

INTERNAL OBJECT ACTIONS IN HOMOLOGICAL CATEGORIES

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ABSTRACT. Let G and A be objects of a finitely cocomplete homological category \mathbb{C} . We define a notion of an (internal) action of G on A which is functorially equivalent with a point in \mathbb{C} over G , i.e. a split extension in \mathbb{C} with kernel A and cokernel G . This notion and its study are based on a preliminary investigation of cross-effects of functors in a general categorical context. These also allow us to define higher categorical commutators. We show that any proper subobject of an object E (i.e., a kernel of some map on E in \mathbb{C}) admits a “conjugation” action of E , generalizing the conjugation action of E on itself defined by Bourn and Janelidze. If \mathbb{C} is semi-abelian, we show that for subobjects X, Y of some object A , X is proper in the supremum of X and Y if and only if X is stable under the restriction to Y of the conjugation action of A on itself. This amounts to an elementary proof of Bourn and Janelidze’s functorial equivalence between points over G in \mathbb{C} and algebras over a certain monad \mathbb{T}_G on \mathbb{C} . The two axioms of such an algebra can be replaced by three others, in terms of cross-effects, two of which generalize the usual properties of an action of one group on another.

MSC: 18A05, 18A20, 18A22 ; *key words:* action, semi-direct product, conjugation, proper subobject, commutator, homological category, semi-abelian category, algebra over monad.

For two objects G and A of a category \mathbb{C} , an *action* of G on A morally should be some minimal data equivalent to a split extension with kernel A and cokernel G (also called a point of \mathbb{C} over G), which then may be called the semi-direct product of A and G along the given data. For semi-abelian categories \mathbb{C} , an elegant solution of the problem to exhibit such minimal data was first provided by Bourn and Janelidze in [5], by defining an action to be an algebra over a certain monad \mathbb{T}_G on \mathbb{C} . This notion then was further studied by various authors, in particular in terms of monoidal categories by Borceux, Janelidze and Kelly in [3].

In this paper, we define a notion of action of G on A in terms of even “smaller” data than a \mathbb{T}_G -algebra, namely as being a map $(A|G) \rightarrow A$ satisfying a certain property, where $(A|G)$ is the kernel of the canonical map from the sum $A + G$ to the product $A \times G$. An action in this sense is functorially equivalent to a point over G even in a non-exact context, namely in finitely cocomplete homological categories \mathbb{C} (which are of special interest to the authors, as, for example, certain categories of filtered groups are of this type, see the example at the end of section 3). Moreover, our notion allows to directly generalize certain basic concepts from groups and Lie-algebras, such as a conjugation action of an object on any proper subobject, and to formalize the fact that the semi-direct product along some action can be viewed as its universal transformation into a conjugation action. This leads to a very useful characterization of proper subobjects in semi-abelian categories, namely as those being stable under the conjugation action (this result was recently obtained

independantly by Mantovani and Metere in [10]); and this property characterizes semi-abelian categories among finitely cocomplete homological ones. More generally, we define a notion of one subobject normalizing another one, in terms of the conjugation action, which is equivalent to the latter being proper in the supremum of both when \mathbb{C} is semi-abelian (Theorem 23).

These facts lead to many applications: e.g., based on a detailed comparison of our actions with \mathbb{T}_G -algebras, they allow to reprove the equivalence between these algebras and points over G when \mathbb{C} is semi-abelian, in an elementary way without using Beck's criterion. More applications are given in our forthcoming study of extensions with non-abelian kernel and of higher commutators of subobjects as introduced in this paper (Definition 19). It is based on the notion of *cross-effect* of a functor which is fundamental in algebraic topology; it was introduced, for functors between abelian categories, by Eilenberg and MacLane [6], and adapted to functors with values in the category of groups by Baues and Pirashvili [1], see also [7] for further properties. In section 1 of this paper, we define cross-effects in a general categorical context and exhibit some elementary properties, which constitute our main tools: in fact, the object $(A|G)$ can be viewed as the second cross-effect of the identity functor of \mathbb{C} . In particular, we use the cross-effect machinery to cut the axioms of a \mathbb{T}_G -algebra in three pieces, two of which again look like associativity conditions, on the terms $((A|G)|A)$ and $((A|G)|G)$, and have nice interpretations in the category of groups: the first one says that G acts by endomorphisms of A , and the second then expresses the usual associativity condition for the action of G on the underlying set of A . Thus the third condition is void in the category of groups, but probably not in general. It involves a triple cross-effect and actually looks quite odd; we did not yet succeed to give it a more convenient form.

BASIC CONVENTIONS

Throughout this paper, \mathbb{C} denotes a finitely cocomplete homological category, cf. [2]. When merely working in a pointed category with finite sums, we denote the canonical inclusion $X_k \rightarrow X_1 + \dots + X_n$ by i_{X_k} or by i_k , and its canonical retraction by r_{X_k} or by r_k ; dually, when working in a pointed category with finite products, we denote the canonical projection $X_1 \times \dots \times X_n \rightarrow X_k$ by π_k and its canonical section by σ_k . When the category is pointed and has finite products and finite sums, the canonical map $X_1 + \dots + X_n \rightarrow X_1 \times \dots \times X_n$ is denoted by $b_{X_1 \dots X_n}$ or simply by b .

1. CROSS EFFECTS OF FUNCTORS

The fundamental concept of cross-effect of a functor between abelian categories is due to Eilenberg and MacLane [6]; it was adapted to functors with values in the category of groups in [1] and further studied in [7]. In this paper and subsequent work we will tend to show, however, that the concept of cross-effect provides a powerful in the theory of homological and semi-abelian categories, so we keep our hypothesis' as general as possible. We here content ourselves to prove just some elementary properties of cross-effects in this framework; a more thorough investigation is carried out in [8].

Let $F : \mathbb{D} \rightarrow \mathbb{E}$ be a functor where \mathbb{D} is a pointed category with finite sums and \mathbb{E} is a pointed finitely complete category.

Definition 1. For any nonzero natural number n , the n -th cross effect of F is defined to be the functor $cr_n(F) : \mathbb{D}^{\times n} \rightarrow \mathbb{E}$ inductively defined by:

$$cr_1(F)(X) = Ker(F(0) : F(X) \rightarrow F(0));$$

$$cr_2(F)(X, Y) = Ker((F(r_X), F(r_Y))^t : F(X + Y) \rightarrow F(X) \times F(Y));$$

$$cr_n(F)(X_1, X_2, \dots, X_n) = cr_2(cr_{n-1}(F)(-, X_3, \dots, X_n))(X_1, X_2), \quad \text{for } n \geq 3.$$

The definition of cr_1 and cr_2 (hence of cr_n) on morphisms is obvious.

One often abbreviates $cr_n(F)(X_1, X_2, \dots, X_n) = F(X_1 | \dots | X_n)$. For the identity functor $Id_{\mathbb{C}}$ of \mathbb{C} and objects X_1, \dots, X_n in \mathbb{C} we write $(X_1 | \dots | X_n) = Id_{\mathbb{C}}(X_1 | \dots | X_n)$, so that for instance, by the very definition, $(X|Y|Z)$ is the kernel of the canonical map from $(X + Y|Z)$ to $(X|Z) \times (Y|Z)$.

Proposition 2. The functors cr_n have the following properties:

(1) There is a natural injection

$$\iota_{X_1, \dots, X_n}^F : cr_n(F)(X_1, \dots, X_n) \hookrightarrow F(X_1 + \dots + X_n)$$

(2) The functor cr_n is multireduced, i.e. $cr_n(F)(X_1, \dots, X_n) = 0$ if $X_k = 0$ for some $k = 1, \dots, n$.

(3) The bifunctor $cr_2(F)$ is symmetric.

In fact, we show in [8] that the functor $cr_n(F)$ is symmetric for all n , but we do not need this here.

When no confusion can arise we often just write ι instead of $\iota_{X_1, \dots, X_n}^F$.

Proof. Assertion (3) is obvious, the other two are easily proved by induction. For (2), just observe that if $X = 0$, also $cr_2(X, Y) = 0$ since the map $(F(r_X), F(r_Y))^t : F(0 + Y) \rightarrow F(0) \times F(Y)$ admits the second projection followed by $F(i_Y)$ as a retraction. \square

The following facts are key tools in handling cross effects.

Proposition 3. Suppose in addition that \mathbb{E} is homological and that F preserves regular epimorphisms. Then for all objects A in \mathbb{D} the functor $F(A| -) : \mathbb{D} \rightarrow \mathbb{E}$ also preserves regular epimorphisms.

Proof. Let $f : X \twoheadrightarrow Y$ be a regular epimorphism. Consider the following commutatif diagram where $k : Ker(F(f)) \twoheadrightarrow F(X)$ is a kernel of $F(f)$ and α is induced by $(F(r_X), F(r_Y))^t$. :

$$\begin{array}{ccccc} & & F(A|X) & \xrightarrow{F(1|f)} & F(A|Y) \\ & & \downarrow \iota & & \downarrow \iota \\ & & F(A+X) & \xrightarrow{F(1+f)} & F(A+Y) \longrightarrow 0 \\ \text{Ker}(F(1+f)) \hookrightarrow & & & & \\ \downarrow \alpha & & \downarrow (Fr_A, Fr_X)^t & & \downarrow (Fr_A, Fr_Y)^t \\ 0 \longrightarrow & \text{Ker}(F(f)) & \xrightarrow{(0,k)^t} & F(A) \times F(X) & \xrightarrow{1 \times F(f)} & F(A) \times F(Y) \end{array}$$

The columns are exact by definition of the cross-effect, and the rows are exact, too; for the middle row this follows from the hypothesis on F since $1 + f$ is a regular epimorphism. Thus the snake lemma provides an exact sequence

$$F(A|X) \xrightarrow{F(1+f)} F(A|Y) \longrightarrow \text{Coker}(\alpha)$$

We claim that $\text{Coker}(\alpha) = 0$: in fact, the map $F(i_X)k: \text{Ker}(F(f)) \rightarrow F(A + X)$ factors through j and thus provides a section s of α , indeed: $F(1 + f)F(i_Y)k = F(i_Y)F(f)k = 0$, and

$$(0, k)^t \alpha s = (Fr_A, Fr_X)^t j s = (Fr_A, Fr_X)^t F(i_X)k = (0, 1)^t k = (0, k)^t$$

whence $\alpha s = 1$ since $(0, k)^t$ is monic. □

Proposition 4. *Suppose in addition that \mathbb{D} is protomodular, \mathbb{E} is homological and that F preserves regular epimorphisms. Let $f: X \rightarrow Y$ be a map in \mathbb{D} with splitting $s: Y \rightarrow X$, i.e. such that $fs = 1$. Let $k: K \rightarrow X$ be a kernel of f and let $A \in \text{Ob}(\mathbb{D})$. Then the map*

$$\langle \iota', F(k|1), F(s|1) \rangle : F(K|Y|A) + F(K|A) + F(Y|A) \rightarrow F(X|A)$$

is a regular epimorphism where ι' is the composite map

$$F(K|Y|A) = F(-|A)(K|Y) \xrightarrow{\iota} F(K + Y|A) \xrightarrow{F(\langle k, s \rangle | 1)} F(X|A)$$

where more precisely $\iota = \iota_{K, Y}^{F(-|A)}$.

Proof. Consider the following commutative diagram whose right-hand square is a pullback:

$$\begin{array}{ccccc} 0 & \longrightarrow & F(K|Y|A) & \xrightarrow{(\iota, 0)^t} & P & \xrightarrow{p_2} & F(K|A) + F(Y|A) \\ & & \parallel & & \downarrow p_1 & & \downarrow (r_1, r_2)^t \\ 0 & \longrightarrow & F(K|Y|A) & \xrightarrow{\iota} & F(K + Y|A) & \xrightarrow{r} & F(K|A) \times F(Y|A) \end{array}$$

Here $r = (F(r_K|1), F(r_Y|1))^t$, $r_1 = r_{F(K|A)}$ and $r_2 = r_{F(Y|A)}$. The bottom row is exact by definition of the cross-effect, hence the top row is also exact. Let $\sigma: F(K|A) + F(Y|A) \rightarrow P$ be the map such that $p_1 \sigma = \langle F(i_K|1), F(i_Y|1) \rangle$ and $p_2 \sigma = 1$. Then σ is a splitting of p_2 , whence by protomodularity of \mathbb{E} the map

$$\langle (\iota, 0)^t, \sigma \rangle : F(K|Y|A) + F(K|A) + F(Y|A) \longrightarrow P$$

is a regular epimorphism. We then successively conclude that the following maps also are regular epimorphisms:

- $(r_1, r_2)^t$ by protomodularity of \mathbb{E} ;
- p_1 by regularity of \mathbb{E} ;
- $\langle k, s \rangle$ by protomodularity of \mathbb{D} ;
- $F(\langle k, s \rangle | 1)$ by Proposition 3;
- the composite map $F(\langle k, s \rangle | 1)p_1 \langle (\iota, 0)^t, \sigma \rangle$.

