Spacetime Dependent Lagrangians and the Vacuum Expectation Value of the Higgs field

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Abstract

The spacetime dependent lagrangian formalism of references [1-2] is used to obtain a classical solution of Yang-Mills theory. This is then used to obtain an estimate of the vacuum expectation value of the Higgs field , *viz.* $\phi_a = A/e$, where A is a constant and e is the Yang-Mills coupling (related to the usual electric charge). The solution can also accommodate non-commuting coordinates on the boundary of the theory which may be used to construct $D-$brane actions.

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The spacetimetime dependent lagrangian formalism [1-2] gives an alternative way to deal with electromagnetic duality [3], weak-strong duality [4] and electro- gravity duality [5]. Here this method will be used to obtain an estimate of the vacuum expectation value of the Higgs field in terms of the electric charge e and a constant. The motivation of the present work comes from Ref.[2b] where an analogue of the Bogomolny bound has been obtained for the Barriola-Vilenkin [6] gravitational monopole. We will also show that the 't Hooft ansatz for obtaining the "t Hooft-Polyakov monopole solution is sufficiently general to lead to other solutions containing coordinates near the boundary that do not commute. Moreover, we will show that the 't Hooft ansatz for the gauge field is sufficient to yield a solution for the Higgs field for $r \to \infty$ without the necessity of any further ansatz for ϕ. We first briefly review the relevant material of [1].

Let the lagrangian L' be a function of fields η_ρ, their derivatives $\eta_{\rho,\nu}$ *and the spacetime coordinates* x_ν, i.e. $L' = L'(\eta_\rho, \eta_{\rho,\nu}, x_\nu)$. Variational principle [12] yields :

$$\int dV \left(\partial_\eta L' - \partial_\mu \partial_{\partial_\mu \eta} L' \right) = 0 \tag{1}$$

Assuming a separation of variables : $L'(\eta_\sigma, \eta_{\sigma,\nu}, ..x_\nu) = L(\eta_\sigma, \eta_{\sigma,\nu})\Lambda(x_\nu)$
($\Lambda(x_\nu)$ is the x_ν dependent part and is a finite non-vanishing function) gives

$$\int dV \left(\partial_\eta (L\Lambda) - \partial_\mu \partial_{\partial_\mu \eta} (L\Lambda) \right) = 0 \tag{2}$$

We will be confined to classical solutions of theories where the fields do not couple to gravity. Then Λ is not dynamical and is a finite, non-vanishing function of x_ν multiplying the primitive lagrangian L. It is like an external field and equations of motion for Λ meaningless. Duality invariance is related to finiteness of Λ. When equations of motion are duality invariant,

finiteness of Λ on the spatial boundary at infinity leads to new solutions for the fields. Poincare invariance and duality invariance is achieved through same behaviour of Λ. The finite behaviour of Λ on the boundary encodes the exotic solutions of the theory within the boundary. In this way we are reminded of the holographic principle.

Consider the Georgi-Glashow model with [3]

$$L = [-(1/4)G_a^{\mu\nu}G_{a\ \mu\nu} + (1/2)(D^\mu\phi)_a(D_\mu\phi)_a - V(\phi)] \qquad (3a)$$

where usually one takes $V(\phi) = (\lambda/4)(\phi^a\phi^a - a^2)^2$. The gauge group is $SO(3), a, b, c$ are $SO(3)$ indices, with the generators τ^a satisfying $[\tau^a, \tau^b] = i\epsilon^{abc}\tau^c$. Gauge fields $W_\mu = W_\mu^a\tau^a$ and the field strength is $G_a^{\mu\nu} = \partial^\mu W_a^\nu - \partial^\nu W_a^\mu - e\epsilon_{abc}W_b^\mu W_c^\nu$, $\tilde{G}_a^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}G_{a\ \rho\sigma}$; and the matter fields ϕ are in the adjoint representation of $SO(3)$. The equations of motion are

$$(D^\nu G_{a\ \mu\nu})_a = \partial^\mu\phi_a - e\epsilon_{abc}W_b^\mu\phi_c \qquad (3b)$$

