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Estimates of solutions for the parabolic p-Laplacian equation with measure via parabolic nonlinear potentials

Vitali Liskevich
Department of Mathematics
Swansea University
Swansea SA2 8PP, UK
v.a.liskevich@swan.ac.uk

Igor I. Skrypnik
Institute of Applied
Mathematics and Mechanics
Donetsk 83114, Ukraine
iskrypnik@iamm.donbass.com

Zeev Sobol
Department of Mathematics
Swansea University
Swansea SA2 8PP, UK
z.sobol@swan.ac.uk

Abstract

For weak solutions to the evolutional p-Laplace equation with a time-dependent Radon measure on the right hand side we obtain pointwise estimates via a nonlinear parabolic potential.

1 Introduction and main results

In this note we give a parabolic extension of a by now classical result by Kilpeläinen-Malý estimates [9], who proved pointwise estimates for solutions to quasi-linear p-Laplace type elliptic equations with measure in the right hand side, in terms of the (truncated) non-linear Wolff potential $W^{\mu}_{\beta,p}(x,R)$ of the measure,

$$W^{\mu}_{\beta,p}(x,\rho)=\sum_{j=0}^{\infty}\left(\frac{\mu(B_{\rho_j}(x))}{\rho_j^{N-\beta p}}\right)^{\frac{1}{p-1}},\ \rho_j:=2^{-j}\rho,\ j=0,1,2,\dots \tag{1.1}$$

These estimates were subsequently extended to fully nonlinear equations by Labutin [10] and fully nonlinear and subelliptic quasi-linear equations by Trudinger and Wang [17]. The pointwise estimates proved to be extremely useful in various regularity and solvability problems for quasilinear and fully

nonlinear equations [9, 10, 14, 15, 17]. For the parabolic equations the corresponding result was recently given in [5, 6] for the case $p = 2$, and by the authors in [12] for the case $p > 2$ and the measure on the right hand side depending on the spatial variable only. One of the main difficulties in the time dependent measure case is that of identifying the right analogue of the elliptic Wolff potential corresponding to p-Laplacian.

It is the aim of this note to introduce a parabolic version of the Wolff potential and in terms of this newly defined potential to establish pointwise estimates for solutions to parabolic equations in the degenerate case $p \geq 2$ with the time-dependent measures on the right hand side. The form of the parabolic potential introduced in the note is such that it reduces to the truncated Wolff potential if the measure does not depend on time, and it reduces to the truncated Riesz potential in the case $p = 2$, so we recover the corresponding result in [5, 6].

We are concerned with weak solutions for the divergence type quasi-linear parabolic equations

$$u_t - \Delta_p u = \mu \quad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a domain and $T > 0$, and μ is an $\mathbb{R}^{N+1}$-valued (non-negative) Radon measure on Ω_T. To this end we introduce a parabolic analog of the non-linear Wolff potentials.

Before formulating the main results, let us remind the reader of the definition of a weak solution to equation (1.2).

We say that u is a weak solution to (1.2) if $u \in V(\Omega_T) := C([0, T]; L^2_{loc}(\Omega)) \cap L^p_{loc}(0, T; W^{1,p}_{loc}(\Omega))$ and for any sub-domain $\Omega' \Subset \Omega$ and any interval $I = [t_1, t_2] \subset (0, T)$ the integral identity

$$\begin{aligned} \int_\Omega u(t)\theta(t)dx\Big|_{t_1}^{t_2} + \iint_{\Omega\times I} |\nabla u|^{p-2}\nabla u \nabla\theta \, dx \, dt \\ = \iint_{\Omega\times I} \theta \, d\mu + \iint_{\Omega\times I} u\partial_t\theta dx \, dt \end{aligned} \tag{1.3}$$

for any $\theta \in C^1_c(\Omega_T)$.

The crucial role in our results is played by parabolic generalization of the truncated Wolff potential, which is defined below.

Parabolic Wolff potentials. Let μ be a positive measure on Ω_T and $(x_0, t_0) \in \Omega_T$. For $\rho, s > 0$, let $Q_{\rho,s} := B_\rho(x_0) \times (t_0 - s, t_0 + s)$. For $\rho > 0$ define

$$D_p(\rho) := \inf_{\tau>0} \left\{ i_p(\tau) + \tfrac{1}{2(p-1)^{p-1}} \rho^{-N} \mu(Q_{\rho,\tau\rho^p}) \right\}, \tag{1.4}$$

where

$$i_p(\tau) := \begin{cases} (p-2)\tau^{-\frac{1}{p-2}}, & p > 2; \\ \begin{cases} +\infty, & \tau \in (0,1), \\ 0, & \tau \geq 1, \end{cases} & p = 2. \end{cases} \tag{1.5}$$

Observe that $i_p(\tau)$ is continuous in p for every $\tau > 0$. Also note that the above infimum is attained at some $\tau \in (0, \infty]$ since the function under the infimum is continuous in τ. Moreover, $D_2(\rho) = \frac{1}{2}\rho^{-N}\mu(Q_{\rho,\rho^2})$.

Now let, for $\rho > 0$ and for $j = 0, 1, 2, \ldots$ set $\rho_j := 2^{-j}\rho$. We define the parabolic potential for a measure μ as follows:

$$P_p^\mu(x_0, t_0; \rho) := \sum_{j=0}^{\infty} D_p(\rho_j). \tag{1.6}$$

In particular, there exists $\gamma > 1$ such that

$$\frac{1}{\gamma} P_2^\mu(x_0, t_0; r) \leq \int_0^r \rho^{-N}\mu(Q_{\rho,\rho^2})\frac{d\rho}{\rho} \leq \gamma P_2^\mu(x_0, t_0; r), \tag{1.7}$$

so that for $p = 2$ the introduced potential is equivalent to the truncated Riesz potential used in the estimates in [5, 6]. Note that, for a time-independent μ charging all balls centered at x_0, the minimum in the definition of $D_p(\rho)$ is attained at $\tau = (\frac{1}{(p-1)^{p-1}}\rho^{p-N}\mu(B_\rho))^{\frac{p-2}{p-1}}$. So

$$D_p(\rho) = \left[\rho^{p-N}\mu(B_\rho)\right]^{\frac{1}{p-1}}, \quad P_p^\mu(x_0, t_0; \rho) = W_p^\mu(x_0, \rho), \tag{1.8}$$

so that in this case the introduced potential reduces to the non-linear Wolff potential. Moreover, with $\tau(\rho)$ defined as follows:

$$\tau(\rho) = \tau_\mu(\rho; x_0, t_0) := \left(\rho^{-N}\mu(Q_{\rho,\rho^p})\right)^{-\frac{p-2}{p-1}},$$

it is easy to see that there exists $\gamma = \gamma_p > 0$ such that, for all $\rho > 0$,

$$D_p(\rho) \leq \gamma\left(\rho^{-N}\mu(Q_{\rho,\rho^p})\right)^{\frac{1}{p-1}} + \gamma\rho^{-N}\mu(Q_{\rho,\tau(\rho)\rho^p})$$

and that

$$P_p^\mu(x_0, t_0; \rho) \leq \gamma\sum_{j=0}^{\infty}\left\{\left(\rho_j^{-N}\mu(Q_{\rho_j,\rho_j^p})\right)^{\frac{1}{p-1}} + \rho_j^{-N}\mu(Q_{\rho_j,\tau(\rho_j)\rho_j^p})\right\}.$$

