

Universality in Quantum Computation

David Deutsch, Adriano Barenco, and Artur Ekert

Clarendon Laboratory, Department of Physics

University of Oxford, Oxford, OX1 3PU, UK

February 1, 2008

Abstract

We show that in quantum computation almost every gate that operates on two or more bits is a universal gate. We discuss various physical considerations bearing on the proper definition of universality for computational components such as logic gates.

To be published in Proc.R.Soc.London A, June 1995.

Introduction

It has been known for several years that the theory of quantum computers — i.e. machines that rely on characteristically quantum phenomena to perform computations [1] — is substantially different from the classical theory of computation, which is essentially the theory of the universal Turing machine. We may identify three important differences. Firstly, the properties of quantum computers are not postulated *in abstracto* but are deduced entirely from the laws of physics. Logically, this was already true of the classical theory, as Landauer [2] has pointed out, but the intuitive nature of the classically-available computational operations, and the millennia-long history of their study, allowed pioneers such as Turing, Church, Post and Gödel to capture the correct classical theory by intuition alone, and falsely to assume that its foundations were self-evident or at least purely abstract. (There is an analogy here with geometry, another branch of physics that was formerly regarded as belonging to mathematics.) Secondly, quantum computers can perform certain classical tasks, such as factorisation [3], using quantum-mechanical algorithms [4] which have no classical analogues and can be overwhelmingly more efficient than any known classical algorithm. Thirdly, quantum computers can perform new computational tasks, such as

quantum cryptography [5], which are beyond the repertoire of any classical computer. (The class of quantum computable *functions* is precisely the set of classical recursive functions. However, unlike in the classical case, not all quantum computations can be re-interpreted as function evaluations.)

Computation may be defined as the systematic creation of symbols (the “output”) which, under a given method of interpretation, have abstract properties that were specified in other symbols (the “input”). “Symbols” here are physical objects. A *universal* set of components is one that is adequate for the building of computers to perform any physically possible computation. A *universal computer* is a single machine that can perform any physically possible computation. The concept of universality for computers on the one hand and for components on the other, and the concept of computation itself, are all closely linked. For if the solution to a problem could be created by a certain physically possible computer, but there were no systematic method of building that computer, then the solution would not necessarily be “computable” in any useful sense. But if there were a systematic method of building a computer to solve each such problem, then the factory that manufactured such computers to order would in effect be a universal computer. And if there is a universal computer, there must be a universal set of components, namely the components that are needed to build it. Similarly, if there is a finite universal set of components then a factory for manufacturing and assembling them into given configurations would also be a universal computer.

Both the classical and the quantum theory of computation admit universal computers. But the ability of the respective universal computers to perform any computation that any other machine could perform under the respective laws of physics, could in the classical case only be conjectured (the Church-Turing conjecture). In the quantum theory it can be proved [1], at least for quantum systems of finite volume. This is one of the many ways in which the quantum theory of computation has turned out to be inherently simpler than its classical predecessor.

In this paper we concentrate on universality for components, and in particular for quantum logical gates. These are the active components of *quantum computational networks* [6], which are computers in which quantum two-state systems (i.e. quantum bits or *qubits*) are carried inertly from one gate to another. In an n -bit quantum gate, n qubits undergo a coherent interaction. Barenco [7] has shown that any two-bit gate $\mathbf{A}(\phi, \alpha, \theta)$ that effects a unitary transformation of the form

$$A(\phi, \alpha, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha} \cos \theta & -ie^{i(\alpha-\phi)} \sin \theta \\ 0 & 0 & -ie^{i(\alpha+\phi)} \sin \theta & e^{i\alpha} \cos \theta \end{pmatrix}, \quad (1)$$

on the state of two qubits is universal, where the representation (1) is in terms of the

computation basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, and α , ϕ , and θ are irrational multiples of π and of each other.

