Transversely projective holomorphic foliations with singularities

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Abstract

In this paper we study the classification holomorphic foliations with singularities. The main hypothesis is the existence of a projective transverse structure outside of an analytic invariant subset of codimension one. We prove classification results for germs of foliations and for foliations in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ in terms of logarithmic and Riccati foliations. Our main result reads as follows: *Let $\mathcal{F}$ be a foliation on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ and with a projective transverse structure in the complement of an invariant algebraic curve $\Lambda \subset \overline{\mathbb{C}} \times \overline{\mathbb{C}}$. Assume that the singularities of $\mathcal{F}$ in Λ are non-resonant generalized curves. Then $\mathcal{F}$ is a logarithmic foliation or it is a rational pull-back of a Riccati foliation.*

Contents

2000 *Mathematics Subject Classification.* Primary 32E10; Secondary 32S65, 37F75, 32M25.

Key words and phrases. Holomorphic foliation, projective transverse structure, holonomy group, Riccati foliation.

The author wishes to thank Professor César Camacho for many valuable conversations, for coauthoring various results in this paper and for his constant and generous support.

1 Transversely projective foliations with singularities

Let M be a complex surface. A one-dimensional holomorphic *foliation with singularities* on M is a pair $\mathcal{F} = (\mathcal{F}_0, \operatorname{sing}(\mathcal{F}))$ where $\operatorname{sing}(\mathcal{F}) \subset M$ is a discrete set and $\mathcal{F}_0$ is a holomorphic foliation in the open set $M \setminus \operatorname{sing}(\mathcal{F})$. It is natural to assume that there is no extension of $\mathcal{F}_0$ to a point in $\operatorname{sing}(\mathcal{F})$. We call $\operatorname{sing}(\mathcal{F})$ the *singular set* of $\mathcal{F}$. By definition the leaves of $\mathcal{F}$ are the leaves of $\mathcal{F}_0$. By a standard application of Hartogs' extension theorem a foliation is given in a small neighborhood of a singularity by a holomorphic one-form with a singularity at the given singularity.

From now on in this paper by *foliation* we shall mean a holomorphic foliation with singularities in a complex dimension two space. A foliation $\mathcal{F}$ is called *transversely projective* if the underlying "non-singular" foliation $\mathcal{F}_0 =: \mathcal{F}\big|_{M\setminus\operatorname{sing}(\mathcal{F})}$ is transversely projective. This means that there is an open cover $\bigcup\limits_{j\in J} U_j = M \setminus \operatorname{sing}(\mathcal{F})$ such that in each U_j the foliation is given by a submersion $f_j \colon U_j \to \overline{\mathbf{C}}$ and if $U_i \cap U_j \neq \emptyset$ then we have $f_i = f_{ij} \circ f_j$ in $U_i \cap U_j$ where $f_{ij} \colon U_i \cap U_j \to \mathrm{SL}(2,\mathbf{C})$ is locally constant. Thus, on each intersection $U_i \cap U_j \neq \emptyset$, we have $f_i = \frac{a_{ij} f_j + b_{ij}}{c_{ij} f_j + d_{ij}}$ for some locally constant functions $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ with $a_{ij}d_{ij} - b_{ij}c_{ij} = 1$. Basic references for transversely affine and transversely projective foliations (in the nonsingular case) are found in [9].

As observed in [16] *the singularities of a foliation admitting a projective transverse structure are all of type $df = 0$ for some local meromorphic function.* In this work we will be considering foliations which are transversely projective in the complement of *codimension one invariant divisors*. Such divisors may, a priori, exhibit singularities which do not admit meromorphic first integrals.

Next we introduce our main model.

Example 1.1 (Riccati Foliations, cf. [16] Example 1.1 page 190)**.** We fix affine coordinates $(x,y) \in \mathbf{C}^2$ and consider a polynomial vector field $X(x,y) = p(x)\,\frac{\partial}{\partial x} + \big(a(x)y^2 + b(x)y + c(x)\big)\,\frac{\partial}{\partial y}$ on $\mathbf{C}^2$. Then X defines a *Riccati foliation* on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ as follows: if we change coordinates via $u = \frac{1}{x}\,,\ \ v = \frac{1}{y}$ then we obtain $X(x,v) = p(x)\,\frac{\partial}{\partial x} - \big(a(x) + b(x)v + c(x)v^2\big)\,\frac{\partial}{\partial v}\cdot$ Similarly for

$$X(u,y) = u^{-n}[\tilde{p}(u)\,\frac{\partial}{\partial u} + \big(\tilde{a}(u)y^2 + \tilde{b}(u)y + \tilde{c}(u)\big)\frac{\partial}{\partial y}] \text{ and}$$
$$X(u,v) = u^{-n}[\tilde{p}(u)\,\frac{\partial}{\partial u} + \big(\tilde{a}(u) + \tilde{b}(u)v + \tilde{c}(u)v^2\big)\frac{\partial}{\partial v}]$$

The similarity of these four expressions shows that Ω defines a holomorphic foliation $\mathcal{R}$ with isolated singularities on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ and having a geometric behavior as follows:
(i) $\mathcal{R}$ is transverse to the fibers $\{a\} \times \overline{\mathbf{C}}$ except for invariant fibers which are given in $\mathbf{C}^2$ by $\{p(x) = 0\}$.
(ii) If $\Lambda = \bigcup\limits_{j=1}^{r} \{a_j\} \times \overline{\mathbf{C}}$ is the set of invariant fibers then $\mathcal{R}$ is transversely projective in $(\overline{\mathbf{C}} \times \overline{\mathbf{C}})\setminus\Lambda$. Indeed, $\mathcal{R}|_{(\overline{\mathbf{C}}\times\overline{\mathbf{C}})\setminus\Lambda}$ is conjugate to the suspension of a representation $\varphi \colon \pi_1(\overline{\mathbf{C}}\setminus \bigcup\limits_{j=1}^{r}\{a_j\}) \to \mathbb{P}\,\mathrm{SL}(2,\mathbf{C})$.
(iii) For a generic choice of the coefficients $a(x), b(x), c(x), p(x) \in \mathbf{C}[x]$ the singularities of $\mathcal{R}$ on $\overline{\mathbf{C}}\times\overline{\mathbf{C}}$ are hyperbolic, Λ is the only algebraic invariant set and therefore for each singularity

$q \in \text{sing}(\mathcal{R}) \subset \Lambda$ there is a local separatrix of $\mathcal{R}$ transverse to Λ passing through q.

Now we consider the canonical way of passing from $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ to $\mathbf{CP}^2$ by a bi-rational map $\sigma\colon \overline{\mathbf{C}} \times \overline{\mathbf{C}} \to \mathbf{CP}^2$ obtained as a sequence of birational maps. The resulting foliation $\mathcal{F} = \sigma_*(\mathcal{R}) = (\sigma^{-1})^*(\mathcal{R})$ induced by $\mathcal{R}$ on $\mathbf{CP}^2$ has the following characteristics:
(i') $\mathcal{F}$ is transversely projective in $\mathbf{CP^2}\backslash\Lambda$ where $\Lambda \subset \mathbf{CP}^2$ is the union of a finite number of projective lines of the form $\bigcup\limits_{j=1}^{r} \overline{\{x = a_j\}} \subset \mathbf{CP}^2$ in a suitable affine chart $(x, y) \in \mathbf{C}^2 \subset \mathbf{CP}^2$.
(ii') For a generic choice of the coefficients of Ω, the singularities of $\mathcal{F}$ in Λ are hyperbolic except for one single dicritical singularity $q_\infty\colon (x = \infty, y = 0) \in \mathbf{CP}^2$ which after one blow-up originates a nonsingular foliation transverse to the projective line except for a single tangency point.

2 Projective structures and differential forms

Let $\mathcal{F}$ be a codimension one holomorphic foliation with singular set $\text{sing}(\mathcal{F})$ of codimension ≥ 2 on a complex manifold M. The existence of a projective transverse structure for $\mathcal{F}$ is equivalent to the existence of suitable triples of differential forms as follows (see [16] Section 3, page 193):

Proposition 2.1 ([16], Proposition 1.1 page 190)**.** *Assume that $\mathcal{F}$ is given by an integrable holomorphic one-form Ω on M and suppose that there exists a holomorphic one-form η on M such that (Proj.1) $d\Omega = \eta \wedge \Omega$. Then $\mathcal{F}$ is transversely projective on M if and only if there exists a holomorphic one-form ξ on M such that (Proj.2) $d\eta = \Omega \wedge \xi$ and (Proj.3) $d\xi = \xi \wedge \eta$.*

Notice that also ξ defines a foliation with a projective transverse structure given by the triple $(\xi, -\eta, \Omega)$; we will usually denote this *transverse foliation* by $\mathcal{F}^\perp$. This motivates the following definition:

Definition 2.2 (projective triple)**.** Given holomorphic one-forms (respectively, meromorphic one-forms) Ω, η and ξ on M we shall say that (Ω, η, ξ) is a *holomorphic projective triple* (respectively, a *meromorphic projective triple*) if they satisfy relations *(Proj.1)*, *(Proj.2)* and *(Proj.3)* above.

With this notion Proposition 2.1 says that $\mathcal{F}$ is transversely projective on M if and only if the holomorphic pair $(\Omega,\, \eta)$ may be completed to a holomorphic projective triple. According to [16] we may perform modifications in a projective triple as follows:

Proposition 2.3. (i) *Given a meromorphic projective triple (Ω, η, ξ) and meromorphic functions g, h on M we can define a new meromorphic projective triple as follows:*

(Mod.1) $\Omega' = g\,\Omega$

(Mod.2) $\eta' = \eta + \frac{dg}{g} + h\,\Omega$

(Mod.3) $\xi' = \frac{1}{g}\left(\xi - dh - h\eta - \frac{h^2}{2}\,\Omega\right)$

(ii) *Two holomorphic projective triples (Ω, η, ξ) and (Ω', η', ξ') define the same projective transverse structure for a given foliation $\mathcal{F}$ if and only if we have (Mod.1), (Mod.2) and (Mod.3) for some holomorphic functions g, h with g non-vanishing.*

(iii) *Let (Ω, η, ξ) and (Ω, η, ξ') be meromorphic projective triples. Then $\xi' = \xi + F\,\Omega$ for some meromorphic function F in M with $d\,\Omega = -\frac{1}{2}\,\frac{dF}{F} \wedge \Omega$.*

This last proposition implies that suitable meromorphic projective triples also define projective transverse structures. We can rewrite condition (iii) on F as $d(\sqrt{F}\,\Omega) = 0$. This implies that if the projective triples (Ω, η, ξ) and (Ω, η, ξ') are not identical then the foliation defined by Ω is transversely affine outside the codimension one analytical invariant subset $\Lambda = \{F = 0\} \cup \{F = \infty\}$. ([16]).

Definition 2.4. A meromorphic projective triple (Ω', η', ξ') is *geometric* (or also *true*) if it can be written locally as in *(Mod.1)*, *(Mod.2)* and *(Mod.3)* for some (locally defined) holomorphic projective triple (Ω, η, ξ) and some (locally defined) meromorphic functions.

