arXiv:quant-ph/9903058v2 9 Apr 1999

Excited Binomial States and Excited Negative Binomial States of the Radiation Field and Some of their Statistical Properties

December 19, 2013

Xiao-Guang Wang[a1] and **Hong-Chen Fu**[b]
a Laboratory of Optical Physics,Institute of Physics,Chinese Academy of Sciences,Beijing,100080,P.R.China
b Institute of Theoretical Physics, Northeast Normal University,Changchun,130024,P.R.China

Abstract: We introduce excited binomial states and excited negative binomial states of the radiation field by repeated application of the photon creation operator on binomial states and negative binomial states. They reduce to Fock states and excited coherent states in certain limits and can be viewed as intermediate states between Fock states and coherent states. We find that both the excited binomial states and excited negative binomial states can be exactly normalized in terms of hypergeometric functions. Base on this interesting character, some of the statistical properties are discussed.

Keywords: excited binomial states, excited negative binomial states, intermediate states, statistical properties.

PACS numbers:42.50.Dv,03.65.Db,32.80.Pj,42.50.Vk

[1]e-mail:xyw@aphy.iphy.ac.cn

1 Introduction

The interesting non-classical states which are engendered by excitations on particular quantum states have been examined. For the first time these states were introduced by Agarwal and Tara as the excited coherent states (ECS)[1]. The ECS exhibit remarkable non-classical properties such as sub-Poisonnian photon statistics and squeezing in one of the quadratures of the radiation field, etc. Several other excited quantum states, even and odd ECS[2], excited squeezed states [3-6] and excited thermal states[7,8] have been discussed in the literature. It was shown that some of the excited quantum states can be prepared in the processes of the field-atom interaction in a cavity[1] or via conditional measurement on a beam splitter[9].

In this work, we introduce the excited binomial states (EBS) and excited negative binomial states (ENBS) of the radiation field by repeated application of the photon creation operator on binomial states (BS)[10-15] and negative binomial states (NBS)[16-22]. We examine the mathematical and physical properties of such states. They interpolate between Fock states and coherent states in the sense that they reduce to Fock states and coherent states in different limits. An interesting property is found that both the EBS and ENBS are exactly normalized in terms of hypergeometric functions[23]. These are demonstrated in section 2. In section 3, we study the sub-Poissonian statistics and squeezing effects of these states. The conclusion is given in section 4.

2 EBS and ENBS

We introduce the EBS $|k,\eta,M\rangle$ and ENBS $|k,\eta,M\rangle^{-}$ defined by

$$|k,\eta,M\rangle = \mathcal{N}(k,\eta,M)a^{\dagger k}|\eta,M\rangle, \tag{1}$$

$$|k,\eta,M\rangle^{-} = \mathcal{N}^{-}(k,\eta,M)a^{\dagger k}|\eta,M\rangle^{-}, \tag{2}$$

where

$$|\eta,M\rangle = \sum_{n=0}^{M} C_n(\eta,M)|n\rangle = \sum_{n=0}^{M}\binom{M}{n}^{1/2}\eta^n(1-\eta^2)^{(M-n)/2}|n\rangle, \tag{3}$$

$$|\eta,M\rangle^{-} = \sum_{n=0}^{\infty} C_n^{-}(\eta,M)|n\rangle = \sum_{n=0}^{\infty}\binom{M+n-1}{n}^{1/2}\eta^n(1-\eta^2)^{M/2}|n\rangle \tag{4}$$

are the BS and NBS, respectively. Here k,n and M are integers, η is real number, $|n\rangle$ is Fock state , $a^{\dagger}$ and its conjugate a are boson creation and annihilation operators, $\mathcal{N}(k,\eta,M)$ and $\mathcal{N}^{-}(k,\eta,M)$ are normalization constants of the EBS and ENBS ,respectively.