The fact that the laws of physics support computational universality is a profound property of Nature. Since any computational task that is repeatable or checkable may be regarded as the simulation of one physical process by another, all computer programs may be regarded as symbolic representations of some of the laws of physics, specialised to apply to specific processes. Therefore the limits of computability coincide with the limits of science itself. If the laws of physics did not support computational universality, they would be decreeing their own un-knowability. Since they do support it, it would have been strangely anthropocentric if universality had turned out to be a property of a very narrowly-defined class of interactions (such as (1)). For then the physical processes in which such interactions occurred, presumably including certain human artefacts, would have required a fundamentally more general mathematical description than most other physical processes in the universe. But it turns out that the opposite is the case. Almost every class of physical processes must instantiate the same, standard set of mathematical relationships, namely those that are quantum computable. For we shall prove that universality is not confined to the special class (1) of gates, but that almost all two-bit quantum gates are universal. This confirms, and betters, the conjecture of Deutsch [6] that almost all *three*-bit quantum gates are universal.

Proof that almost all 2-bit gates are universal

Consider a two-bit quantum gate \mathbf{U} that effects a unitary transformation U of $U(4)$. We shall prove that a generic \mathbf{U} is universal, i.e. that the set of transformations in $U(4)$ corresponding to non-universal gates is of lower dimensionality than $U(4)$ itself, where $U(4)$ is considered as a 16-dimensional manifold with the natural metric $\sqrt{1 - \frac{1}{4}\text{Re}(\text{Tr}(P^\dagger Q))}$. Define the *repertoire* of \mathbf{U} as the set of gates whose effects on their input qubits can be approximated with arbitrary accuracy by networks containing only \mathbf{U} -gates.

U has the form

$$U = e^{i\hat{H}_1}, \quad (2)$$

where \hat{H}_1 is a Hermitian operator. U and its generator \hat{H}_1 are diagonal in the same basis, so n successive applications of the gate on the same pair of qubits effects the unitary operation U^n given in the diagonal basis by

$$\begin{pmatrix} e^{in\phi_1} & 0 & 0 & 0 \\ 0 & e^{in\phi_2} & 0 & 0 \\ 0 & 0 & e^{in\phi_3} & 0 \\ 0 & 0 & 0 & e^{in\phi_4} \end{pmatrix}, \quad (3)$$

where the ϕ_j are the eigenvalues of \hat{H}_1 . If, as will be the case for generic \mathbf{U} , the ϕ_j are irrational multiples of π and of each other, n can be chosen so that the four expressions

$$\tilde{\phi}_j = n\phi_j \bmod 2\pi \quad j = 1 \dots 4 \quad (4)$$

approximate arbitrarily well any set of values $\tilde{\phi}_j$ in the interval $[0, 2\pi[$. In particular, for any real λ and ϵ , there exists an integer n such that $\tilde{\phi}_j = (\lambda\phi_j \bmod 2\pi) + O(\epsilon)$ for $j = 1 \dots 4$. In other words, any operation of the form

$$U^\lambda = e^{i\lambda\hat{H}_1} \quad (5)$$

is in our repertoire.

A second gate $\tilde{\mathbf{U}}$ defined by

$$\tilde{U} = TUT = e^{i\hat{H}_2} \quad (6)$$

is obtained directly from \mathbf{U} by exchanging the two input qubits just before they enter the gate and the two output qubits just after they emerge. \mathbf{T} (“twist”) is the unitary operation corresponding to each of these exchanges. In the computation basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ it has the representation

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (7)$$

The generator $\hat{H}_2 = T\hat{H}_1T$ of $\tilde{\mathbf{U}}$ is linearly independent of \hat{H}_1 provided that \hat{H}_1 does not commute with T , which again, generically, it does not.