But

$$\begin{aligned} F(\langle k, s \rangle | 1)p_1 \langle (\iota, 0)^t, \sigma \rangle &= F(\langle k, s \rangle | 1) \langle \iota, F(i_K|1), F(i_Y|1) \rangle \\ &= \langle \iota', F(k|1), F(s|1) \rangle, \end{aligned}$$

whence the assertion. □

2. GENERAL PROPERTIES OF INTERNAL OBJECT ACTIONS

Recall that the category of (internal) *actions* of an object G of \mathbb{C} can be equivalently defined as being

- The category of split extensions of G , whose objects are short split exact sequences

$$0 \longrightarrow A \longrightarrow E \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{q} \end{array} G \longrightarrow 0$$

(the sequence is exact and s is a spitting of q), and such an object of this category

is called an action of G on A . A map from $0 \longrightarrow A \longrightarrow E \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{q} \end{array} G \longrightarrow 0$ to

$0 \longrightarrow A' \longrightarrow E' \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{q'} \end{array} G \longrightarrow 0$ being the data of two maps $a : A \rightarrow A'$ and

$b : E \rightarrow E'$ making the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{q} \end{array} & G \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \parallel \\ 0 & \longrightarrow & A' & \longrightarrow & E' & \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{q'} \end{array} & G \longrightarrow 0 \end{array}$$

- The category $\text{Pt}_G(\mathbb{C})$ of G -points of \mathbb{C} , i.e. the category $(\mathbb{C}/G) \setminus (1_G)$, which can be described as the category of objects E of \mathbb{C} together with a map $q : E \rightarrow G$ and a section $s : G \rightarrow E$ of q . A map b between two such objects (E, q, s) and (E', q', s') is a map $b : E \rightarrow E'$ making the following diagram commute:

$$\begin{array}{ccc} E & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{q} \end{array} & G \\ \downarrow b & & \parallel \\ E' & \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{q'} \end{array} & G \end{array}$$

- if \mathbb{C} is moreover exact, hence semi abelian, these two obviously equivalent categories are also equivalent to the category of Eilenberg-Moore algebras over the monad $\mathbb{T}_G : \mathbb{C} \rightarrow \mathbb{C}$ sending an object A to the kernel $T_G(A)$ of the natural retraction $A + G \rightarrow G$, according to Bourn and Janelidze, see [5]).

The monadic aspect may of course be studied in any pointed finitely complete category with finite sums, as in [3] where $T_G(A)$ is considered as a subobject of $G + A$ rather than of $A + G$, and is denoted by GbA .

Recall that $\text{Pt}_G(\mathbb{C})$ is a subcategory of a more general category $\text{Pt}(\mathbb{C})$, whose objects are points (on variable objects G), and a map between two points (G, E, q, s) and (G', E', q', s') is a pair of maps $a : G \rightarrow G'$ and $b : E \rightarrow E'$ making the following diagram commute:

$$\begin{array}{ccc} E & \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{q} \end{array} & G \\ \downarrow b & & \downarrow a \\ E' & \begin{array}{c} \xleftarrow{s'} \\ \xrightarrow{q'} \end{array} & G' \end{array}$$

We introduce another way to define a category of G -actions, using the (second) cross-effect of the identity functor of \mathbb{C} ; this new category is always equivalent to the category of points, hence to the category of algebras described above when \mathbb{C} is exact; but we also will exhibit the link between our definition and this category of algebras in the non-exact case. We start by relating the cross-effect of the identity functor to the functor T_G .

Lemma 5. *Let us denote by k (or $k_{A,G}$) the inclusion of $T_G(A)$ in $A + G$. Then there exists an inclusion j (or $j_{A,G}$) of $(A|G)$ in $T_G(A)$, yielding a split short exact sequence*

$$0 \longrightarrow (A|G) \xrightarrow{j} T_G(A) \xrightarrow[r_A.k]{\eta_A} A \longrightarrow 0$$

where η is the unit of the monad \mathbb{T}_G , i.e. η_A is the unique map $A \rightarrow T_G(A)$ such that $k.\eta_A = i_A$, which exists and is unique since $r_G.i_A = 0$.

Proof. Since $r_G.\iota = \pi_G.b.\iota = 0$, we get a unique $j : (A|G) \rightarrow T_G(A)$ such that $k.j = \iota$. It remains to show that j is the kernel of $r_A.k$. First, $r_A.k.j = r_A.\iota = \pi_A.b.\iota = 0$. Second, let $f : X \rightarrow T_G(A)$ be such that $r_A.k.f = 0$, then since one also obviously has $r_G.k.f = 0$, one has $b.k.f = 0$, so kf factorizes through $(A|G)$, say $kf = \iota\bar{f} = k.j.\bar{f}$, hence $f = j.\bar{f}$ and we get a factorization of f by $(A|G)$, which is unique since j is a monomorphism. \square

Corollary 6. *A map $\xi : T_G(A) \rightarrow A$ satisfying the unit axiom (i.e., $\xi\eta_A = 1$) is uniquely determined by its restriction ψ (along j) to $(A|G)$.*

Proof. Consider two maps ξ and $\xi' : T_G(A) \rightarrow A$ satisfying this unit axiom, i.e. $\xi.\eta_A : \xi'.\eta_A = 1_A$ having same restriction ψ to $(A|G)$, i.e. $\psi = \xi.j = \xi'.j$. Then $\xi = \xi'$ since by Lemma 5 and by protomodularity the pair (η, j) is (strongly) epimorphic. \square

Corollary 6 explains why we will characterize our actions in terms of maps $(A|G) \rightarrow A$ rather than $T_G(A) \rightarrow A$.

Recall that when $\xi : T_G(A) \rightarrow A$ is an action of G on A (in the sense of [3]), the semi-direct product of A and G along ξ is the coequalizer of $\langle k, i_G \rangle$ and $\xi + 1 : T_G(A) + G \rightarrow A + G$ (as defined in [3], but using our terminology and putting the G 's on the right). This definition arises naturally from the general theory of monads, but it is clear that this coequalizer is also the coequalizer of k and $i_A\xi$. Here again, we show that the knowledge of the restriction $\psi : (A|G) \rightarrow G$ of ξ to $(A|G)$ along $j : (A|G) \rightarrow T_G(A)$ suffices to determine this coequalizer:

Proposition 7. *Consider a map $\xi : T_G(A) \rightarrow A$ satisfying the unit axiom of a \mathbb{T}_G -algebra, and consider $\psi = \xi.j : (A|G) \rightarrow A$. Then the coequalizer of k and $i_A\xi$ (hence the semi-direct product of A and G along ξ , if moreover ξ is a \mathbb{T}_G -algebra, considering the observation here above) is also the coequaliser of $\iota (= k.j)$ and $i_A.\psi$.*

Proof. We have to show that for any map $h : A + G \rightarrow X$ one has $hk = hi_A\xi$ if and only if $hkj = hi_A\psi = hi_A\xi.j$. And of course only the sufficient condition must be proved; but it follows immediately from the fact that the pair (j, η) is (strongly) epimorphic by Lemma 5. \square

The following proposition underlines the key role of the object $(A|G)$:

Proposition 8. *Let G and A be objects of \mathbb{C} . Consider a morphism $\psi : (A|G) \rightarrow A$, and let $q_\psi : A + G \rightarrow Q_\psi$ be the coequalizer of ι and $i_1.\psi$. Let l_ψ be the composite $q_\psi.i_A$. Then:*

- 1) r_G coequalizes $\iota_{A,G}$ and $r_A.\psi$, giving rise to a unique extension $p_\psi : Q_\psi \rightarrow G$, such that $p_\psi q_\psi = r_G$;
- 2) The morphism $s_\psi = q_\psi.i_G : G \rightarrow Q_\psi$ is a section of p_ψ ;
- 3) The sequence $A \xrightarrow{l_\psi} Q_\psi \xrightarrow{p_\psi} G$ is exact, hence the sequence

$$0 \longrightarrow A \xrightarrow{l_\psi} Q_\psi \begin{array}{c} \xleftarrow{s_\psi} \\ \xrightarrow{p_\psi} \end{array} G \longrightarrow 0$$

is split short exact if and only if $q_\psi.i_A$ is a monomorphism.

Proof. 1. One has: $r_G.\iota = \pi_G.b_{A,G}.\iota = 0$ and $r_G.i_A.\psi = 0.\psi = 0$.

2) $p_\psi.s_\psi = p_\psi q_\psi.i_G = r_g.i_G = 1_G$.

3) Let $k' : \text{Ker}(q_\psi) \rightarrow A + G$ be the kernel of q_ψ . One has: $r_G.k' = p_\psi.q_\psi.k = 0$, hence $\text{Ker}(q_\psi)$ is a subobject of $T_G(A)$. So by Noether's first isomorphism theorem, it is proper, and

$$\frac{\frac{A+G}{\text{Ker}(q_\psi)}}{\frac{T_G(A)}{\text{Ker}(q_\psi)}} = \frac{A+G}{T_G(A)} = G$$

meaning more precisely that we have the following diagram, where all sequences are short exact:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}(q_\psi) & \longrightarrow & T_G(A) & \xrightarrow{q'} & \text{Ker}(p_\psi) \longrightarrow 0 \\ & & \parallel & & \downarrow k & & \downarrow k'' \\ 0 & \longrightarrow & \text{Ker}(q_\psi) & \xrightarrow{k'} & A + G & \xrightarrow{q_\psi} & Q_\psi \longrightarrow 0 \\ & & \downarrow & & \downarrow r_G & & \downarrow p_\psi \\ 0 & \longrightarrow & 0 & \longrightarrow & G & \xlongequal{\quad} & G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Now one has: $p_\psi.q_\psi.i_A = r_G.i_A = 0$, so $q_\psi.i_A$ factorizes by $\text{Ker}(p_\psi)$, say $q_\psi.i_A = k''.v$ for some unique $v : A \rightarrow \text{Ker}(p_\psi)$. So it remains to show that v is a regular epimorphism to get the result.

By Lemma 5 and by protomodularity of \mathbb{C} , the map $\langle j, \eta_A \rangle : (A|G) + A \rightarrow T_G(A)$ is a regular epimorphism, and so is q' , hence so is $q'. \langle j, \eta_A \rangle$. It suffices then to show that $q'. \langle i, \eta_A \rangle = v. \langle \psi, 1_A \rangle$, which will show that v is a regular epimorphism as required. Of course, it suffices to show that $k''.q'. \langle j, \eta_A \rangle = k''.v. \langle \psi, 1_A \rangle$, or equivalently that $q_\psi.k. \langle j, \eta_A \rangle = q_\psi.i_A \langle \psi, 1_A \rangle$. By preceding by the inclusions of $(A|G)$ and A in their sum, this amounts to show that $q_\psi.k.j = q_\psi.i_A.\psi$ and that $q_\psi.k.\eta_A = q_\psi.i_A.1_A = q_\psi.i_A$. First, $q_\psi.k.i = q_\psi.\iota = q_\psi.i_A.\psi$ by the very definition of q_ψ . Second, $k.\eta_A = i_A$, which gives the result. \square

Example: Let \mathbb{C} be the category of groups. In order to simplify notations here, we will feel free to consider the inclusions of A and G in their sum as set inclusions, so that for instance, for $a \in A$ and $g \in G$ the notation $[g, a] = gag^{-1}a^{-1}$ denotes unambiguously the commutator of g and a , seen as elements of $A + G$. It is well known (see for instance [9]) that for two groups A, G , the cross effect $(A|G)$ is freely generated by the elements $[g, a]$, $(g, a) \in G^* \times A^*$, where X^* denotes the set of nonzero elements of the group X . More formally, for a set S denote by $\mathcal{F}(S)$ the free group with basis S . Then the homomorphism $\mathcal{F}(G^* \times A^*) \rightarrow A + G$ sending a basis element (g, a) to $[g, a]$ maps isomorphically onto $(A|G)$ ¹. Indeed, the fact that this map maps surjectively onto $(A|G)$ can be easily seen as follows: any element of $A + G$ can be written as a product of the form $(\prod_{i=1}^n [g_i, a_i]^{z_i}) \cdot a \cdot g$, where the z_i 's are integers, g and the g_i 's are in G and a and the a_i 's are in A . Moreover, $b_{A,G}((\prod_{i=1}^n [g_i, a_i]^{z_i}) \cdot a \cdot g) = (a, g)$, showing the uniqueness of a and g and showing that the kernel of $b_{A,G}$ is the set of elements of $A + G$ having the form $\prod_{i=1}^n [g_i, a_i]^{z_i}$, hence that we get an epimorphism $\mathcal{F}(G^* \times A^*) \rightarrow (A|G)$. That this epimorphism is an isomorphism is less easy to show.

