

COMPLETIONS, REVERSALS, AND DUALITY FOR TROPICAL VARIETIES

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ABSTRACT. We state and prove an identity for polynomials over the max-plus algebra, which shows that any polynomial divides a product of binomials. Interpreted in tropical geometry, any tropical variety W can be completed to a union of hypersurfaces. In certain situations, W has a “reversal” variety, which together with W already yields the union of hypersurfaces; this phenomenon also is explained in terms of the algebraic structure.

INTRODUCTION

Tropical mathematics has been developed mainly over the tropical semiring $\mathbb{T}_{\max} = \mathbb{R} \cup \{-\infty\}$, whose addition and multiplication are respectively the operations of maximum and summation,

$$a \oplus b = \max\{a, b\}, \quad a \odot b = a + b,$$

cf. [2, 4, 8, 9, 13, 14]. Factorization in polynomials over \mathbb{T}_{\max} is notoriously difficult, cf. [6, 7]. One reason is that different polynomials in $\mathbb{T}[\lambda_1, \dots, \lambda_n]$ (viewed as functions from $\mathbb{T}^{(n)}$ to \mathbb{T}) may act as the same function over the max-plus algebra, when the values of one monomial are dominated by other monomials. Thus, we write $f \stackrel{\sim}{\sim} g$ to denote that polynomials f and g correspond to the same function. Our main result in this paper is the following new identity of polynomials in $\mathbb{T}[\lambda_1, \dots, \lambda_n]$:

Theorem 0.1. *Suppose $f = \sum_{i=1}^m f_i \in \mathbb{T}[\lambda_1, \dots, \lambda_n]$, for $m \geq 2$. Then*

$$(1) \quad \prod_{i < j} (f_i + f_j) \stackrel{\sim}{\sim} \left(\sum_i f_i \right) \left(\sum_{i < j} f_i f_j \right) \cdots \left(\sum_i \prod_{j \neq i} f_j \right).$$

(The right side is written as a product of $m - 1$ terms.)

This result could be viewed as an extreme failure of unique factorization, since *every* polynomial f which is a sum of at least three distinct monomials is part of a factorization that is not unique. Specifically, if f_i are the monomials of f , then f is a factor of the product $\prod_{i \neq j} (f_i + f_j)$, as was seen in [6, Theorem 12.4]. On the other hand, Theorem 0.1 has a positive geometric interpretation – Every tropical variety W can be “completed” to a variety $\mathcal{P}(W)$ comprised of various k -dimensional planes, which in turn can be decomposed into a union of varieties that can be interpreted via (1).

Equation (1) also gives rise to “reversals” of tropical varieties and a duality in tropical geometry. The motivation for the former came from Mikael Passare’s talk, Mathematisches Forschungsinstitut Oberwolfach, December 2007.

1. THE TROPICAL POLYNOMIAL ALGEBRA

Since our basic result is algebraic, we need to consider the underlying algebraic structure of the max-plus algebra \mathbb{T} . It is easy to see that \mathbb{T} is a semiring (which by definition satisfies all of the axioms of an associative ring except for existence of negatives), where $-\infty$ is the zero element of \mathbb{T} . In fact, the existence of negatives fails spectacularly, since $a \oplus b = \max\{a, b\}$ can never be $-\infty$ unless $a = b = -\infty$. We refer to [3] as a standard reference on semirings. Many familiar notions in ring theory (subrings, ideals,

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polynomials, etc.) carry over almost word for word to semirings, although the lack of additive negatives makes the construction of a factor semiring much less useful. In particular, given any semiring R , we can form the semiring of polynomials $R[\lambda]$, whose addition and multiplication are induced by addition and multiplication of the coefficients, where $\alpha_i, \beta_j \in R$:

$$\begin{aligned} \left(\sum_i \alpha_i \lambda^i\right) + \left(\sum_i \beta_i \lambda^i\right) &= \sum_i (\alpha_i + \beta_i) \lambda^i; \\ \left(\sum_i \alpha_i \lambda^i\right) \left(\sum_j \beta_j \lambda^j\right) &= \sum_{i+j=k} \alpha_i \beta_{k-j} \lambda^k. \end{aligned}$$

Here we have reverted to the usual algebraic notation, although we are always working over \mathbb{T} . Thus, $f = \sum_i \alpha_i \lambda^i$ denotes $\bigoplus_{i \in \mathbb{N}} \alpha_i \odot \lambda^i$, and the substitution $f(a)$ denotes $\bigoplus_{i \in \mathbb{N}} \alpha_i \odot a^i$, where $a^i = a \odot \cdots \odot a$, taken i times.