As an immediate consequence we obtain:

Proposition 2.5. *A geometric projective triple (Ω', η', ξ') defines a transversely projective foliation $\mathcal{F}$ given by Ω' on M.*

Example 2.6 (Riccati Foliations - revisited)**.** Fix affine coordinates $(x, y) \in \mathbf{C}^2$ and consider a polynomial one-form $\Omega = p(x)dy - (y^2\, c(x) - yb(x) - a(x))dx$. Then Ω defines a *Riccati foliation* $\mathcal{R}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ as seen in Example 1.1 above. Now we study the Lie Algebra associated to this example. Put $\eta = 2\frac{dy}{y} + \frac{p'+b}{p}\,dx + \frac{2a}{yp}\,dx$ and $\xi = \frac{-2a}{y^2p^2}\,dx$. Then (Ω, η, ξ) satisfies the projective relations stated in Proposition 2.1. This shows that $\mathcal{F}$ is transversely projective in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ minus the algebraic subset $\overline{\{x \in \mathbf{C} \mid p(x) = 0\} \times \mathbf{C} \cup \mathbf{C} \times \{y = 0\}}$. But since in the case $a(x) \not\equiv 0$, only the subset $\Lambda = \{p(x) = 0\} \times \overline{\mathbf{C}}$ is $\mathcal{F}$ invariant it follows that the transverse projective structure extends to $\overline{\mathbf{C}} \times \overline{\mathbf{C}} \backslash \Lambda$. Indeed according to Proposition 2.3 if we define $g = \frac{-1}{p(x)y}$ then $\eta' = \eta + 2g\Omega = \frac{p'-b+2yc}{p}\,dx$ and $\xi' = \xi - 2dg - 2g\eta - 2g^2\Omega = \frac{2c}{p^2}\,dx$; define a triple (Ω, η', ξ') holomorphic in $(\overline{\mathbf{C}} \times \overline{\mathbf{C}}) \setminus \Lambda$ which gives a projective structure for $\mathcal{F}$ in this affine set. This projective structure coincides with the one given in $(\overline{\mathbf{C}} \times \overline{\mathbf{C}}) \backslash (\Lambda \cup \overline{\mathbf{C}} \times \{y = 0\})$ by (Ω, η, ξ). The one-form η is closed if and only if $a \equiv 0$. Therefore $\mathcal{F}$ is transversely affine in $\overline{\mathbf{C}} \times \overline{\mathbf{C}} \backslash (\Lambda \cup \overline{\mathbf{C}} \times \{y = 0\})$ if the projective line $\{y = 0\}$ is invariant. The forms (Ω, η', ξ') define a rational projective triple and *the projective transverse structure of the foliation $\mathcal{F}^\perp$ defined by ξ extends from $\mathbf{C}^2 \backslash \Lambda$ to $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$.* Indeed, $\mathcal{F}_\xi$ admits a rational first integral. We will see this is a general fact, under suitable hypothesis on the singularities of the foliation $\mathcal{F}$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$, admitting a projective transverse structure in the complementary of an algebraic one dimensional invariant subset $\Lambda \subset \overline{\mathbf{C}} \times \overline{\mathbf{C}}$.

As a kind of converse of the above example we have:

Proposition 2.7 ([16] Theorem 4.1 page 197)**.** *Let $\mathcal{F}$ be a foliation on a bidisc $U \subset \mathbf{C}^2$ or a projective surface U admitting a meromorphic projective triple (Ω, η, ξ) defined in U. If ξ admits a meromorphic first integral in U then $\mathcal{F}$ is a meromorphic pull-back of a Riccati foliation.*

Proof. If we write $\xi = g\,dR$ for some meromorphic function g then we may replace the meromorphic triple (Ω, η, ξ) by (Ω', η', ξ') where $\Omega' = g\Omega$, $\eta' = \eta + \frac{dg}{g}$ and $\xi' = \frac{1}{g}\,\xi = dR$. The relations $d\Omega' = \eta' \wedge \xi'$, $d\eta' = \Omega' \wedge \xi'$, $d\xi' = \xi \wedge \eta'$ imply that $\eta' = HdR$ for some meromorphic function H. Now we define $\omega := \frac{H^2}{2}\,\xi' - H\eta' + dH = \frac{1}{2}\,H^2dR + dH$ one-form such that $d\omega = -HdH \wedge dR$. On the other hand $\eta' \wedge \omega = HdR \wedge dH = -HdH \wedge dR$. Thus $d\omega = \eta' \wedge \omega$. We also have $d\eta' = dH \wedge dR = (-\frac{1}{2}\,H^2dR + dH) \wedge dR = \omega \wedge \xi'$. The meromorphic triple (ω, η', ξ') satisfies the projective relations $d\omega = \eta' \wedge \omega$, $d\eta' = \omega \wedge \xi'$, $d\xi' = \xi' \wedge \eta'$ and therefore by Proposition 2.3 (iii) we conclude that $\Omega' = \omega + F.\xi'$ for some meromorphic function F such that $d\xi' = \xi' \wedge \frac{1}{2}\frac{dF}{F}\cdot$ This implies $dF \wedge dR \equiv 0$. By the classical Stein Factorization theorem we may assume from the beginning that R has connected fibers and therefore $dF \wedge dR \equiv 0$ implies $F = \varphi(R)$ for some one-variable meromorphic function $\varphi(z) \in \mathbf{C}(z)$. We obtain therefore $\Omega' = -\frac{1}{2}\,H^2dR + dH + \varphi(R)dR == dH - (\frac{1}{2}\,H^2 - \varphi(R))dR$. If we define a rational map $\sigma\colon \mathbf{CP}^2 \dashrightarrow \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ by $\sigma(x,y) = \big(R(x,y), H(x,y)\big)$ on $\mathbf{C}^2$ then clearly $\Omega' = \sigma^*(dy - (\frac{1}{2}\,y^2 - \varphi(x))dx)$ and therefore $\mathcal{F}$ is the pull-back $\mathcal{F} = \sigma^*(\mathcal{R})$ of the Riccati foliation $\mathcal{R}$ given on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ by the rational one-form $\Omega_\varphi := dy - (\frac{1}{2}\,y^2 - \varphi(x))dx$. $\square$

3 Irreducible singularities

Let $\omega = a(x,y)dx + b(x,y)dy$ be a holomorphic one-form defined in a neighborhood $0 \in U \in \mathbf{C}^2$. We say that $0 \in \mathbf{C}^2$ is a *singular* point of ω if $a(0,0) = b(0,0) = 0$, and a *regular* point otherwise. We say that $0 \in \mathbf{C}^2$ is an *irreducible* singular point of ω if the eigenvalues λ_1, λ_2 of the linear part of the corresponding dual vector field $X = -b(x,y)\frac{\partial}{\partial x} + a(x,y)\frac{\partial}{\partial y}$ at $0 \in \mathbf{C}^2$ satisfy one of the following conditions:

(1) $\lambda_1.\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$

(2) either $\lambda_1 \neq 0$ and $\lambda_2 = 0$, or viceversa.

In case (1) there are two invariant curves tangent to the eigenvectors corresponding to λ_1 and λ_2. In case (2) there is an invariant curve tangent at $0 \in \mathbf{C}^2$ to the eigenspace corresponding to λ_1. These curves are called *separatrices* of the foliation.

Suppose that $0 \in \mathbf{C}^2$ is either a regular point or an irreducible singularity of a foliation $\mathcal{I}$. Then in suitable local coordinates (x,y) in a neighborhood $0 \in U \in \mathbf{C}^2$ of the origin, we have the following local normal forms for the one-forms defining this foliation ([4]):

(Reg) $dy = 0$, whenever $0 \in \mathbf{C}^2$ is a regular point of $\mathcal{I}$.

and whenever $0 \in \mathbf{C}^2$ is an irreducible singularity of $\tilde{\mathcal{F}}$, then either

(Irr.1) $xdy - \lambda ydx + \omega_2(x,y) = 0$ where $\lambda \in \mathbf{C}\backslash\mathbb{Q}_+$, $\omega_2(x,y)$ is a holomorphic one-form with a zero of order ≥ 2 at $(0,0)$. This is called *non-degenerate singularity.* Such a singularity is *resonant* if $\lambda \in \mathbb{Q}_-$ and *hyperbolic* if $\lambda \notin \mathbb{R}$, or

(Irr.2) $y^{t+1}dx - [x(1+\lambda y^t) + A(x,y)]dy = 0$, where $\lambda \in \mathbf{C}$, $t \in \mathbb{N} = \{1,2,3,\dots\}$ and $A(x,y)$ is a holomorphic function with a zero of order $\geq t+2$ at $(0,0)$. This is called *saddle-node singularity.* The *strong manifold* of the saddle-node is given by $\{y = 0\}$. If the singularity admits another separatrix then it is necessarily smooth and transverse to the strong manifold, it can be taken as the other coordinate axis and will be called *central* manifold of the saddle-node.

Therefore, for a suitable choice of the coordinates, we have $\{y = 0\} \subset \operatorname{sep}(\mathcal{I}, U) \subset \{xy = 0\}$, where $\operatorname{sep}(\mathcal{I}, U)$ denotes the union of separatrices of $\mathcal{I}$ through $0 \in \mathbf{C}^2$.

Definition 3.1 ([19], Def. II.4.1)**.** Let $\mathcal{F}$ be a germ of a holomorphic foliation at the origin $0 \in \mathbf{C}^2$. We say that $\mathcal{F}$ is *transversely projective of moderate growth* if it admits a meromorphic projective triple (Ω, η, ξ) defined in a neighborhood of the origin.

We recall the following fundamental result from [19]:

In the case where U is a projective manifold all the meromorphic objects are rational and therefore $\varphi(z)$ is also a rational function.

Theorem 3.2 (Touzet, [19] Theorem II.4.2 and Theorem II.3.1)**.** *A germ of irreducible singularity $\mathcal{F}$ at the origin $0 \in \mathbf{C}^2$ which is of resonant type or saddle-node type is projective of moderate growth if, and only if, the germ is a pull-back of a Riccati foliation by a meromorphic map. A non-degenerate non-resonant singularity $xdy - \lambda ydx + \Omega_2(x,y) = 0$, $\lambda \in \mathbf{C} \setminus \mathbb{Q}_+$, is analytically linearizable if and only if the corresponding foliation $\mathcal{F}$ is transversely projective in $U \setminus \text{sep}(\mathcal{F}, U)$ for some neighborhood U of the singularity.*

Remark 3.3. The proof of the first part of the above theorem is based on the study and classification of the Martinet-Ramis cocycles ([12, 13]) of the singularity expressed in terms of some classifying holonomy map of a separatrix of the singularity. For a resonant singularity any of the two separatrices has a *classifying holonomy* (*i.e.*, the analytical conjugacy class of the singularity germ is determined by the analytical conjugacy class of the holonomy map of the separatrix) and for a saddle-node it is necessary to consider the strong separatrix holonomy map. Thus we conclude that the proof given in [19] works if we only assume the existence of a meromorphic projective triple (Ω', η', ξ') on a neighborhood U_0 of $\Lambda \setminus (0,0)$, where $\Lambda \subset \text{sep}(\mathcal{F}, U)$ is any separatrix in the resonant case, and the strong separatrix if the origin is a saddle-node.