So a map $\psi : (A|G) \rightarrow A$ can be seen as a set-theoretic application $G^* \times A^* \rightarrow A$. The link with usual actions is the following. Let $\phi : G \times A \rightarrow A$ be an action in the usual sense. Then one also may consider $\phi' : G^* \times A^* \rightarrow A$ defined by $\phi'(g, a) = \phi(g, a) \cdot a^{-1}$, and $\psi : \mathcal{F}(G^* \times A^*) = (A|G) \rightarrow A$ its extension as a group morphism. Then it is easy to see that the coequalizer of $i_A \cdot \psi$ and ι is the canonical epimorphism from $A + G$ to the usual semi-direct product $A \rtimes_{\phi} G$. Conversely, if $\psi : \mathcal{F}(G^* \times A^*) = (A|G) \rightarrow A$ is such that $q_{\psi} \cdot i_A$ (as constructed above) is a monomorphism, then it gives rise to a

split extension $0 \longrightarrow A \xrightarrow{q_{\psi} \cdot i_A} Q_{\psi} \begin{array}{c} \xleftarrow{s_{\psi}} \\ \xrightarrow{p_{\psi}} \end{array} G \longrightarrow 0$, hence to an action $\phi : G \times A \rightarrow A$ in the usual sense. Then it can be shown that for nonzero $g \in G$ and $a \in A$ one has $\psi(g, a) = \phi(g, a) \cdot a^{-1} = \phi'(g, a)$. The resulting equivalence between maps ϕ or ϕ' and suitable homomorphisms ψ as above motivates the following definition for any finitely cocomplete homological category:

Definition 9. An action (of an object G on an object A) is the data of two objects A and G and of a map $\psi : (A|G) \rightarrow A$ such that $q_{\psi} \cdot i_A$ is a monomorphism. A map of actions: $(A, G, \psi) \rightarrow (A', G', \psi')$ is an ordered pair (a, g) where $a : A \rightarrow A'$ and $g : G \rightarrow G'$ are maps in \mathbb{C} making the following diagram commute:

$$\begin{array}{ccc} (A|G) & \xrightarrow{(a|g)} & (A'|G') \\ \psi \downarrow & & \psi' \downarrow \\ A & \xrightarrow{a} & A' \end{array}$$

If (A, G, ψ) is an action, then we denote by $A \rtimes_{\psi} G$ the quotient Q_{ψ} as defined above.

This obviously defines a category which we denote by $\text{Act}(\mathbb{C})$. It is equipped with a forgetful functor to \mathbb{C} , sending the object (A, G, ψ) to the object G , and the map (a, g) to g . This makes of $\text{Act}(\mathbb{C})$ a fibration on \mathbb{C} (as is easily shown, but also as a consequence of Proposition 12 below). The fiber on G of this fibration is denoted by $\text{Act}_G(\mathbb{C})$.

¹Of course, it would seem more natural to present this isomorphism, obviously equivalently, as an isomorphism $\mathcal{F}(A^* \times G^*) \rightarrow (A|G)$. But for technical reasons which will appear clearly, $\mathcal{F}(G^* \times A^*) \rightarrow (A|G)$ is more convenient.

As can be expected, this notion of semi-direct product satisfies a universal property (see Proposition 17 below for another version which more directly generalizes the usual one in the category of groups):

Proposition 10. *Let C be an object of \mathbb{C} and $f_A : A \rightarrow C$, $f_G : G \rightarrow C$ be two morphisms such that the following diagram commutes:*

$$\begin{array}{ccc} (A|G) & \xrightarrow{\iota_{A,G}} & A + G \\ \psi \downarrow & & \downarrow \langle f_A, f_G \rangle \\ A & \xrightarrow{f_A} & C \end{array}$$

Then there exists a unique $f : A \rtimes_{\psi} G \rightarrow C$ such that $f.l_{\psi} = f_A$ and $f.s_{\psi} = f_G$:

$$\begin{array}{ccc} A & \xrightarrow{f_A} & C \\ & \searrow l_{\psi} & \uparrow \exists! f \\ & A \rtimes_{\psi} G & \xrightarrow{\dots} C \\ & \nearrow s_{\psi} & \\ G & \xrightarrow{f_G} & C \end{array}$$

Proof. Consider the map $\langle f_A, f_G \rangle : A + G \rightarrow C$. One has: $\langle f_A, f_G \rangle . \iota_{A,G} = f_A . \psi = \langle f_A, f_G \rangle . i_A . \psi$. So, since $A \rtimes_{\psi} G$ is the coequalizer of $\iota_{A,G}$ and $i_A . \psi$, there exists a unique $f : A \rtimes_{\psi} G \rightarrow C$ such that $f.q_{\psi} = \langle f_A, f_G \rangle$:

$$\begin{array}{ccccc} (A|G) & \xrightarrow{\iota_{A,G}} & A + G & & \\ \psi \downarrow & \nearrow i_A & \downarrow q_{\psi} & \searrow \langle f_A, f_G \rangle & \\ A & \xrightarrow{l_{\psi}} & A \rtimes_{\psi} G & \xrightarrow{\dots} & C \end{array}$$

Then $f.l_{\psi} = f.q_{\psi}.i_A = \langle f_A, f_G \rangle . i_A = f_A$, and $f.s_{\psi} = f.q_{\psi}.i_G = \langle f_G, i_G \rangle . i_G = f_G$. And since \mathbb{C} is protomodular, the family (l_{ψ}, s_{ψ}) is (strongly) epimorphic, which ensures uniqueness of f with the required property. \square

We may now compare our category of actions to the category of Eilenberg-Moore algebras on \mathbb{T}_G , for fixed G :

Proposition 11. *Any action $\psi : (A|G) \rightarrow A$ extends uniquely to an Eilenberg-Moore algebra $\xi : T_G(A) \rightarrow A$; this gives rise to a full and faithful functor $\Xi_G : Act_G(\mathbb{C}) \rightarrow \mathbb{C}^{\mathbb{T}_G}$. Moreover, Ξ_G is “injective on objects” so that if \mathbb{X}_G is the full subcategory of the objects of $\mathbb{C}^{\mathbb{T}_G}$ which are images by Ξ_G of objects ψ of $Act_G(\mathbb{C})$, then Ξ_G is an isomorphism of categories between $Act_G(\mathbb{C})$ and \mathbb{X}_G .*

Proof. If ψ is an action in our sense, then we may construct $q_{\psi} : A + G \rightarrow A \rtimes_{\psi} G$ the coequalizer of $\iota_{A,G}$ and $i_A . \psi$, and $p_{\psi} : A \rtimes_{\psi} G \rightarrow G$ like in Proposition 8. Then since ψ is an action, $l_{\psi} = q_{\psi}.i_{\psi}$ is the kernel of p_{ψ} . One has $p_{\psi}.q_{\psi}.k = r_G.k = 0$, hence there exists a unique $\xi : T_G(A) \rightarrow A$ such that $q_{\psi}.k = l_{\psi}.\xi$. One has $\xi.j_{A,G} = \psi$, since

$l_\psi \psi = q_\psi i_A \psi = q_\psi \iota = q_\psi k j = l_\psi \xi$ and l_ψ is a monomorphism. We put $\Xi_G(\psi) = (A, \xi)$. If $\psi' : (A'|G) \rightarrow A'$ is another G -action, and $a : A \rightarrow A'$ a map in \mathbb{C} , then the diagram

$$\begin{array}{ccc} T_G A & \xrightarrow{T_G a} & T_G A' \\ \xi \downarrow & & \downarrow \xi' \\ A & \xrightarrow{a} & A' \end{array}$$

commutes if and only if a is a map of actions, again since (η, j) is a (strongly) epimorphic pair. This shows that if one can ensure that ξ and ξ' are Eilenberg-Moore algebras over \mathbb{T}_G and a is a map of actions, then a also is a morphism of algebras. We thus obtain a full and faithful functor $\Xi_G : \text{Act}_G(\mathbb{C}) \rightarrow \mathbb{C}^{\mathbb{T}_G}$, defined on objects as above, and on maps by $\Xi_G(a) = a$. Now it is easy to see that ξ is nothing but the pair $(R(A \rtimes_\psi G), R(\epsilon_{A \rtimes_\psi G}))$, where R is the right adjoint of the adjunction between $\text{Pt}_G(\mathbb{C})$ and \mathbb{C} giving rise to the monad \mathbb{T}_G , i.e. R is the kernel functor, and ϵ is its counit. But this is precisely the construction of the comparison functor $\mathcal{J}_G : \text{Pt}_G(\mathbb{C}) \rightarrow \mathbb{C}^{\mathbb{T}_G}$ of the adjunction, and in particular it takes values in $\mathbb{C}^{\mathbb{T}_G}$. Note also that $\xi.j = \psi$ (since it is true, followed by the mono l_ψ), showing that Ξ_G is “injective on objects”. \square

Proposition 12. *If $\psi : (A|G) \rightarrow A$ is an action, then the construction of $A \rtimes_\psi G$ coincides with the one of $G \rtimes (A, \Xi_G(\psi))$ in [3], hence is functorial. The comparison adjunction $(F', G', \eta', \epsilon') : \mathbb{C}^{\mathbb{T}_G} \rightarrow \text{Pt}_G(\mathbb{C})$ in [3] (where F' essentially is the semi-direct product functor and G' is the functor \mathcal{J}_G above) restricts to an equivalence between \mathbb{X}_G and $\text{Pt}_G(\mathbb{C})$. Hence precomposition with Ξ_G provides an equivalence of categories between $\text{Act}(\mathbb{C})$ and $\text{Pt}(\mathbb{C})$, which is compatible with the forgetful functors, so that $\text{Act}(\mathbb{C})$ is a fibration whose fibres are the categories $\text{Act}_G(\mathbb{C})$, and the “inverse” functors $\Psi_G : \text{Pt}_G(\mathbb{C}) \rightarrow \text{Act}_G(\mathbb{C})$ are such that $\Xi_G.\Psi_G = \mathcal{J}_G$, the comparison functor of the adjunction. .*

Proof. The first assertion is an immediate consequence of Proposition 7.

Let us now prove the assertion about the comparison adjunction. First, let us consider a \mathbb{T}_G -algebra $\xi : T_G(A) \rightarrow A$ which is in \mathbb{X}_G , i.e. which is the extension of a (unique) action $\psi : (A|G) \rightarrow G$. We show that η'_ξ is an isomorphism between ξ and $\mathcal{J}_G(F'(\xi))$. Consider the following diagram :

$$\begin{array}{ccccc} (A|G) & \xrightarrow{j_{A,G}} & T_G A & \xrightarrow{k_{A,G}} & A + G \\ & \searrow \psi & \downarrow \xi & \nearrow i_A & \downarrow q_\psi \\ & & A & \xrightarrow{l_\psi} & A \rtimes_\psi G \xrightleftharpoons[p_\psi]{s_\psi} G \end{array}$$

Then in view of all what precedes, $(A \rtimes_\psi G, p_\psi, s_\psi) = F'(\xi)$ and since l_ψ is the kernel of p_ψ (because ψ is an action), $G'(F'(\xi))$ is the unique arrow h from $T_G A$ to A such that $l_\psi h = q_\psi.k_{A,G}$, i.e. ξ .

Secondly, we show that G' has values in \mathbb{X}_G , i.e. that for any object

$$X \xrightleftharpoons[p]{s} G$$

the algebra $\mathcal{J}_G(X, p, s)$ is (the extension to $T_G(A)$ of) an action (A, G, ψ) . Moreover we show that it is such that $A \rtimes_{\psi} G$ is isomorphic to (X, G, p, s) , the G -part of the isomorphism being the identity : this in fact shows that $\epsilon'_{(X, p, s)}$ is an isomorphism.

Let A be $\text{Ker}(p)$ and $l : A \rightarrow X$ be $\text{ker}(p)$. Consider the following diagram, where p_G denotes the projection on G in the product $A \times G$. It is commutative, since $p < l, s > = r_G$ (one may check it by preceding these morphisms by the canonical injections i_A and i_G):

$$\begin{array}{ccccc}
 (A|G) & \xrightarrow{\iota_{A,G}} & A + G & \xrightarrow{b_{A,G}} & A \times G \\
 \exists! \psi \downarrow & \nearrow i_A & \downarrow \langle l, s \rangle & \swarrow i_G & \downarrow p_G \\
 A & \xrightarrow{l} & X & \xrightarrow{p} & G
 \end{array}$$

Since $\iota_{A,G}$ is the kernel of $b_{A,G}$ and since $p_G \cdot b_{A,G} = p \cdot \langle l, s \rangle$, one has: $p \cdot \langle l, s \rangle \cdot \iota_{A,G} = p_G \cdot b_{A,G} \cdot \iota_{A,G} = 0$; so since l is the kernel of p , there exists a unique $\psi : (A|G) \rightarrow A$ such that $l \cdot \psi = \langle l, s \rangle \cdot \iota_{A,G}$. We claim that (A, G, ψ) is an action. Since $l = \langle l, s \rangle \cdot i_A$ is a monomorphism, it suffices to show that $\langle l, s \rangle$, which is known to be a regular epimorphism, is the coequalizer of $\iota_{A,G}$ and $i_A \cdot \psi$.