In two limits, $\eta \to 1$ and $\eta \to 0$, the BS reduce to Fock state $|M\rangle = |1, M\rangle$ and vacuum state $|0\rangle = |0, M\rangle$, and the EBS to Fock states $|M+k\rangle$ and $|k\rangle$, respectively. In a different limit of $M \to \infty, \eta \to 0$ with $\eta^2 M = \alpha^2$ fixed(α is real) , the BS reduce to coherent states $|\alpha\rangle$ and the EBS to the ECS. In the limit of $\eta \to 0$, the NBS reduce to vacuum state and ENBS to the Fock state $|k\rangle$. When $M \to \infty, \eta \to 0$ with $\eta^2 M = \alpha^2$ fixed, the NBS reduce to coherent states and the ENBS to the ECS. Thus, both the EBS and ENBS can be reduced to the Fock states and ECS. Note that the ECS can be reduced to coherent states, the two excited states can be recognized as intermediate states between Fock states and coherent states.

In determining the normalization constant of the EBS, we calculate the expectation value of operator $a^k a^{\dagger k}$ on the BS,

$$\begin{aligned} B(k,\eta,M) &= \langle \eta, M | a^k a^{\dagger k} | \eta, M \rangle \\ &= M!(1-\eta^2)^M \sum_{n=0}^{M} \frac{(n+k)!}{n!n!(M-n)!} \left(\frac{\eta^2}{1-\eta^2}\right)^n \\ &= M!\eta^{2M} \sum_{n=0}^{M} \frac{(M+k-n)!}{n!(M-n)!(M-n)!} \left(\frac{1-\eta^2}{\eta^2}\right)^n \\ &= \eta^{2M}[(M+k)!/M!]_2F_1(-M,-M;-M-k;(\eta^2-1)/\eta^2), \end{aligned} \tag{5}$$

where ${}_2F_1(\alpha,\beta;\gamma;x)$ is the hypergeometric function[23]. The normalization constant $\mathcal{N}(k,\eta,M) = 1/\sqrt{B(k,\eta,M)}$.

Using the same procedure as above, we give the expectation value of operator $a^k a^{\dagger k}$ on NBS

$$\begin{aligned} B^-(k,\eta,M) &= {}^-\langle \eta, M | a^k a^{\dagger k} | \eta, M \rangle^- \\ &= \frac{(1-\eta^2)^M}{(M-1)!} \sum_{n=0}^{\infty} \frac{(M+n-1)!(n+k)!}{n!n!} \eta^{2n}. \end{aligned} \tag{6}$$

It is a infinite sum. The normalization constant of the ENBS $\mathcal{N}^-(k,\eta,M) = 1/\sqrt{B^-(k,\eta,M)}$.

Now we try to give a different form of $B^-(k,\eta,M)$ as a finite sum. The normal ordering of the operator $a^k a^{\dagger k}$ is

$$a^k a^{\dagger k} = \sum_{l=0}^{k} \frac{k!k! a^{\dagger(k-l)} a^{k-l}}{l!(k-l)!(k-l)!}. \tag{7}$$

It is easy to evaluate from Eq.(4) that

$$a^k |\eta, M\rangle^- = \left(\frac{\eta}{\sqrt{1-\eta^2}}\right)^k \sqrt{\frac{(M+k-1)!}{(M-1)!}} |\eta, M+k\rangle^- \tag{8}$$

$${}^-\langle \eta, M | a^{\dagger k} a^k | \eta, M \rangle^- = \left(\frac{\eta^2}{1-\eta^2}\right)^k \frac{(M+k-1)!}{(M-1)!}. \tag{9}$$

From Eq.(7) and Eq.(9), we have

$$B^{-}(k,\eta,M)=\sum_{l=0}^{k}\frac{k!k!(M+k-l-1)!}{l!(k-l)!(k-l)!(M-1)!}\left(\frac{\eta^2}{1-\eta^2}\right)^{k-l}. \tag{10}$$

The above equation can be written in terms of hypergeometric functions as

$$\begin{aligned} B^{-}(k,\eta,M) &= \left(\frac{\eta^2}{1-\eta^2}\right)^{k}\frac{(M+k-1)!}{(M-1)!} \\ &\quad {}_2F_1(-k,-k;-M-k+1;(\eta^2-1)/\eta^2). \end{aligned} \tag{11}$$

It is interesting that the normalization constants of both the EBS and ENBS can be expressed in terms of hypergeometric function. The expressions are useful in the following investigations of non-classcal properties of the two excited states.