4 Separatrices and resolution of singularities

Suppose $\mathcal{F}$ is a complex one-dimensional foliation defined on an open neighborhood $0 \in U \subset \mathbf{C}^2$. The *resolution* process of $\mathcal{F}$ at $0 \in \mathbf{C}^2$ consists of a finite number of successive blow-ups originating a foliation with only irreducible singularities. This process can be described as follows. The blow-up of $\mathcal{F}$ at $0 \in \mathbf{C}^2$ is $(U_0, \pi_0, D_0, \mathcal{F}_0)$ where $\pi_0 : U_0 \to U$ is the usual blow-up map. Then, U_0 is a complex 2-manifold, $D_0 = \pi_0^{-1}(0) \subset U_0$ is an embedded projective line called the *exceptional divisor*, and the restriction of the map π_0 to $U_0 \setminus D_0$ is a biholomorphism from $U_0 \setminus D_0$ to $U \setminus \{0\}$. Moreover $\mathcal{F}_0$ is the analytic foliation on U_0 obtained by extension to D_0 of $(\pi_0|_{U_0 \setminus D})^* \mathcal{F}$, *i.e.*, the pull-back foliation $\pi_)^*(\mathcal{F})$. If D_0 is tangent to $\mathcal{F}_0$, i.e. D_0 is a leaf plus a finite number of singularities, we say that D_0 is *non-dicritical*. Otherwise, D_0 is transverse to $\mathcal{F}_0$ everywhere except at a finite number of points, singularities or tangency points of $\mathcal{F}_0$ with D_0. In this last case we say that D_0 is *dicritical*.

Proceeding by induction we define the step $\underline{0}$ as the first blow-up $(U_0, \pi_0, D_0, \mathcal{F}_0)$. We assume that $(U_k, \pi_k, D_k, \mathcal{F}_k)$ has been already defined, where $\pi_k : U_k \to U$ is a holomorphic map, such that $D_k = \pi_k^{-1}(0)$ is a divisor, union of a finite number of embedded projective lines with normal crossing. The crossing points of D_k are called *corners*. The restriction of π_k to $U_k \setminus D_k$ is a biholomorphism from $U_k \setminus D_k$ to $U \setminus \{0\}$. The foliation $\mathcal{F}_k$ on U_k is the pull-back of $\mathcal{F}$ by the map π_k. The Resolution theorem of Seidenberg [18] guarantees that after a finite number of blow-up´s all corners obtained in this process will be either irreducible singular points or regular points. As final product we get a complex surface $\widetilde{U}$ and a proper holomorphic map $\pi \colon \widetilde{U} \to U$, which is a finite composition of quadratic blow-ups, such that the *exceptional divisor* $D = \pi^{-1}(0)$ is a normal crossing divisor without triple points. Also D is a finite union of projective lines $D = \cup_{j=1}^m \mathbb{P}_j$, $\mathbb{P}_j \simeq \mathbf{C}P(1)$ with negative self-intersection in $\widetilde{U}$. The pull-back foliation $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ is a foliation with isolated singularities, $\text{sing}(\widetilde{\mathcal{F}}) \subset D$, consisting of irreducible singularities. Any component $\mathbb{P}_j \subset D$ is either $\tilde{\mathcal{F}}$-invariant or, in the dicritical case, everywhere transverse to $\widetilde{\mathcal{F}}$.

Let now $\mathcal{F}$ be a foliation with isolated singularities on a complex manifold M. Given an analytic invariant curve $\Lambda \subset U$ we may perform the *resolution of singularities of $\mathcal{F}$ in Λ* obtaining a proper holomorphic map $\pi \colon \tilde{M} \to M$ and a foliation $\tilde{\mathcal{F}} = \pi^*(\mathcal{F})$ such that the singularities of $\tilde{\mathcal{F}}$ in $\pi^{-1}(\Lambda)$ are all irreducible. Denote by $\tilde{\Lambda} \subset \tilde{M}$ the *strict transform* of Λ, defined as $\tilde{\Lambda} := \overline{\pi^{-1}(\Lambda \setminus (\Lambda \cap \mathrm{sing}(\mathcal{F}))}$. Then $\tilde{\Lambda}$ is $\tilde{\mathcal{F}}$-invariant. The *large transform* of Λ is by definition $\pi^{-1}(\Lambda) = \tilde{\Lambda} \cup D$, the union of the strict transform $\tilde{\Lambda}$ and the *exceptional divisor* $D = \pi^{-1}(\mathrm{sing}(\mathcal{F}))$.

Consider now an arbitrary germ of an analytic foliation $\mathcal{F}$ at an isolated singularity $0 \in \mathbf{C}^2$. A *separatrix* of $\mathcal{F}$ at $0 \in \mathbf{C}^2$ is the germ at $0 \in \mathbf{C}^2$ of an irreducible analytic curve which is invariant by $\mathcal{F}$. Let $\mathcal{F}|_U$ be a representative of the foliation defined in a neighborhood U of $0 \in \mathbf{C}^2$, the separatrix is the union of a leaf of $\mathcal{F}|_U$ and the singular point $0 \in \mathbf{C}^2$. By Newton-Puiseux parametrization theorem, if U is small enough, there is an analytic injective map $f\colon \mathbb{D} \to U$ from the unit disk $\mathbb{D} \subset \mathbf{C}$ onto the separatrix, mapping the origin to $0 \in \mathbf{C}^2$, and nonsingular outside the origin $0 \in \mathbb{D}$. Therefore a separatrix locally has the topology of a punctured disk.

We shall say that the separatrix is *resonant* if for any loop in the punctured disk that represents a generator of the homotopy of the leaf, the corresponding holonomy map is a resonant diffeomorphism. Choose a holomorphic vector field X which generates the foliation $\mathcal{F}|_U$, and has an isolated singularity at $0 \in \mathbf{C}^2$. Then, the separatrix is resonant if the loop γ generating the homotopy of the leaf in the separatrix satisfies $\exp \int_\gamma \mathrm{tr}(DX)$ is a root of the unity.

By the resolution of singularities we conclude that a separatrix Γ of $\mathcal{F}$ is the projection $\Gamma = \pi(\tilde{\Gamma})$ of a curve $\tilde{\Gamma}$ invariant by $\tilde{\mathcal{F}}$ and transverse to the exceptional divisor $\pi^{-1}(0)$. We shall say that Γ is a *dicritical separatrix* if $\tilde{\Gamma}$ meets the resolution divisor at a non-singular point. Equivalently, $\Gamma = \pi(\tilde{\Gamma})$ is non-dicritical if $\tilde{\Gamma}$ is the separatrix of some singularity of $\tilde{\mathcal{F}}$.

5 Extension through singularities

The following results are proved in [6] and imply the existence of a globally defined projective triple in the situation we are dealing with:

Proposition 5.1. *Let $\mathcal{I}$ be a holomorphic foliation in a neighborhood V of the origin $0 \in \mathbf{C}^2$ given by the holomorphic one-form Ω admitting a meromorphic one-form η in V with $d\Omega = \eta \wedge \Omega$. Suppose that $\mathcal{I}$ has an irreducible singularity at the origin and is transversely projective in $U \setminus \mathrm{sep}(\mathcal{I}, U)$ for some neighborhood $U \subset V$ of the origin where $\mathcal{I}$ has an expression in irreducible normal form. Then given a meromorphic-one form ξ defined in $U \setminus \mathrm{sep}(\mathcal{I}, U)$ such that (Ω, η, ξ) is a geometric projective triple in $U \setminus \mathrm{sep}(\mathcal{I}, U)$, we have:*

(1) *If the origin is a non-degenerate non-resonant singularity then ξ extends as a meromorphic one-form to U.*

(2) *If ξ extends as a meromorphic one-form to $S^* = S - \{0\}$, for some separatrix $S \subset \mathrm{sep}(\mathcal{I}, U)$ which is not a central manifold in case the singularity is a saddle-node, then ξ extends as a meromorphic one-form to U.*

This proposition and the Globalization theorem in [6] give:

Proposition 5.2. *Let $\mathcal{F}$ a holomorphic foliation defined in a neighborhood V of $0 \in \mathbf{C}^2$ with an isolated singularity at the origin. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \mathrm{sep}(\mathcal{F}, U)$ for some neighborhood $U \subset V$ of the origin where $\mathcal{F}$ is given by a holomorphic one-form Ω admitting a meromorphic one-form η such that $d\Omega = \eta \wedge \Omega$ in U. Given a meromorphic-one form ξ defined in $U \setminus \mathrm{sep}(\mathcal{F}, U)$ such that (Ω, η, ξ) is a geometric projective triple, then the* *one-form ξ extends to U provided that at any resonant separatrix Γ the form ξ extends to a neighborhood of an annulus $A \subset \Gamma$ around the singularity.*

6 Extension to codimension one divisors

In this section we investigate the extension of meromorphic projective triples to a codimension one divisor, invariant or not by the foliation.

Lemma 6.1 (extension through a point)**.** *Let (Ω, η, ξ) be a meromorphic projective triple on a complex surface M^2, and $\Lambda \subset M$ an irreducible analytic subset of dimension one. Suppose that the triple defines a projective transverse structure outside Λ. If there is a point $q \in \Lambda$ and a neighborhood $q \in U \subset M$ to which the projective structure extends, then this projective structure extends to M.*

Proof. We consider the local case where the foliation $\mathcal{F}$ is given by a holomorphic one-form Ω in an open subset $W \subset \mathbf{C^n}$ with isolated zeros and admitting a meromorphic one-form η on W satisfying $d\Omega = \eta \wedge \Omega$. We can assume that Ω and η have poles in general position with respect to Λ.

For $U \subset W$ small enough we can find a holomorphic submersion $y\colon U \to \mathbf{C}$ and meromorphic functions g, h in U such that

$$\Omega = gdy, \eta = \frac{dg}{g} + hdy,\ \xi = -\frac{1}{g}\big[dh + \frac{h^2}{2}dy\big] + \ell gdy$$

where

$$d(\sqrt{\ell}gdy) = 0.$$

Thus, $\sqrt{\ell}g = \varphi(y)$ for some meromorphic function $\varphi(z)$ and therefore $\ell = \frac{\varphi^2(y)}{g^2}$. Hence we have

$$\Omega = gdy, \eta = \frac{dg}{g} + hdy,\ \xi = -\frac{1}{g}\big[dh + \frac{h^2}{2}dy\big] + \frac{\varphi^2(y)}{g}dy$$

We investigate under which conditions we can write

$$\Omega = \tilde{g}d\tilde{y}, \eta = \frac{d\tilde{g}}{\tilde{g}} + \tilde{h}d\tilde{y},\ \xi = -\frac{1}{\tilde{g}}\big[d\tilde{h} + \frac{\tilde{h}^2}{2}d\tilde{y}\big]$$

for some suitable meromorphic functions $\tilde{g}, \tilde{h}, \tilde{y}$.