Consider $q_{\psi} : A + G \rightarrow Q_{\psi}$ the coequalizer of $\iota_{A,G}$ and $i_A \cdot \psi$, and p_{ψ} and s_{ψ} defined as in Proposition 8 above. We will show that $q_{\psi} \cdot i_A$ is a monomorphism, thus showing that is an action, and that the G -point $(Q_{\psi}, p_{\psi}, s_{\psi})$, which is nothing but $A \rtimes_{\psi} G$, is isomorphic to (X, p, s) .

First of all, $\langle l, s \rangle \cdot i_A \cdot \psi = l \cdot \psi = \langle l, s \rangle \cdot \iota_{A,G}$ hence, since q_{ψ} is the coequalizer of $i_A \cdot \psi$ and ι , there exists a unique morphism $e : Q_{\psi} \rightarrow X$ such that $q_{\psi} \cdot e = \langle l, s \rangle$:

$$\begin{array}{ccc}
 & A + G & \\
 i_A \nearrow & \downarrow \langle l, s \rangle & \\
 A & \xrightarrow{l} & X \\
 q_{\psi} \cdot i_A \searrow & & \uparrow e \\
 & Q &
 \end{array}$$

and of course, then $e \cdot q_{\psi} \cdot i_A = \langle l, s \rangle \cdot i_A = l$, so this diagram commutes. But then, since l is the kernel of p , it is a monomorphism, hence so is $q_{\psi} \cdot i_A$. But then, by Proposition 8.3) above, $q_{\psi} \cdot i_A = \text{ker } p_{\psi}$.

Then consider the following diagram:

$$\begin{array}{ccccccc}
 & & A + G & \xrightleftharpoons[i_G]{r_G} & G & & \\
 & & \downarrow q_{\psi} & & \downarrow s_{\psi} & & \\
 1 & \longrightarrow & A & \xrightarrow{q_{\psi} \cdot i_A} & Q_{\psi} & \xrightarrow{p_{\psi}} & G \longrightarrow 1 \\
 & & \downarrow e & & \downarrow s & & \\
 1 & \longrightarrow & A & \xrightarrow{l} & X & \xrightarrow{p} & G \longrightarrow 1
 \end{array}$$

The only part of this diagram which has not been shown to commute is the bottom right-hand square. But one has $p_\psi \cdot q_\psi = r_G = p \cdot \langle l, s \rangle = p \cdot e \cdot q_\psi$, hence $p \cdot e = p_\psi$ since q_ψ is a (regular) epimorphism. And $e \cdot s_\psi = e \cdot q_\psi \cdot i_G = \langle l, s \rangle \cdot i_G = s$. Hence all conditions of the Short Split Five Lemma are satisfied, so e is an isomorphism, and thus defines an isomorphism $(e, 1_G)$ in $\text{Pt}(\mathcal{C})$.

Note that we then have implicitly constructed (by composition with the inverse of Ξ_G) a functor $\text{Pt}(\mathcal{C}) \rightarrow \text{Act}(\mathcal{C})$. We denote it by Ψ (and its restriction to the fibers on G by Ψ_G). For a point $X \xrightleftharpoons[p]{s} G$, $\Psi(X, G, p, s)$ is an action on $A = \text{Ker } p$. It is easy to verify

that if $X' \xrightleftharpoons[p']{s'} G'$ is another point and (x, g) is a morphism of points between them, i.e. a pair of morphisms making the following diagram commute

$$\begin{array}{ccc} X & \xrightleftharpoons[p]{s} & G \\ x \downarrow & & \downarrow g \\ X' & \xrightleftharpoons[p']{s'} & G' \end{array}$$

then $\Psi(x, g)$ is the unique map $a : A \rightarrow A'$ making the following diagram commute

$$\begin{array}{ccc} A = \text{Ker } p & \xrightarrow{\text{ker } p} & X \\ a \downarrow & & \downarrow x \\ A' = \text{Ker } p' & \xrightarrow{\text{ker } p'} & X' \end{array}$$

which indeed is a map of actions between $\Psi(X, G, p, s)$ and $\Psi(X', G', p', s')$.

Finally, the very constructions of Ψ_G and Ξ_G ensure that $\Xi_G \Psi_G = \mathcal{J}_G$. \square

Examples

1) Recall that the conjugation action of an object E of \mathbb{C} on itself is defined in [5], as a split extension on E , to be the short exact sequence $0 \rightarrow E \xrightarrow{\sigma_1} E \times E \xrightarrow{\pi_2} E \rightarrow 0$ with the splitting $\Delta : E \rightarrow E \times E$ being the diagonal map. By Proposition 12, it corresponds to some action $(E|E) \rightarrow E$. We show that this corresponding action, which will be denoted by c_2^E in the sequel, is $\nabla_E \cdot k_{E,E}$, where $\nabla_E : E + E \rightarrow E$ is the codiagonal, and that the corresponding algebra $\Xi_E(c_2^E)$ is $\nabla_E \cdot k_{E,E}$. Since $\psi = \Xi_E(\psi) \cdot k_{E,E}$, it suffices to prove the second assertion, and by the construction of Ξ_E it suffices to show that this map makes the following diagram commute:

$$\begin{array}{ccc} T_E E & \xrightarrow{k_{E,E}} & E + E \\ k_{E,E} \downarrow & & \downarrow \langle \sigma_1, \Delta_E \rangle \\ E + E & & \\ \nabla_E \downarrow & & \downarrow \\ E & \xrightarrow{\sigma_1} & E \times E \end{array}$$

Followed by π_1 , one gets $\pi_1 \cdot \sigma_1 \cdot \nabla_E \cdot k_{E,E} = \nabla_E \cdot k_{E,E}$ on the one hand, and $\pi_1 \cdot \langle \sigma_1, \Delta_E \rangle \cdot k_{E,E} = \langle \pi_1 \cdot \sigma_1, \pi_1 \cdot \Delta_E \rangle \cdot k_{E,E} = \langle 1_E, 1_E \rangle \cdot k_{E,E} = \nabla_E \cdot k_{E,E}$ on the other hand. And

followed by π_2 , one gets $\pi_2.\sigma_1.\nabla_E.k_{E,E} = 0.\nabla_E.k_{E,E} = 0$ on the one hand, and $\pi_2.\langle \sigma_1, \Delta_E \rangle .k_{E,E} = \langle \pi_2.\sigma_1, \pi_1.\Delta_E \rangle .k_{E,E} = \langle 0, 1_E \rangle .k_{E,E} = r_2.k_{E,E} = 0$ on the other hand.

Of course, this construction $c_2^{(-)}$ is functorial. More precisely, $c_2^{(-)}$ may be considered as a functor $\mathbb{C} \rightarrow \text{Act}(\mathbb{C})$, or as a natural transform $S \rightarrow 1_{\mathbb{C}}$, where $S : \mathbb{C} \rightarrow \mathbb{C}$ is defined by $F(E) = (E|E)$ and $S(f) = (f|f)$, both meaning that the following diagram commutes:

$$\begin{array}{ccc} (E|E) & \xrightarrow{(f|f)} & (F|F) \\ c_2^E \downarrow & & \downarrow c_2^F \\ E & \xrightarrow{f} & F \end{array}$$

which follows immediately from naturality of ι and of ∇ .

Then the split extension

$$0 \longrightarrow E \xrightarrow{i_1} E \rtimes_{c_2^E} E \xrightarrow{p_2} E \longrightarrow 0$$

is (canonically isomorphic to) the following one:

$$0 \longrightarrow E \xrightarrow{i_1} E \times E \xrightarrow{p_2} E \longrightarrow 0$$

by the very definition of c_2 . We will generalize this construction to a conjugation action of an object on any proper subobject in the following paragraph.

2) The split short exact sequence of Lemma 5:

$$0 \longrightarrow (A|G) \xrightarrow{j} T_G(A) \xrightarrow[r_A.k]{\eta_A} A \longrightarrow 0$$

corresponds to an action ψ of A on $(A|G)$ such that $T_G(A) = (A|G) \rtimes_{\psi} A$, and the following one:

$$0 \longrightarrow T_G(A) \xrightarrow{k} A + G \xrightarrow[r_G]{i_G} G \longrightarrow 0$$

to an action ψ' of G on $T_G(A)$ such that $A + G = T_G \rtimes_{\psi'} G$, hence one may write $A + G = ((A|G) \rtimes_{\psi} A) \rtimes_{\psi'} G$.

Proposition 13. *Let $\psi : (A|G) \rightarrow A$ be an action in \mathbb{C} , and let $B \xrightarrow{b} A$ and $h : H \rightarrow G$ be two subobjects. Suppose that B is H -stable under ψ , i.e. the map $\psi(b|h) : (B|H) \rightarrow A$ factors through a map $\psi' : (B|H) \rightarrow B$ such that $b\psi' = \psi(b|h)$. Then ψ' is an action of H on B .*

Proof. Consider the following diagram of solid arrows, where q' is the coequalizer of $\iota_{B,H}$ and $i_B\psi'$; all squares and the bottom triangles are commutative, and the upper triangles are coequalized by q' and $q = q_{\psi}$ respectively:

$$\begin{array}{ccc}
& B + H & \xrightarrow{b+h} & A + G \\
\iota_{B,H} \nearrow & \downarrow q & \iota_{A,G} \nearrow & \downarrow q \\
(B|H) & \xrightarrow{(b|h)} & (A|G) & \\
\downarrow \psi' & \downarrow q' & \downarrow \psi & \\
& Q & \xrightarrow{f} & A \rtimes_{\psi} G \\
& \downarrow q' \cdot i_B & & \downarrow l_{\psi} \\
B & \xrightarrow{b} & A &
\end{array}$$

We have to show that $q'i_B$ is a monomorphism. Since $q(b+h)i_B\psi' = qi_A b\psi' = qi_A\psi(b|h) = q\iota_{A,G}(b|h) = q(b+h)\iota_{B,H}$, there is a unique $f : Q \rightarrow A \rtimes_{\psi} G$ such that $q(b+h) = fq'$. Then $l_{\psi}b = qi_A b = q(b+h)i_B = fq'i_B$. Hence, since b and l_{ψ} are monomorphisms, so is $q'i_B$. \square

Note that under these conditions, Q' is $B \rtimes_{\psi'} H$ and f is nothing but $b \rtimes h$; and it is easy to see that it is a monomorphism too.

Finally, we need to exhibit the kernel and image of the semi-direct product of compatible maps. Here, and several times in the sequel, we need the following obvious lemma.

Lemma 14. *Consider a commutative diagram*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow q & & \downarrow i \\
Q & \xrightarrow{j} & Z
\end{array}$$

in any category where q is a regular epimorphism and i is a monomorphism. Then j factors through i .

Proposition 15. *Let $\psi : (A|G) \rightarrow A$ and $\psi' : (B|H) \rightarrow B$ be actions in \mathbb{C} , and let $f : A \rightarrow B$ and $g : G \rightarrow H$ in \mathbb{C} such that (f, g) is a map $\psi \rightarrow \psi'$ in $\text{Act}(\mathbb{C})$, i.e. the diagram*

$$(2.1) \quad \begin{array}{ccc}
(A|G) & \xrightarrow{\psi} & A \\
\downarrow (f|g) & & \downarrow f \\
(B|H) & \xrightarrow{\psi'} & B
\end{array}$$

commutes. Then

$$\ker(f \rtimes g) = \ker(f) \rtimes \ker(g) : \text{Ker}(f) \rtimes_{\tilde{\psi}} \text{Ker}(g) \rightarrow A \rtimes_{\psi} G$$

where $\tilde{\psi}$ is given by restricting to $\text{Ker}(g)$ the G -action on $\text{Ker}(f)$ coming from the fact that $\text{Ker}(f)$ is stable under ψ , see Proposition 13. Analogously,

$$\text{im}(f \rtimes g) = \text{im}(f) \rtimes \text{im}(g) : \text{Im}(f) \rtimes_{\tilde{\psi}'} \text{Im}(g) \rightarrow B \rtimes_{\psi'} H$$

Here $\tilde{\psi}' : (\text{Im}(f)|\text{Im}(g)) \rightarrow \text{Im}(f)$ is the unique map such that $\text{im}(f)\tilde{\psi}' = \psi'(\text{im}(f)|\text{im}(g))$

where $A \xrightarrow{\tilde{f}} \text{Im}(f) \xrightarrow{\text{im } f} B$ denotes an image factorization of f and similarly for g .