Note that if μ is a time-independent measure then there exists $\gamma > 1$ such that

$$\frac{1}{\gamma}W(x_0, \rho) \leq \sum_{j=0}^{\infty}\left\{\left(\rho_j^{-N}\mu(Q_{\rho_j,\rho_j^p})\right)^{\frac{1}{p-1}} + \rho_j^{-N}\mu(Q_{\rho_j,\tau(\rho_j)\rho_j^p})\right\} \leq \gamma W(x_0, \rho).$$

The main result of this paper is the following theorem.

Theorem 1.1. *Let u be a weak solution to equation* (1.2). *Then, for every $\lambda \in (0, \min\{\frac{1}{p-1}, \frac{1}{N}\}]$ there exists $\gamma > 0$ depending on p, N, c_0, c_1 and λ, such that for every Lebesgue point $(y,s) \in \Omega_T$ of $u_\pm$ and $\rho, \theta > 0$ such that $Q_{\rho,\theta} := \{x : |x-y| \le \rho\} \times [s-\theta, s+\theta] \subset \Omega_T$, with an additional assumption that $\rho^2 \le \theta$ in case $p = 2$, one has*

$$u_\pm(y,s) \le \gamma \Bigg\{ \varepsilon_{\rho,\theta} + \left(\frac{1}{\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_\pm^{(1+\lambda)(p-1)} dxdt \right)^{\frac{1}{1+\lambda(p-1)}} + P_p^{\mu_\pm}(y,s;\rho) \Bigg\},$$

with

$$\varepsilon_{\rho,\theta} := \begin{cases} \rho^{\frac{p}{p-2}} \theta^{-\frac{1}{p-2}}, & p > 2, \\ 0, & p = 2. \end{cases}$$

The estimate above is not homogeneous in u which is usual for such type of equations [2, 4]. The proof of Theorem 1.1 is based on a suitable modifications of De Giorgi's iteration technique [1] following the adaptation of Kilpeläinen-Malý technique [9] to parabolic equations with ideas from [11, 16].

Corollary 1.2. *Let u be a weak solution to equation* (1.2). *Assume that, for all $\Omega' \Subset \Omega$ and $I \Subset (0,T)$,*

$$\lim_{\rho \to 0} \sup_{(x,t) \in \Omega' \times I} P_p^{|\mu|}(x,t;\rho) < \infty.$$

Then $u \in L^\infty_{loc}(\Omega_T)$.

Remark 1.3. In case $\mu(dx,dt) = \mu(x,t)dxdt$ we can estimate $P_p^{|\mu|}$ by the Lebesgue and Lorentz norms as follows.

1. Let $\mu \in L^r\big(0,T; L^q(\Omega)\big)$ for $r > 1$ and $q > \frac{N}{p}$. Then

$$\rho^{-N} \mu(Q_{\rho,\rho^p\tau}) \le \gamma \tau^{1-\frac{1}{r}} \rho^{p-\frac{p}{r}-\frac{N}{q}} \|\mu\|_{q,r}$$

and

$$D_p(\rho) \le \gamma \left[\rho^{p-\frac{p}{r}-\frac{N}{q}} \|\mu\|_{q,r} \right]^{\frac{1}{p-1-\frac{1}{r}(p-2)}}.$$

Hence, if $\frac{1}{r} + \frac{N}{pq} < 1$ then

$$\sup_{x,t,\rho} P_p^{|\mu|}(x,t;\rho) \le \gamma \|\mu\|_{q,r}^{\frac{1}{p-1-\frac{1}{r}(p-2)}}.$$

In particular, we recover a classical condition on local boundedness of the solution u (see, e.g., [3, Remark 0.1]).

By the same argument one proves that, for $\mu \in L^q\big(\Omega; L^r(0,T)\big)$ with $r>1$ and $q>\frac{N}{p}$ such that $\frac{1}{r}+\frac{N}{pq}<1$, the following estimate holds:

$$\sup_{x,t,\rho} P_p^{|\mu|}(x,t;\rho) \le \gamma \|\mu\|_{r,q}^{\frac{1}{p-1-\frac{1}{r}(p-2)}}.$$

2. The latter estimates can be refined in terms of the Lorentz norms. Recall that, for a measurable function f, the non-increasing rearrangement f^* and its average f^{**} are defined as follows:

$$f^*(s) := \inf\{t:\ |\{|f|(\xi)>t\}| \le t\}, \quad f^{**}(s) := \frac{1}{s}\int_0^s f^*(\sigma)d\sigma$$

and that the spaces $L^{q,\alpha}$, $0<q,\alpha\le\infty$ are defined by the following translation-invariant metrics:

$$\|f\|_{q,\alpha} := \begin{cases} \left[\int_0^\infty \left(s^{\frac{1}{q}} f^{**}(s)\right)^\alpha \frac{ds}{s}\right]^{\frac{1}{\alpha}}, & 0<q,\alpha<\infty, \\ \sup_{s>0} s^{\frac{1}{q}} f^{**}(s), & 0<q\le\infty,\ \alpha=\infty. \end{cases}$$

It is clear that

$$\int_E f(\xi)d\xi \le \int_0^{|E|} f^*(s)ds = |E| f^{**}(|E|) \le |E|^{1-\frac{1}{r}}\|f\|_{r,\infty}.$$