3 Non-classical properties

We expand the EBS and ENBS in terms of Fock states as

$$\begin{aligned} |k,\eta,M\rangle &= \sum_{n=k}^{M+k} D_n(k,\eta,M)|n\rangle \\ &= \mathcal{N}(k,\eta,M)\sum_{n=k}^{M+k} C_{n-k}(\eta,M)\sqrt{\frac{n!}{(n-k)!}}|n\rangle \end{aligned} \tag{12}$$

$$\begin{aligned} |k,\eta,M\rangle^{-} &= \sum_{n=k}^{\infty} D_n^{-}(k,\eta,M)|n\rangle \\ &= \mathcal{N}^{-}(k,\eta,M)\sum_{n=k}^{\infty} C_{n-k}^{-}(\eta,M)\sqrt{\frac{n!}{(n-k)!}}|n\rangle. \end{aligned} \tag{13}$$

It can be seen that the Fock states $|0\rangle,|1\rangle,...,|k-1\rangle$ are removed from the BS and NBS.

3.1 Photon statistics

The mean photon number of the EBS are given by

$$\begin{aligned} \langle k,\eta,M|a^{\dagger}a|k,\eta,M\rangle &= \langle k,\eta,M|aa^{\dagger}|k,\eta,M\rangle-1 \\ &= \frac{B(k+1,\eta,M)}{B(k,\eta,M)}-1. \end{aligned} \tag{14}$$

The mean value of the operator $(a^{\dagger}a)^2$ can be obtained by expressing it in the anti-normally ordered form $(a^{\dagger}a)^2=a^2a^{\dagger 2}-3aa^{\dagger}+1$ as

$$\langle k,\eta,M|(a^{\dagger}a)^2|k,\eta,M\rangle=\frac{B(k+2,\eta,M)}{B(k,\eta,M)}-3\frac{B(k+1,\eta,M)}{B(k,\eta,M)}+1. \tag{15}$$

Mandel's Q parameter defined by

$$Q = \frac{\langle (a^\dagger a)^2 \rangle - \langle a^\dagger a \rangle^2}{\langle a^\dagger a \rangle} - 1 \tag{16}$$

measures the deviation from the Poisson distribution (the coherent states, $Q = 0$). If $Q < 0 (> 0)$, the field is called sub(super)-Poissonian, respectively.

From Eq.(14),(15) and (16), we can easily derive Mandel's Q parameter

$$\begin{aligned} Q &= [B(k+2,\eta,M)B(k,\eta,M) - B(k+1,\eta,M)B(k,\eta,M) - \\ &\quad B(k+1,\eta,M)^2]/[B(k+1,\eta,M)B(k,\eta,M) - B(k,\eta,M)^2] - 1. \end{aligned} \tag{17}$$

Let $B \to B^-$ in Eq.(14), (15) and (17), we can obtain the expectation values of $a^\dagger a, (a^\dagger a)^2$ on the ENBS and the corresponding Q parameter.

In Fig.3 we plot the Q parameter of the EBS as a function of η for different values of k. For $k = 0$, the EBS become the BS and $Q = -\eta^2$[13]. Q parameters of the EBS $|k, 0, M\rangle$ and $|k, 1, M\rangle$ are equal to -1 for $k \neq 0$. This is easily understood for the states $|k, \eta, M\rangle$ reduce to Fock states $|k\rangle$ and $|k + M\rangle$ in the limits $\eta \to 0$ and $\eta \to 1$ and the Q parameter of Fock states $|n\rangle (n \neq 0)$ are equal to -1. We see that the field in the EBS exhibit a significant amount of sub-Poissonian statistics. The Q parameters of the ENBS are displayed in Fig.2 as a function of η for different values of k.For $k = 0$, the ENBS becomes the NBS. The Q parameter of the NBS is equal to $\eta^2/(1-\eta^2)$[20] and the NBS always show super-Poissonian statistics except for $\eta = 0$. For $k \neq 0$ and $\eta \to 0$, the ENBS $|k, 0, M\rangle^-$ becomes Fock states $|k\rangle$ and $Q = -1$. Like the EBS, the ENBS also shows a significant amount of sub-Possonian statistics. From Fig.1 and Fig.2, we find that the sub-Poissonian statistics of both the EBS and ENBS are enhanced as k increases.