Proof. First note that diagram (2.1) assures the existence of $\tilde{\psi}$ and $\tilde{\psi}'$: stability of $\text{Ker}(f)$ under ψ is easily deduced; to see that $\psi'(\text{im}(f)|\text{im}(g))$ indeed factors through $\text{im}(f)$, apply Lemma 14 to the factorization $(f|g) = (\text{im}(f)|\text{im}(g))(\tilde{f}|\tilde{g})$, noting that $(\tilde{f}|\tilde{g})$ is a regular epimorphism by Propositions 3 and 2.(3).

Now it is clear that

$$(\ker(f), \ker(g)) : (\text{Ker}(f), \text{Ker}(g), \tilde{\psi}) \longrightarrow (A, G, \psi)$$

is a kernel of the map (f, g) in $\text{Act}(\mathbb{C})$, whence

$$(\ker(g), \ker(f) \rtimes \ker(g)) : (\text{Ker}(g), \text{Ker}(f) \rtimes_{\psi'} \text{Ker}(g), p_{\psi'}, s_{\psi'}) \rightarrow (G, A \rtimes_{\psi} G, p_{\psi}, s_{\psi})$$

is a kernel of $f \rtimes g$ in $\text{Pt}(\mathbb{C})$, by the equivalence $\text{Act}(\mathbb{C}) \sim \text{Pt}(\mathbb{C})$. To show that $\ker(f) \rtimes \ker(g)$ is a kernel of $f \rtimes g$ in \mathbb{C} , let $x : X \rightarrow A \rtimes_{\psi} G$ be a map in \mathbb{C} such that $(f \rtimes g)x = 0$.

Then $p_{\psi}x$ factors as $X \xrightarrow{\tilde{x}} \text{Ker}(g) \xrightarrow{\ker(f)} G$ for some map \tilde{x} , and

$$\begin{array}{ccc} X + \text{Ker}(g) & \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{\langle \tilde{x}, 1 \rangle} \end{array} & \text{Ker}(g) \\ \begin{array}{c} \downarrow \langle x, s_{\psi}, \ker(g) \rangle \\ \downarrow \ker(g) \end{array} & & \downarrow \ker(g) \\ A \rtimes_{\psi} G & \begin{array}{c} \xleftarrow{s_{\psi}} \\ \xrightarrow{p_{\psi}} \end{array} & G \end{array}$$

is a map in $\text{Pt}(\mathbb{C})$ whose postcomposition with $(g, f \rtimes g)$ is trivial, hence factors through $(\ker(g), \ker(f) \rtimes \ker(g))$. Consequently x factors through $\ker(f) \rtimes \ker(g)$, as desired.

To see that $\text{im}(f) \rtimes \text{im}(g)$ is an image of $f \rtimes g$, it suffices to note that $f \rtimes g = (\text{im}(f) \rtimes \text{im}(g))(\tilde{f} \rtimes \tilde{g})$ where $\tilde{f} \rtimes \tilde{g}$ is a regular epimorphism and $\text{im}(f) \rtimes \text{im}(g)$ is a monomorphism by the short five lemma for regular epimorphisms and for monomorphisms, resp. \square

3. CONJUGATION ACTION OF AN OBJECT ON A PROPER SUBOBJECT

Using our alternative description of internal actions we now introduce a general notion of conjugation action of an object E on any of its proper subobjects, as follows.

Proposition 16. *Let $n : N \rightarrow E$ be a proper subobject in \mathbb{C} . Then there is an action $c^{N,E} : (N|E) \rightarrow N$ of E on N such that $nc^{N,E} = c_2^E(n|1)$. We call $c^{N,E}$ the conjugation action of E on N . It is natural with respect to pair maps $(E, N) \rightarrow (E', N')$, i.e. maps $f : E \rightarrow E'$ in \mathbb{C} such that $f(N) \subset N'$ for a given subobject N' of E' .*

Proof. Let $\pi : E \rightarrow G$ be a cokernel of n . Then we have the following diagram of plain arrows which commutes by naturality of commutator maps:

$$\begin{array}{ccccc} (N|E) & \xrightarrow{(n|1)} & (E|E) & \xrightarrow{(\pi|\pi)} & (G|G) \\ \downarrow c^{N,E} & & \downarrow c_2^E & & \downarrow c_2^G \\ N & \xrightarrow{n} & E & \xrightarrow{\pi} & G \end{array}$$

Thus $\pi c_2^E(n|1) = c_2^G(\pi|\pi)(n|1) = c_2^G(\pi n|\pi) = (0|\pi) = 0$ since the functor $(-|-)$ is bireduced by Proposition 2 (2), whence $c_2^E(n|1)$ factors through n , thus providing the desired map $c^{N,E}$. By Proposition 13 it is an action. Its naturality is immediate from naturality of c_2^E . \square

Note that $c^{E,E}$ coincides with the conjugation action of E on itself in Example 1 following Proposition 11, so our definition of a conjugation action of E on N generalizes the one for $N = E$ in [5].

We now give two properties which litterally generalize certain standard facts in the theory of groups or Lie algebras.

First we quote a reformulation of the *universal property* of the semi-direct product:

Proposition 17. *Let $\psi : (A|G) \rightarrow A$ be an action in \mathbb{C} , and let $A \xrightarrow{f} X \xleftarrow{g} G$ be maps in \mathbb{C} . Then there exists a map $h = \langle f, g \rangle : A \rtimes_{\psi} G \rightarrow X$ such that $hl_{\psi} = f$ and $hs_{\psi} = g$ iff the following square commutes:*

$$(3.1) \quad \begin{array}{ccc} (A|G) & \xrightarrow{\psi} & A \\ (f|g) \downarrow & & \downarrow f \\ (X|X) & \xrightarrow{c_2^X} & X \end{array}$$

Moreover, if h exists it is unique.

Proof. Uniqueness follows from the fact that (l_{ψ}, s_{ψ}) is a (strongly) epimorphic pair. To prove existence, by the definition of $A \rtimes_{\psi} G$ as a quotient of $A + G$, we must show that commutativity of diagram (3.1) is equivalent with the relation $\langle f, g \rangle \iota = \langle f, g \rangle i_A \psi$. But $\langle f, g \rangle \iota = \nabla(f+g)\iota = \nabla\iota(f|g) = c_2^X(f|g)$. On the other hand, $\langle f, g \rangle i_A \psi = f\psi$, whence the assertion. \square

In particular, Proposition 17 shows that the semi-direct product can be viewed as a universal transformation of an abstract action into a conjugation action:

Corollary 18. *An action $\psi : (A|G) \rightarrow A$ in \mathbb{C} coincides with the restriction to G of the conjugation action of $A \rtimes_{\psi} G$ on A , or formally, $c^{A, A \rtimes_{\psi} G}(1|s_{\psi}) = \psi$.*

Proof. In Proposition 17, take $X = A \rtimes_{\psi} G$, $f = l_{\psi}$, $g = s_{\psi}$ and $h = 1$. We get $l_{\psi}\psi = c_2^X(l_{\psi}|s_{\psi}) = c_2^X(l_{\psi}|1)(1|s_{\psi}) = l_{\psi}c^{A, X}(1|s_{\psi})$, whence the assertion since l_{ψ} is monic. \square

To make further progress we define commutators of subobjects in terms of the cross-effects of the identity functor; this actually is the starting point of a new approach to categorical commutator calculus which is further developped in [8].

For $n \geq 1$ and an object X in \mathbb{C} , let $\nabla_X^n = \langle 1, \dots, 1 \rangle : nX = X + \dots + X \rightarrow X$ be the canonical folding map.

Definition 19. *The n -fold commutator map of an object X of \mathbb{C} is the natural composite map*

$$c_n^X : (X|\dots|X) \xrightarrow{\iota} X + \dots + X \xrightarrow{\nabla_X^n} X$$

Moreover, if $x_i : X_i \hookrightarrow X$ are subobjects of X , define their commutator to be the following subobject of X :

$$[X_1, \dots, X_n] = \text{Im}((X_1|\dots|X_n) \xrightarrow{(x_1|\dots|x_n)} (X|\dots|X) \xrightarrow{c_n^X} X).$$

Remark 20. *With the notations of definition 19, the Huq commutator of X_1, X_2 is the proper closure of our commutator $[X_1, X_2]$ defined above (where the proper closure of a subobject is defined to be the kernel of its cokernel). Thus, if X_1, X_2 generate X (i.e. if*

the map $\langle x_1, x_2 \rangle: X_1 + X_2 \rightarrow X$ is a regular epimorphism), then $[X_1, X_2]$ coincides with the commutators of X_1, X_2 of Hug and of Smith (the Smith case being due to Everaert and van der Linden (private communication)). In an unpublished note [4], Dominique Bourn gives a detailed proof of this fact in the special case when X_1 and X_2 are considered as subobjects of their sum $X_1 + X_2$.

The link with the subject of this paper is that stability under the conjugation action can be expressed in terms of commutators:

Lemma 21. *Let $X \xrightarrow{x} A \xleftarrow{y} Y$ be subobjects of an object of \mathbb{C} . Then X is Y -stable under the conjugation action of A on itself (see Proposition 13) iff $[X, Y] \subset X$, i.e. the injection of $[X, Y]$ into A factors through x .*

Proof. Consider the following commutative diagram of solid arrows:

$$\begin{array}{ccc}
 (X|Y) & \xrightarrow{(x|y)} & (A|A) \\
 \downarrow q & & \downarrow c_2^A \\
 & \searrow & \\
 & [X, Y] & \\
 \downarrow \psi & \swarrow u & \downarrow \\
 X & \xrightarrow{x} & A
 \end{array}$$

Then by the respective definitions, X is Y -stable iff a map ψ as indicated exists and renders the diagram commutative; and $[X, Y] \subset X$ iff a map u as indicated exists and renders the diagram commutative. But these conditions are equivalent by Lemma 14 since q is a regular epimorphism and x is monic. \square

Definition 22. *Let $X \xrightarrow{x} A \xleftarrow{y} Y$ be subobjects of an object of \mathbb{C} . We say that Y normalizes X if X is Y -stable under the conjugation action of A on itself (i.e. if $[X, Y] \subset X$, in view of Lemma 21).*

The following is an extremely useful criterion of normality in semi-abelian categories, as will be shown in the sequel and in subsequent work.

Theorem 23. *Suppose that \mathbb{C} is semi-abelian. Then the following properties are equivalent for subobjects X, Y of A as above:*

- (1) Y normalizes X .
- (2) X is a proper subobject of $X \vee Y$, the subobject generated by X and Y (i.e. the image of the map $\langle x, y \rangle: X + Y \rightarrow A$).
- (3) The object $X \cap Y$ is proper in Y and the sequence

$$0 \rightarrow X \xrightarrow{x'} X \vee Y \xrightarrow{q^r y} Y/X \cap Y \rightarrow 0$$

is short exact where x' is the factorization of x through $X \vee Y$ and $q: Y \rightarrow Y/X \cap Y$ is the projection.

If one of these conditions (1)-(3) is satisfied we write $X \vee Y = [X]Y = Y[X]$.

We note that the implication (1) \Rightarrow (3) is a crucial ingredient in our commutator theory in [8].

Proof: The equivalence of (2) and (3) is immediate using the second Noether isomorphism theorem. Moreover, (2) implies (1) by Proposition 16, so it remains to show that (1) implies (2). By Proposition 13 we obtain an action ψ of Y on X , and the universal property (Proposition 17) of the semi-direct product implies that there is a map $\overline{\langle x, y \rangle} : X \rtimes_{\psi} Y \rightarrow A$. Now $\text{Im}(\overline{\langle x, y \rangle}) = \text{Im}(\langle x, y \rangle) = X \vee Y$ since by construction $X \rtimes_{\psi} Y$ is a quotient of $X + Y$. But $\text{Im}(l_{\psi})$ is proper in $X \rtimes_{\psi} Y$, whence $\text{Im}(\overline{\langle x, y \rangle} l_{\psi}) = \text{Im}(x) = X$ is proper in $\text{Im}(\overline{\langle x, y \rangle}) = X \vee Y$ by exactness of \mathbb{C} . \square

Corollary 24. *A finitely cocomplete homological category \mathbb{C} is semi-abelian (i.e. Barr exact) iff the following condition holds:*

(P) *A subobject X of an object A of \mathbb{C} is proper in A iff it is stable under the conjugation action of A , i.e. if $[X, A] \subset X$.*

The result that property (P) holds in semi-abelian categories was recently also obtained independantly by Mantovani and Metere [10].