Now note that if all operations generated by a pair of Hermitian operators \hat{P} and \hat{Q} are in a given repertoire, and α and β are real, then every operation generated by $\alpha\hat{P} + \beta\hat{Q}$ is also in the repertoire [12]. This is because

$$e^{i(\alpha\hat{P} + \beta\hat{Q})} = \lim_{n \rightarrow \infty} \left(e^{i\alpha\hat{P}/n} e^{i\beta\hat{Q}/n} \right)^n. \quad (8)$$

Likewise, every operation generated by the commutator $i[\hat{P}, \hat{Q}]$ is in the repertoire because

$$e^{i[\hat{P}, \hat{Q}]} = \lim_{n \rightarrow \infty} \left(e^{-i\hat{P}/\sqrt{n}} e^{i\hat{Q}/\sqrt{n}} e^{i\hat{P}/\sqrt{n}} e^{-i\hat{Q}/\sqrt{n}} \right)^n. \quad (9)$$

Thus we can use the generator

$$\hat{H}_3 = i[\hat{H}_1, \hat{H}_2] \quad (10)$$

to generate a third class of operations in our repertoire. All the operations generated by arbitrary linear combinations of \hat{H}_1 , \hat{H}_2 and \hat{H}_3 are in the repertoire too. Physically

these are all obtained by acting on a single pair of qubits with a long chain of \mathbf{U} -gates, some connected directly and others by a twisted pair of wires.

This procedure can be repeated and new generators obtained by commuting ones that have already been derived. If at any stage sixteen linearly independent generators \hat{H}_j have been constructed in this way from \mathbf{U} , the universality of \mathbf{U} is established.

Consider the scheme

$$\left. \begin{aligned} \hat{H}_2 &= T\hat{H}_1T \\ \hat{H}_j &= i[\hat{H}_1, \hat{H}_{j-1}] \quad j = 3 \dots 14 \\ \hat{H}_{15} &= i[\hat{H}_2, \hat{H}_3] \\ \hat{H}_{16} &= i[\hat{H}_2, \hat{H}_5]. \end{aligned} \right\} \quad (11)$$

Linear independence of the sixteen generators \hat{H}_j is equivalent to the non-vanishing of the determinant of a 16×16 matrix consisting of the coefficients of the decomposition of the \hat{H}_j in an orthonormal basis. Suppose that, for a particular gate \mathbf{U} , this determinant does vanish. We must show that such gates lie in a lower- (less than 16-) dimensional sub-manifold of $U(4)$. To this end, consider a universal gate \mathbf{A} of the form (1). Straightforward but tedious calculation verifies that the generators $\hat{H}_1^A \dots \hat{H}_{16}^A$ formed according to the scheme (11) *are* linearly independent. The simpler scheme defined by the first two lines of Eq.(11) with $j = 3 \dots 16$ does not have this property. Then consider the one-parameter family of generators $\hat{H}_j(k)$ formed according to the same scheme (11), but with

$$\hat{H}_1(k) = \hat{H}_1 + k(\hat{H}_1^A - \hat{H}_1). \quad (12)$$

Let $\Delta(k)$ be the corresponding determinant. This is a polynomial of degree 100 in k which, in the unfavourable case we are considering, vanishes at $k=0$ where $\hat{H}_1(0) = \hat{H}_1$. But this polynomial cannot be identically zero, since we have checked that $\Delta(1) \neq 0$. Therefore it can have at most 99 zeros in addition to the one at $k = 0$. For every other value of k , the gate generated by the corresponding $\hat{H}_1(k)$ is universal, and in particular there is an interval around $k = 0$ in which every generator $\hat{H}_1(k)$ other than \hat{H}_1 generates universal gates.

A similar argument applies to an entire 16-dimensional neighbourhood of the generator \hat{H}_1 . The generators in a sufficiently small neighbourhood can be parametrised by sixteen coordinates, which can be chosen in the manner of (12) so that a generator of \mathbf{A} lies at a finite point in the coordinate space. The determinant formed according to the scheme (11) from each of these generators is a polynomial in each of these sixteen coordinates — so it is an analytic function which, even if it vanishes at \hat{H}_1 , is not identically zero. Hence it can at worst vanish on a 15-dimensional sub-manifold (more precisely on a 15-dimensional variety) of the neighbourhood of \hat{H}_1 , and our result is proved.