3.2 Squeezing effects

From Eq.(12) and (13), the expectation values of operator a and a^2 on the EBS and ENBS are obtained as

$$\begin{aligned} \langle k,\eta,M|a|k,\eta,M\rangle &= \sum_{n=k}^{M+k-1} \sqrt{n+1} D_n(k,\eta,M) D_{n+1}(k,\eta,M) \\ \langle k,\eta,M|a^2|k,\eta,M\rangle &= \sum_{n=k}^{M+k-2} \sqrt{(n+2)(n+1)} D_n(k,\eta,M) D_{n+2}(k,\eta,M) \\ {}^-\langle k,\eta,M|a|k,\eta,M\rangle^- &= \sum_{n=k}^{\infty} \sqrt{n+1} D_n^-(k,\eta,M) D_{n+1}^-(k,\eta,M) \\ {}^-\langle k,\eta,M|a^2|k,\eta,M\rangle^- &= \sum_{n=k}^{\infty} \sqrt{(n+2)(n+1)} D_n^-(k,\eta,M) D_{n+2}^-(k,\eta,M). \end{aligned} \tag{18}$$

Define the quadrature operators x (coordinate) and p(momentum) by

$$x = \frac{1}{2}(a^\dagger + a), p = \frac{i}{2}(a^\dagger - a) \tag{19}$$

In the present case $\langle a \rangle$ and $\langle a^2 \rangle$ are real. Thus, the variances of x and p are

$$\begin{aligned} Var(x) &= \frac{1}{4} + \frac{1}{2}(\langle a^\dagger a \rangle + \langle a^2 \rangle - 2\langle a \rangle^2), \\ Var(p) &= \frac{1}{4} + \frac{1}{2}(\langle a^\dagger a \rangle - \langle a^2 \rangle). \end{aligned} \tag{20}$$

Combining Eq.(14),(18) and (20), we can study the squeezing effects of the EBS and ENBS.

In Fig.3 we show the quantity $Var(x)$ of the EBS as a function of η for different values of k .The squeezing occurs in the quadrature component x over a wide range of parameters. As k increases, the range of squeezing becomes narrow and the degree of squeezing is reduced. Fig.4 gives the quantity $Var(p)$ of the ENBS as a function of η for different values of k. The ENBS exhibit squeezing in the quadrature component p .As the EBS, the range of squeezing becomes narrow and the degree of squeezing is reduced as k increases.

4 Conclusions

We have introduced and investigated the EBS and ENBS and found the following:

(1) The EBS and ENBS can be reduced to the Fock states and ECS. As intermediate states, both the EBS and ENBS interpolate between the Fock states and coherent states.

(2)The EBS and ENBS are exactly normalized in terms of hypergeometric functions.

(3)The EBS always shows sub-Poissonian statistics. NBS always shows super-Poissonian statistics, but the ENBS can be sub-Poissonian. The sub-Poissonian statistics are enhanced as k increases for both the EBS and ENBS. In contrary to this, the squeezing properties of the two excited states are reduced and the range of squeezing becomes narrower as k increases. The squeezing occurs in the quadrature component x of the EBS and in the quadrature component p of the ENBS over a wide range of parameters.

The BS and NBS can be generated in some nonlinear processes[11,12,20]. Then, by a similar production mechanism of the ECS[1], let the excited atoms pass through a cavity and provide the field in the cavity is initially in a binomial state or a negative binomial state, one can produce the EBS and ENBS.

Acknowledgement: This work is supported in part by the National Science Foundation of China.

References

[1] G.S.Agarwal and K.Tara,Phys.Rev.A43(1991)492.

[2] V.V.Dodonov, Ya.A.Korennoy, V.I.Man'ko, and Y.A.Moukhin,Quantum Semiclass.Opt.8(1996)413.

