigoplus_{g\in G_e} A_g).$$

as unital K-algebras. □

Theorem 4.5.
$$C_0 \simeq \bigoplus_{e\in G_0}(\bigoplus_{f\in G_0} M_{n_{e,f}}(\bigoplus_{g\in G_e} A_g)),$$
as unital K-algebras.

Proof. It follows by Lemmas 4.1 and 4.2, and Proposition 4.4. □

5. Final remarks

Let G be a finite groupoid, R a unital K-algebra and $\beta = (\{E_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ an action of G on R such that every E_e, $e \in G_0$, is unital and $R = \bigoplus_{e\in G_0} E_e$. By Proposition 2.2 R is a KG-module algebra and we can consider the smash product algebra as in [15], which is constructed as follows:

Given a weak Hopf algebra H, denote by H_t the target counital subalgebra of H defined by $H_t := \{h \in H \mid \varepsilon_t(h) = h\} = \varepsilon_t(H)$, where $\varepsilon_t(h) = \varepsilon(1_1h)1_2$, for every $h \in H$. The algebra H has a natural structure of a left H_t-module via multiplication, and any H-module algebra X is a right H_t-module via the antipode S of H, that is, $x \cdot z := S(x) \cdot x$, for all $x \in X$ and $z \in H_t$. Hence, we can take the K-module $X \otimes_{H_t} H$, which has an structure of a unital K-algebra induced by the multiplication $(x \otimes h)(y \otimes l) = x(h_1 \cdot y) \otimes h_2l$, for all $x, y \in X$ and $h, l \in H$. Its identity element is $1_X \otimes 1_H$. Notice that $X \otimes_{H_t} H$ also is a left H^*-module algebra via $h^* \cdot (x \otimes l) = x \otimes (h^* \rightharpoonup l)$, for all $h^* \in H^*$, $l \in H$ and $x \in X$, so we can also consider the K-algebra $(X \bigotimes_{H_t} H) \bigotimes_{H_t^*} H^*$.

Our intent in this section is to present a natural exact sequence of K-algebras relating $B = (R \star_\beta G)\#KG^*$, as considered in the subsection 2.4, to $A = (R \otimes_{H_t} H) \otimes_{H_t^*} H^*$, in the case that $H = KG$. Observe that KG (resp., KG^*) is a weak Hopf algebra, with antipode given by $S(u_g) = u_{g^{-1}}$ (resp., $S(v_g) = v_{g^{-1}}$). We start with the following proposition. Recall that $T_e = \{g \in G \mid r(g) = e\}$, for all $e \in G_0$.

Proposition 5.1. *There exist a unital subalgebra C of B containing B_0 as subalgebra, and an ideal D of B such that $B = C \bigoplus D$ and $BD = DB = 0$.*

Proof. Notice that
$$B = \bigoplus_{g,h\in G} E_g\delta_g\#v_h = C \bigoplus D, \text{ where } C = \bigoplus_{d(g)=r(h)} E_g\delta_g\#v_h \text{ and } D = \bigoplus_{d(g)\neq r(h)} E_g\delta_g\#v_h.$$
Furthermore, $B_0 = \bigoplus_{r(g)=d(g)=r(h)} E_g\delta_g\#v_h$ is a direct summand of C. It is a routine calculation to check that C (resp., B_0) is a unital subalgebra of B (resp., C) with identity element $\sum_{e\in G_0} 1_e\delta_e\# \sum_{g\in T_e} v_g$ (see Lemma 3.2(i)), as well as D is an ideal of B. We saw in the subsection 2.5 that $BD = 0$. It follows by similar arguments that also $DB = 0$. □

Theorem 5.2. *The natural map $\varphi : B \to A$, given by $a_g\delta_g\#v_h \mapsto a_g \otimes u_g \otimes v_h$, induces the following exact sequence of K-algebras*
$$0 \longrightarrow D \longrightarrow B \longrightarrow \varphi(C) \longrightarrow 0.$$
In particular, B_0 is isomorphic to a subalgebra of A.

To prove this theorem we need first to describe the elements of KG_t and KG_t^*.

Lemma 5.3.
$$KG_t = \bigoplus_{e\in G_0} Ku_e \quad \textit{and} \quad KG_t^* = \sum_{e\in G_0} K(\sum_{h\in T_e} v_h).$$

Proof. An element $x = \sum_{g\in G} \lambda_g u_g \in KG$ satisfies $\varepsilon_t(x) = x$ if and only if

$$\begin{aligned}\sum_{g\in G} \lambda_g u_g &= \sum_{g\in G} \lambda_g \varepsilon_t(u_g) = \sum_{g\in G} \lambda_g \sum_{e\in G_0} \varepsilon(u_e u_g) u_e \\ &= \sum_{g\in G} \lambda_g \varepsilon(u_{r(g)g}) u_{r(g)} = \sum_{g\in G} \lambda_g u_{r(g)},\end{aligned}$$

if and only if $\lambda_g = 0$, para all $g \notin G_0$.