Let $\mu \in L^{q,\alpha}\big(\Omega; L^{r,\infty}(0,T)\big)$, with $r>\frac{p-2}{p-1}$, $q=\frac{N}{p-\frac{p}{r}}$ and $\alpha=\frac{1}{p-1-\frac{1}{r}(p-2)}$. Then we estimate

$$\begin{aligned} \tfrac{1}{2}\int_{-\tau\rho^p}^{\tau\rho^p} \mu(x,t)dt \le (\tau\rho^p)^{1-\frac{1}{r}}\|\mu\|_{r,\infty}(x) \text{ and } \tfrac{1}{2}\rho^{-N}\mu(Q_{\rho,\rho^p\tau}) \\ \le \frac{1}{\omega_N}\tau^{1-\frac{1}{r}}\rho^{p-\frac{p}{r}}\|\mu\|_{r,\infty}^{**}(\omega_N\rho^N), \end{aligned}$$

where ω_N denotes the volume of a unit ball in $\mathbb{R}^N$. Hence

$$D_p(\rho) \le \gamma\left[\rho^{p-\frac{p}{r}}\|\mu\|_{r,\infty}^{**}(\omega_N\rho^N)\right]^\alpha$$

and

$$\begin{aligned} \sup_{x,t} P_p^{|\mu|}(x,t;\rho) &\le \gamma\int_0^\rho \left[s^{p-\frac{p}{r}}\|\mu\|_{r,\infty}^{**}(\omega_N s^N)\right]^\alpha \frac{ds}{s} \\ &= \gamma\int_0^{\omega_N\rho^N} \left[s^{\frac{p-\frac{p}{r}}{N}}\|\mu\|_{r,\infty}^{**}(s)\right]^\alpha \frac{ds}{s} \le \|\mu\|_{(r,\infty),(q,\alpha)}^\alpha. \end{aligned}$$

The rest of the paper contains the proof of Theorem 1.1.

2 Proof of Theorem 1.1

We start with some auxiliary integral estimates for the solutions of (1.2) which are formulated in the next lemma. Let

$$\varepsilon_p := \begin{cases} (p-2)^{p-2}, & p > 2; \\ 1, & p = 2. \end{cases}$$

Note that ε_p is continuous and that $\varepsilon_p \geq e^{-\frac{1}{e}} > \frac{1}{2}$. For $\lambda \in (0,1)$ we define

$$G(s) := s_+^2 \wedge s_+ \text{ and } \psi(s) := (1+s_+)^{1-\frac{1+\lambda}{p}} - 1 \asymp s_+ \wedge s_+^{\frac{p-1-\lambda}{p}}. \tag{2.1}$$

For $\delta > 0$ and $0 < \rho < R$ define,

$$I_\rho^{(\delta)}(s) := (s - \varepsilon_p \delta^{2-p}\rho^p,\, s + \varepsilon_p \delta^{2-p}\rho^p), \quad Q_\rho^{(\delta)}(y,s) = B_\rho(y) \times I_\rho^{(\delta)}(s).$$

In the sequel, γ stands for a constant which depends only on N, p, c_0, c_1 and λ, and which may vary from line to line.

Lemma 2.1. *Let $\lambda \leq \frac{1}{p-1}$ and $m \geq p$. Then there exists a constant $\gamma > 0$ depending only on N, p, c_0, c_1, λ and m, such that, for every solution u to (1.2) in Ω_T, every $l, \delta > 0$, and $(y,s) \in \Omega_T$ such that the cylinder $Q_\rho^{(\delta)}(y,s) \subset \Omega_T$, and every $\xi \in C_c^\infty(Q_\rho^{(\delta)}(y,s))$ such that $0 \leq \xi \leq 1$ and $|\xi_t| \leq 8\delta^{p-2}\rho^{-p}$ and $|\nabla \xi| \leq 4\rho^{-1}$, the following estimate holds.*

$$\begin{aligned} &\sup_{t \in I_\rho^{(\delta)}(s)} \frac{1}{\rho^N} \int_{B_\rho(y)} G\left(\frac{u-l}{\delta}\right) \xi(x,t)^m dx \\ &+ \frac{\delta^{p-2}}{\rho^N} \iint_L \left|\nabla \psi\left(\frac{u-l}{\delta}\right)\right|^p \xi^m dx\, dt \\ &\leq \gamma \frac{\delta^{p-2}}{\rho^{p+N}} \iint_L G\left(\frac{u-l}{\delta}\right) \xi^{m-1} dx\, dt \\ &+ \gamma \frac{\delta^{p-2}}{\rho^{p+N}} \iint_L \left(\frac{u-l}{\delta}\right)^{(1+\lambda)(p-1)} \xi^{m-p} dx\, dt + \gamma \frac{1}{\delta \rho^N} \mu_+\left(Q_\rho^{(\delta)}(y,s)\right), \end{aligned} \tag{2.2}$$

where $L = Q_\rho^{(\delta)}(y,s) \cap \{u > l\}$, $L(t) = L \cap \{\tau = t\}$.

Proof. For shortness, we write $B := B_\rho(y)$, $I := I_\rho^{(\delta)}(s)$ and $Q := Q_\rho^{(\delta)}(y,s)$. We also denote $I(t) := I \cap (0,t)$ and $Q(t) := B \times I(t)$.

Let

$$\begin{aligned} \phi(s) &:= \int_0^{s_+} (1+\tau)^{-1-\lambda} d\tau \asymp s_+ \wedge 1 \asymp \frac{s_+}{1+s_+} \\ \text{and } \Phi(s) &:= \int_0^s \phi(\tau) d\tau \asymp G(s) = s_+^2 \wedge s_+. \end{aligned} \tag{2.3}$$

Let m_ε and M_σ denote symmetric mollifiers in t and in x, respectively. Note that $m_\varepsilon M_\sigma$ is a contraction in $L^q(Q)$ and $C\big(I;L^q(B)\big)$ for all $q\in[1,\infty]$ and that $m_\varepsilon M_\sigma\to I$ as $\varepsilon,\sigma\to 0$ in the strong operator topology of the aforementioned spaces for $q\in[1,\infty)$. Also, $m_\varepsilon M_\sigma\theta\to\theta$ a.e. on Q as $\varepsilon,\sigma\to 0$. Further on, for a function θ we denote $\theta_\varepsilon:=m_\varepsilon M_\varepsilon\theta$.