Imposing the above equations we obtain

$$\begin{cases} gdy = \tilde{g}d\tilde{y} \\ \frac{dg}{g} + hdy = \frac{d\tilde{g}}{\tilde{g}} + \tilde{h}d\tilde{y} \\ -\frac{1}{g}\big[dh + \frac{h^2}{2}dy\big] + \frac{\varphi^2(y)}{g}dy = -\frac{1}{\tilde{g}}\big[d\tilde{h} + \frac{\tilde{h}^2}{2}d\tilde{y}\big] \end{cases} \tag{1}$$

We shall refer to equations in (1) as *main equations*. From $gdy = \tilde{g}d\tilde{y}$ we obtain $g = r(y)\tilde{g}$ for some meromorphic function $r(y)$. This implies $d\tilde{y} = r(y)dy$ and then $\frac{dg}{g} + hdy = \frac{d\tilde{g}}{\tilde{g}} + \frac{r'(y)}{r(y)}dy + hdy$ so that replacing in the second main equation we obtain $\frac{d\tilde{g}}{\tilde{g}} + \tilde{h}d\tilde{y} = \frac{d\tilde{g}}{\tilde{g}} + \frac{r'(y)}{r(y)}dy + hdy$ and then $\frac{r'(y)}{r(y)}dy + hdy = \tilde{h}d\tilde{y} = \tilde{h}r(y)dy$. This last equation rewrites

$$\frac{r'(y)}{r(y)} + h = \tilde{h} r(y) \tag{2}$$

and the final form

$$\tilde{h} = \frac{1}{r(y)} \Big[\frac{r'(y)}{r(y)} + h \Big] \tag{3}$$

Let us turn our attention to the third main equation. From this we obtain

$$\frac{1}{g} \big[dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy \big] = \frac{1}{\tilde{g}} \big[d\tilde{h} + \frac{\tilde{h}^2}{2} d\tilde{y} \big]$$

Then

$$\frac{\tilde{g}}{g} \big[dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy \big] = d\tilde{h} + \frac{\tilde{h}^2}{2} d\tilde{y}$$

$$\frac{1}{r(y)} \big[dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy \big] = d\tilde{h} + \frac{\tilde{h}^2}{2} d\tilde{y}$$

$$\frac{1}{r(y)} \big[dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy \big] = d\tilde{h} + \frac{\tilde{h}^2}{2} r(y) dy$$

$$dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy = r(y) \big[d\Big(\frac{1}{r(y)} \big(\frac{r'(y)}{r(y)} + h \big) \Big) + \frac{1}{2r(y)^2} \big(\frac{r'(y)}{r(y)} + h \big)^2 r(y) dy \big]$$

$$dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy = r(y) \big[d\Big(\frac{1}{r(y)} \big(\frac{r'(y)}{r(y)} + h \big) \Big) + \frac{1}{2} \frac{1}{r(y)} \big(\frac{r'(y)}{r(y)} + h \big)^2 dy \big]$$

$$dh + \big(\frac{h^2}{2} - \varphi^2(y) \big) dy = \frac{1}{2} \big(\frac{r'(y) + h}{r(y)} \big)^2 dy - \frac{r'(y)}{r(y)} \big(\frac{r'(y)}{r(y)} + h \big) dy + d\big(\frac{r'(y)}{r(y)} + h \big)$$

This last equation is equivalent to

$$-\varphi^2(y) = -\frac{1}{2} \big(\frac{r'(y)}{r(y)} \big)^2 + \big(\frac{r'(y)}{r(y)} \big)' \tag{4}$$

Let us put

$$s(y) := \frac{r'(y)}{r(y)}$$

Then equation (4) rewrites

$$s' - \frac{1}{2} s^2 = -\varphi^2 \tag{5}$$

So, the original question is reduced to find conditions under which the equation above has a holomorphic solution. This is the case, for instance if φ is holomorphic. Now we need to return to equation $\frac{r'(y)}{r(y)} = s(y)$ and study its solutions. It is clear from integration that there is a holomorphic solution, which must be given by $r(y) = e^{\int s(y) dy}$, if and only if the given data $s(y)$ is either holomorphic or meromorphic with a simple pole and integral positive residue at $y = 0$.

First case. If $s(y)$ has a simple pole at $y=0$. We may assume for simplicity that $s(y)=a/y$ for some $a\in\mathbf{C}^*$. In this case from the differential equation $s'-s^2/2=-\varphi^2$ we obtain $\varphi=\frac{\sqrt{2a-a^2}}{y}$. Integrating $r(y)=e^{\int s(y)dy}$ we obtain $r(y)=y^a$. Since $r(y)=g/\tilde{g}$ we have that $r(y)$ is holomorphic without zeros. In particular we cannot have $a\neq 0$, contradiction.
Second case. If $s(y)$ has a pole of order $m+1\geq 2$ at $y=0$. In this case we can assume that $s(y)=a/y^{m+1}$ for some $m\geq 1$ and integration gives $r(y)=e^{-\frac{a}{my^m}}$ which is not meromorphic at the origin, contradiction.
Third case. If $s(y)$ is holomorphic at $y=0$. In this case we write $s(y)=ay^m$ for some $m\geq 0$. We obtain $r(y)=e^{\frac{a}{m+1}y^{m+1}}$ which is holomorphic and non-vanishing.

Let us now finish the proof. Because the projective structure extends to U the equation (1) has a holomorphic solution and this implies that $\varphi(y)$ is holomorphic according to the above considerations. As a consequence the one-form ξ is also holomorphic in U and therefore admits a holomorphic extension to $\Lambda\setminus[(\Omega)_\infty\cup(\eta)_\infty]$. Hence, the projective structure extends to $\Lambda\setminus[(\Omega)_\infty\cup(\eta)_\infty]$ and then to Λ.

□

Lemma 6.2. *Let (Ω,η,ξ) be a meromorphic projective triple in a complex surface M. Assume that the triple defines a projective transverse structure for $\mathcal{F}$ in $M\setminus\Lambda$ for some invariant codimension one analytic subset $\Lambda\subset M$. Let ξ' be a meromorphic one-form in M such that (Ω,η,ξ') is also a projective triple. Then Λ is ξ-invariant if and only if it is ξ'-invariant.*

Proof. We fix a local coordinate system $(x,y)\in U$ centered at a point $p\in M$ such that $\mathcal{F}$ is given in these coordinates by $\Omega=gdy$ and Λ by $\{y=0\}$. We may write $\xi'=\xi+\ell\Omega$ where $d(\sqrt{\ell}\Omega)=0$. Then we have $\ell=\frac{\varphi^2(y)}{g}$ for some meromorphic function $\varphi(z)$. Assume by contradiction that Λ is not ξ-invariant but Λ is ξ'-invariant. We may assume that the polar set of ξ has no irreducible component contained in Λ and therefore $\varphi(y)$ and g have no poles on $\{y=0\}$. Write $\xi'=Adx+Bdy$ with holomorphic coefficients $A(x,y)$, $B(x,y)$. Since Λ is ξ'-invariant we have $A(x,y)=y\,A_1(x,y)$ for some holomorphic function $A_1(x,y)$. Then from $\xi'=\xi+\ell\Omega$ we get $\xi=yA_1(x,y)dx+(B(x,y)-\frac{\varphi^2(y)}{g})dy$. Since A_1 and $B(x,y)-\frac{\varphi^2(y)}{g}$ have no poles in $\{y=0\}$ we conclude from the above expression that Λ is ξ-invariant, contradiction. □

Lemma 6.3 (non-invariant divisor, [6])**.** *Let be given a holomorphic foliation $\mathcal{F}$ on a complex surface M. Suppose that $\mathcal{F}$ is given by a meromorphic integrable one-form Ω which admits a meromorphic one-form η on M such that $d\Omega=\eta\wedge\Omega$. If $\mathcal{F}$ is transversely projective in $M\setminus\Lambda$ for some non-invariant irreducible analytic subset $\Lambda\subset M$ of codimension one then $\mathcal{F}$ is transversely projective in M. Indeed, the projective transverse structure for $\mathcal{F}$ in $M\setminus\Lambda$ extends to M as a projective transverse structure for $\mathcal{F}$.*

Proof. Our argumentation is local, *i.e.*, we consider a small neighborhood U of a generic point $q\in\Lambda$ where $\mathcal{F}$ is transverse to Λ. Thus, since Λ is not invariant by $\mathcal{F}$, performing changes as $\Omega'=g_1\Omega$ and $\eta'=\eta+\frac{dg_1}{g_1}$ we can assume that Ω and η have poles in general position with respect to Λ in U. The existence of a projective transverse structure for $\mathcal{F}$ in $M\setminus\Lambda$ then gives a meromorphic one-form ξ in $M\setminus\Lambda$ such (Ω,η,ξ) is a geometric projective triple in $M\setminus\Lambda$. For U small enough we can assume that for suitable local coordinates $(x,y)\in U$ we have $\Lambda\cap U=\{x=0\}$ and also

$$\Omega=gdy,\eta=\frac{dg}{g}+hdy$$

for some holomorphic function $g, h\colon U \to \mathbf{C}$ with $1/g$ also holomorphic in U. Then we have

$$\xi = -\frac{1}{g}\big[dh + \frac{h^2}{2}dy\big]$$

where

$$d(\sqrt{\ell} g dy) = 0$$

Thus, $\sqrt{\ell} g = \varphi(y)$ for some meromorphic function $\varphi(y)$ defined for $x \neq 0$ and therefore for $x = 0$. This shows that ξ extends to U as a *holomorphic one-form* and then the projective structure extends to U. This shows that the transverse structure extends to Λ. $\square$

7 Germs of foliations and foliations on projective spaces

Let $\mathcal{F}$ be a holomorphic foliation of codimension one on $\mathbf{CP}^2$ having singular set $\operatorname{sing}(\mathcal{F}) \subsetneq \mathbf{CP}^2$. As it is well-known we can assume that $\operatorname{sing}(\mathcal{F})$ is of codimension ≥ 2 and $\mathcal{F}$ is given in any affine space $\mathbf{C}^2 \subset \mathbf{CP}^2$ with coordinates (x, y), by a polynomial one-form $\Omega(x, y) = A(x, y)dx + B(x, y)dy$ with $\operatorname{sing}(\mathcal{F}) \cap \mathbf{C}^2 = \operatorname{sing}(\Omega)$. In particular $\operatorname{sing}(\mathcal{F}) \subset \mathbf{CP}^2$ is a nonempty finite set of points. Given any algebraic subset $\Lambda \subset \mathbf{CP}^2$ of dimension one we can therefore always obtain a meromorphic (rational) one-form Ω on $\mathbf{CP}^2$ such that Ω defines $\mathcal{F}$, $(\Omega)_\infty$ is non-invariant and in general position (indeed, we can assume that $(\Omega)_\infty$ is any projective line in $\mathbf{CP}^2$). Also if we take $\eta_0 = \frac{B_x}{B}\,dx + \frac{A_y}{A}\,dy$ then we obtain a rational one-form such that $d\Omega = \eta_0 \wedge \Omega$ and with polar set given by $(\eta_0)_\infty = \{(x, y) \in \mathbf{C}^2 : A(x, y) = 0\} \cup \{(x, y) \in \mathbf{C}^2 : B(x, y) = 0\} \cup (\Omega)_\infty$. In particular, $(\eta_0)_\infty \cap \mathbf{C}^2$ has order one and the "residue" of η_0 along any component T of $(\Omega)_\infty$ equals $-k$ where k is the order of T as a set of poles of Ω. Any rational one-form η such that $d\Omega = \eta \wedge \Omega$ writes $\eta = \eta_0 + h\Omega$ for some rational function h. We obtain in this way one-forms η with appropriately located set of poles, with respect to $\mathcal{F}$, and applying Proposition 2.1 and 2.3 we obtain:

Proposition 7.1. *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbf{CP}^2$. Assume that $\mathcal{F}$ is transversely projective in $\mathbf{CP}^2 \backslash \Lambda$ for some algebraic subset Λ of dimension one. Then $\mathcal{F}$ has a projective triple (Ω, η, ξ) on $\mathbf{CP}^2 \setminus \Lambda$ where Ω and η are rational one-forms and ξ is meromorphic on $\mathbf{CP}^2 \backslash \Lambda$. In particular ξ defines a transverse foliation $\mathcal{F}^\perp$ to $\mathcal{F}$ on $\mathbf{CP}^2 \backslash \Lambda$ having a projective transverse structure.*

This proposition admits a natural local version, *i.e.*, a version for germs of foliations at the origin $0 \in \mathbf{C^2}$ where the curve Λ is replaced by a finite set of local branches of separatrices of the foliation through the singularity.