An element $x = \sum_{g\in G} \lambda_g v_g \in KG^*$ satisfies $\varepsilon_t(x) = x$ if and only if

$$\begin{aligned}\sum_{g\in G} \lambda_g v_g &= \sum_{g\in G} \lambda_g \varepsilon_t(v_g) \\ &= \sum_{g\in G} \lambda_g (\sum_{h\in G} \sum_{\substack{l\in G \\ d(l)=d(h)}} \varepsilon(v_{hl^{-1}} v_g) v_l) \\ &= \sum_{g\in G} \lambda_g (\sum_{\substack{h\in G \\ r(h)=r(g)}} \varepsilon(v_g) v_{g^{-1}h}) \\ &= \sum_{g\in G_0} \lambda_g (\sum_{h\in T_g} v_h)\end{aligned}$$

if and only if $\lambda_h = \lambda_g$, for all $g \in G_0$ and $h \in T_g$. □

Proof of Theorem 5.2:

It is straightforward to check that the map $\varphi : B \to A$, given by $\varphi(a_g\delta_g \# v_h) = a_g \otimes u_g \otimes v_h$, is a well defined homomorphism of K-algebras. Furthermore, the preunit $1_{R\star_\beta G}\#1_{KG^*}$ of B is taken by φ onto the identity element $1_R \otimes 1_{KG} \otimes 1_{KG^*}$ of A. Indeed,

$$\varphi(1_{R\star_\beta G}\#1_{KG^*}) = \varphi(\sum_{e\in G_0} 1_e\delta_e \# \sum_{g\in G} v_g) = \sum_{e\in G_0}\sum_{g\in G} 1_e\#u_e\#v_g.$$

And, on the other hand, since $KG_t = \bigoplus_{e\in G_0} Ku_e$ (Lemma 5.3), we have

$$\begin{aligned}1_R\#1_{KG}\#1_{KG^*} &= \sum_{e\in G_0} 1_e\# \sum_{f\in G_0} u_f \# \sum_{g\in G} v_g = \sum_{e,f\in G_0} 1_e.u_f\#u_f\#\sum_{g\in G} v_g \\ &= \sum_{e,f\in G_0} S(u_f).1_e\#u_f\#\sum_{g\in G} v_g = \sum_{e,f\in G_0} u_f.1_e\#u_f\#\sum_{g\in G} v_g \\ &= \sum_{e,f\in G_0} \beta_f(1_e 1_f)\#u_f\#\sum_{g\in G} v_g = \sum_{e\in G_0} 1_e\#u_e\#\sum_{g\in G} v_g,\end{aligned}$$

This implies that the ideal D of B is contained in the kernel of φ because $(1_{R\star_\beta G}\#1_{KG^*})D = 0$ (Proposition 5.1) and so

$$0 = \varphi((1_{R\star_\beta G}\#1_{KG^*})D) = \varphi(1_{R\star_\beta G}\#1_{KG^*})\varphi(D) = 1_A\varphi(D) = \varphi(D).$$

From this we also have $\varphi(B) = \varphi(C) = \bigoplus_{d(g)=r(h)} E_g \bigotimes_{KG_t} Ku_g \bigotimes_{KG_t^*} Kv_h$.

To end this proof it is enough to show that the K-algebras $\varphi(C)$ and B/D are isomorphic. For this, take the map $\psi : \bigoplus_{d(g)=r(h)} E_g \times Ku_g \times Kv_h \to B/D$, given by $\psi(a_g, u_g, v_h) = \overline{a_g\delta_g\#v_h}$.

Notice that for $x = \sum_{e\in G_0} \lambda_e u_e \in KG_t$ we have

$$\begin{aligned}\psi(a_g.x, u_g, v_h) &= \psi(S(x).a_g, u_g, v_h) = \psi((\sum_{e\in G_0} \lambda_e u_e).a_g, u_g, v_h)\\ &= \psi(\sum_{e\in G_0} \lambda_e \beta_e(a_g 1_e), u_g, v_h)\\ &= \psi(\lambda_{r(g)} a_g, u_g, v_h) = \overline{\lambda_{r(g)} a_g \delta_g \# v_h}\end{aligned}$$

and

$$\psi(a_g, xu_g, v_h) = \psi(a_g, \sum_{e\in G_0} \lambda_e u_e u_g, v_h) = \psi(a_g, \lambda_{r(g)} u_g, v_h) = \overline{\lambda_{r(g)} a_g \delta_g \# v_h}.$$

Hence, ψ is KG_t-balanced.