Formally iterating the polynomial construction n times enables one to define the polynomial semiring $\mathbb{T}[\Lambda] = \mathbb{T}[\lambda_1, \dots, \lambda_n]$ in n commuting indeterminates $\lambda_1, \dots, \lambda_n$. Elements of $\mathbb{T}[\lambda_1, \dots, \lambda_n]$ are called **tropical polynomials**. A tropical polynomial which is a sum of precisely two different monomials is called a **binomial**.

Summarizing, any tropical polynomial can be written as

$$(2) \quad f = \bigoplus_{\mathbf{i} \in \Omega} \alpha_{\mathbf{i}} \odot \Lambda^{\mathbf{i}} \in \mathbb{T}[\lambda_1, \dots, \lambda_n] \setminus \{-\infty\},$$

where $\Omega \subset \mathbb{Z}^n$ is a finite nonempty set of n -tuples $\mathbf{i} = (i_1, \dots, i_n)$ with nonnegative coordinates, $\alpha_{\mathbf{i}} \in \mathbb{T}$ for all $\mathbf{i} \in \Omega$, and $\Lambda^{\mathbf{i}}$ stands for $\lambda_1^{i_1} \odot \cdots \odot \lambda_n^{i_n}$.

1.1. The upper essential polynomial semiring. Any tropical polynomial $f \in \mathbb{T}[\lambda_1, \dots, \lambda_n] \setminus \{-\infty\}$ determines a piecewise linear convex function $\tilde{f} : \mathbb{R}^{(n)} \rightarrow \mathbb{R}$, defined by:

$$(3) \quad \tilde{f}(\mathbf{a}) = \max_{\mathbf{i} \in \Omega} \{\langle \mathbf{a}, \mathbf{i} \rangle + \alpha_{\mathbf{i}}\}, \quad \mathbf{a} \in \mathbb{R}^{(n)},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard scalar product. The map $f \mapsto \tilde{f}$ is not 1:1. This map can be viewed naturally as a homomorphism of semirings, but we do not pursue that path here; instead, we look for a canonical polynomial representing each of these functions.

Definition 1.1. A polynomial g is **dominated by** a polynomial f if $g(\mathbf{a}) \leq f(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{T}^{(n)}$. Suppose $f = \bigoplus \alpha_i \odot \Lambda^i$, and $h = \alpha_j \odot \Lambda^j$ is a monomial of f , and write $f_h = \bigoplus_{i \neq j} \alpha_i \odot \Lambda^i$. We say that the monomial h is **(upper) inessential** if h is dominated by f_h (or, in other words, $f(\mathbf{a}) = f_h(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{T}^{(n)}$); otherwise h is said to be **(upper) essential**. Note that $h(\mathbf{a}) \leq f(\mathbf{a})$ for each inessential monomial h and all $\mathbf{a} \in \mathbb{T}^{(n)}$. The **(upper) essential part** \tilde{f} of a polynomial $f = \bigoplus \alpha_i \odot \Lambda^i$ is the sum of all essential monomials of f , while its **inessential part** f^i consists of the sum of all inessential monomials of f . When $f = \tilde{f}$, we call f an **essential polynomial**.

(Note that any monomial by itself, considered as a polynomial, is essential.)