It also follows that every generator of a *universal* gate is surrounded by a neighbourhood containing only such generators.

Analogous results hold for n -bit gates for all $n > 2$.

(Elements of the above proof have been derived independently by Lloyd [13].)

Which gates are *not* universal?

Clearly there can be no 1-bit universal gate because a 1-bit gate, and indeed any number of 1-bit gates, cannot place two initially un-entangled qubits into an entangled state. Likewise no classical gate can be universal because, by definition, a classical gate evolves computation-basis states to other computation-basis states and never to superpositions of them. Similarly, no gate that had that property with respect to *any* fixed bases in the state spaces of the qubits that it acted upon, could be universal. Such gates are properly called “classical” also. The question arises whether there are any other non-universal gates.

Our proof leaves that question open because at various stages we have imposed generic conditions on parameters to prove the universality of certain classes of gates. In most cases, however, failure to meet those conditions is no guarantee of *non*-universality. For example if, for a particular gate, the scheme (11) does not yield sixteen linearly independent generators, there may be other schemes that do. Even if there are none, that does not rule out constructions involving more general networks than the simple chains we have considered, in which instances of the gate could be composed to make a universal computer. Likewise, if the parameters ϕ_j violate the irrationality conditions, the corresponding gate may nevertheless be universal. For instance the gate $\mathbf{A}(\pi, \pi/2, \theta)$ is known [8] to be universal even though the second parameter is a rational fraction of π .

We conjecture that the non-universal gates are precisely

- the 1-bit gates and collections of 1-bit gates; and
- the classical gates.

If true, this would reveal an interesting connection between the existence of a “classical level” in physics (i.e. a regime in which classical physics is a good approximation to quantum physics) and the existence of classical computation as a closed and stable regime within quantum computation.

How to define universality for components

We believe that the quantum computational network model, with its active “gates”, passive “wires” and moving “qubits”, is a robust and reliable idealisation for analysing a wide class of possible technologies for quantum computation. It might well cover *all possible* technologies, but we wish to stress that this is neither proven nor self-evident. The proper definition of “universality”, like everything else in the quantum theory of computation, depends on what the laws of physics are. Therefore we shall now point out some of the assumptions we have made about the physics and technology of computation, which will have to be justified or perhaps amended by future, deeper analyses.

One of our most fundamental assumptions is that the most general possible computations can indeed be performed by machines — i.e. by well-defined physical systems which can be constructed to order and which maintain their identity during computations. This assumption is presumably secure, given the necessity for *unitary* operations in quantum computation. Unitarity can only be maintained in systems from which the rest of the universe is isolated.

It is not quite so clear that these systems must in turn be composed of well-defined, albeit interacting, subsystems, i.e. computational *components* such as gates. All we can say is that it is hard to conceive of a technology to manufacture complex computing machines other than from simpler subsystems which are themselves computing machines. That the computational state should be carried by *two-state* quantum systems rather than three- or higher-state ones is obviously not a necessary assumption, but all our conclusions still hold when straightforwardly generalised to computers and components that use more complex information carriers, so long as their state spaces are of finite dimension.

In the theory of quantum computational networks, a gate is considered to be universal if instances of it are the only computational components required to build a universal computer. That does not mean that those gates would be the only *physical* components of such a computer. At the very least there must also be “wires”, or some other means of presenting the qubits to the gates at the right times and in the right combinations. And there must be input and output devices and presumably many other components that form the environment in which the gates and bits interact in the necessary ways. Yet however indispensable such components are, they perform no strictly *computational* function in that they do not change the state of the computation. For instance, a wire may be regarded as a gate, but it is the trivial “identity gate” which does not affect the quantum state of the qubits that pass through it. Only components that affect the computational state count as gates in discussions of universality. However, the distinction between “computational” and “non-computational” operations can only be made relative to a given physical and technological implementation.