Proof. The condition is necessary by Theorem 23 (take $Y = A$); let us prove that it is sufficient: to show that \mathbb{C} is exact, let $X \triangleright^x \rightarrow A \xrightarrow{q} \twoheadrightarrow B$ be a proper subobject and a regular epimorphism in \mathbb{C} . Let $X \xrightarrow{q_X} \twoheadrightarrow qX \triangleright^{\bar{x}} \rightarrow B$ be an image factorization of qx . Then we get a commutative diagram

$$\begin{array}{ccccc}
 (X|A) & \xrightarrow{c^{A,X}} & X & & \\
 \downarrow & \searrow (1|x) & \downarrow & \searrow x & \\
 (q|q_X) & & (A|A) & \xrightarrow{q_X} & A \\
 \downarrow & & \downarrow & & \downarrow q \\
 (B|qX) & & (q|q) & & qX \\
 \downarrow & \searrow (1|\bar{x}) & \downarrow & \searrow \bar{x} & \\
 & & (B|B) & \xrightarrow{c_2^B} & B
 \end{array}$$

It implies that $c_2^B(1|\bar{x})(q|q_X) = \bar{x}q_X c^{A,X}$, whence $c_2^B(1|\bar{x})(q|q_X)$ factors through \bar{x} . But $(q|q_X)$ is a regular epimorphism by Propositions 3 and 2.(3), whence $c_2^B(1|\bar{x})$ also factors through \bar{x} , compare the proof of Lemma 22. Consequently qX is stable under the conjugation action of B , thus it is proper by hypothesis. \square

Note that since in a semi-abelian category “proper subobject” is equivalent to “normal subobject”, the equivalence of (1) and (2) in Theorem 23 may obviously be stated as the equivalence of (1) and the following

(2’): *X is a normal subobject of $X \vee Y$.*

The proof of (1) \Rightarrow (2) in a semi-abelian category indeed provides a proof of (1) \Rightarrow (2’) in any finitely cocomplete homological category, since in any such category the image of any proper subobject (or even normal subobject) by a regular epimorphism is a normal subobject. We do not know if the converse (2’) \Rightarrow (1) is true in any such category, but the following example shows that the latter implication may be true even if the category

is not exact (hence the existence of the conjugation action of an object X on a normal subobject N does not imply in general that this subobject is proper, by corollary 24).

First note that (2') \Rightarrow (1) holds in a finitely cocomplete homological category if and only if it holds in the special case when $Y = A$ (and $y = 1_A$), i.e. if and only if any object acts by conjugation on any of its normal subobjects. Indeed, suppose that this is true, and let $x : X \rightarrow A$ and $y : Y \rightarrow A$ be subobjects such that X is normal in $X \vee Y$. Then $X \vee Y$ acts on X by conjugation, by hypothesis. So we get an action $\psi : (X|X \vee Y) \rightarrow X$ which is the restriction on X of the conjugation action of $X \vee Y$ on itself. Then by precomposition with $(1|y')$ (where y' denotes the inclusion of Y in $X \vee Y$) we get a map $(X|Y) \rightarrow X$ which is the restriction to $(X|Y)$ of the conjugation action of $X \vee Y$ on itself, but also of A on itself.

Consider the category \mathbb{C} whose objects are ordered pairs $\mathcal{G} = (G, B)$ where G is a group and B a subgroup. A map $f : \mathcal{G} = (G, B) \rightarrow \mathcal{H} = (H, C)$ is a group homomorphism $f : G \rightarrow H$ such that $f(B) \subseteq C$. We show that in this category, any object \mathcal{G} acts by conjugation on any of its normal subobjects. This category is (finitely) cocomplete homological and is provided with two “forgetful” functors U_1 and U_2 to the category of groups: $U_1(G, B) = G$, $U_1(f) = f$, $U_2(G, B) = B$, $U_2(f) = f|_B$. Both U_1 and U_2 preserve finite limits and coproducts. So for instance, $(\mathcal{G}|\mathcal{H}) = ((G|H), (B|C))$ (where in the second member of the equality the $(\cdot|\cdot)$'s are computed in the category of groups). However, only U_1 preserves coequalizers: for instance, the cokernel of the inclusion of (N, C) in (G, B) , where N is, say, a normal subgroup of G and C a subgroup of B (possibly, but not necessarily normal in B) is independent of C , it is $(G/N, B/(N \cap B))$. In particular, (N, C) is not the kernel of its cokernel unless $C = N \cup B$, whence \mathbb{C} is not semi-abelian. Now, an equivalence relation on (G, B) in \mathbb{C} is (the inclusion into $(G \times G, B \times B)$ of) a pair (R, S) , where R is a congruence on G , S is a congruence on B , and $S \subseteq R$. Like in any pointed protomodular category, a normal subobject of (G, B) is “the equivalence class of 0 for such an equivalence relation”, i.e. the inverse image of R by the inclusion σ_1 (or equivalently σ_2) of (G, B) in $(G, B) \times (G, B)$, i.e. the pair $\mathcal{N} = ([e]_R, [e]_S)$ where e is the unit of G and $[e]_R$ its equivalence class (in G) for R , and $[e]_S$ in B for S). The object $(\mathcal{G}|\mathcal{G})$ is the pair $((G|G), (B|B))$; the elements of $(G|G)$ are products (in $G + G$) of the form $\prod_i [g_i, g'_i]_+^{z_i}$ with g_i and g'_i are in the two different copies of G , and where $[-, -]_+$ denotes commutators in $G + G$; and $(B|B)$ is the same with the g_i 's and g'_i 's in (the two copies of) B . The diagram that has to be filled up to get the result is the following:

$$\begin{array}{ccc} (\mathcal{N}|\mathcal{G}) = (([e]_R|G), ([e]_S|A)) & \longrightarrow & (\mathcal{G}|\mathcal{G}) = ((G|G), (A|A)) \\ \vdots \downarrow & & \downarrow c_2^{\mathcal{G}} \\ \mathcal{N} = ([e]_R, [e]_S) & \longrightarrow & \mathcal{G} = (G, A) \end{array}$$

where the horizontal arrows are the obvious inclusions. Moreover, $c_2^{\mathcal{G}}(\prod_i [g_i, g'_i]_+^{z_i}) = \prod_i [g_i, g'_i]^{z_i}$, where $[-, -]$ denotes the ordinary commutator in G . So what has to be proved is that if the g'_i 's are in $[e]_R$ then $\prod_i [g_i, g'_i]^{z_i}$ is in $[e]_R$, and that if moreover the g'_i 's are in $[e]_S$ (hence in A) and the g_i 's are in A , then $\prod_i [g_i, g'_i]^{z_i}$ is in $[e]_S$. This is obvious, since R and S are congruences on G and on A respectively.

4. OTHER CHARACTERIZATIONS OF ACTIONS

In this paragraph \mathbb{C} remains a finitely cocomplete homological category, even if, at some places, we shall pay special attention to the semi-abelian case. The functor Ξ_G of Proposition 11 associates to any action ψ (in our sense) of an object G of \mathbb{C} on another object A an algebra (A, ξ) over the monad \mathbb{T}_G , which moreover makes the following diagram commute.

$$\begin{array}{ccc} (A|G) & \xrightarrow{j} & T_G(A) \\ & \searrow \psi & \swarrow \xi \\ & & A \end{array}$$

Using our preceding results, we here give necessary and sufficient conditions for an arbitrary morphism $\psi : (A|G) \rightarrow A$ to have such an extension to an algebra over \mathbb{T}_G , so they are necessary for ψ to be an action. Hence if the category \mathbb{C} is such that the comparison functor \mathcal{J}_G is an equivalence of categories between $\text{Pt}_G(\mathbb{C})$ and $\mathbb{C}^{\mathbb{T}_G}$ (in particular if \mathbb{C} is semi-abelian) they are also sufficient, but we also give a direct proof of this fact in a semi-abelian category, providing an alternative proof (without using Beck's criterion) of the fact that \mathcal{J}_G is an equivalence of categories in this case.

The object G is fixed throughout this section; we therefore simplify the notations, denoting by \mathbb{T} the monad, by T the underlying endofunctor of \mathbb{C} , and by η and μ the unit and the multiplication of \mathbb{T} . We divide the question of the existence of an extension of a map $\psi : (A|G) \rightarrow A$ to an algebra over \mathbb{T} in two parts:

Proposition 25. *Let ψ be a map $(A|G) \rightarrow A$ in \mathbb{C} . Then the following conditions are equivalent:*

1. ψ can be extended along j to a map $\xi : T(A) \rightarrow A$ satisfying the unit axiom of a \mathbb{T} -algebra, i.e. $\eta_A \cdot \xi = 1_A$.
2. The following diagram commutes:

$$(4.1) \quad \begin{array}{ccc} ((A|G)|A) & \xrightarrow{\psi_0} & (A|G) \\ (\psi|1_A) \downarrow & & \downarrow \psi \\ (A|A) & \xrightarrow{c_2^A} & A \end{array}$$

where ψ_0 is the action of A on $(A|G)$ which arises from the fact that $T(A)$ is the semi-direct product of $(A|G)$ and A (see Lemma 5 and Example 2 in section 2). Recall that in view of Corollary 18, ψ_0 is the restriction to A of the conjugation action of $T(A)$ on $(A|G)$, which itself is the restriction to $T(A)$ of the conjugation action of $A+G$ on $(A|G)$.

Moreover under these conditions, the resulting map ξ also satisfies $l_\psi \cdot \xi = q_\psi \cdot k$, where $q_\psi : A + G \rightarrow Q_\psi$ is the coequalizer of $\iota_{A,G}$ and $i_A \cdot \psi$ and l_ψ is the composite $q_\psi \cdot i_A$ as in Proposition 8.

Proof. That some ξ extends ψ along j with $\xi \cdot \eta_A = 1_A$ can be translated into the following diagram:

$$\begin{array}{ccccc}
(A|G) & \xrightarrow{j} & T(A) & \xleftarrow{\eta_A} & A \\
& \searrow \psi & \downarrow \xi & \swarrow & \\
& & A & &
\end{array}$$

so by Proposition 17, it is immediate that such an extension exists if and only diagram (4.1) commutes. Moreover, since the pair (j, η_A) is a (strongly) epimorphic family, in order to prove that $l_\psi \cdot \xi = q_\psi \cdot k$, it suffices to verify that $l_\psi \cdot \xi \cdot j = q_\psi \cdot k \cdot j$ and $l_\psi \cdot \xi \cdot \eta_A = q_\psi \cdot k \cdot \eta_A$. The first equality amounts to $q_\psi \cdot i_A \cdot \psi = q_\psi \cdot l$, which is true because q_ψ is the coequalizer of these two maps, and the second to $q_\psi \cdot i_A \cdot \xi \cdot \eta_A = q_\psi \cdot k \cdot \eta_A$, which is true because $\xi \cdot \eta_A = 1_A$ and $k \cdot \eta_A = i_A$.

□

Example: In the category of groups, consider a homomorphism $\psi : (A|G) \rightarrow A$. Since $(A|G)$ is the free group on $A^* \times G^*$, ψ is the extension of a unique set-application $\phi' : A^* \times G^* \rightarrow A$. Let us define $\phi : A \times G \rightarrow A$ by $\phi(e_G, a) = a$, $\phi(g, e_A) = e_A$ (where e_G and e_A denote the units², and $\phi(g, a) = \phi'(g, a) \cdot a$ for non-trivial g, a (so that of course $\psi([g, a]) = \phi'(g, a) = \phi(g, a) \cdot a^{-1}$ for nonzero g, a). We know that ψ is an action in our sense if and only if ϕ is an action in the usual sense. More generally, and rather starting our from ϕ , one may determine the condition on ϕ which insures that ψ satisfies the equivalent conditions of Proposition 25. In view of the proof of this proposition, it is more convenient to apply the classical universal property of the semi-direct product than condition 2. Considering that the action ψ_0 of Proposition 25 corresponds to a classical action ϕ_0 , the property that is needed to insure the existence of ξ is: for any $x \in (A|G)$, $\psi(\phi_0(a, x)) = a \cdot \psi(x) \cdot a^{-1}$ (note that we indeed work with ψ on one hand, and with ϕ_0 on the other hand). But since $\phi_0(a, -)$ and conjugation by a both are group morphisms, it suffices to prove this for an x of the form $[g, a']$. And $\phi_0(a, [g, a']) = a \cdot [g, a'] \cdot a^{-1}$ (in $T(A)$, or in $A + G$). But, working in $A + G$, using the well-known formula $x \cdot [y, z] = [x, y] \cdot [y, xz] \cdot x$ (or equivalently $[x, [y, z]] = [x, y][y, xz][z, y]$), one gets: $a \cdot [g, a'] \cdot a^{-1} = [a, g][g, aa'] \cdot a \cdot a^{-1} = [a, g]^{-1}[g, aa']$. So since ψ is a morphism and since $\psi([g, a]) = \phi(g, a) \cdot a^{-1}$, one gets $\psi(\phi_0(a, [g, a'])) = \psi([g, a])^{-1} \cdot \psi([g, aa']) = a(\phi(g, a))^{-1} \phi(g, aa')(aa')^{-1}$ and $a\psi[g, a']a^{-1} = a\phi(g, a')a'^{-1}a^{-1} = a\phi(g, a')(aa')^{-1}$. So these terms are equal if and only if $(\phi(g, a))^{-1} \phi(g, aa') = \phi(g, a')$, or $\phi(g, aa') = \phi(g, a)\phi(g, a')$, i.e. $\phi(g, -)$ is an endomorphism of A .