We choose $\theta^{(\varepsilon)}:=\frac{1}{\delta}\left[\phi\left(\frac{u_\varepsilon-l}{\delta}\right)\xi^m\right]_\varepsilon$ as a test function in (1.3). Then we have that

$$\begin{aligned}\int_B u(t)\theta^{(\varepsilon)}(t)dx+\iint_{Q(t)}|\nabla u|^{p-2}(\nabla u)\nabla\theta^{(\varepsilon)}dx\,dt\\ =\iint_{Q(t)}\theta^{(\varepsilon)}d\mu+\iint_{Q(t)}u\partial_t\theta^{(\varepsilon)}dx\,dt.\end{aligned}\tag{2.4}$$

Note that $\theta^{(\varepsilon)}\to\theta:=\frac{1}{\delta}\phi\left(\frac{u-l}{\delta}\right)\xi^m$ in $C(I;L^q(B))\cap L^p(I;\overset{\circ}{W}{}^{1,p}(B))$ as $\varepsilon\to 0$ for all $q\in[1,\infty)$ since ϕ is a bounded continuous function. Hence

$$\begin{aligned}\int_B u(t)\theta^{(\varepsilon)}(t)dx+\iint_{Q(t)}|\nabla u|^{p-2}(\nabla u)\nabla\theta^{(\varepsilon)}dx\,dt\\ \to\int_B u(t)\theta dx+\iint_{Q(t)}|\nabla u|^{p-2}(\nabla u)\nabla\theta dx\,dt\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{2.5}$$

Since $m_\varepsilon M_\sigma$ is a contraction in $L^\infty(Q)$, we have that $\theta^{(\varepsilon)}\le\sup\phi=\frac{1}{\delta\lambda}$. Therefore we obtain that

$$\iint_{Q(t)}\theta^{(\varepsilon)}d\mu\le\tfrac{1}{\delta\lambda}\mu_+\big(Q(t)\big).\tag{2.6}$$

Now we consider the last integral on the right hand side of (2.4). Since $m_\varepsilon M_\sigma$ is a self-adjoint operator commuting with the derivative,

$$\begin{aligned}&\iint_{Q(t)}u\partial_t\theta^{(\varepsilon)}dx\,dt=\int_B u_\varepsilon(t)\frac{1}{\delta}\phi\left(\frac{u_\varepsilon(t)-l}{\delta}\right)\xi^m(t)dx\\ &-\iint_{Q(t)}(\partial_t u_\varepsilon)\frac{1}{\delta}\phi\left(\frac{u_\varepsilon-l}{\delta}\right)\xi^m dx\,dt\\ &=\int_B u_\varepsilon(t)\frac{1}{\delta}\phi\left(\frac{u_\varepsilon(t)-l}{\delta}\right)\xi^m(t)dx-\iint_{Q(t)}\xi^m\partial_t\Phi\left(\frac{u_\varepsilon-l}{\delta}\right)dx\,dt\\ &=\int_B u_\varepsilon(t)\frac{1}{\delta}\phi\left(\frac{u_\varepsilon(t)-l}{\delta}\right)\xi^m(t)dx-\int_B\Phi\left(\frac{u_\varepsilon(t)-l}{\delta}\right)\xi^m(t)dx\\ &+m\iint_{Q(t)}\Phi\left(\frac{u_\varepsilon-l}{\delta}\right)\xi^{m-1}\xi_t dx\,dt.\end{aligned}$$

Since Φ is a Lipschitz continuous function, we conclude that

$$\begin{aligned}\iint_{Q(t)} u\partial_t\theta^{(\varepsilon)}dx\,dt \to \int_B u(t)\theta(t)dx - \int_B \Phi\left(\frac{u(t)-l}{\delta}\right)\xi^m(t)dx \\ +m\iint_Q \Phi\left(\frac{u-l}{\delta}\right)\xi^{m-1}\xi_t dx\,dt \quad \text{as } \varepsilon\to 0.\end{aligned} \tag{2.7}$$

Collecting (2.4)–(2.7) we obtain the following inequality:

$$\begin{aligned}&\int_B \Phi\left(\frac{u(t)-l}{\delta}\right)\xi^m(t)dx + \iint_{Q(t)} |\nabla u|^{p-2}(\nabla u)\nabla\theta dx\,dt \\ &\le m\iint_Q \Phi\left(\frac{u-l}{\delta}\right)\xi^{m-1}\xi_t dx\,dt + \frac{1}{\delta\lambda}\mu_+(Q).\end{aligned}$$

Taking the supremum in t, we obtain

$$\begin{aligned}&\sup_{t\in I}\int_B \Phi\left(\frac{u(t)-l}{\delta}\right)\xi^m(t)dx + \iint_Q |\nabla u|^{p-2}\nabla u\nabla\theta dx\,dt \\ &\le m\iint_Q \Phi\left(\frac{u-l}{\delta}\right)\xi^{m-1}\xi_t dx\,dt + \frac{1}{\delta\lambda}\mu_+(Q).\end{aligned} \tag{2.8}$$

Now we estimate the second term on the left hand side of (2.8) as follows.

$$\begin{aligned}&\iint_Q |\nabla u|^{p-2}\nabla u\nabla\theta dx\,dt \ge \frac{1}{\delta^2}\iint_L \left(1+\frac{u-l}{\delta}\right)^{-1-\lambda}|\nabla u|^p\xi^m dx\,dt \\ &-\gamma\frac{1}{\lambda\delta}\iint_L |\nabla u|^{p-1}\left(1+\frac{u-l}{\delta}\right)^{-1}\left(\frac{u-l}{\delta}\right)|\nabla\xi|\xi^{m-1}dx\,dt \\ &\ge \frac{1}{2\delta^2}\iint_L \left(1+\frac{u-l}{\delta}\right)^{-1-\lambda}|\nabla u|^p\xi^m dx\,dt \\ &-\gamma\frac{\delta^{p-2}m^p}{\lambda^p}\iint_L \left(1+\frac{u-l}{\delta}\right)^{\lambda(p-1)-1}\left(\frac{u-l}{\delta}\right)^p|\nabla\xi|^p\xi^{m-p}dx\,dt.\end{aligned} \tag{2.9}$$

Observe now that $G\le\Phi\le\frac{1}{\lambda}G$ and $\psi'(s)=(1+s)^{-\frac{1+\lambda}{p}}$, that $|\xi_t|\le 4\delta^{p-2}\rho^{-p}$ and $|\nabla\xi|\le 4\rho^{-1}$, and that $(1+s)^{\lambda(p-1)-1}s^p\le s^{(1+\lambda)(p-1)}$ since $\lambda(p-1)\le 1$.

Hence we conclude from (2.7) and (2.9) that

$$\sup_{t\in I}\int_{L(t)} G\left(\frac{u(t)-l}{\delta}\right)\xi^m(t)dx + \delta^{p-2}\iint_L \left|\nabla\psi\left(\frac{u-l}{\delta}\right)\right|^p \xi^m dx\,dt$$
$$\le \gamma\frac{\delta^{p-2}}{\rho^p}\iint_L G\left(\frac{u-l}{\delta}\right)\xi^{m-1}dx\,dt + \gamma\frac{\delta^{p-2}}{\rho^p}\iint_L \left(\frac{u-l}{\delta}\right)^{(1+\lambda)(p-1)}\xi^{m-p}dx\,dt$$
$$+\gamma\frac{1}{\delta}\mu_+(Q).$$

□

Remark 2.2. The constant γ in (2.2) is proportional to a power of $m\max\phi = \frac{m}{\lambda}$, where ϕ is defined in (2.3). In particular, it blows up as $\lambda \downarrow 0$.