We recall that a germ of a foliation singularity at the origin $0 \in \mathbf{C}^2$ is a *generalized curve* if it is non-dicritical and exhibits no saddle-node in its resolution by blow-ups ([3]). The generalized curve is *resonant* if *all* singularities are of resonant type, otherwise it is called *non-resonant.*

For this type of singularity we have the following extension lemma:

Lemma 7.2. *Let $\mathcal{F}$ be a germ of a non-resonant generalized curve at the origin $0 \in \mathbf{C}^2$. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ and let (Ω, η, ξ) be a meromorphic triple in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ with Ω holomorphic in U, η meromorphic in U and ξ meromorphic in $U \setminus \operatorname{sep}(\mathcal{F}, U)$. Then the one-form ξ extends to U as a meromorphic one-form.*

Proof. This lemma follows from Proposition 5.2. $\square$

We extend the notion of generalized curve in a natural way by allowing dicritical components, but no saddle-nodes, in the resolution process. Such singularities will be called *extended generalized curves*. Let now $\mathcal{F}$ be a foliation in U where U is either a bidisc centered at the origin, or a projective surface. Consider an invariant subset $\Lambda \subset U$ analytic of dimension one. Denote by $\pi\colon \tilde{U} \to U$ the resolution morphism of the singularities of $\mathcal{F}$ in Λ. We say that *the singularities in Λ are non-resonant extended generalized curves if each connected component of invariant of the resolution divisor $\pi^{-1}(\Lambda)$ contains some non-resonant (non-degenerate) singularity*. In a natural extension of the arguments in the proof of Lemma 7.2 we obtain:

Lemma 7.3. *Let $\mathcal{F}$ be a germ of a foliation at the origin $0 \in \mathbf{C}^2$. Suppose that $\mathcal{F}$ is transversely projective in $U \setminus \Lambda$ where $\Lambda \subset \operatorname{sep}(\mathcal{F}, U)$ is a finite set of local branches. Assume that the singularity $0 \in \Lambda$ is a non-resonant extended generalized curves Let (Ω, η, ξ) be a meromorphic triple in $U \setminus \operatorname{sep}(\mathcal{F}, U)$ with Ω holomorphic in U, η meromorphic in U and ξ meromorphic in $U \setminus \operatorname{sep}(\mathcal{F}, U)$. Then the one-form ξ extends to U as a meromorphic one-form.*

This lemma and Proposition 7.1 promptly give:

Theorem 7.4. *Let $\mathcal{F}$ be a holomorphic foliation on U where U is either the projective plane $\mathbf{CP^2}$ or a bidisc centered at the origin $0 \in \mathbf{C^2}$. Assume that $\mathcal{F}$ is transversely projective in $U \setminus \Lambda$ where $\Lambda \subset U$ is either an algebraic invariant curve in the projective plane or a finite union of local branches of separatrices of $\mathcal{F}$ through the origin. Suppose that the singularities of $\mathcal{F}$ in Λ are non-resonant extended generalized curves. Then $\mathcal{F}$ admits a meromorphic projective triple (Ω, η, ξ) defined in U, which defines the projective transverse structure in $U \setminus \Lambda$.*

8 Monodromy

In this section we follow original ideas from [15] in the same vein as in [17]. Let $\mathcal{F}$ be a holomorphic foliation with singularities on a complex surface M and $X \subset M$ an invariant codimension one analytic subset such that $\mathcal{F}$ is transversely projective in $M \backslash X$. According to [9] the foliation $\mathcal{F}\big|_{M \backslash X}$ admits a *development*, i.e., there is a Galoisian covering $p\colon P \to M \backslash X$ where p is holomorphic, a homomorphism $h\colon \pi_1(M \backslash X) \to \mathrm{SL}(2, \mathbf{C})$ and a holomorphic submersion $\Phi\colon P \to \mathbf{CP}^1$ such that:

(i) Φ is h-equivariant.

(ii) $p^*\big(\mathcal{F}\big|_{M \backslash X}\big)$ is the foliation defined by the submersion Φ.

Remark 8.1. The construction of the development in [9] requires the foliation to be non-singular. In our case, it is not necessary to require that $\operatorname{sing}(\mathcal{F}) \cap (M \backslash X) = \emptyset$. Indeed, by Hartogs' Extension Theorem any holomorphic map from $(M \backslash X) \backslash (\operatorname{sing}(\mathcal{F}) \cap (M \backslash X))$ to $\mathbf{CP^1}$ extends uniquely to a holomorphic map from $M \backslash X$ to $\mathbf{CP^1}$. Also, since $\operatorname{codim}_{\mathbf{C}} \operatorname{sing} \mathcal{F} = 2$ the inclusion $\pi_1((M \backslash X) \backslash ((M \backslash X) \cap \operatorname{sing}(\mathcal{F})) \to \pi_1(M \backslash X)$ is an isomorphism. Nevertheless, for our purposes it is enough to consider the case $\operatorname{sing}(\mathcal{F}) \subset X$.

Using the notion of development we can introduce the notion of *monodromy* of the projective transverse structure of $\mathcal{F}\big|_{M \backslash X}$ as follows:

Fix a base point $m_0 \in M\backslash X$ and a local determination f_{m_0} of the submersion Φ in a small ball B_{m_0} centered at m_0 (we have the following commutative diagram)

$$\begin{array}{ccccc} P & \supset & p^{-1}(B_{m_0}) & & \Phi\big|_{p^{-1}(B_{m_0})} \\ p\downarrow & & p\big|_{p^{-1}(B_{m_0})}\downarrow & \searrow & \\ M\backslash X & \supset & B_{m_0} & \xrightarrow{f_{m_0}} & \mathbf{C}P(1) \end{array}$$

Notice that $p^{-1}(B_{m_0}) = \bigcup\limits_{\alpha\in\mathcal{A}} U_\alpha$, $p\big|_{U_\alpha} : U_\alpha \to B_{m_0}$ is a biholomorphism for each $\alpha \in \mathcal{A}$.
By construction, the total space of the covering $p\colon P \to M\backslash X$ is obtained by analytic continuation of f_{m_0} along all the elements in $\pi_1(M\backslash X, m_0)$.
The fiber $p^{-1}(m_0)$ is the set of all local determinations f_{m_0} at m_0. We can, by the general theory of transitive covering spaces, identify the group $\mathrm{Aut}(P,p)$ of deck transformations of $p\colon P \to M\backslash X$ to the quotient $\pi_1(M\backslash X; m_0)/p_\#\pi_1(P; f_{m_0})$. This is the *monodromy group* of $\mathcal{F}\big|_{M\backslash X}$ which will be denoted by $\mathrm{Mon}(\mathcal{F}, X)$.
The *monodromy map* is the natural projection

$$\rho\colon \pi_1(M\backslash X; m_0) \longrightarrow \pi_1(M\backslash X; m_0)\ /p_\#\pi_1(P; f_{m_0}) = \mathrm{Mon}(\mathcal{F}, X)$$

Our first remark is the following:

Lemma 8.2. *The monodromy group* $\mathrm{Mon}(\mathcal{F}, X)$ *is naturally isomorphic to a subgroup of* $\mathrm{SL}(2, \mathbf{C})$.

Proof. This is clear since $\mathcal{F}\big|_{M\backslash X}$ is transversely projective on $M\backslash X$. ☐

9 Holonomy versus monodromy

Here we keep on following arguments originally in [15] and mimed in [17]. We proceed to study the holonomy of each irreducible component of X. It is enough to assume that X is the union of a smooth compact curve Λ and local analytic separatrices $\mathrm{sep}(\mathcal{F}, \Lambda)$ of $\mathcal{F}$ transverse to Λ; $X = \Lambda \cup \mathrm{sep}(\mathcal{F}, \Lambda)$, all of them smooth invariant and without triple points. We suppose that $\mathrm{sing}(\mathcal{F}) \cap \Lambda \neq \emptyset$, each singular point in Λ is irreducible and, if it admits two separatrices then one is transverse to Λ). In this case we can consider a C^∞ retraction $r\colon W \to \Lambda$ from some tubular neighborhood W of Λ on M onto Λ such that, $\forall\, m \in \Lambda$ the fiber $r^{-1}(m)$ is either a disc transverse to $\mathcal{F}$ or a local branch of $\mathrm{sep}(\mathcal{F}, \Lambda)$ at $m \in \mathrm{sing}(\mathcal{F})$. We set $V = W\backslash(X \cap W)$ to obtain a C^∞ fibration $r\big|_V : V \to \Lambda\backslash\,\mathrm{sing}(\mathcal{F})$ by punctured discs over $\Lambda\backslash\,\mathrm{sing}(\mathcal{F})$. Since $\pi_2(\Lambda\backslash\,\mathrm{sing}(\mathcal{F})) = 0$ the homotopy exact sequence of the above fibration gives the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(V, \tilde{m}_0) \xrightarrow{\tau} \pi_1(\Lambda\backslash\,\mathrm{sing}(\mathcal{F}); m_0) \longrightarrow 0$$

where $\tilde{m}_0 \in V$ is a base point and $m_0 \in \Lambda\backslash\,\mathrm{sing}(\mathcal{F})$ is its projection and $\tau = (r\big|_V)_\#$.
Now we consider the restriction of the covering space P to V; indeed for our purposes we may assume that $W = M$ and $V = M\backslash X$ so that we are just considering the space P itself. Let ρ be the monodromy map

$$\rho\colon \pi_1(V; \tilde{m}_0) \longrightarrow \pi_1(V; \tilde{m}_0)/p_\#(\pi_1(p^{-1}(V); f_{\tilde{m}_0})) =: \mathrm{Mon}(\mathcal{F}, V)$$