[3] Z.Zhang and H.Fan, Phys.Lett.A165(1992)14.

[4] D.-G.Welsch,M.Dakna.L.knöll and T.Opatrny,
e-print quant-ph/9708018.

[5] Zs.Kis, P.Adam, J.Jansky, Phys.Lett. A188(1994)16.

[6] V.I.Man'ko and A.Wübsche, Quantum Semiclass.Opt.9(1997)381.

[7] G.S.Agarwal and K.Tara,Phys.Rev.A46(1992)485.

[8] G.N.Jones, J.Haight, and C.T.Lee, Quantum Semiclass.Opt.9(1997)411.

[9] M.Dakna, L.knöll and D.-G.Welsch, Opt.Commun. 145(1998)309.

[10] D.Stoler,B.E.A.Saleh and M.C.Teich,Opt.Acta.32(1985)345.

[11] C.T.Lee,Phys.Rev.A31(1985)1213.

[12] G.Dattoli,J.Gallardo and A.Torre,J.Opt.Soc.Am.B2(1987)185.

[13] A.V.Barranco and J.Roversi,Phys.Rev.A50(1994)5233.

[14] H.Y.Fan, Commun.Theor.Phys.24(1995)483.

[15] H.C.Fu and R.Sasaki,J.Phys.A:Math.Gen.29(1996)5637.

[16] K.Matsuo,Phys.Rev.A41(1990)519.

[17] V.Vourdas and R.F.Bishop,Phys.Rev.A51(1995)2353.

[18] R.Srinivasan and C.T.Lee,Phys.Lett.A218(1996)151.

[19] A.Joshi and A.S.F.Obada,J.Phys.A:Math.Gen.30(1997)81.

[20] H.C.Fu and R.Sasaki,J.Phys.Soc.Japan,66(1997)1989.

[21] H.C.Fu and R.Sasaki,J.Math.Phys.38(1997)3968.

[22] S.M.Barnett, J.Mod.Opt. 45(1998)2201.

[23] I.S.Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).

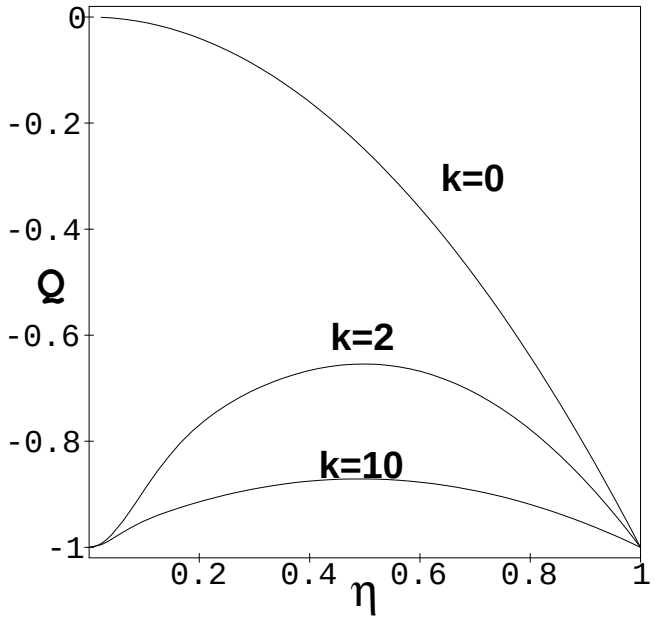

Figure 1: Q parameter of EBS as a function of η for different values of $k(M = 10)$.