Let (y,s) be an arbitrary point in Ω_T. Fix $\rho,\theta > 0$ such that $\rho < \operatorname{dist}(y,\partial\Omega)$ and $\theta < \min\{s, T-s\}$. For $p = 2$ assume, in addition, that $\rho^2 \le \theta$. Fix $\delta_{\rho,\theta}$:

$$\delta_{\rho,\theta} = \begin{cases} \left(\varepsilon_p\rho^p\theta^{-1}\right)^{\frac{1}{p-2}}, & p > 2,\\ 0, & p = 2.\end{cases}$$

Fix $m \ge 2p$ and $\xi \in C_c^\infty(B_1(0)\times(-1,1))$, such that $0 \le \xi \le 1$, $\xi(x,t) = 1$ on $B_{\frac12}(0)\times(-\frac12,\frac12)$, and $|\nabla\xi| < 4$, $|\partial_t\xi| < 4$.

Fix a number $\varkappa \in (0,1)$ depending on N, p, c_1, c_2 and λ, which will be specified later.

For $j = 0,1,2,\dots$ positive numbers l_j and δ_j are defined inductively as follows. We set $\delta_{-1} = 2\delta_{\rho,\theta}$ and $l_0 = 0$ and, for $j = 0,1,2,3,\dots$, given δ_{j-1} and l_j, we define δ_j and l_{j+1} as follows. We denote $\rho_j := \rho 2^{-j}$, $B_j := B_{\rho_j}(y)$ and

$$\tau_j := \sup\left\{\tau : i_p(\tau) + \tfrac{1}{2(p-1)^{p-1}}\rho^{-N}\mu(Q_{\rho_j,\tau\rho_j^p}) = D_p(\rho_j)\right\},$$

where $D(\rho_j)$ is as in (1.4). For $\delta \ge \hat\delta_j$ with

$$\hat\delta_j := (\tfrac12\delta_{j-1}) \vee i_p(\tau_j), \tag{2.10}$$

we define

$$I_j^\delta := (s-\delta^{2-p}\rho_j^p\varepsilon_p, s+\delta^{2-p}\rho_j^p\varepsilon_p),\ \ Q_j^\delta := B_j\times I_j^\delta,\ \ L_j^\delta := \{(x,t)\in Q_j^\delta :\, u(x,t) > l_j\}$$

and, for $t \in I_j^\delta$,

$$L_j(t) := \{x \in B_j : u(x,t) > l_j\}.$$

Then denote

$$\xi_{j,\delta}(x,t) := \xi\left(\frac{x-y}{\rho_j}, \frac{t-s}{\delta^{2-p}\rho_j^p\varepsilon_p}\right).$$

Note that $\xi_{j,\delta} \in C_c^\infty(Q_j)$ and $\xi_{j,\delta}(x,t) = 1$ for $(x,t) \in \frac{1}{2}Q_j^\delta$, with the derivative estimates $|\nabla \xi_{j,\delta}| \le 4\rho_j^{-1}$, $|\partial_t \xi_{j,\delta}| \le 4\delta^{p-2}\rho_j^{-p}\varepsilon_p^{-1} \le 8\delta^{p-2}\rho_j^{-p}$.

Set

$$A_j(\delta) = \frac{\delta^{p-2}}{\varepsilon_p \rho_j^{N+p}} \iint_{L_j^\delta} \left(\frac{u-l_j}{\delta}\right)^{(1+\lambda)(p-1)} \xi_{j,\delta}^{m-p} dx\, dt + \sup_{t \in I_j^\delta} \frac{1}{\rho_j^N} \int_{L_j(t)} G\left(\frac{u-l_j}{\delta}\right) \xi_{j,\delta}^m dx. \tag{2.11}$$

For $j = 0, 1, 2, \dots$, if

$$A_j(\hat{\delta}_j) \le \varkappa, \tag{2.12}$$

we set $\delta_j = \hat{\delta}_j$ and $l_{j+1} = l_j + \delta_j$.

Note that $A_j(\delta)$ is continuous as a function of δ and $A_j(\delta) \to 0$ as $\delta \to \infty$. So if

$$A_j(\hat{\delta}_j) > \varkappa, \tag{2.13}$$

there exists $\hat{\delta} > \hat{\delta}_j$ such that $A_j(\hat{\delta}) = \varkappa$. In this case we set $\delta_j = \hat{\delta}$ and $l_{j+1} = l_j + \delta_j$.

With fixed δ_j, we set $I_j := I_j^{\delta_j}$, $Q_j := Q_j^{\delta_j}$, $L_j := L_j^{\delta_j}$ and $\xi_j := \xi_{j,\delta_j}$.

The following proposition is a key in the Kilpeläinen-Malý technique [9].

Proposition 2.3. *One can choose $\varkappa > 0$ such that there exists $\gamma \ge 1$ depending on the data, such that*

$$\delta_j \le \tfrac{1}{2}\delta_{j-1} + \gamma D_p(\rho_j), \tag{2.14}$$

for $j = 1, 2, 3, \dots$, and, for $j = 0$,

$$\delta_0 \le \delta_{\rho,\theta} + \gamma \left(\frac{1}{\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_+^{(1+\lambda)(p-1)} \right)^{\frac{1}{1+\lambda(p-1)}} + \gamma D_p(\rho). \tag{2.15}$$

The proof of Proposition 2.3 is split into several lemmas.

Lemma 2.4. *For $j = 1, 2, 3, \dots$, we have*

$$Q_j \subset \tfrac{1}{2}Q_{j-1} \ j = 1, 2, 3 \dots, \text{ and } Q_j \subset Q_{\rho,\theta} \ j = 0, 1, 2, \dots, \tag{2.16}$$

so in particular $\xi_{j-1} \equiv 1$ on Q_j, $j = 1, 2, 3 \dots$;

$$Q_j \subset Q_{\rho_j, \tau_j \rho_j^p}, \ j = 0, 1, 2, \dots. \tag{2.17}$$

$$\frac{\delta_j^{p-2}}{\varepsilon_p \rho_j^{p+N}} |L_j| \le \sup_{t \in I_j} \frac{|L_j(t)|}{\rho_j^N} \le 2^N \varkappa, \ j = 1, 2, \dots \tag{2.18}$$

and

$$\sup_{t\in I_j}\frac{1}{\rho_j^N}\int_{L_j(t)}\frac{u(x,t)-l_j}{\delta_j}dx\le 2^{N+1}\varkappa,\ j=1,2,\dots. \tag{2.19}$$

$$\frac{\delta_j^{p-2}}{\varepsilon_p\rho_j^{N+p}}\int_{L_j}\left(\frac{u(x,t)-l_j}{\delta_j}\right)^{(1+\lambda)(p-1)}dx\,dt\le 2^{N+(1+\lambda)(p-1)}\varkappa,\ j=1,2,\dots. \tag{2.20}$$