$$(D^\mu D_\mu\phi)_a = -\partial_{\phi^a}V \qquad (3c)$$

There are also the Bianchi identities

$$D^\mu\tilde{G}_{a\ \mu\nu} = 0 \qquad (3d)$$

Duality invariance means that $D^\mu G_{a\ \mu\nu} = 0$. The energy density is:

$$\Theta_{00} = (1/2)[(E_a^i)^2 + (B_a^i)^2 + (D^0\phi_a)^2 + (D^i\phi_a)^2 + V(\phi)] \qquad (4)$$

where the non abelian electric and magnetic fields are defined respectively as: $E_a^i = -G_a^{0i}$ and $B_a^i = -(1/2)\epsilon_{jk}^i G_a^{jk}$

The energy density $\Theta_{00} \geq 0$ and the vacuum configuration is

$$G_a^{\mu\nu} = 0 \ ; \ D_\mu\phi = 0 \ ; \ V(\phi) = 0 \qquad (5)$$

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For this theory L' is then

$$L' = L\Lambda = [-(1/4)G_a^{\mu\nu}G_{a\ \mu\nu} + (1/2)(D^\mu\phi)_a(D_\mu\phi)_a - V(\phi)]\Lambda(x_\nu) \qquad (6)$$

Equations of motion using (2) are :

$$\Lambda(D^\mu G_{a\ \mu\nu}) + (\partial^\mu\Lambda)G_{a\ \mu\nu} + \Lambda e\epsilon_{abc}(\partial_\nu\phi)_b(\phi)_c - \Lambda e^2\epsilon_{abc}\epsilon_{bc'd'}W_{\nu\ c'}\phi_c\phi_{d'} = 0$$
$$(7a)$$

$$(D^\mu D_\mu\phi)_a\Lambda + (D_\mu\phi)_a\partial_\mu\Lambda = -(\partial_{\phi^a}V)\Lambda \qquad (7b)$$

and the Bianchi identities are:

$$D^\mu\tilde{G}_{a\ \mu\nu} = 0 \qquad (7c)$$

Requiring duality invariance (i.e.$D^\mu G_{a\ \mu\nu} = 0$) gives

$$(\partial^\mu\Lambda)G_{a\ \mu\nu} = -\Lambda e\epsilon_{abc}(D_\nu\phi)_b(\phi)_c \qquad (8)$$

For $\Lambda = \Lambda(r)$ we have

$$\Lambda_\infty = \Lambda_0 exp[-e\int_0^\infty dr\left((\epsilon_{abc}(D_\nu\phi)_b\phi_c)(\partial^i rG_{a\ i\nu})^{-1}\right)] \qquad (9)$$

where Λ_p is the value of Λ at $r = p$; a, ν are fixed; and there is a sum over indices i, b and c. Λ_∞ must be finite. Choose this to be the constant unity.This may be realised in various ways, the simplest being $(D_\nu(\phi)_b \Rightarrow 0$, $(\phi)_c \Rightarrow finite$, and the product $(D_\nu\phi)_b(\phi)_c$ falls off faster than $G_{a\ i\nu}$ for large r. Then a constant value for Λ is perfectly consistent with (7b) and the conditions become analogous to the Higgs' vacuum condition for the t'Hooft-Polyakov monopole solutions where the duality invariance of the equations of motion and Bianchi identities are attained at large r by demanding $(D_\mu\phi)_a \Rightarrow 0$ and $\phi_a \Rightarrow a\delta_{a3}$ at large r. Note that our results are perfectly consistent with the usual choice for the Higgs' potential $V(\phi)$ even though nothing has

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been assumed regarding this. Thus, the t'Hooft-Polyakov monopole solutions follow naturally in our formalism. We now discuss two other interesting possibilities.