Different laws of physics, or different technologies, would lead us to draw the line differently. Consider, for example, the proposed quantum-dot-based technology [9] in which qubits are stored as the states of individual electrons trapped at fixed locations (“dots”) and interacting only with their nearest neighbours in an array. The form of the interaction (i.e. the type of gate) is determined by externally applied electric fields and radiation. There are no physical “wires” to move qubits into adjacent positions so that they can undergo a gate-type interaction. Instead this is achieved by successively swapping the states of adjacent dots, each swap involving three elementary “controlled-not” gate operations [9]. Thus the computational state of each qubit is transferred unchanged from one physical dot to another, so in that sense such an operation, taken as a whole, is computationally trivial. Nevertheless the only way of realising it is as the net effect of several non-trivial computations. Therefore in quantum dot technology, the operations that move qubits around *are* computational operations.

So we see that in principle a gate may be universal or not according to the physics and technology with which it is realised, and in particular, universality for quantum dot technology has to be defined slightly differently from universality for quantum networks. We may define a universal 2-bit *operation* \mathbf{U} on a quantum dot array as follows: \mathbf{U} is universal if for every integer n , any unitary transformation of n qubits can be effected with arbitrary precision by successive applications of \mathbf{U} to suitable pairs of adjacent quantum dots. The proof we have given above must be adapted to show that in quantum-dot and related cellular-automaton-like technologies, almost all 2-bit operations are universal: The key difference is that we cannot assume *ab initio* that the “twist” operation \mathbf{T} is in the repertoire. We can only assume that the given operation \mathbf{A} can be applied to an arbitrary pair of adjacent dots. But that means in particular that it can be applied in two senses to a given pair, regarding them in either order as the “first” and “second” input bits of the gate. So \mathbf{TAT} is automatically in the repertoire even though \mathbf{T} itself is not (initially). It then follows from the proof we have given that every 2-bit gate, and in particular the controlled-not gate, is in the repertoire. From this, as we have said, both \mathbf{T} and the “wire” or qubit-moving operation can be constructed, and our result follows.

Another way of expressing the distinction between computational and non-computational operations is through the concept of *composition*. Compositions are operations which, though necessary in the construction and operation of computing machines, do not affect the computational state (even transiently). We compose *components* when we join one to another to form a more complex machine. We compose *computational operations* when we use the output of one sub-computation as the input of another. As we have seen, presenting the appropriate qubits as the inputs of a gate is mere composition in a quantum network technology, but not in a quantum dot technology. Strictly speaking, definitions of universality refer to computations of arbitrary length

and complexity, using arbitrarily large amounts of storage. This means that we must contemplate mechanisms for providing a supply of qubits in a standard state (the equivalent of Turing’s unlimited “blank tape”), and for maintaining the computer in operation for an arbitrarily long period. Before we can certify that a given component or operation is universal, we must satisfy ourselves that these mechanisms do not themselves affect the state of the computation.

In addition to “blank tape”, some computations require “sources” of qubits in fixed states. The constructions in our proof do not require such sources; moreover any source can be constructed from the “blank tape” by suitable operations from our universal repertoire. However, this too need not be true in every technology. It is possible that there are technologies in which a given component or operation is universal if sources of qubits in given states are available, but not otherwise.

If any quantum computation, required to produce its output state with a given accuracy, is intractable when it is performed entirely by composing instances of a universal operation \mathbf{U} , but is tractable using a larger, finite set of elementary operations, then the universality of \mathbf{U} may well be unphysical. In this paper we have so far paid no attention to issues of complexity, and the specific constructions in our proofs are very inefficient. But we should expect efficient constructions to exist. For suppose that we want to effect a given n -bit unitary operation \mathbf{X} , with probability no less than $1 - \epsilon$, using a universal n -bit unitary operation \mathbf{U} . From \mathbf{U} and suitable compositions we can, in $O(n)$ steps, create a second operation \mathbf{V} that does not commute with \mathbf{U} (for instance \mathbf{TUT} when $n = 2$). The total number of k -step sequences of \mathbf{U} ’s and \mathbf{V} ’s is 2^k , and generically, exponentially many of them are different. Now, the manifold $U(2^n)$ of n -bit unitary operations, with metric $\sqrt{1 - 2^{-n}\text{Re}(\text{Tr}(P^\dagger Q))}$, is compact. Therefore the average error probability ϵ when the best of the 2^k sequences is used to approximate \mathbf{X} must fall exponentially with k , and conversely the number of steps (or gates) required to perform \mathbf{X} with probability $1 - \epsilon$ is polynomial in $\log \epsilon$. This heuristic argument about *average* efficiencies does not cover the worst case, which strictly speaking is the relevant one. Nevertheless it gives us excellent reason to believe that polynomial efficiency is possible even in the worst case and in general technologies (see also Yao [10]).