There exists $\gamma>0$ such that, for $j=1,2,\dots,$

$$\frac{\delta_j^{p-2}}{\rho_j^N}\iint_{L_j}\left|\nabla\psi\left(\frac{u-l_j}{\delta_j}\right)\right|^p\xi^m dx\,dt\le\gamma\varkappa+\gamma\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j). \tag{2.21}$$

Proof. The imbedding (2.16)-(2.17) follows from the choice $\delta_j\ge\hat\delta_j$, with $\hat\delta_j$ defined in (2.10). Indeed, since $\delta_j\ge\frac12\delta_{j-1}$, one has $\delta_j^{2-p}\rho_j^p\le\frac14\delta_{j-1}^{2-p}\rho_{j-1}^p$. Hence (2.16). Similarly, $\delta_j\ge i_p(\tau_j)$ implies $\varepsilon_p\delta_j^{2-p}\rho_j^p\le\tau_j\rho_j^p$. Hence (2.17).

To prove (2.18), observe that, for $(x,t)\in L_j$ one has

$$\frac{u(x,t)-l_{j-1}}{\delta_{j-1}}=1+\frac{u(x,t)-l_j}{\delta_{j-1}}\ge 1. \tag{2.22}$$

Since $\xi_{j-1}=1$ on Q_j and $I_j\subset I_{j-1}$ and $L_j(t)\subset L_{j-1}(t)$ for $t\in I_j$, we obtain

$$\begin{aligned}\sup_{t\in I_j}\frac{|L_j(t)|}{\rho_j^N}&\le\rho_j^{-N}\sup_{t\in I_j}\int_{L_j(t)}G\left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)\xi_{j-1}^m dx\\&\le 2^N\rho_{j-1}^{-N}\sup_{t\in I_{j-1}}\int_{L_{j-1}(t)}G\left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)\xi_{j-1}^m dx\le 2^N\varkappa,\end{aligned} \tag{2.23}$$

which proves (2.18). To verify (2.19), note that $G(s)+1>s$ for $s\ge0$. Then, since $\delta_j\ge\frac12\delta_{j-1}$, one has, for $(x,t)\in L_j$,

$$\frac{u(x,t)-l_j}{\delta_j}\le 2\frac{u(x,t)-l_j}{\delta_{j-1}}=2\frac{u(x,t)-l_{j-1}}{\delta_{j-1}}-2\le 2G\left(\frac{u(x,t)-l_{j-1}}{\delta_{j-1}}\right). \tag{2.24}$$

So, by the same argument as in (2.23),

$$\begin{aligned}&\sup_{t\in I_j}\frac{1}{\rho_j^N}\int_{L_j(t)}\left(\frac{u(t)-l_j}{\delta_j}\right)dx\le\sup_{t\in I_j}\frac{2}{\rho_j^N}\int_{L_j(t)}G\left(\frac{u(t)-l_{j-1}}{\delta_{j-1}}\right)\xi_{j-1}^m dx\\&\le\sup_{t\in I_{j-1}}\frac{2^{N+1}}{\rho_{j-1}^N}\int_{L_{j-1}(t)}G\left(\frac{u(t)-l_{j-1}}{\delta_{j-1}}\right)\xi_{j-1}^m dx\le 2^{N+1}\varkappa.\end{aligned}$$

The estimate (2.20) follows from the next observation:

$$\begin{aligned}\delta_j^{p-2}\left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(p-1)} &= \left(\frac{\delta_j}{\delta_{j-1}}\right)^{p-2-(1+\lambda)(p-1)}\delta_{j-1}^{p-2}\left(\frac{u-l_j}{\delta_{j-1}}\right)^{(1+\lambda)(p-1)}\\ &\le 2^{1+\lambda(p-1)}\delta_{j-1}^{p-2}\left(\frac{u-l_{j-1}}{\delta_{j-1}}\right)^{(1+\lambda)(p-1)}.\end{aligned}$$

To conclude (2.21) from (2.2) one has to estimate the first term in the right hand side of the latter. To do this, it suffices to observe that $G(s)\le s$ and apply (2.19). □

Lemma 2.5. *For every $\varepsilon>0$ there exist $\gamma_1(\varepsilon),\gamma_2(\varepsilon)>0$ such that, for $j=1,2,3,\dots$,*

$$\begin{aligned}&\frac{\delta_j^{p-2}}{\rho_j^{N+p}}\iint_{L_j}\left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(p-1)}\xi_j^{m-p}dx\,dt\\ &\qquad\le \varepsilon\varkappa+\gamma_1(\varepsilon)\varkappa^{\frac{p}{N}}\left(\varkappa+\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j)\right)\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}&\sup_{t\in I_j}\frac{1}{\rho_j^N}\int_{L_j(t)}G\left(\frac{u(t)-l_j}{\delta_j}\right)\xi_j^m dx\\ &\le\varepsilon\varkappa+\gamma_2(\varepsilon)\varkappa^{\frac{p}{N}}\left(\varkappa+\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j)\right)+\gamma\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j).\end{aligned}\tag{2.26}$$

Proof. For shortness we denote

$$w_j:=\psi\left(\frac{u-l_j}{\delta_j}\right).$$

Note that, for every $\varepsilon>0$, there exists $\gamma(\varepsilon)>0$ such that $s^{(1+\lambda)(p-1)}\le 2^{-N}\varepsilon+\gamma(\varepsilon)\psi^{p+\frac{\lambda p^2}{p-1-\lambda}}(s)$. Hence, by (2.18),

$$\begin{aligned}&\frac{\delta_j^{p-2}}{\rho_j^{N+p}}\iint_{L_j}\left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(p-1)}\xi_j^{m-p}dx\,dt\\ &\le\varepsilon\varkappa+\gamma(\varepsilon)\frac{\delta_j^{p-2}}{\rho_j^{N+p}}\iint_{L_j}w_j^{p+\frac{\lambda p^2}{p-1-\lambda}}\xi_j^{m-p}dx\,dt.\end{aligned}\tag{2.27}$$