Using this definition we say that two polynomials f and g are **essentially equivalent**, written $f \simeq g$, if $\tilde{f} = \tilde{g}$, that is if $f(\mathbf{a}) = g(\mathbf{a})$ for each $\mathbf{a} \in \mathbb{T}^{(n)}$. Clearly, \simeq is an equivalence relation which, for convenience, we call **e -equivalence**. We always consider factorization up to e -equivalence; in other words, we say g **divides** f if $gh \simeq f$ for some polynomial h . (Otherwise one could make any polynomial irreducible by adding some inessential monomial.)

Definition 1.2. The **essential polynomial semiring**, $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, of $\mathbb{T}[\lambda_1, \dots, \lambda_n]$ is the set of essential polynomials, where addition and multiplication are defined by taking the essential part of the respective sum or product in $\mathbb{T}[\lambda_1, \dots, \lambda_n]$. In other words, if \oplus and \odot are the respective operations in $\mathbb{T}[\lambda_1, \dots, \lambda_n]$, we define

$$f + g = \widetilde{f \oplus g}, \quad fg = \widetilde{f \odot g}$$

to be the corresponding operations in the essential polynomial semiring $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$.

We identify the essential polynomial semiring $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ with the isomorphic semiring of polynomial functions $\{\tilde{f} : \mathbb{T}^{(n)} \rightarrow \mathbb{T}\}$. Abusing notation slightly, we still write elements of $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ as polynomials, although strictly speaking, they are equivalence classes of polynomials.

Since the meaning should be clear from the context, we use the same notation, $+$ and \cdot , for the operations of the essential polynomial semiring and for $\mathbb{T}[\lambda_1, \dots, \lambda_n]$.

The next observation shows how inessential terms often may arise.

Lemma 1.3. *Assume $f = f_1 + f_2 + f_3 \in \mathbb{T}[\lambda_1, \dots, \lambda_n]$, where*

$$f_1 = \lambda_1^{j_1+k} \lambda_2^{j_2-k} \lambda_3^{j_3} \dots \lambda_m^{j_m}, \quad f_2 = \lambda_1^{j_1} \lambda_2^{j_2} \lambda_3^{j_3} \dots \lambda_n^{j_n}, \quad f_3 = \lambda_1^{j_1-k} \lambda_2^{j_2+k} \lambda_3^{j_3} \dots \lambda_n^{j_n},$$

are monomials and $k \leq \min\{j_1, j_2\}$ is a natural number. Then f_2 is inessential for f .

Proof. Pick $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{T}^{(n)}$ and assume $a_1 > a_2$; then $f(\mathbf{a}) = f_1(\mathbf{a}) > f_2(\mathbf{a}), f_3(\mathbf{a})$. Conversely, if $a_2 > a_1$, then $f(\mathbf{a}) = f_3(\mathbf{a}) > f_1(\mathbf{a}), f_2(\mathbf{a})$. When $a_1 = a_2$, $f(\mathbf{a}) = f_1(\mathbf{a}) + f_3(\mathbf{a}) = f_1(\mathbf{a}) + f_2(\mathbf{a}) + f_3(\mathbf{a})$. In every case, f_2 is inessential for f . \square

For the next lemma, we let \mathcal{I} denote the set of all m -tuples $\mathbf{i} = (i_1, \dots, i_m)$ for which each $0 \leq i_u < m$ and $\sum_{u=1}^m i_u = \binom{m}{2}$. For any $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}$ and $0 \leq j \leq m-1$, we define the j -index $\iota_j(\mathbf{i})$ to be the number of i_u 's that equal j ; define $\iota(\mathbf{i}) = (\iota_{m-1}(\mathbf{i}), \dots, \iota_0(\mathbf{i}))$.

Let S_m denote the set of permutations of $(0, 1, \dots, m-1)$. Thus, $\mathbf{i} \in S_m$ iff $\iota(\mathbf{i}) = (1, 1, \dots, 1)$.

We say \mathbf{i} is **admissible** if for each number k , the sum of the largest k components of \mathbf{i} is at most $(m-1) + \dots + (m-k) = km - \frac{k(k+1)}{2}$.