We now give a characterization of actions in terms of extensions to $T(A)$:

Proposition 26. *Let ψ be any map $(A|G) \rightarrow A$. Then, using the same notations as above, the following properties are equivalent:*

- 1) ψ is an action, i.e. l_ψ is a monomorphism;
- 2) ψ satisfies the equivalent conditions of Proposition 25 and the arising $\xi : T(A) \rightarrow A$ is such that $\text{Ker } \xi$ is proper in $A + G$.

Proof. 1) implies 2): By the proof of Proposition 11, if ψ is an action, then it has an extension $\xi : TA \rightarrow A$ which satisfies $\xi \cdot j = \psi$ and $q_\psi \cdot k = l_\psi \cdot \psi$ (with our usual notations). We claim that $\text{Ker } q_\psi = \text{Ker } \xi$. More precisely, consider $\text{ker } q_\psi : \text{Ker } q_\psi \rightarrow A + G$, we claim that $\text{ker } q_\psi$ factors through $k : TA \rightarrow A + G$ and that the factorization is the kernel

²We prefer to denote the units this way, to avoid confusion with identities; but when dealing with maps, we denote by 0 the zero maps, and by 1_X the identity on X

of ξ . Indeed, in the following diagram, one has $p_G \cdot \ker q_\psi = 0$, so that there exists a unique $m : \text{Ker } q_\psi \rightarrow TA$ such that $km = \ker q_\psi$. Then $\xi \cdot m = 0$ since $l_\psi \cdot \xi \cdot m = 0$ and l_ψ is a monomorphism. And if $f : X \rightarrow TA$ is such that $\xi \cdot f = 0$, then $q_\psi \cdot k \cdot f = l_\psi \cdot \xi \cdot f = 0$, so there exists a unique $x : X \rightarrow \text{Ker } q_\psi$ such that $\ker(q_\psi) \cdot x = kf$. But this means $kmx = kf$, so $mx = f$ since k is a monomorphism.

$$\begin{array}{ccccccc}
& & X & & 0 & & \\
& & \searrow \text{dotted } x & & \downarrow & & \\
& & \text{Ker } q_\psi & \xlongequal{\quad} & \text{Ker } q_\psi & & \\
& & \downarrow \text{dotted } m & & \downarrow \text{ker } q_\psi & & \\
0 & \longrightarrow & TA & \xrightarrow{k} & A+G & \xrightleftharpoons[p_G]{s_G} & G \longrightarrow 0 \\
& & \downarrow \xi & & \downarrow q_\psi & & \parallel \\
0 & \longrightarrow & A & \xrightarrow{l_\psi} & Q_\psi & \xrightleftharpoons[p_\psi]{s_\psi} & G \longrightarrow 0 \\
& & & & \downarrow & & \\
& & & & 0 & &
\end{array}$$

Conversely, suppose that $\psi : (A|G) \rightarrow A$ satisfies the conditions of Proposition 25. First note that since ξ is split by η one has a split short exact sequence

$$0 \longrightarrow \text{Ker } \xi \xrightarrow{\text{ker } \xi} T_G A \xrightleftharpoons[\xi]{\eta} A \longrightarrow 0,$$

and in particular $T_G A / \text{Ker } \xi = A$. If moreover $\text{Ker } \xi$ is proper in $A+G$, then by Noether's first isomorphism theorem $T_G A / \text{Ker } \xi$ (i.e. A) is proper in $A+G / \text{Ker } \xi$ and

$$\frac{\frac{A+G}{\text{Ker } \xi}}{\frac{T_G A}{\text{Ker } \xi}} \cong \frac{A+G}{T_G A} \cong G$$

This means that one has the following commutative diagram where all horizontal and vertical lines are short exact sequences:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker } \xi & \xrightarrow{\text{ker } \xi} & T_G A & \xrightleftharpoons[\xi]{\eta} & A \longrightarrow 0 \\
& & \parallel & & \downarrow k & & \downarrow f \\
0 & \longrightarrow & \text{Ker } \xi & \xrightarrow{k \cdot \text{ker } \xi} & A+G & \xrightarrow{q'} & \frac{A+G}{\text{Ker } \xi} \longrightarrow 0 \\
& & \downarrow & & \downarrow r_G & & \downarrow g \\
0 & \longrightarrow & 0 & \longrightarrow & G & \xlongequal{\quad} & G \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

But moreover $f = q'k\eta$. Indeed, it suffices to show that $f\xi = q'k\eta\xi$. This is immediate, because $q'k\eta\xi = f\xi\eta\xi = f\xi$.

But then q' coequalizes ι and $i_A\psi$. Indeed, $q'\iota = q'kj = f\xi j = q'k\eta\xi j = q'i_A\psi$. And since q_ψ is the coequalizer of ι and $i_A\psi$, it follows that there exists a unique $h : Q_\psi \rightarrow (A + G)/\text{Ker } \xi$ such that $hq_\psi = q'$. But then $hl_\psi = hq_\psi i_A = q' i_A = q'k\eta = f$ which is a monomorphism, hence l_ψ is a monomorphism. \square

Lemma 27. Consider $\mu_A : T(TA) \rightarrow TA$ the multiplication of the monad and $j_{TA,G}$ the inclusion of $(TA|G)$ in $T(TA)$. Then the composite $\mu_A \cdot j_{TA,G}$ is the restriction to G of the conjugation action of $A + G$ on TA (which exists by Proposition 16 since TA is a proper subobject of $A + G$).

Proof. By Proposition 13, this restriction is the unique map $c : (TA|G) \rightarrow TA$ making the following diagram commute:

$$\begin{array}{ccc} (TA|G) & \xrightarrow{(k_{A,G}|i_G)} & (A + G|A + G) \\ c \downarrow & & \downarrow c_{A+G}^2 \\ TA & \xrightarrow{k_{A,G}} & A + G \end{array}$$

so we have to show that $c = \mu_A \cdot j_{TA,G}$ makes it commute. This follows from the commutativity of all components of the following diagram:

$$\begin{array}{ccccc} (TA|G) & \xrightarrow{(k_{A,G}|i_G)} & (A + G, A + G) & & \\ j_{TA,G} \downarrow & \searrow \iota_{TA,G} & \downarrow \iota_{A+G,A+G} & & \\ T(TA) & \xrightarrow{k_{TA,G}} & (TA) + G & \xrightarrow{k_{A,G}+i_G} & (A + G) + (A + G) \\ \mu_A \downarrow & & \searrow \langle k_{A,G}, i_G \rangle & & \downarrow \nabla_{A+G} \\ TA & \xrightarrow{k_{A,G}} & A + G & & \end{array}$$

$\curvearrowright c_{A+G}^2$

\square

We denote this restriction by $c^{TA,A+G} \upharpoonright_G$ or c when no confusion is possible. Note that since it is true in general that the multiplication μ_A of a monad \mathbb{T} is always a \mathbb{T} -algebra over TA , it is obvious that if the category is semi-abelian and if \mathbb{T} is \mathbb{T}_G , then μ_A is a G -action over TA in the sense of [BJ], hence $\mu_A \cdot j_{TA,G}$ is an action in our sense: Lemma 27 generalizes this to the non-exact case. It will also be useful in the sequel to note that c factors through $(A|G)$:

Lemma 28. Under the hypothesis of Lemma 27, the map c factors through the inclusion $j_{A,G}$ of $(A|G)$ in TA .

Proof. Since $j_{A,G}$ is the kernel of $r_A \cdot k_{A,G}$ by Lemma 5, it suffices to show that $r_A \cdot k_{A,G} \cdot \mu_A \cdot j_{TA,G} = 0$. But by definition of μ_A , the right hand square in the following diagram commutes:

$$\begin{array}{ccccc}
(TA|G) & \xrightarrow{j_{TA,G}} & TTA & \xrightarrow{k_{TA,G}} & TA + G \\
\downarrow \text{dotted} & & \downarrow \mu_A & & \downarrow \langle k_{A,G}, i_G \rangle \\
(A|G) & \xrightarrow{j_{A,G}} & TA & \xrightarrow{k_{A,G}} & A + G \xrightleftharpoons[r_G]{i_G} G \\
& & & & \downarrow r_A \\
& & & & A
\end{array}$$

so $r_A k_{A,G} \mu_A j_{TA,G} = r_A \langle k_A, i_G \rangle k_{TA,G} = \langle r_A k_A, r_A i_G \rangle k_{TA,G} = 0 \cdot k_{TA,G} = 0$. \square

Example: In the category of groups, an element of TA is an element of $A + G$ of the form $(\prod_i [g_i, a_i]) \cdot a$. We denote by $[-, -]$ the commutators in $A + G$, and by $\llbracket -, - \rrbracket$ the commutators in $(A + G) + G$. Hence an generator of $(TA|G)$ has the form $\llbracket g, (\prod_i [g_i, a_i]) \cdot a \rrbracket$ (with g in the second copy of G , and g_i in the first one), and $c(\llbracket g, (\prod_i [g_i, a_i]) \cdot a \rrbracket) = [g, (\prod_i [g_i \cdot a_i]) \cdot a]$ where both commutators are considered in $A + G$. Lemma 28 is illustrated by the fact that this element is in $(A|G)$.

Proposition 29. *A map $\xi : TA \rightarrow A$ which satisfies the unit axiom of a \mathbb{T} -algebra also satisfies the associativity axiom if and only if the following diagram commutes:*

$$\begin{array}{ccc}
(TA|G) & \xrightarrow{c^{TA, A+G}|_G} & TA \\
(\xi|_{1_G}) \downarrow & & \downarrow \xi \\
(A|G) & \xrightarrow{\xi \cdot j_{A,G}} & A
\end{array}$$

Hence if a map $\psi : (A|G) \rightarrow A$ satisfies the conditions of Proposition 25, with extension ξ to TA , then (A, ξ) is a \mathbb{T} -algebra if and only if the following diagram commutes:

$$(4.2) \quad \begin{array}{ccc}
(TA|G) & \xrightarrow{c^{TA, A+G}|_G} & TA \\
(\xi|_{1_G}) \downarrow & & \downarrow \xi \\
(A|G) & \xrightarrow{\psi} & A
\end{array}$$

Proof. Recall that the associativity condition for ξ is commutativity of the following diagram:

$$\begin{array}{ccc}
T(TA) & \xrightarrow{\mu_A} & TA \\
T\xi \downarrow & & \downarrow \xi \\
TA & \xrightarrow{\xi} & A
\end{array}$$

Since the pair $\{j_{TA,G} : (TA|G) \rightarrow T(TA), \eta_{TA} : TA \rightarrow T(TA)\}$ is (strongly) epimorphic this condition is equivalent to the two equations $\xi \cdot \mu_A \cdot \eta_{TA} = \xi \cdot T\xi \cdot \eta_{TA}$ and $\xi \cdot \mu_A \cdot j_{TA,G} = \xi \cdot T\xi \cdot j_{TA,G}$. The first one is automatically satisfied, because $\xi \cdot T\xi \cdot \eta_{TA} = \xi \cdot \eta_A \cdot \xi = \xi$ since η is a natural transformation and ξ satisfies the unit axiom; and $\xi \cdot \mu_A \cdot \eta_{TA} = \xi$ since $\mu_A \cdot \eta_{TA} = 1_{TA}$. Considering that in the following diagram the left-hand square commutes,

the second condition is equivalent to the commutativity of the outer rectangle; but in view of Lemma 27 this amounts to the required commutativity.

$$\begin{array}{ccccc}
 (TA|G) & \xrightarrow{j_{TA,G}} & T(TA) & \xrightarrow{\mu_{A,G}} & TA \\
 (\xi|1_G) \downarrow & & T\xi \downarrow & & \downarrow \xi \\
 (A|G) & \xrightarrow{j_{A,G}} & TA & \xrightarrow{\xi} & A
 \end{array}$$

□

The following result shows that in a semi-abelian category, the conditions of Proposition 29 are also sufficient for some $\psi : (A|G) \rightarrow A$ to be an action. This provides an alternative proof of the fact that in any semi-abelian category, all \mathbb{T}_G -algebras are actions. We point out that this proof is based on Proposition 26 and the characterization of proper subobjects by stability under the conjugation action in Corollary 24, instead of Beck's criterion as in [5].