The second term on the right hand side of (2.27) is estimated by using the Hölder inequality first (note that $\lambda\le\frac{1}{N}$), and then the Sobolev inequality,

as follows

$$
\begin{aligned}
&\frac{\delta_j^{p-2}}{\rho_j^{N+p}} \iint_{L_j} w_j^{p+\frac{\lambda p^2}{p-1-\lambda}} \xi_j^{m-p} dx dt \\
&\leq \frac{\delta_j^{p-2}}{\rho_j^p} \int_{I_j} \left(\frac{|L_j(t)|}{\rho_j^N} \right)^{p(\frac{1}{N}-\lambda)} \left(\frac{1}{\rho_j^N} \int_{L_j(t)} w_j^{\frac{p}{p-1-\lambda}} dx \right)^{\lambda p} \times \\
&\times \left(\frac{1}{\rho_j^N} \int_{L_j(t)} (w_j \xi_j)^{\frac{pN}{N-p}} \right)^{\frac{N-p}{N}} dt \qquad (2.28) \\
&\leq \gamma \left(\sup_{t \in I_j} \frac{|L_j(t)|}{\rho_j^N} \right)^{p(\frac{1}{N}-\lambda)} \left(\sup_{t \in I_j} \frac{1}{\rho_j^N} \int_{L_j(t)} w_j^{\frac{p}{p-1-\lambda}} dx \right)^{\lambda p} \times \\
&\times \left(\frac{\delta_j^{p-2}}{\rho_j^N} \iint_{L_j} |\nabla (w_j \xi_j)|^p \, dx \, dt \right).
\end{aligned}
$$

Since $\psi(s)^{\frac{p}{p-1-\lambda}} \leq \gamma s$ for $s \geq 0$, the first two factors in the right hand side of (2.28) are estimated in (2.18)-(2.19) so that we obtain

$$
\begin{aligned}
&\frac{\delta_j^{p-2}}{\rho_j^{N+p}} \iint_{L_j} \left(\frac{u - l_j}{\delta_j} \right)^{(1+\lambda)(p-1)} \xi_j^{m-p} dx \, dt \\
&\qquad \leq \varepsilon \varkappa + \gamma(\varepsilon) \varkappa^{\frac{p}{N}} \frac{\delta_j^{p-2}}{\rho_j^N} \iint_{L_j} |\nabla (w_j \xi_j)|^p \, dx \, dt \\
&\qquad \leq \varepsilon \varkappa + \gamma(\varepsilon) \varkappa^{\frac{p}{N}} \frac{\delta_j^{p-2}}{\rho_j^N} \iint_{L_j} |\nabla w_j|^p \xi_j^p dx \, dt \\
&\qquad + \gamma(\varepsilon) \varkappa^{\frac{p}{N}} \frac{\delta_j^{p-2}}{\rho_j^{N+p}} \iint_{L_j} w_j^p dx \, dt.
\end{aligned}
$$

The second term on the right hand side of the last inequality is estimated in (2.21). Then, the inequality $\psi^p(s) \leq \gamma(1 + s^{(1+\lambda)(p-1)})$ and (2.18) and (2.20) imply that

$$
\frac{\delta_j^{p-2}}{\rho_j^{N+p}} \iint_{L_j} w_j^p dx \, dt \leq \gamma \varkappa.
$$

Hence (2.25) follows.

To conclude (2.26) from (2.2) and (2.25), we have to estimate the first term in the right hand side of (2.2). Note that, for every $\varepsilon > 0$ there exists $\hat{\gamma}(\varepsilon) > 0$ such that $G(s) \leq 2^{-N-1}\varepsilon + \hat{\gamma}(\varepsilon) s^{(1+\lambda)(p-1)}$. Then (2.2) and (2.18)

imply that

$$\sup_{t\in I_j}\frac{1}{\rho_j^N}\int_{L_j(t)} G\left(\frac{u(t)-l_j}{\delta_j}\right)\xi_j^m dx$$
$$\le \tfrac{1}{2}\varepsilon\varkappa + (\gamma+\hat\gamma(\varepsilon))\frac{\delta_j^{p-2}}{\rho_j^{N+p}}\iint_{L_j}\left(\frac{u-l_j}{\delta_j}\right)^{(1+\lambda)(p-1)}\xi_j^{m-p}dx\,dt + \gamma\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j),$$

with $\gamma>0$ as in (2.2). Choose now $\varepsilon_1>0$ such that $\varepsilon_1(\gamma+\hat\gamma(\varepsilon))\le\frac{1}{2}\varepsilon$. Then applying (2.25) with ε_1 in place of ε, we obtain (2.26) with $\gamma_2(\varepsilon):=(\gamma+\hat\gamma(\varepsilon))\gamma_1(\varepsilon_1)$. □

Proof of Proposition 2.3. It suffices to prove (2.14)-(2.15) in case $\delta_j>\hat\delta_j$. Otherwise the estimates are evident as $\delta_j=\hat\delta_j$ implies that $\delta_j=\frac{1}{2}\delta_{j-1}$ (recall that $\frac{1}{2}\delta_{-1}=\delta_{\rho,\theta}$) or $\delta_j=i_p(\tau_j)$. Note that $\delta_j>\hat\delta_j$ guarantees that $A_j(\delta_j)=\varkappa$.

First we prove (2.14), that is, consider the case $j=1,2,3,\dots$. Then it follows from Lemma 2.3 that, for every $\varepsilon>0$, there exists $\gamma(\varepsilon)>0$ such that

$$\varkappa\le\varepsilon\varkappa+\gamma(\varepsilon)\varkappa^{\frac{p}{N}}\left(\varkappa+\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j)\right)+\gamma\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j). \tag{2.29}$$

Now choose $\varepsilon=\frac{1}{2}$ and $\varkappa$ such that $\gamma(\frac{1}{2})\varkappa^{\frac{p}{N}}<\frac{1}{4}$. Then it follows from (2.29) that there exists $\gamma>0$ such that

$$\frac{1}{\delta_j\rho_j^N}\mu_+(Q_j)\ge\gamma\varkappa,\quad\text{hence}\quad\delta_j\le\frac{1}{\gamma\varkappa}\frac{1}{\rho_j^N}\mu_+(Q_j).$$

By (2.17), $\mu_+(Q_j)\le\mu_+(Q_{\rho_j,\tau_j\rho_j^p})$ so

$$\delta_j\le\tfrac{1}{2}\delta_{j-1}+i_p(\tau_j)+\gamma\rho_j^{-N}\mu_+(Q_{\rho_j,\tau_j\rho_j^p})\le\tfrac{1}{2}\delta_{j-1}+\gamma D_p(\rho_j).$$

So (2.14) is shown.