Remark 1.4. *Lexicographically, $\iota(\mathbf{i}) \leq (1, 1, \dots, 1)$ for each admissible $\mathbf{i} \in \mathcal{I}$. (Indeed, the sum of the largest two components of \mathbf{i} is at most $2m-3$, which means that at most one component is $m-1$, so the first component of $\iota(\mathbf{i})$ is at most 1. We are done unless it is 1, and conclude by induction on m .)*

For each $\mathbf{i} \in \mathcal{I}$, we define the monomial

$$h_{\mathbf{i}} = \lambda_1^{i_1} \dots \lambda_m^{i_m} = \Lambda^{\mathbf{i}}.$$

For any permutation $\sigma \in S_m$, we denote

$$h_{\sigma} = \lambda_1^{\sigma(0)} \dots \lambda_{m-1}^{\sigma(m-1)} = \Lambda^{\sigma}.$$

Lemma 1.5. *The polynomial $p = \sum_{\sigma \in S_m} h_{\sigma}$ dominates $h_{\mathbf{i}}$ for each admissible $\mathbf{i} \in \mathcal{I}$.*

Proof. The proof is by reverse induction on the lexicographic order of $\iota(\mathbf{i})$. The assertion holds by hypothesis when $\iota(\mathbf{i}) = (1, 1, \dots, 1)$. In general, if $\iota(\mathbf{i}) < (1, 1, \dots, 1)$, then some j -index is 0, which implies that for some $j' < j$, the j' -index $\iota_{j'}(\mathbf{i}) \geq 2$; in other words, \mathbf{i} has components $i_s = i_t = j'$ for suitable $s \neq t$.

Take \mathbf{i}' to be the m -tuple in which $i_s = j' + 1$ and $i_t = j' - 1$ (with all other components the same as for \mathbf{i}), and let \mathbf{i}'' be the m -tuple in which $i_s = j' - 1$ and $i_t = j' + 1$. By Lemma 1.3, $h_{\mathbf{i}}$ is dominated by $h_{\mathbf{i}'} + h_{\mathbf{i}''}$.

We claim that $h_{\mathbf{i}'}$ and $h_{\mathbf{i}''}$ are admissible and $\leq (1, 1, \dots, 1)$. Indeed, this is clear for $j' < j-1$, since then $\iota_j(\mathbf{i}') = \iota_j(\mathbf{i}'') = 0$. Thus, we may assume that $j' = j-1$. By definition of admissibility, $\iota_{j-1}(\mathbf{i}) \leq 2$, because if $\iota_{j-1}(\mathbf{i}) \geq 3$, one would have the sum of the largest $k = m-j+2$ components of \mathbf{i} would be

$$(m-1) + \dots + (j+1) + 0 + 3(j-2),$$

which is greater than $km - \frac{k(k+1)}{2}$. On the other hand, by definition of j' , we have $\iota_{j-1}(\mathbf{i}) \geq 2$, so $\iota_{j-1}(\mathbf{i}) = 2$. Since \mathbf{i}' increases one of the exponent of i_s from $j-1$ to j , we see that $\iota_j(\mathbf{i}') = 1$ and $\iota_{j-1}(\mathbf{i}') = 0$, proving $\iota(\mathbf{i}') < (1, 1, \dots, 1)$, as desired. Clearly $\iota(\mathbf{i}'') = \iota(\mathbf{i}')$, since the roles of s and t are interchanged, so $h_{\mathbf{i}''}$ also is admissible.

Clearly, $\iota(\mathbf{i}') = \iota(\mathbf{i}'')$ is of higher lexicographic order than $\iota(\mathbf{i})$ (since $\iota_{j'+1}(\mathbf{i}') = \iota_{j'+1}(\mathbf{i}) + 1$), so, by reverse induction, $h_{\mathbf{i}'}$ and $h_{\mathbf{i}''}$ are both dominated by p , implying $h_{\mathbf{i}}$ is dominated by p . \square