Denote by $\mathrm{Mon}(\mathcal{F},\Lambda)$ the quotient of $\mathrm{Mon}(\mathcal{F},V)$ by the (normal) subgroup $\mathrm{Ker}(\tau)\cong\mathbb{Z}$. Then there is a unique morphism $[\rho]$ such that the diagram commutes:

$$\begin{array}{ccccc} 0\longrightarrow\mathbb{Z} & \to & \pi_1(V;\tilde{m}_0) & \longrightarrow & \pi_1(\Lambda\backslash\operatorname{sing}(\mathcal{F});m_0)\to 0\\ & \searrow & \rho\downarrow & & [\rho]\downarrow\\ & & \mathrm{Mon}(\mathcal{F},V) & \longrightarrow & \mathrm{Mon}(\mathcal{F},\Lambda)\to 0 \end{array}$$

The morphism $[\rho]$ is a monodromy of $\mathcal{F}\big|_V$ seen as follows:
given any element $[\gamma]\in\pi_1(\Lambda\backslash\operatorname{sing}(\mathcal{F});m_0)$ the monodromy $[\rho]([\gamma])$ is the analytic continuation of the local first integral f_{m_0} along γ and its holonomy lifting. This gives:

Lemma 9.1. *There exists a surjective group homomorphism* $\alpha\colon \mathrm{Hol}(\mathcal{F},\Lambda)\longrightarrow\mathrm{Mon}(\mathcal{F},\Lambda)$ *such that the diagram commutes*

$$\begin{array}{ccc} & \pi_1(\Lambda\backslash\operatorname{sing}(\mathcal{F})) & \\ \mathrm{Hol}\ \swarrow & & \searrow\ [\rho]\\ \mathrm{Hol}(\mathcal{F}.\Lambda) & \xrightarrow{\alpha} & \mathrm{Mon}(\mathcal{F};\Lambda) \end{array}$$

where $\mathrm{Hol}\colon\pi_1(\Lambda\backslash\operatorname{sing}(\mathcal{F}))\longrightarrow\mathrm{Hol}(\mathcal{F};\Lambda)$ *is the holonomy morphism of the leaf* $\Lambda\backslash\operatorname{sing}(\mathcal{F})$ *of* $\mathcal{F}$*, and* $[\rho]\colon\pi_1(\Lambda\backslash\operatorname{sing}(\mathcal{F}))\longrightarrow\mathrm{Mon}(\mathcal{F};\Lambda)$ *is as above.*

The kernel of α is the subgroup $\mathrm{Ker}(\alpha)<\mathrm{Hol}(\mathcal{F};\Lambda)$ of those diffeomorphisms keeping fixed any element $\ell(z)$ of the fiber of $p\big|_V\colon V\to\Lambda\backslash\operatorname{sing}(\mathcal{F})$ over $m_o\in\Lambda\backslash\operatorname{sing}(\mathcal{F})$. Therefore $\mathrm{Ker}(\alpha)$ is a subgroup of the *invariance group* of ℓ, $\mathrm{Inv}(\ell,z)$, defined as follows $\mathrm{Inv}(\ell,z)=\{h\in\mathrm{Diff}(\mathbf{C},0);\ell\circ h\equiv\ell\}$, in the sense that if $p_\ell\colon V_\ell\to\mathbb{D}^*$ is the covering space of the punctured disc $\mathbb{D}^*=\mathbb{D}\backslash\{0\}$ associated to ℓ then $\ell\circ h\equiv\ell$ *means that* $\forall\, m\in\mathbb{D}^*$, $\forall\,\ell_m\in p_\ell^{-1}(m)$, $\exists\,\ell_{h(m)}\in p_\ell^{-1}(h(m))$, $\ell_{h(m)}\circ h=\ell_m$.
In particular, to any element $h\in\mathrm{Inv}(\ell,z)$ there is associated a pair $(\tilde{h},h)$ where $\tilde{h}$ is the lifting of h to the covering space V_ℓ defined by $\tilde{h}\colon\ell_m\mapsto\ell_{h(m)}$.
Another lemma we need is:

Lemma 9.2. *Let* $0\to G\to H\to K\to 0$ *be an exact sequence of groups. Then* H *is solvable if, and only if,* G *and* K *are solvable.*

From the above discussion we have an exact sequence

$$0\longrightarrow\mathrm{Ker}(\alpha)\longrightarrow\mathrm{Hol}(\mathcal{F},\Lambda)\xrightarrow{\alpha}\mathrm{Mon}(\mathcal{F},\Lambda)\longrightarrow 0$$

We claim that $\mathrm{Inv}(\ell,z)$ is solvable. Indeed, suppose the contrary. By Nakai's Density Lemma [14] the orbits of a non-solvable subgroup of $\mathrm{Diff}(\mathbf{C},0)$ are locally dense in a neighborhood Γ of the origin. Let therefore $m\in\Gamma$ be a point and $\Gamma_m\subset\Gamma\setminus\{0\}$ be a small sector with vertex at the origin, such that the orbit of m in Γ_m is dense in Γ_m. Denote by ℓ_{Γ_m} a local determination of ℓ in Γ_m. Then ℓ_{Γ_m} is constant along the orbits of $\mathrm{Inv}(\ell,z)$ in Γ_m and the orbit of m is dense in Γ_m so that ℓ_{Γ_m} is constant in Γ_m. By analytic continuation ℓ and the first integral Φ are constant yielding a contradiction. Thus the group $\mathrm{Inv}(\ell,z)$ is solvable and therefore embeds in $\mathrm{SL}(2,\mathbf{C})$. Hence $\mathrm{Hol}(\mathcal{F},\Lambda)\big/\mathrm{Ker}(\alpha)\simeq\mathrm{Mon}(\mathcal{F},\Lambda)$ embeds in $\mathrm{SL}(2,\mathbf{C})$ but $\mathrm{Hol}(\mathcal{F},\Lambda)$ embeds in $\mathrm{Diff}(\mathbf{C},0)$, as well as $\mathrm{Ker}(\alpha)$ embeds in $\mathrm{Inv}(\ell)$ which is a subgroup of $\mathrm{Diff}(\mathbf{C},0)$ and therefore $\mathrm{Hol}(\mathcal{F},\Lambda)\big/\mathrm{Ker}(\alpha)$ is isomorphic to a subgroup of $\mathrm{SL}(2,\mathbf{C})$ with a fixed point. This implies that indeed, $\mathrm{Hol}(\mathcal{F},\Lambda)\big/\mathrm{Ker}(\alpha)$ is solvable and conjugate to a subgroup of $\mathrm{Aff}(\mathbf{C},0)$. Therefore $\mathrm{Mon}(\mathcal{F},\Lambda)$ is solvable and by Lemma 9.2 the holonomy group $\mathrm{Hol}(\mathcal{F},\Lambda)$ is solvable.

Summarizing the above discussion we have:

Theorem 9.3. *Let $\mathcal{F}$ be a holomorphic foliation on a complex surface M, $X \subset M$ a closed analytic invariant curve and assume that $\mathcal{F}$ is transversely projective in $M \backslash X$. Let $\Lambda \subset X$ be an irreducible component of X. We suppose that each singular point in Λ is irreducible and exhibits a separatrix transverse to Λ. Then the holonomy group $\operatorname{Hol}(\mathcal{F}, \Lambda)$ of the leaf $\Lambda \backslash (\operatorname{sing}(\mathcal{F}) \cap \Lambda)$ of $\mathcal{F}$ is a solvable group.*

10 Classification of transversely projective foliations

We consider now an application of the above study to the classification of foliations with projective transverse structure.

Theorem 10.1. *Let $\mathcal{F}$ be a germ of holomorphic foliation at the origin $0 \in \mathbf{C}^2$. Suppose that*

(i) *$\mathcal{F}$ is a germ of a non-resonant generalized curve and can be reduced with a single blow-up.*

(ii) *$\mathcal{F}$ is transversely projective outside of the set $\operatorname{sep}(\mathcal{F}, 0)$ of local separatrices of $\mathcal{F}$ through 0.*

Then $\mathcal{F}$ is given by a logarithmic one-form in a neighborhood of the origin or it is a meromorphic pull-back of a germ of a Bernoulli type foliation $\mathcal{R}\colon \alpha(x)dy - (y^2\beta_0(x) + y\beta_1(x))dx = 0$ where α, β_0, β_1, β_2 are meromorphic in some neighborhood of the origin.

We shall need the following well-known technical result.

Lemma 10.2. *Let $G < \operatorname{Diff}(\mathbf{C}, 0)$ be a solvable non-abelian subgroup of germs of holomorphic diffeomorphisms fixing the origin $0 \in \mathbf{C}$.*

(i) *If the group of commutators $[G, G]$ is not cyclic then G is analytically conjugate to a subgroup of $\mathbb{H}_k = \left\{z \mapsto \frac{az}{\sqrt[k]{1+bz^k}}\right\}$ for some $k \in \mathbb{N}$.*

(ii) *If there is some $f \in G$ of the form $f(z) = e^{2\pi i\lambda}\, z + \dots$ with $\lambda \in \mathbf{C} \backslash \mathbb{Q}$ then f is analytically linearizable in a coordinate that also embeds G in $\mathbb{H}_k$.*

Proof. (i) is in [8]. Given $f \in G$ as in (ii) then by (i) we can write $f(z) = \frac{e^{2\pi i\lambda}\, z}{\sqrt[k]{1+bz^k}}$ for some $k \in \mathbb{N}$, $b \in \mathbf{C}$. Since $\lambda \in \mathbf{C} \backslash \mathbb{Q}$ the homography $H(z) = \frac{e^{2\pi i\lambda}\, z}{1+bz}$ is conjugate by another homography to its linear part $z \mapsto e^{2\pi i\lambda}\, z$ and therefore f is analytically linearizable. □

Proof of Theorem 10.1. Let $\mathcal{F}$ be given in an open subset $0 \in U \subset \mathbf{C}^2$ and put $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ in $\widetilde{U} = \pi^{-1}(U)$ where $\pi\colon \widetilde{\mathbf{C}}^2_0 \to \mathbf{C}^2$ is the blow-up of $\mathbf{C}^2$ at $0 \in \mathbf{C}^2$. Then the exceptional divisor $\Lambda = \pi^{-1}(0)$ is a compact invariant curve and we have $\operatorname{sep}(\widetilde{\mathcal{F}}, \Lambda) = \overline{\pi^{-1}(\operatorname{Sep}(\mathcal{F}, 0) \backslash \{0\})} = \pi^{-1}(\operatorname{Sep}(\mathcal{F}, 0)) \backslash \Lambda$ in $\widetilde{U}$. Therefore, each singularity of $\tilde{\mathcal{F}}$ in Λ is irreducible and exhibits a separatrix transverse to Λ. Now, by hypothesis (ii) the pull-back foliation $\widetilde{\mathcal{F}}$ is transversely projective in $\widetilde{U} \backslash \widetilde{X}$ where $\widetilde{X} = \Lambda \cup \operatorname{Sep}(\widetilde{\mathcal{F}}, \Lambda)$. According to Theorem 9.3 this implies that the holonomy group $\operatorname{Hol}(\widetilde{\mathcal{F}}, \Lambda)$ of the leaf $\Lambda \backslash \operatorname{sing}(\widetilde{\mathcal{F}})$ of $\widetilde{\mathcal{F}}$ is solvable. By the non-resonant hypothesis this group contains some element of the form $f(z) = e^{2\pi i\lambda}\, z + \dots$ with $\lambda \in \mathbf{C} \backslash \mathbb{Q}$.