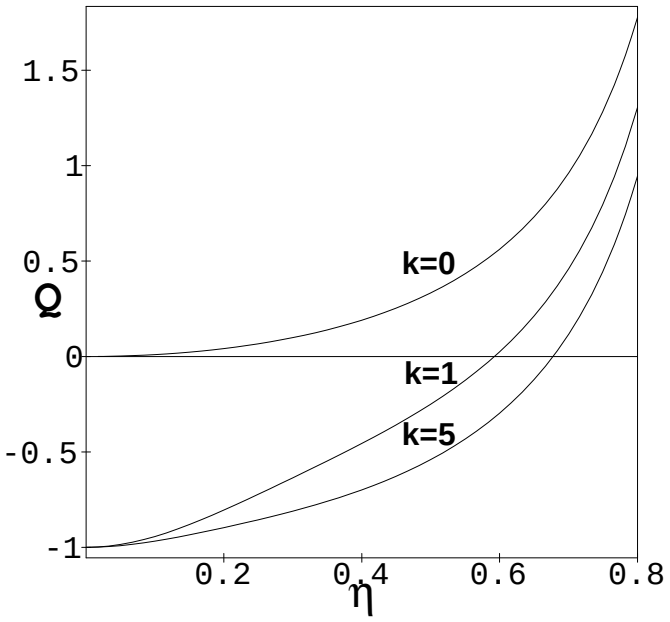

Figure 2: Q parameter of ENBS as a function of η for different values of $k(M = 3)$.

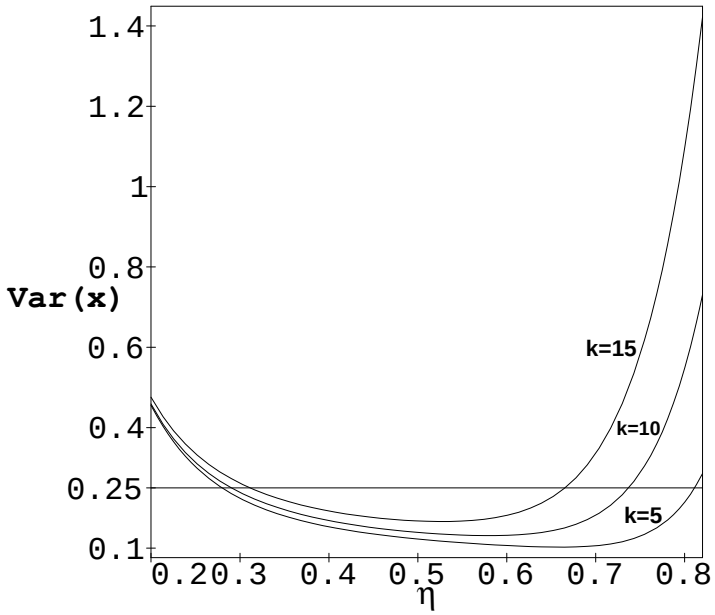

Figure 3: The variance the quatrature component x of EBS as a function of η for different values of $k(M = 10)$.

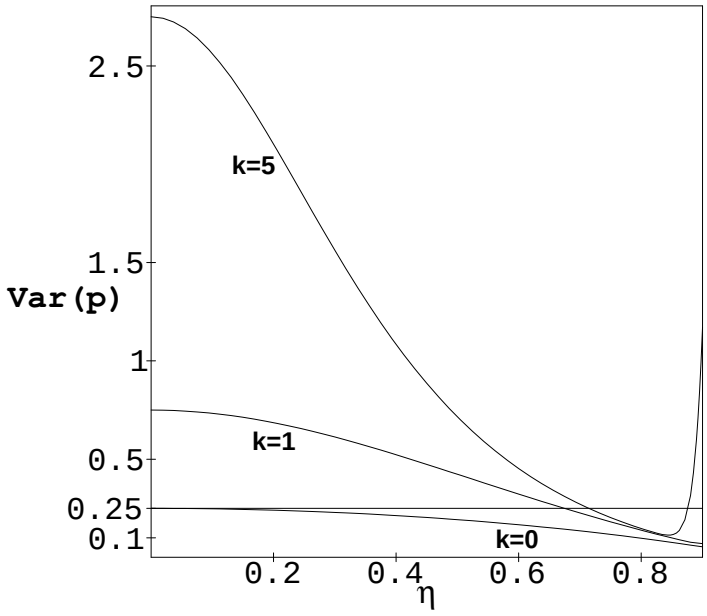

Figure 4: The variance the quatrature component p of ENBS as a function of η for different values of $k(M = 2)$

.