Case I

$$(\epsilon_{abc}(D_\nu\phi)_b\phi_c) \Rightarrow 0 \tag{10}$$

(i.e. the duality condition $(D^\mu G_{\nu\mu})_a = 0$) and falls off faster than $G_a\ _{i\nu}$ for large r (a and ν are fixed). A solution is when

$$D_\nu\phi = \alpha_\nu\phi \tag{11}$$

where α_ν can be any Lorentz four vector field that is consistent with all the relevant equations of motion and the minimum energy requirements. The minimum energy requirements are satisfied because it is straightforward to verify that the gauge fields W_a^μ do not change. This is seen by taking the cross product of ϕ (ϕ is a $SO(3)$ vector) with equation (11). We again arrive at the well known results of Corrigan *et al* [7], *viz.* $W^\mu = (1/a^2e)\phi\wedge\partial^\mu\phi+(1/a)\phi A^\mu$, where A^μ is arbitrary.

As the gauge fields W_a^μ do not change, so even with this solution we obtain the same gauge field solutions as before and so minimum energy requiremet is automatically satisfied. However, this new solution allows us to obtain an estimate of the vacuum expectation value of the Higgs field and to this we now proceed. Let $\alpha_\nu = (0, \alpha_i) \equiv \alpha(r)\hat{r}$, where $\hat{r}$ is the unit radial vector. So the Bogomolny condition is $B_a^i = D^i\phi_a = \alpha^i\phi_a$ and the Higgs vacuum condition obtained from equation (7b) (for $r \to \infty, \Lambda$ is a constant , say unity) is

$$[D^i(\alpha_i\phi)]_a = -\partial_{\phi^a}V = 0 \tag{12a}$$

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i.e. we are at a minima of the potential V. If $\phi_a \neq 0$, $(12a)$ implies

$$div\ \vec{\alpha} + \vec{\alpha}^2 = 0 \tag{12b}$$

and the solution is

$$\alpha(r) = 1/(cr^2 - r)\ ,\alpha^i = r^i/(cr^3 - r^2) \tag{13}$$

where we take the constant c to be negative. Let us now take the 't Hooft ansatz for the gauge field, *viz.*

$$W_a^0 = 0\ \ ; W_a^i = -\epsilon_{aik} r^k [1 - K(aer)]/(er^2) \tag{14}$$

where the function $K(aer)$ has been well studied [3] and goes to zero at $r \to \infty$. Then the electric field vanishes while $G_{a\ jk}, B_a^i$ are

$$G_{a\ jk} = (1/er^2)[2\epsilon_{ajk}(1 - K) + \epsilon_{akl}r^l\partial_j K - \epsilon_{ajl}r^l\partial_k K]$$

$$+(1/er^4)[2(1-K)(\epsilon_{akl}r^l r_j - \epsilon_{ajl}r^l r_k) + (1-K)^2(\delta_{aj}\epsilon_{ckl}r^c r^l - \epsilon_{jkl}r_a r^l)] \tag{15a}$$

$$B_a^i = (1/er^4)[(1-K)^2 r_a r^i - 2(1-K)r^i r_a] - (1/er^2)[r^i\partial_a K - \delta_a^i r^m\partial_m K] \tag{15b}$$

Now $B_a^i = D^i\phi_a = \alpha^i\phi_a$.Therefore,taking $c = -A$ so that A is positive we have

$$\phi_a = \frac{(1 + Ar)}{(er^2)}[2(1 - K)r_a - (1 - K)^2 r_a + r^2\partial_a K - r_a r^m\partial_m K] \tag{15c}$$

It is easily seen that $(15c)$ reduces to the 't Hooft ansatz for ϕ_a for $A = 0$ and $r \to \infty$. Thus we have obtained an expression for ϕ without assuming any ansatz. This had never been possible before. There is another interesting outcome.For $r \to \infty$ we have $K \to 0$ and so

$$\phi_a \to \frac{Ar^a}{er} + \frac{r^a}{er^2} = \frac{A}{e}\hat{r}^a + \frac{\hat{r}}{er} \to \frac{A}{e}\hat{r}^a \tag{16}$$

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for $r \to \infty$. But $\phi_a = a\delta_{a3}$ for $r \to \infty$. Therefore $a = A/e$. This is the principal result of this work.