Remark 1.6. *Although we focus on the image of $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ under the natural map*

$$\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n] \rightarrow \{\tilde{f} : \mathbb{T}^{(n)} \rightarrow \mathbb{T}\},$$

this map loses the (upper) inessential part of a polynomial. If we consider instead the min-plus algebra $\mathbb{T}^* = \mathbb{T}_{\min}$, in which max is replaced by min, then the essential monomials become the ones taking on minimal values; let us call these **lower essential**. Then we get a map

$$\tilde{\mathbb{T}}^*[\lambda_1, \dots, \lambda_n] \longrightarrow \{\tilde{f} : \mathbb{T}^{*(n)} \rightarrow \mathbb{T}^*\},$$

where now we define addition by taking the minimum value instead of the maximum value, and thus lose the lower inessential part of each polynomial. We preserve more information by considering both together, i.e., $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n] \times \tilde{\mathbb{T}}^*[\lambda_1, \dots, \lambda_n]$, viewed in

$$\{\tilde{f} : \mathbb{T}^{(n)} \longrightarrow \mathbb{T}\} \times \{\tilde{f} : \mathbb{T}^{*(n)} \rightarrow \mathbb{T}^*\}.$$

(Even so, one still loses information such as f_2 in Lemma 1.3.) Since this viewpoint leads to more complicated notation, we put it aside for the time being, but return to it later.

1.2. The tropical Vandermonde matrix. Our main tool in proving Theorem 0.1 is the **tropical Vandermonde matrix** V_f of an essential polynomial $f = \sum_{i=1}^m f_i \in \tilde{\mathbb{T}}[x_1, \dots, x_n]$, define as the $m \times m$ matrix with entries $v_{ij} = f_i^{j-1}$. Since the determinant is not available in tropical algebra (because it involves negative signs), one uses the permanent

$$\text{per}(V_f) = \sum_{\sigma \in S_m} f_1^{\sigma(0)} \cdots f_m^{\sigma(m-1)},$$

where m denotes the number of (essential) monomials in f . We can compute the permanent in two ways:

Lemma 1.7. *If $V_f = (\lambda_i^{j-1})$ is an $m \times m$ Vandermonde matrix (for $f = \sum \lambda_i$), then*

- (1) $\text{per}(V_f) \simeq \prod_{i < j} (\lambda_i + \lambda_j)$ and,
- (2) $\text{per}(V_f) \simeq (\sum_i \lambda_i)(\sum_{i < j} \lambda_i \lambda_j) \cdots (\sum_i \prod_{j \neq i} \lambda_j)$.

Proof. Let $p = \text{per}(V_f)$; then p is a homogenous polynomial of degree $\frac{m(m-1)}{2}$ in the m indeterminates $\lambda_1, \dots, \lambda_m$. Moreover, p is a sum of the $m!$ monomials h_σ , each corresponding to a single permutation $\sigma \in S_m$; thus p is the polynomial of Lemma 1.5, which says that p dominates $h_{\mathbf{i}}$ for each admissible $\mathbf{i} \in \mathcal{I}$.

But it is easy to see that each monomial of $q_1 = \prod_{i < j} (\lambda_i + \lambda_j)$ has the form $h_{\mathbf{i}}$ where \mathbf{i} is admissible (since the extreme case is for some λ_i to have degree $m-1$, in which case the next indeterminate has degree at most $m-2$, and so forth), and thus $h_{\mathbf{i}}$ is dominated by p . Since each monomial of p appears in q_1 , we get $p \simeq q_1$.

Likewise, each monomial of $q_2 = (\sum_i \lambda_i) (\sum_{i < j} \lambda_i \lambda_j) \cdots (\sum_i \prod_{j \neq i} \lambda_j)$ clearly has the form $h_{\mathbf{i}}$ where \mathbf{i} is admissible, and each monomial of p appears in q_2 , implying $p \simeq q_2$. \square

1.3. Proof of Theorem 0.1. The proof of Theorem 0.1 now becomes quite transparent:

Proof. Specialize λ_i to f_i and apply Lemma 1.7. \square

Algebraically, Theorem 0.1 shows that the factorization of $\text{per}(V_f) \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ into irreducible polynomials is not unique.