Furthermore, for $x = \sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_l) \in KG_t^*$ we also have

$$\begin{aligned}\psi(a_g, u_g.x, v_h) &= \psi(a_g, S(x).u_g, v_h)\\ &= \psi(a_g, S(\sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_l)).u_g, v_h)\\ &= \psi(a_g, (\sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_{l^{-1}})).u_g, v_h)\\ &= \psi(a_g, \sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_{l^{-1}}.u_g), v_h)\\ &= \psi(a_g, \lambda_{d(g)} u_g, v_h)\\ &= \psi(a_g, \lambda_{d(g)} u_g, v_h)\\ &= \overline{\lambda_{d(g)} a_g \delta_g \# v_h}\end{aligned}$$

and

$$\begin{aligned}\psi(a_g, u_g, x.v_h) &= \psi(a_g, u_g, (\sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_l))v_h)\\ &= \psi(a_g, u_g, \sum_{e\in G_0} \lambda_e (\sum_{l\in T_e} v_l v_h))\\ &= \psi(a_g, u_g, \lambda_{r(h)} v_h)\\ &= \psi(a_g, u_g, \lambda_{r(h)}) v_h\\ &= \overline{\lambda_{r(h)} a_g \delta_g \# v_h}\\ &= \overline{\lambda_{d(g)} a_g \delta_g \# v_h},\end{aligned}$$

Thus, ψ also is KG_t^*-balanced. Therefore, ψ induces a K-linear map $\widetilde{\psi}$ from $\varphi(C)$ into B/D. It is immediate to see that this map is the inverse of the K-algebra homomorphism $\widetilde{\varphi}$ from B/D onto $\varphi(C)$ induced by φ. □

References

[1] D. Bagio, D. Flôres and A. Paques, *Partial actions of ordered groupoids on rings*, J. Algebra Appl. 9 (2010), 501-517.

[2] D. Bagio and A. Paques, *Partial groupoid actions: globalization, Morita theory and Galois theory*, Comm. Algebra 40 (2012), 3658-3678.

[3] M. van den Bergh, *A duality theorem for Hopf algebras*, Methods in Ring Theory, (Antwerp, 1983), 517-522.

[4] R. Blattner and S. Montgomery, *A duality theorem for Hopf module algebras*, J. Algebra 95 (1985), 153-172.

[5] G. Böhm, *Doi-Hopf modules over weak Hopf algebras*, Comm. Algebra 28 (2000), 4687-4698.

[6] G. Böhm, F.Nill and K. Szlachányi, *Weak Hopf algebra I. Integral theory and $C^\star$-structure*, J. Algebra 221 (1999), 385-438.

[7] S. Caenepeel and E. De Groot, *Modules over weak entwining structures*, Contemporary Math. 267 (2000), 31-54.
[8] S. Caenepeel and E. De Groot, *Galois theory for weak Hopf algebras*, Rev. Roumaine Math. Pures Appl. 52 (2007), 151-176.
[9] M. Cohen and S. Montgomery, *Group-graded rings, smash products and group actions*, Trans. AMS 282 (1984), 237-258.
[10] M. Dokuchaev and R. Exel, *Associativity of crossed products by partial actions, enveloping actions and partial representations*, Trans. AMS 357 (2005), 1931-1952.
[11] N.D. Gilbert, *Actions and expansions of ordered groupoids*, J. Pure Appl. Algebra 198 (2005), 175-195.
[12] M.V. Lawson, *Inverse Semigroups. The Theory of Partial Symmetries*, World Scientific Pub. Co, London, 1998.
[13] G. Liu and F. Li, *On strongly groupoid graded rings and the corresponding Clifford Theorem*, Algebra Colloquium, 13 (2006), 181-196.
[14] P. Lundström, *Separable groupoid rings*, Comm. Algebra 34 (2006), 3029-3041.
[15] D. Nikshych, *A duality theorem for quantum groups*, Contemporary Math. 267 (2000), 237-243.
[16] D. S. Passman, *The Algebraic Structure of Finite Groups*, Wiley-Interscience, NY, 1977.

Departamento de Matemática, Universidade Federal de Santa Maria, 97105-900, Santa Maria, RS, Brazil
E-mail address: flores@ufsm.br

Instituto de Matemática, Universidade Federal do Rio Grande do Sul, 91509-900, Porto Alegre, RS, Brazil
E-mail address: paques@mat.ufrgs.br