Proposition 30. *Let \mathbb{C} be semi abelian. Let $\psi : (A|G) \rightarrow A$ be a map satisfying the equivalent conditions of Proposition 25. Then ψ is an action if and only if the diagram (4.2) commutes.*

Proof. The condition is necessary in view of Proposition 29 (even if the category is not semi-abelian), since if ψ is an action then its extension ξ is an algebra. Conversely, suppose \mathbb{C} is semi-abelian and suppose that ψ satisfies the conditions of Propositions 25 and 29. To show that ψ is an action, it suffices by Proposition 26 to show that $\text{Ker } \xi$ is proper in $A+G$. Since \mathbb{C} is semi-abelian, it suffices by Corollary 24 to show that $\text{Ker } \xi$ is stable under the conjugation action of $A+G$, i.e. that there exists a map $c'' : (\text{Ker } \xi|A+G) \rightarrow \text{Ker } \xi$ making the following diagram commute:

$$\begin{array}{ccc}
 (\text{Ker } \xi|A+G) & \xrightarrow{(k.\text{ker } \xi|1)} & (A+G|A+G) \\
 c'' \downarrow \text{dotted} & & \downarrow c_2^{A+G} \\
 \text{Ker } \xi & \xrightarrow{k.\text{ker } \xi} & A+G
 \end{array}$$

But since TA is proper in $A+G$ one has the conjugation action $c^{TA,A+G}$ of $A+G$ over TA which makes the following diagram commute:

$$\begin{array}{ccc}
 (TA|A+G) & \xrightarrow{(k|1)} & (A+G|A+G) \\
 c^{TA,A+G} \downarrow & & \downarrow c_2^{A+G} \\
 TA & \xrightarrow{k} & A+G
 \end{array}$$

so that one gets the result if one can find a map c'' such that

$$\begin{array}{ccc}
 (\text{Ker } \xi|A+G) & \xrightarrow{(\text{ker } \xi|1)} & (TA|A+G) \\
 c'' \downarrow \text{dotted} & & \downarrow c^{TA,A+G} \\
 \text{Ker } \xi & \xrightarrow{\text{ker } \xi} & TA
 \end{array}$$

commutes, i.e. if $\xi.c^{TA, A+G}(\ker \xi|1) = 0$. But

$$\begin{aligned} \xi.c^{TA, A+G}(\ker \xi|1) &= \psi.(\xi|1_G).(\ker \xi|1) \quad \text{by hypothesis} \\ &= \psi.(\xi.\ker \xi|1) \\ &= \psi.(0|1) \\ &= 0 \quad \text{by Proposition 2.(2).} \end{aligned}$$

□

We finish with a proposition which characterizes the maps $\psi : (A|G) \rightarrow A$ which can be extended to \mathbb{T} -algebras, hence which are actions if the category is semi-abelian. Of course, the diagrams involved in this proposition are more complicated than the axioms of an \mathbb{T} -algebra. The interest is that these axioms are exclusively expressed in terms of cross effects. Moreover, it appears that the second axiom is a simplification of the associativity axiom: in view of Lemma 27 which shows that μ is a conjugation action, it can be seen as the associativity axiom restricted to the endofunctor $(-|G)$ instead of T_G . And the third axiom expresses that one then must add a condition involving a higher cross effect. It turns out, however, that in the category of groups the third condition may be skipped.

Proposition 31. *Let ψ be a map $(A|G) \rightarrow A$. Then ψ can be extended to a \mathbb{T} -algebra if and only if the following three diagrams commute:*

$$(4.3) \quad \begin{array}{ccc} ((A|G)|A) & \xrightarrow{c^{(A|G), A+G}|_A} & (A|G) \\ (\psi|1_A) \downarrow & & \downarrow \psi \\ (A|A) & \xrightarrow{c_2^A} & A \end{array}$$

$$(4.4) \quad \begin{array}{ccc} ((A|G)|G) & \xrightarrow{c^{(A|G), A+G}|_G} & (A|G) \\ (\psi|1_G) \downarrow & & \downarrow \psi \\ (A|G) & \xrightarrow{\psi} & A \end{array}$$

$$(4.5) \quad \begin{array}{ccc} ((A|G)|A|G) & \xrightarrow{\iota} & ((A|G) + A|G) \\ \downarrow \iota & & \downarrow \langle j_{A,G}, \eta_A \rangle|_{1G} \\ (A|G) & & (TA|G) \\ \downarrow \langle \psi, 1_A \rangle|_{1G} & & \downarrow c' \\ ((A|G) + A|G) & & (A|G) \\ \downarrow \langle \psi, 1_A \rangle|_{1G} & & \downarrow \psi \\ (A|G) & \xrightarrow{\psi} & A \end{array}$$

where c' is the factorization of c in Lemma 28.

Proof. Since the first diagram is (4.1) we know from Proposition 25 that it commutes iff ψ has an extension ξ which satisfies the unit axiom for an algebra. Supposing that this holds, it remains to show that then the associativity axiom is equivalent to the commutativity of the two latter diagrams. We apply Proposition 4 to $F = Id_{\mathbb{C}}$, $A = G$, $X = TA$, $K = (A|G)$, $Y = A$, $k = j_{A,G}$, $f = r_A \cdot k_{A,G}$, $s = \eta_A$. Then the family $(\iota', (j_{A,G}|1), (\eta_A|1))$ of morphisms with codomain $(TA|G)$ is jointly epimorphic, so diagram (4.2) commutes iff its precompositions with these three maps commute. To simplify the notation, we will denote by c all conjugation morphisms: naturality of c makes this simplification unambiguous.

First we precompose diagram (4.2) with $(\eta_A|1_G)$:

$$\begin{array}{ccc}
 (A|G) & \xrightarrow{\dots j_{A,G}} & TA \\
 \downarrow (\eta_A|1_G) & \searrow & \downarrow \xi \\
 (TA|G) & \xrightarrow{c} & TA \\
 \downarrow (\xi|1_G) & & \downarrow \xi \\
 (A|G) & \xrightarrow{\psi} & A
 \end{array}$$

$(\xi \cdot \eta_A|1_G) = 1_{(A|G)}$

If one shows that the upper triangle commutes, i.e. $c \cdot (\eta_A|1_G) = j_{A,G}$ then one may conclude that the whole outer diagram commutes, so that the resulting condition is void here. In fact, since $k_{A,G}$ is a monomorphism, it suffices to show that $k_{A,G} \cdot c \cdot (\eta_A|1_G) = k_{A,G} \cdot j_{A,G} = \iota_{A,G}$. But by Proposition 13, one has $k_{A,G} \cdot c = c_2^{A+G} \cdot (k_{A,G}|i_G)$, so one has:

$$\begin{aligned}
 & k_{A,G} \cdot c \cdot (\eta_A|1_G) \\
 &= c_2^{A+G} \cdot (k_{A,G}|i_G) \cdot (\eta_A|1_G) \\
 &= c_2^{A+G} \cdot (k_{A,G} \cdot \eta_A|i_G) \\
 &= c_2^{A+G} \cdot (i_A|i_G) \\
 &= \iota_{A,G} \text{ because the following diagram commutes:}
 \end{aligned}$$

$$\begin{array}{ccc}
 (A|G) & \xrightarrow{(i_A|i_G)} & (A+G|A+G) \\
 \downarrow \iota_{A,G} & & \downarrow \iota_{A+G,A+G} \\
 A+G & \xrightarrow{i_A+i_G} & (A+G) + (A+G) \\
 & \searrow & \downarrow \nabla_{A+G} \\
 & & A+G
 \end{array}$$

Secondly, we we precompose diagram (4.2) with $(j_{A,G}|1)$:

$$\begin{array}{ccc}
 ((A|G)|G) & \xrightarrow{c} & (A|G) \\
 \downarrow (j_{A,G}|1_G) & & \downarrow j_{A,G} \\
 (TA|G) & \xrightarrow{c} & TA \\
 \downarrow (\xi|1_G) & & \downarrow \xi \\
 (A|G) & \xrightarrow{\psi} & A
 \end{array}$$

$(\xi \cdot j_{A,G}|1_G) = (\psi|1_G)$

All parts of this diagram except the bottom square are known to commute, so diagram (4.2) precomposed with $(j_{A,G}|1)$ commutes iff diagram (4.4) commutes.

Finally, we precompose diagram (4.2) with $\iota' = (\langle j_{A,G}, \eta_A \rangle | 1_G) \cdot \iota$, ι here being the injection of $((A|G)|A|G)$ into $((A|G) + A|G)$:

$$\begin{array}{ccc}
 ((A|G)|A|G) & & \\
 \downarrow \iota & & \\
 ((A|G) + A|G) & \xrightarrow{c'(\langle j_{A,G}, \eta_A \rangle | 1_G)} & (A|G) \\
 \downarrow \langle j_{A,G}, \eta_A \rangle | 1_G & & \downarrow j_{A,G} \\
 (TA|G) & \xrightarrow{c} & TA \\
 \downarrow (\xi | 1_G) & & \downarrow \xi \\
 (A|G) & \xrightarrow{\psi} & A
 \end{array}$$

$(\langle \psi, 1_A \rangle | 1_G)$ is indicated by a curved arrow from $(TA|G)$ to $(A|G)$.

The left-hand triangle commutes since $(\xi | 1)(\langle j_{A,G}, \eta_A \rangle | 1_G) = (\langle \xi j_{A,G}, \xi \eta_A \rangle | 1_G) = (\langle \psi, 1_A \rangle | 1_G)$, and the upper square commutes by definition of c' . Hence diagram (4.2) precomposed with ι' commutes iff diagram (4.5) commutes, which achieves the proof. \square

Example: In the category of groups, consider a map $\psi : (A|G) \rightarrow A$ and the corresponding $\phi : G \times A \rightarrow G^3$. We know that the first axiom in Proposition 31 expresses that the $\phi(g, -)$ are group endomorphisms of A . We now examine the second diagram. Considering that $(X|Y)$ is the subgroup of $Y + X$ generated by the $[x, y]$'s (with x, y different from the units), a generator of $((A|G)|G)$ has the form $\llbracket g, \prod_{i=1}^n [g_i, a_i]^{z_i} \rrbracket$, where the outer commutator $\llbracket \dots \rrbracket$ is considered in $(A + G) + G$, while the inner one $[\dots]$ is in $A + G$. First consider the case $n = 1$ with $z_1 = 1$. For a generator $\llbracket g, [g_1, a_1] \rrbracket$ one has $c(\llbracket g, [g_1, a_1] \rrbracket) = [g, [g_1, a_1]] = [gg_1, a_1][a_1, g][a_1, g_1] = [gg_1, a_1][g, a_1]^{-1}[g_1, a_1]^{-1}$, the second equality arising from the formula $[x, [y, z]] = [xy, z][z, x][z, y]$. So one gets $\psi(c(\llbracket g, [g_1, a_1] \rrbracket)) = \psi([gg_1, a_1][g, a_1]^{-1}[g_1, a_1]^{-1}) = \psi([gg_1, a_1])(\psi([g, a_1])^{-1}\psi([g_1, a_1])^{-1}) = \phi(gg_1, a_1)a_1^{-1} \cdot (\phi(g, a_1) \cdot a_1^{-1})^{-1} \cdot (\phi(g_1, a_1) \cdot a_1^{-1})^{-1} = \phi(gg_1, a_1) \cdot (\phi(g, a_1))^{-1} \cdot a_1 \cdot (\phi(g_1, a_1))^{-1}$ on the one hand. On the other hand, one gets $\psi((\psi | 1_G)(\llbracket g, [g_1, a_1] \rrbracket)) = \psi([g, \psi([g_1, a_1])]) = \psi([g, \phi(g, a_1)a_1^{-1}]) = \phi(g, \phi(g, a_1)a_1^{-1})(\phi(g, a_1)a_1^{-1})^{-1} = \phi(g, \phi(g, a_1)a_1^{-1})a_1(\phi(g, a_1))^{-1}$. So the equation $\psi(c(\llbracket g, [g_1, a_1] \rrbracket)) = \psi((\psi | 1_G)(\llbracket g, [g_1, a_1] \rrbracket))$ is verified if and only if so is the equation $\phi(gg_1, a_1) \cdot (\phi(g, a_1))^{-1} = \phi(g, \phi(g_1, a_1) \cdot a_1^{-1})$. Now if one assumes moreover that the first diagram commutes, i.e. that the $\phi(g, -)$'s are endomorphisms of A , then this amounts to the relation $\phi(gg_1, a_1) \cdot (\phi(g, a_1))^{-1} = \phi(g, \phi(g_1, a_1)) \cdot (\phi(g, a_1))^{-1}$, i.e. $\phi(gg_1, a_1) = \phi(g, \phi(g_1, a_1))$. So, assuming the commutativity of the first diagram in Proposition 31, the commutativity of the second one implies that ϕ is a group action in the usual sense; and of course, conversely, if ϕ is an action in the usual sense, then the corresponding ψ is an action in our sense, hence the three diagrams of Proposition 31 commute.

³Note that if one defines ϕ from ψ , one can put $\phi(g, a) = \psi([g, a]) \cdot a$, even if $[g, a]$ is the unit, i.e. g is the unit of G or a is the unit of A : this insures that $\phi(1, a) = a$ and $\phi(g, 1) = 1$, and of course $\psi[g, a] = \phi(g, a) \cdot a^{-1}$. So there is no need for a special discussion for the case when $gg_1 = 1$ in what follows.

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