Example 1.8. *Suppose $f = \lambda_1^i + \lambda_2^j + \alpha$, where $\alpha \in \mathbb{R}$, is a polynomial in $\mathbb{T}[\lambda_1, \lambda_2]$. Then*

$$V_f = \begin{pmatrix} 0 & \alpha & \alpha^2 \\ 0 & \lambda_1^i & \lambda_1^{2i} \\ 0 & \lambda_2^j & \lambda_2^{2j} \end{pmatrix} \quad \text{and}$$

$$\text{per}(V_f) \simeq (\lambda_1^i + \lambda_2^j + \alpha)(\alpha \lambda_1^i + \alpha \lambda_2^j + \lambda_1^i \lambda_2^j) \simeq (\lambda_1^i + \lambda_2^j)(\lambda_1^i + \alpha)(\lambda_2^j + \alpha).$$

This yields two different tropical factorizations of $\text{per}(V_f)$ into irreducible polynomials. (The right factorization is a binomial factorization.)

In tropical algebra, perhaps “unique factorization” is the wrong emphasis, but rather we should emphasize factorization of $\text{per}(V_f)$ into binomials. We pursue this avenue in the next section.

We define the sums

$$(4) \quad \begin{aligned} f^{(1)} &= \sum_i f_i, \\ f^{(2)} &= \sum_{i < j} f_i f_j, \\ f^{(3)} &= \sum_{i < j < k} f_i f_j f_k, \\ &\vdots \\ f^{(m-1)} &= \sum_i \prod_{j \neq i} f_j, \end{aligned}$$

and write f^{trn} for $f^{(m-1)}$ which we call the **transpose polynomial** of f .

Also, we write \bar{f} for $\prod_{i < j} (f_i + f_j)$. Under this notation, one can rephrase Theorem 0.1 in the language of the essential tropical semiring.

Corollary 1.9. *Suppose $f = \sum_{i=1}^m f_i$ is a polynomial in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ with monomials f_i . Then*

$$(5) \quad \text{per}(V_f) = \prod_{i=1}^{m-1} f^{(i)} = \prod_{i < j} (f_i + f_j) = \bar{f}.$$

Remark 1.10. (i) *Since the factorization (5) is the mainstay of this paper, let us pause for a moment to consider the degrees of the factors, in the situation where f is completely homogeneous of degree d . Note that $\deg(\bar{f}) = d \binom{m}{2}$, but \bar{f} can be factored into a product of $\binom{m}{2}$ homogeneous binomials, each of degree d . On the other hand, each $f^{(k)}$ is completely homogeneous, of degree $dk \binom{m}{k}$, and need not be reducible.*

(ii) $\overline{(\bar{f})}$ need not be \bar{f} . For example, taking

$$f = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3) = \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3,$$

we have

$$\begin{aligned} \bar{f} &= (\lambda_1^2 + \lambda_1 \lambda_2)(\lambda_1^2 + \lambda_1 \lambda_3)(\lambda_1^2 + \lambda_2 \lambda_3)(\lambda_1 \lambda_2 + \lambda_1 \lambda_3)(\lambda_1 \lambda_2 + \lambda_2 \lambda_3)(\lambda_1 \lambda_3 + \lambda_2 \lambda_3) \\ &= \lambda_1^3 \lambda_2 \lambda_3 (\lambda_1 + \lambda_2)^2 (\lambda_1 + \lambda_3)^2 (\lambda_2 + \lambda_3) (\lambda_1^2 + \lambda_2 \lambda_3). \end{aligned}$$

Computing $\overline{(\bar{f})}$, one easily sees that it has higher degree than f .

(iii) Another example where $\overline{(\bar{f})} \neq \bar{f}$, even when f is symmetric in the λ_i : For $f = \lambda_1 + \lambda_2 + \lambda_3$,

$$\bar{f} = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3),$$

and a similar computation to (ii) yields $\overline{(\bar{f})}$ to be a product of powers of the λ_i together with binomials of the form $\lambda_i + \lambda_j$ and $\lambda_i^2 + \lambda_j \lambda_k$.

2. GEOMETRIC INTERPRETATION

The main definition of a tropical variety is given in [11], as Section 1.2E. It is convenient for us to work with the equivalent definition of tropical varieties in terms of sets of roots of tropical polynomials, in the sense of [14, Section 2]; see also [4, 8, 10, 14, 15].