Now we prove the estimate (2.15) of δ_0. Since $A_0(\delta_0)=\varkappa$, at least one of the following two inequalities holds (recall that $l_0=0$):

$$\tfrac{1}{2}\varkappa\le\frac{\delta_0^{p-2}}{\varepsilon_p\rho^{N+p}}\iint_{Q_0}\left(\frac{u_+}{\delta_0}\right)^{(1+\lambda)(p-1)}dx\,dt,$$

hence

$$\delta_0\le\left(\frac{2}{\varkappa\varepsilon_p\rho^{N+p}}\iint_{Q_{\rho,\theta}}u_+^{(1+\lambda)(p-1)}dx\,dt\right)^{\frac{1}{1+\lambda(p-1)}},$$

or

$$\tfrac{1}{2}\varkappa \le \sup_{t\in I_0} \frac{1}{\rho^N} \int_{B_\rho} G\left(\frac{u_+}{\delta_0}\right) \xi_0^m dx. \tag{2.30}$$

In the former case (2.15) follows immediately, while in the latter one we use (2.2) and the next estimate: for every $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ such that $G(s) \le \varepsilon + \gamma(\varepsilon) s^{(1+\lambda)(p-1)}$. Then (2.30) implies that, there exists $\gamma > 0$ and, for every $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ such that

$$\tfrac{1}{2}\varkappa \le \gamma\varepsilon + \gamma(\varepsilon)\frac{\delta_0^{p-2}}{\rho^{N+p}} \iint_{Q_0} \left(\frac{u_+}{\delta_0}\right)^{(1+\lambda)(p-1)} dx\,dt + \gamma \frac{1}{\delta_0 \rho^N} \mu_+(Q_0).$$

Choose $\varepsilon > 0$ such that $\gamma\varepsilon \le \frac{1}{4}\varkappa$. Then, for some (other) $\gamma > 0$,

$$\gamma\varkappa \le \frac{\delta_0^{p-2}}{\rho^{N+p}} \iint_{Q_0} \left(\frac{u_+}{\delta_0}\right)^{(1+\lambda)(p-1)} dx\,dt + \frac{1}{\delta_0 \rho^N} \mu_+(Q_0).$$

Thus at least one of the following two inequalities holds:

$$\tfrac{1}{2}\gamma\varkappa \le \frac{\delta_0^{p-2}}{\rho^{N+p}} \iint_{Q_0} \left(\frac{u_+}{\delta_0}\right)^{(1+\lambda)(p-1)} dx\,dt,$$

hence

$$\delta_0 \le \left(\frac{2}{\gamma\varkappa\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_+^{(1+\lambda)(p-1)} dx\,dt\right)^{\frac{1}{1+\lambda(p-1)}},$$

or

$$\tfrac{1}{2}\gamma\varkappa \le \frac{1}{\delta_0\rho^N}\mu_+(Q_0), \quad \text{hence} \quad \delta_0 \le \frac{2}{\gamma\varkappa}\frac{1}{\rho^N}\mu_+(Q_0).$$

Note that $\mu_+(Q_0) \le \mu_+(Q_{\rho,\tau_0\rho^p})$, due to (2.17). Hence

$$\begin{aligned}\delta_0 \le& \delta_{\rho,\theta} + \gamma\left(\frac{1}{\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_+^{(1+\lambda)(p-1)} dx\,dt\right)^{\frac{1}{1+\lambda(p-1)}} + i_p(\tau_0) + \gamma\mu_+(Q_{\rho,\tau_0\rho^p}) \\ \le& \delta_{\rho,\theta} + \gamma\left(\frac{1}{\rho^{N+p}} \iint_{Q(\rho)} u_+^{(1+\lambda)(p-1)} dx\,dt\right)^{\frac{1}{1+\lambda(p-1)}} + \gamma D_p(\rho).\end{aligned}$$

So (2.15) holds. □

Corollary 2.6. *The sequence (l_j) is bounded above and*

$$l_j \nearrow l_\infty \le 2\delta_{\rho,\theta} + \gamma\left\{\left(\frac{1}{\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_+^{(1+\lambda)(p-1)} dxdt\right)^{\frac{1}{1+\lambda(p-1)}} + P_p^{\mu_+}(y,s;\rho)\right\}.$$

Proof. It follows from Proposition 2.3 and setting $l_0 = 0$ that there exists $\gamma > 0$ such that, for $J = 2, 3, 4, \dots,$

$$l_J = \sum_{j=0}^{J-1} \delta_j \le \delta_0 + \tfrac{1}{2} \sum_{j=0}^{J-2} \delta_j + \gamma \sum_{j=1}^{J-1} D_p(\rho_j)$$

$$\le \tfrac{1}{2} l_{J-1} + \delta_{\rho,\theta} + \gamma \left(\frac{1}{\rho^{N+p}} \iint_{Q_{\rho,\theta}} u_+^{(1+\lambda)(p-1)} dx dt \right)^{\frac{1}{1+\lambda(p-1)}} + \gamma \sum_{j=0}^{J-1} D_p(\rho_j).$$

Since $l_J > l_{J-1}$, the assertion follows.

□

Proof of Theorem 1.1. Since $\tilde{u} := -u$ satisfies the equation $\partial_t \tilde{u} - \Delta_p \tilde{u} = -\mu$, it suffices to show that $u_+(y,s) \le l_\infty$ whenever $l_\infty < \infty$ and (y,s) is a Lebesgue point for the function u_+.

Note that, by (2.16), $Q_j \downarrow \{(y,s)\}$ as $j \to \infty$. Observe that comparable symmetric cylinders form a basis satisfying the Besicovitch property, by [7, Lemma 1.6] (see also [8, Chap. I, Sec.1, Remark (5)]). Hence, by [7, Theorem 2.4] (see also [8, Chap. II, Sec. 2, Theorem 2.1]), it is a differentiable basis for all functions from $L^1_{loc}(\Omega_T)$. So for a Lebesgue point (y,s) for u_+, one has

$$u_+(y,s) = \lim_{j\to\infty} \frac{1}{|Q_j|} \iint_{Q_j} u_+ dx\, dt \le l_\infty + \limsup_{j\to\infty} \frac{1}{|Q_j|} \iint_{Q_j} (u - l_\infty)_+ dx\, dt$$

$$\le l_\infty + \left(\limsup_{j\to\infty} \frac{1}{|Q_j|} \iint_{Q_j} (u - l_\infty)_+^{(1+\lambda)(p-1)} dx\, dt \right)^{\frac{1}{(1+\lambda)(p-1)}}.$$

On the other hand

$$\frac{1}{|Q_j|} \iint_{Q_j} (u - l_\infty)_+^{(1+\lambda)(p-1)} dx\, dt < \gamma \frac{\delta_j^{p-2}}{\rho_j^{N+p}} \iint_{L_j} (u - l_j)^{(1+\lambda)(p-1)} dx\, dt$$

$$\le \gamma \varkappa \delta_j^{(1+\lambda)(p-1)} \to 0 \text{ as } j \to \infty,$$

since the series $\sum \delta_j < \infty$. Hence the assertion follows. □

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