Case II

α_ν is any Lorentz four vector field as in **I** but which may also carry internal symmetry indices *other than* $SO(3)$ with the generators of the symmetry satisfying some Lie algebra $[T_P, T_Q] = if_{PQR}T_R$. Let us take the group to be $SU(2)$. i.e. say, $\alpha^\nu = \alpha_P^\nu T_P$; $P, Q, R = 1, 2, 3$; T_P being the generators of $SU(2)$. Again choosing $\alpha^\nu = (0, \alpha_P^i T^P)$ with $\vec{\alpha}_P = \alpha_P(r)\hat{r}$ and using the well known properties of the Pauli matrices it is easily seen that the analogue of equation (12) is

$$div \ \vec{\alpha_P} = 0 \tag{17}$$

which has the solution

$$\alpha_P^i = A_P \frac{r^i}{r^3} \tag{18}$$

where A_P are constants. Writing $r_P^i = A_P r^i = r^3 \alpha_P^i$, we can then define new coordinates

$$R^i = r_P^i T_P \ ; [R^i, R^j] \neq 0 \tag{19}$$

and these are non-commuting. Moreover, they carry both Lorentz and internal indices and hence are like gauge fields in some different theory. Note that transverse coordinates (i.e. transverse to the brane and lying in the bulk volume) in D brane theories are often identified with gauge fields [8] and so we can construct such actions with our solutions (19). Under these circumstances, equation (11) should be written as

$$\partial_\mu^P \phi_a - e\epsilon_{abc}(W_\mu^b)^P \phi_c = \alpha_\mu^P \phi_a$$

where the coordinates and their differentials are now matrices and capital

alphabets denote the indices of the new symmetry group. For fixed P, $(W_\mu^b)^P$ may be identified with the old gauge fields W_μ^b.

A point to note is that we have taken the symmetry group for α_ν to be some group other than $SO(3)$. This is to ensure in the simplest possible way that the fields $(W_\mu^b)^P$ *for fixed P* may be identified with the old (i.e.unchanged) gauge fields W_μ^b (P is now a fixed index) and so the minimum energy requirements are satisfied in each sector of $P = 1, 2, 3$. (This is seen by taking the cross product of ϕ with the analogue of equation (11) and proceeding as before).So each sector now contains a monopole. Then we have a configuration that is quite similar to "string" of monopole solutions connecting two D-branes. Such configurations are known in the literature [8]. The other point is that the vacuum expectation value of the Higgs field is proportional to the inverse of the coupling e; and this result has been obtained from the classical solutions. This result is similar to that obtained in Ref.[2b] if we are ready to identify the inverse of Newton's gravitational constant (which is definitely the coupling constant in theories of gravity) as the vacuum expectation value of some field hitherto unknown.

The solutions in equation (11) were hidden in 't Hooft-Polyakov's work. This had been overlooked before for the simple reason because at that point of time one was more concerned in obtaining solutions from the minimum (finite) energy principles. This was perfectly justified. We have obtained the solutions from the requirement of duality invariance which is quite relevant at this point of time. However, we have also shown that the duality requirements automatically contain the minimum energy condition (Λ is also finite for $D_\nu\phi = 0$).

All the results have been obtained at $r \to \infty$. That is, we are at the boundary of the theory. So the finiteness of Λ at the boundary encodes the duality invariance of the theory within the boundary and thus an analogue of the holographic principle [9] seems to be at work. On the boundary there seems to exist *a different gauge field theory* together with non-commuting coordinates.

In conclusion, the spacetime dependent lagrangian formalism in conjunction with the 't Hooft-Polyakov results have yielded an expression for the vacuum expectation value of the Higgs field as A/e. This result is definitely susceptible to experiments. We have also shown that the 't Hooft ansatz for the gauge field is sufficient to obtain an expression for the Higgs field if one uses our formalism. No additional ansatz for ϕ is necessary. The expression obtained reduces to the 't Hooft ansatz for the Higgs field at $r \to \infty$. Finally, we have shown that classical solutions of Yang-Mills theory also contain the germ of non-commuting coordinates residing on the boundary. The structure of these coordinates are like gauge fields and hence are relevant in constructing D-brane actions.

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