Given a tropical polynomial $f \neq -\infty$ in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, we denote by $Z_{\mathbb{T}}(f)$ the set of points $\mathbf{a} \in \mathbb{T}^{(n)}$, on which the value $f(\mathbf{a})$ either equals $-\infty$, or is attained by at least two of the monomials of f ; the set $Z_{\mathbb{T}}(f)$ is called **an affine tropical hypersurface**, whose elements are called **zeros** or **roots** of f . (Considering f as a tropical function, $Z_{\mathbb{T}}(f) \cap \mathbb{R}^{(n)}$ can be viewed as the domain of non-differentiability of f .) Then

$$(6) \quad Z_{\mathbb{T}}(fg) = Z_{\mathbb{T}}(f) \cup Z_{\mathbb{T}}(g).$$

For a finitely generated ideal $A = \langle f_1, \dots, f_s \rangle \subset \mathbb{T}[\lambda_1, \dots, \lambda_n]$, the set

$$Z_{\mathbb{T}}(A) = \bigcap_{f \in A} Z_{\mathbb{T}}(f) \subset \mathbb{T}^{(n)}$$

is called **an affine tropical (algebraic) set**. Clearly, $Z_{\mathbb{T}}(A) = Z_{\mathbb{T}}(f_1) \cap \dots \cap Z_{\mathbb{T}}(f_s)$.

This definition is consistent with the definition given in [14], in view of [11], and is a natural framework for developing the connections between algebra and geometry of polyhedral complexes.

When f contains at least two monomials, the nonempty set $Z(f) = Z_{\mathbb{T}}(f) \cap \mathbb{R}^{(n)}$ is called the **tropical hypersurface** in $\mathbb{R}^{(n)}$ defined by f . Similarly, for a finitely set $A = \{f_1, \dots, f_s\} \subset \mathbb{T}[x_1, \dots, x_n]$, the set $Z(A) = Z_{\mathbb{T}}(A) \cap \mathbb{R}^{(n)}$ is defined to be a **tropical (algebraic) set** in $\mathbb{R}^{(n)}$.

Corollary 2.1. *By the left side of Equation (5), the Newton polytope (see Section 3.3 below) of \bar{f} is a union of hyperplanes. In $\mathbb{R}^{(2)}$ this is called a zonotope, i.e. a Minkowski sum of a set of line segments.*

In other words, Theorem 0.1 says any tropical hypersurface is contained in a union of hyperplanes, which in turn can be decomposed into a union of hypersurfaces of the polynomials on the right side of Equation (5). This leads us to view \bar{f} as some sort of closure of f . But, in view of Remark 1.10(iii), this process could continue further, so we would rather take the polynomial which is the minimal product of binomials that is e -divisible by f , and we call it the **reduced closure** of f . To achieve this, we factor out all binomials first. In other words, writing $f = gh$ where g is the product of the binomial factors of f , the reduced closure is $g\bar{h}$.

3. APPLICATIONS IN TROPICAL GEOMETRY

A **tropical set** is a finite union of convex closed rational (i.e. defined over \mathbb{Q}) polyhedra. The **dimension** of a tropical set is the maximum of the dimensions of these polyhedra. (One can also show that all finite unions of convex closed rational polyhedra of positive codimension are tropical sets.)

A face of a polyhedral complex is **top-dimensional** if it has maximal dimension (with respect to all other faces). A finite polyhedral complex is said to be of **pure dimension** k if each of its faces of dimension $< k$ is contained in a top-dimensional face. Conversely, we say that a face is **bottom-dimensional** if it has minimal dimension (with respect to all other faces).