The “error probability” we have just been discussing is that due to the approximation of a continuous group $U(2^n)$ by finite sets of elements. Practical computing machines are of course also subject to physical sources of error, such as thermal noise, unwanted interactions and imperfect machining of components. Error-correction is itself a form of computation, and error-correction strategies for coherent quantum computations must themselves involve quantum computation [11]. In general, unlike in classical computation, the best error correction strategy depends on the specific computation that is being corrected. Therefore in principle it is possible that a gate or operation might be universal only under the idealisation that errors are absent.

The problem of errors due to imperfect machining is somewhat ameliorated by the results of this paper. For they show that the strategy of building quantum computers need not be, as in the classical case, to design a set of abstract operations and then to try to realise them as accurately as possible. Instead we can find *almost any* 2-qubit operation that can be conveniently and accurately performed and composed, and then be confident that this operation can be used as the elementary operation of a universal quantum computer.

Conclusion

The results of this paper are good evidence that universality in a strong, robust and practical sense is generic in arbitrary quantum computer technologies. But a full proof must await a deeper theoretical integration of physics and computation than is yet available.

Acknowledgements

The authors wish to thank Richard Jozsa and David DiVincenzo for interesting discussions and comments on the subject of this paper. This work was partially supported by the Royal Society. A.B. acknowledges the financial support of the Berrow's fund at Lincoln College, Oxford.

References

- [1] D. Deutsch, Proc. R. Soc. London A **400**, 97 (1985).
- [2] R. Landauer, IBM J. Res. Develop. **5**, 183 (1961).
- [3] P.W. Shor, *Proc. 35th Ann. Symp. on the Foundations of Computer Science* (IEEE Computer Society, Los Alamitos, 1994), p.124.
- [4] D. Deutsch and R. Jozsa, Proc. R. Soc. Lond. A **439**, 553 (1992); E. Bernstein and U. Vazirani, *Proc. 25th ACM Symp. on Theory of Computation*, p.11 (1993).
- [5] S. Wiesner, SIGACT News, **15**, 78 (1983); C. H. Bennett and G. Brassard, in *Proc. IEEE Int. Conference on Computers, Systems and Signal Processing*, (IEEE, New York, 1984).; A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991); C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).
- [6] D. Deutsch, Proc. R. Soc. Lond. A **425**, 73 (1989).

- [7] A. Barenco, University of Oxford preprint (1994).
- [8] T. Sleator and H. Weinfurter, preprint (1994).
- [9] A. Barenco, D. Deutsch, A. Ekert, and R. Jozsa, *Logic gates for quantum computation*, Phys. Rev. Lett. (in print).
- [10] A. Yao, *Proc. 34th IEEE Symp. on Foundation of Computer Science* (IEEE Computer Society Press, Los Alamitos, 1993), pp.352-360.
- [11] A. Berthiaume, D. Deutsch and R. Jozsa, *The stabilization of quantum computations*, in *Proceedings of the Workshop on the Physics and Computation—PhysComp '94* (IEEE Computer Society Press, Dallas, Texas).
- [12] D. DiVincenzo, Phys. Rev. A **50**, 1015 (1995).
- [13] S.Lloyd, preprint (1994).