By Lemma 10.2 this map f is analytically linearizable and in this same coordinate the group $\mathrm{Hol}(\widetilde{\mathcal F},\Lambda)$ is either abelian or it is analytically conjugate to a subgroup of the group

$$\mathbb{H}_k=\Big\{\varphi(z)=\frac{az}{\sqrt[k]{1+bz^k}}\,,a\neq 0\Big\}\quad\text{for some}\quad k\in\mathbb{N}.$$

In the abelian case the foliation ($\widetilde{\mathcal F}$ and therefore) $\mathcal F$ is given by a closed meromorphic one-form with simple poles (see [3]), defined in a neighborhood of the origin and therefore it writes as a logarithmic foliation say

$$\mathcal F:\,\omega=\sum_{j=1}^{t}\lambda_j\,\frac{df_j}{f_j}\,,\ \ \lambda_j\in\mathbf{C}\backslash\{0\},\ \ f_j\in\mathcal O_2\,.$$

Suppose that $\mathrm{Hol}(\widetilde{\mathcal F},\Lambda)$ is solvable and nonabelian. Then by the main result of [5] the foliation $\mathcal F$ is the pull-back by some germ of a holomorphic map $\sigma\colon\mathbf{C}^2,0\to\mathbf{C}^2,0$ of a germ of meromorphic Riccati (Bernoulli type) foliation say

$$\mathcal R:\,\alpha(x)dy-\big(y^2\beta_0(x)+y\beta_1(x)\big)dx=0.$$

This proves Theorem 10.1. □

In the same line of reasoning we can prove:

Theorem 10.3. *Let $\mathcal F$ be a foliation on $\mathbf{CP}^2$. Suppose that:*

(i) *$\mathcal F$ is transversely projective in the complement of an algebraic invariant curve $\Lambda\subset\mathbf{CP}^2$.*

(ii) *Each singularity $p\in\mathrm{sing}(\mathcal F)\cap\Lambda$ is a non-degenerate (non-dicritical) irreducible singularity.*

Then $\mathcal F$ is given by a logarithmic one-form in a neighborhood of the origin or it is a meromorphic pull-back of a germ of a Bernoulli type foliation $\mathcal R\colon\alpha(x)dy-(y^2\beta_0(x)+y\beta_1(x))dx=0$ where α, β_0, β_1, β_2 are meromorphic in some neighborhood of the origin.

Proof. By Theorem 9.3 the holonomy group of the leaf $L=\Lambda\setminus(\mathrm{sing}(\mathcal F)\cap\Lambda)$ is solvable. We claim that some singularity in Λ is non-resonant. Indeed, by the Index theorem ([4]) the sum of all indexes of singularities in Λ is equal to a (natural) positive number, the square of the degree of Λ. This implies that not all indexes are rational negative. Therefore, the holonomy group of (the leaf contained in) Λ contains some non-resonant germ and we may proceed as in the proof of Theorem 10.1 and apply the main results in [5] to conclude that $\mathcal F$ is either given a by logarithmic one-form or by a rational pull-back of a Bernoulli type foliation. □

Remark 10.4. (1) Theorems 10.1 and 10.3 above show that in order to capture the generic foliations in the class of Riccati foliations it is necessary to allow dicritical singularities.

(2) Theorem 10.1 completes an example given in [19] of a germ $\mathcal F$ satisfying (i) and (ii) but which is not a meromorphic pull-back of a Riccati foliation on an algebraic surface. Indeed, the construction given in [19] exhibits $\mathcal F$ having as projective holonomy group G, i.e., the holonomy group $G=\mathrm{Hol}(\widetilde{\mathcal F},D)$, where D is the exceptional divisor of the blow-up, a non-abelian solvable group conjugate to a subgroup of $\mathbb{H}_1=\big\{z\mapsto\frac{\lambda z}{1+\mu z}\big\}$. Our result implies that $\mathcal F$ is a meromorphic pull-back of a germ of meromorphic Bernoulli foliation in a neighborhood of the origin.

(3) In [19] it is also given an example of a foliation $\mathcal{H}$ on a rational surface Y such that $\mathcal{H}$ is transversely projective on $Y\backslash X$ for some algebraic curve $X \subset Y$ and such that $\mathcal{H}$ is *not* birationally equivalent to a Riccati foliation on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. Nevertheless, an analysis of the singular set $\text{sing}(\mathcal{H})$ of $\mathcal{H}$ shows that our non-dicriticalness and nonresonance hypothesis on the singularities of the foliation are not satisfied. Indeed, the singularities are obtained as surface desingularization of quotients of regular foliations by finite groups with fixed points.

The next lemma will be useful in capturing the Riccati case.

Lemma 10.5. *Let $\mathcal{F}$ be a germ of an irreducible singularity at the origin $0 \in \mathbf{C}^2$. Assume that:*

1. *$\mathcal{F}$ admits a meromorphic projective triple (Ω, η, ξ).*
2. *$\mathcal{F}$ is transversely projective in the complement of its local separatrices $\text{sep}(\mathcal{F}, 0)$ which is assumed to be given by $\{x = 0\} \cup \{y = 0\}$.*
3. *The foliation $\mathcal{F}^{\perp} = \mathcal{F}_{\xi}$ is transverse to the axis $\{y = 0\}$.*
4. *The foliation $\mathcal{F}^{\perp} = \mathcal{F}_{\xi}$ is transversely projective in a neighborhood of the origin minus the axis $\{x = 0\}$.*

Then $\mathcal{F}^{\perp}$ is transversely projective in a neighborhood of the origin. Indeed, the projective transverse structure of $\mathcal{F}_{\xi}$ outside the separatrix $\{y = 0\}$ extends as a projective transverse structure for $\mathcal{F}_{\xi}$ to a neighborhood of the singularity.

Proof. We have several possible cases regarding the singularity of the foliation $\mathcal{F}$:
Case 1. $\mathcal{F}$ is non-degenerate non-resonant: In this case, because it is transversely projective in the complement of its set of local separatrices, by [19] (cf. Theorem 3.2) the singularity is analytically linearizable. Let us therefore write $\Omega = g(xdy - \lambda ydx)$ in suitable local coordinates, for some meromorphic function g and $\lambda \in \mathbf{C} \setminus \mathbb{Q}$. By the relations in Proposition 2.3 we may assume that

$$\Omega = \frac{dy}{y} - \lambda\frac{dx}{x}, \;\; \eta = h\Omega = h\left(\frac{dy}{y} - \lambda\frac{dx}{x}\right), \;\; \xi = -dh - \frac{1}{2}h^2\left(\frac{dy}{y} - \lambda\frac{dx}{x}\right) + \ell\left(\frac{dy}{y} - \lambda\frac{dx}{x}\right) \tag{6}$$

where

$$d\left(\sqrt{\ell}(\frac{dy}{y} - \lambda\frac{dx}{x})\right) = 0$$

Since $\lambda \notin \mathbb{Q}$ the form $\left(\frac{dy}{y} - \lambda\frac{dx}{x}\right)$ admits no meromorphic first integral and therefore we must have $\ell = const = c \in \mathbf{C}$. Hence we have

$$\xi = -\frac{2}{h^2 - 2c}dh - \left(\frac{dy}{y} - \lambda\frac{dx}{x}\right)$$

which is a closed meromorphic one-form. Suppose that $c \neq 0$. In this case the one-form above has simple poles. On the other hand, because $\mathcal{F}_{\xi}$ is transverse to the axis $\{y = 0\}$ we conclude that this is not contained in the polar set of ξ and therefore we can integrate ξ as

$$\xi = \alpha\frac{dx}{x} + df(x, y)$$

for some holomorphic function $f(x, y)$. Thus we can write

$$\xi = \alpha \frac{d(xe^{\frac{1}{\alpha}f})}{xe^{\frac{1}{\alpha}f}}$$

This shows that ξ is a regular (nonsingular) foliation in a neighborhood of the singularity, *a fortiori*, it is transversely projective in this neighborhood.

Suppose now that $c = 0$. In this case we have

$$\xi = -\frac{2}{h^2} dh - \left(\frac{dy}{y} - \lambda \frac{dx}{x} \right)$$

The axis $\{y = 0\}$ is contained in the polar set of ξ. Because ξ is closed, its polar set is invariant by the foliation $\mathcal{F}_\xi$. Since by hypothesis $\{y = 0\}$ is not $\mathcal{F}_\xi$-invariant we may exclude the case $c = 0$.

Case 2. $\mathcal{F}$ is non-degenerate resonant: In this case the foliation is a meromorphic pull-back of a Riccati foliation ([19]). We have two possibilities.

(i) If the foliation is transversely affine outside of its set of local separatrices then it is either analytically linearizable as $\Omega = g(xdy - \frac{n}{m} ydx)$ for some $n, m \in \mathbb{N}$, or (cf. [2]) it is analytically conjugate to the germ of foliation given by

$$\omega_{k,l} = k\, x\, dy + l\, y(1 + \frac{\sqrt{-1}}{2\pi} x^l y^k) dx = 0$$

Thus $\mathcal{F}$ is given by the closed one-form

$$\Omega_{k,l} := \frac{1}{x^{l+1} y^{k+1}} \omega_{k,l}$$

Proceeding as in the previous case we may assume that

$$\Omega = \Omega_{k,l},\ \eta = h\Omega = h\Omega_{k,l},\ \ \xi = -dh - \left(\frac{1}{2} h^2 + \ell \right) \Omega_{k,l}$$

for some meromorphic function ℓ satisfying $d(\sqrt{\ell}\Omega_{k,l}) = 0$.

Since $\mathcal{F}$ is not analytically linearizable, it admits no meromorphic first integral. Therefore, because $\Omega_{k,l}$ is closed, we must have $\ell = const$. Thus we may assume that

$$\xi = -\frac{2}{h^2 - 2c} dh - \Omega_{k,l}$$

and therefore $\mathcal{F}_\xi$ is given by a closed meromorphic one-form. At this point the argumentation follows as in the preceding linearizable case. Again we conclude that $\mathcal{F}_\xi$ admits a holomorphic first integral and is transversely projective in a neighborhood of the singularity.

(ii) Assume now that the foliation is not transversely affine outside of the set of local separatrices. Then it is a meromorphic pull-back of a Riccati foliation and writes as

$$\Omega = g(dh - (\frac{1}{2} h^2 - R(f)) df$$

Notice that $\Omega_{k,l} = g_{k,l}\, dy_{k,l}$ where $y_{k,l} = \frac{x^l y^k}{\frac{\sqrt{-1}k}{2\pi} l x^l y^k \log x - 1}$ and $g_{k,l} = -\frac{(\frac{\sqrt{-1}l}{2\pi} x^l y^k \log x - 1)^2}{x^{l-1} y^{k-1}}$.

for some holomorphic function $f(x, y)$, some meromorphic function $R(z)$ and some meromorphic function $h(x, y)$. Because $\mathcal{F}$ is not transversely affine in the complement of its set of local separatrices, we have $R(f) \not\equiv 0$. Moreover, if ℓ is a meromorphic function such that $d(\sqrt{\ell}\Omega) = 0$ then ℓ is constant. This shows that we may assume that $\xi = df$. Hence $\mathcal{F}_\xi$ is transversely projective in a neighborhood of the singularity.

Case 3. **$\mathcal{F}$ is a saddle-node**: If $\mathcal{F}$ admits no affine transverse structure outside of the set of local separatrices then we proceed as in the second case (ii) above. Assume now that $\mathcal{F}$ is transversely affine in the complement of the set of local separatrices. Then the foliation is analytically conjugated to its formal normal form ([2]), that is, it can be written in suitable local coordinates as

$$\Omega = g(x(1 + \lambda y^k)dy - y^{k+1}dx)$$

for some meromorphic function g, some $\lambda \in \mathbf{C}$ and $k \in \mathbb{N}$.

Then we may assume that

$$\Omega = \Omega_{k,\lambda} := \frac{1}{1 + \lambda y^k} y^{k+1}dy - \frac{dx}{x}, \ \eta = h\Omega_{k,\lambda} = h\frac{1}{1 + \lambda y^k} y^{k+1}dy - \frac{dx}{x},$$

and

$$\xi = -dh - \left(\frac{1}{2}h^2 + \ell\right)\Omega_{k,\lambda}$$

for some meromorphic function ℓ satisfying $d(\sqrt{\ell}\Omega_{k,\lambda}) = 0$.

As it is well-known the germ $\mathcal{F}$ admits no meromorphic first integral, therefore because $\Omega_{k,\lambda}$ is closed, we must have $\ell = const. = c$ and then

$$\xi = -\frac{2}{h^2 - 2c}dh - \Omega_{k,\lambda}.$$

This implies again that $\mathcal{F}_\xi$ is given by a closed meromorphic one-form and as above the foliation $\mathcal{F}_\xi$ admits a holomorphic first integral in a neighborhood of the singularity. In particular, $\mathcal{F}_\xi$ is transversely projective in a neighborhood of the singularity. □

Given a foliation $\mathcal{F}$ on $\mathbf{CP}^2$, by an *algebraic leaf* of $\mathcal{F}$ we mean a leaf L of the foliation which is contained in an algebraic curve in $\mathbf{CP}^2$. Thanks to the Identity Principle and to Remmert-Stein extension theorem, a leaf L of $\mathcal{F}$ is algebraic if and only if it accumulates only at singular points of $\mathcal{F}$. In this case the algebraic curve consists of the leaf and such accumulation points. The following remark will be useful:

Lemma 10.6. *Let $\mathcal{F}$ and $\mathcal{F}_1$ be distinct foliations on $\mathbf{CP}^2$. If a leaf L of $\mathcal{F}$ is also a leaf of $\mathcal{F}_1$ then this leaf is algebraic.*

Proof. Suppose $(x(z), y(z)), \ z \in V \subset \mathbf{C}$ is a common solution of the foliations $\mathcal{F}$ and $\mathcal{F}_1$ on $\mathbf{CP}^2$ say: $\mathcal{F}$ is given by $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$ and $\mathcal{F}_1$ by $\frac{dy}{dx} = \frac{P_1(x,y)}{Q_1(x,y)}$ where P, Q and P_1, Q_1 are relatively prime polynomials. Then we have

$$\frac{P(x(z), y(z)}{Q(x(z), y(z))} = \frac{dy/dz}{dx/dz} = \frac{P_1(x(z), y(z))}{Q_1(x(z), y(z))}$$

so that $(PQ_1 - P_1Q)(x(z), y(z)) = 0$. By hypothesis $PQ_1 - P_1Q \not\equiv 0$ so that L satisfies the non-trivial algebraic equation $PQ_1 - P_1Q = 0$. It follows that L is algebraic. □

Theorem 10.7. *Let $\mathcal{F}$ be a foliation on a projective surface M with a projective transverse structure outside of an algebraic curve $\Lambda \subset M$. Let (Ω, η, ξ) be a rational projective triple defining the projective transverse structure outside of the curve Λ. We have the following possibilities:*

1. *Λ contains all the non-dicritical separatrices of $\mathcal{F}$ in Λ.*
2. *$\mathcal{F}^\perp$ coincides with $\mathcal{F}$.*
3. *$\mathcal{F}$ is transversely affine in $M \setminus \Lambda'$ for some algebraic invariant curve $\Lambda' \subset M$ containing Λ.*
4. *The projective transverse structure of $\mathcal{F}_\xi$ extends to M.*

Proof. We perform the resolution of singularities for $\mathcal{F}$ in Λ and obtain a projective manifold $\tilde{M}$, a divisor $E = D \cup \tilde{\Lambda}$, where D is the exceptional divisor and $\tilde{\Lambda}$ is the strict transform of Λ, equipped with a pull-back foliation $\tilde{\mathcal{F}}$ with irreducible singularities in E. The foliation $\tilde{\mathcal{F}}$ is transversely projective in $\tilde{M} \setminus E$. By Lemma 6.3 the projective transverse structure of $\tilde{\mathcal{F}}$ extends to the non-invariant part of D so that, for our purposes we may assume that D is $\tilde{\mathcal{F}}$-invariant, though not necessarily connected. Take a singular point $q \in \tilde{\Lambda} \cap \operatorname{sing}(\tilde{\mathcal{F}})$. Suppose that $\tilde{\mathcal{F}}$ exhibits some local separatrix Γ through q which is not contained in E. If Γ is $\mathcal{F}_\xi$-invariant then by Lemma 10.6 Γ is contained in an algebraic leaf of $\tilde{\mathcal{F}}$ not contained in E. This projects onto an algebraic leaf Λ' of $\mathcal{F}$ not contained in Λ. The projective transverse structure of $\mathcal{F}$ has Λ' as a set of fixed points and therefore $\mathcal{F}$ is transversely affine in $M \setminus (\Lambda \cup \Lambda')$. Assume now that Γ is not $\mathcal{F}_\xi$-invariant. Then we are in the situation considered in Lemma 10.5. By this lemma we conclude that the projective transverse structure of $\mathcal{F}_\xi$ extends from a neighborhood of q minus the local branches of E through q to a projective transverse structure in a neighborhood of q.

□

Remark 10.8 (Logarithmic foliations and invariant curves)**.** Theorem A in [11] gives the following nice characterization of logarithmic foliations: *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X and let S be an invariant compact curve by $\mathcal{F}$. Assume that one of the following conditions hold: (i) $Pic(X)$ is isomorphic to $\mathbb{Z}$ or (ii) $Pic(X)$ is torsion free, $H^1(X, C) = 0$, $S^2 > 0$ and $\sum_{p \in \operatorname{sing}(\mathcal{F}) - S} BB_p(\mathcal{F}) > O$. Then, if every local separatrix of $\mathcal{F}$ through any $p \subset \operatorname{sing}(\mathcal{F}) \cap S$ is a local branch of S and if every singularity of $\mathcal{F}$ in S is a generalized curve, then $\mathcal{F}$ is logarithmic.*

Here, by $BB_p(\mathcal{F})$ we mean the Baum-Bott index associated to the Chern number c_1^2 of the normal sheaf of the foliation ([1]). The author also observes that:
The condition $\sum_{p \in \operatorname{sing}(\mathcal{F}) - S} BB_p(\mathcal{F}) > O$ holds if each singularity of $\mathcal{F}$ in $X \setminus S$ is linearly of Morse type (i.e. f is locally given by the holomorphic 1-form $d(xy) + h.o.t.$). This condition also holds when $\mathcal{F}$ a has local holomorphic first integral around each point of X which is not in S.
Let $\mathcal{F}$ be a holomorphic foliation on $\mathbf{CP}^2$ of degree m, then $\sum_{p \in \operatorname{sing}(\mathcal{F}) - S} BB_p(\mathcal{F}) = (m+2)^2$.

Therefore, the author proves the following extension of the second part of theorem 1 in [7] to compact complex surfaces (cf. [11] Proposition 3.1): *Let $\mathcal{F}$ be a holomorphic foliation on a compact algebraic surface X with $H^1(X, \mathbf{C}) = \mathbf{0}$ and $Pic(X) = \mathbb{Z}$. Let S be an invariant* *compact curve with only nodal type singularities. If* $\sum_{p \in \operatorname{sing}(\mathcal{F}) - S} BB_p(\mathcal{F}) < S^2$, *then* $\mathcal{F}$ *is logarithmic.*

By taking a look at the proof given in [11] we conclude that the conclusion of Theorem A holds for a foliation $\mathcal{F}$ on the complex projective plane $\mathbf{CP}^2$ having an invariant algebraic curve S such that each singularity of $\mathcal{F}$ in S exhibits no saddle-node in its resolution and if S contains each non-dicritical separatrix of each singularity of $\mathcal{F}$ in S.

Theorem 10.9. *Let* $\mathcal{F}$ *be a foliation on* $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ *with a projective transverse structure in the complement of an algebraic curve* $\Lambda \subset \overline{\mathbf{C}} \times \overline{\mathbf{C}}$. *Assume that the singularities of* $\mathcal{F}$ *in* Λ *are all non-resonant generalized curves. Then* $\mathcal{F}$ *is a logarithmic foliation or it is a rational pull-back of a Bernoulli type foliation or it is a rational pull-back of a Riccati foliation.*

Proof. First we apply Theorem 7.4 in order to be able to apply Theorem 10.7. According to Theorem 10.7 we have the following possibilities:

1. Λ contains all the non-dicritical separatrices of $\mathcal{F}$ in Λ.

2. $\mathcal{F}^{\perp}$ coincides with $\mathcal{F}$.

3. $\mathcal{F}$ is transversely affine in $M \setminus \Lambda'$ for some algebraic invariant curve $\Lambda' \subset M$ containing Λ.

4. The projective transverse structure of $\mathcal{F}_\xi$ extends to M.

In case (1), because the singularities are non-dicritical we conclude that the algebraic curve Λ contains all the separatrices through singularities of $\mathcal{F}$ contained in Λ. Since the singularities are assumed to be generalized curves we may apply [11] and conclude that $\mathcal{F}$ is a logarithmic foliation.

In case (2) we have $\xi \wedge \Omega = 0$ so that $d\eta = 0$ and therefore $\mathcal{F}$ is transversely affine in the complement of the algebraic invariant curve given by the polar set of η. In this case, thanks to the main result in [5] and [17] the foliation is a logarithmic foliation or it is a rational pull-back of a Bernoulli foliation.

In case (3) the situation is similar to the one above in case (2) and the same conclusion holds.

Finally, in case (4) the foliation $\mathcal{F}_\xi$ is transversely projective in $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. Since this manifold is simply-connected we conclude that $\mathcal{F}_\xi$ admits a rational first integral. By Proposition 2.7 $\mathcal{F}$ is a rational pull-back of a Riccati foliation. This ends the proof.

□

With a very similar (adapted) proof we have the following variant for foliations in the projective plane:

Theorem 10.10. *Let* $\mathcal{F}$ *be a foliation on* $\mathbf{CP}^2$ *with a projective transverse structure in the complement of an algebraic curve* $\Lambda \subset \mathbf{CP}^2$. *Assume that the singularities of* $\mathcal{F}$ *in* Λ *are all non-resonant extended generalized curves. Then* $\mathcal{F}$ *is a logarithmic foliation or it is a rational pull-back of a Bernoulli type foliation or it is a rational pull-back of a Riccati foliation.*

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