A tropical hypersurface H in $\mathbb{R}^{(n)}$ is then a finite rational polyhedral complex of pure dimension $(n - 1)$ where its the top-dimensional faces δ are equipped with positive integral **weights** $m(\delta)$ so that, for each $(n - 2)$ -dimensional face σ of H the following condition is satisfied, which is called the **balancing condition** (written using the standard operations):

$$(7) \quad \sum_{\sigma \subset \delta} m(\delta)n_{\sigma}(\delta) = 0 ,$$

where δ runs over all $(n - 1)$ -dimensional faces of H containing σ , and $n_{\sigma}(\delta)$ is the primitive integral normal vector to σ lying in the cone centered at σ and directed by δ [1, 14]. The weight, $m(\delta)$, of a face δ is also called the **multiplicity** of δ .

In general, we define a **k -dimensional tropical variety** in $\mathbb{R}^{(n)}$ as a finite rational polyhedral complex of pure dimension k , whose top-dimensional faces are equipped with positive integral weights and satisfy condition (7) for each face of codimension 1. (This definition is given in the sense of [11], which includes that of [14].)

Definition 3.1. *Let $\mathcal{S} = \{S_i \subset \mathbb{T}^{(n)} : i \in I \subset \mathbb{N}\}$, be a finite collection of tropical sets, and let $S_J = \bigcap_{j \in J} S_j$, $J \subseteq I$. Denoting by δ_j the face of maximal dimension in S_j , $j \in J$, containing S_J , we say that \mathcal{S} is **semidisjoint** if for any $J \subseteq I$, $\dim S_J < \delta_j$ for each $j \in J$. We denote the semidisjoint union by \bigsqcup^{semi} .*

Clearly, a disjoint collection of tropical sets is semidisjoint.

In order to distinguish between the standard notation of union and equality of sets to that which include weights we define:

Definition 3.2. *Two tropical varieties $W \subset \mathbb{R}^{(n)}$ and $W' \subset \mathbb{R}^{(n)}$ are said to be **weighted equal**, denoted $W \stackrel{w}{=} W'$, if they are identical as sets and each of their corresponding top dimensional faces has the same weight. The **weighted union** of tropical varieties $U \subset \mathbb{R}^{(n)}$ and $U' \subset \mathbb{R}^{(n)}$, denoted $U \stackrel{w}{\cup} U'$, is defined to be $U \cup U'$ where the weight of a top-dimensional face δ is the sum of the weights of the faces in U and U' that comprise δ .*

This definition of the weighted union satisfies additivity under union, as well as the balancing condition (7). With the definition we also have the relation

$$Z(fg) = Z(f) \stackrel{w}{\cup} Z(g),$$

for any $f, g \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$.

Analogously, we define the semidisjoint union with multiplicity.

Example 3.3. *If $f \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, then $Z(f) = Z(f^m)$ but $Z(f) \stackrel{w}{\neq} Z(f^m)$.*

For the rest of this paper we only consider tropical varieties that are also tropical algebraic sets; namely they can be written as $W = \bigcap H_i$, i.e. a complete intersection, where the H_i are tropical hypersurfaces. Moreover, all tropical hypersurfaces are considered as tropical varieties, i.e. equipped with weights.

3.1. Tropical primitives.

Definition 3.4. *A k -dimensional tropical variety of one face is called a k -dimensional tropical primitive, or tropical primitive, for short, when the variety is a hypersurface.*

Namely, a k -dimensional tropical primitive is a degenerate tropical variety, which in the classical sense is simply a k -dimensional plane having rational slopes. One can easily see that a k -dimensional tropical primitive is an intersection of tropical primitives. By definition, any collection of different primitive hypersurfaces must be semidisjoint.

Remark 3.5. *Any k -dimensional tropical primitive $P \subset \mathbb{R}^{(n)}$ is a tropical variety $Z(A)$, for which $A = \langle p_1, \dots, p_1 \rangle$ is an ideal generated by tropical binomials. Writing $P = \bigcap P_i$, with each P_i an $(n-1)$ -dimensional tropical primitive with rational slopes, say $\frac{t_1}{s_1}, \dots, \frac{t_n}{s_n}$ with each $t_i, s_i \in \mathbb{N}$, we can define the binomial*

$$p_i = \alpha_t \lambda_1^{t_1} \cdots \lambda_n^{t_n} + \alpha_s \lambda_1^{s_1} \cdots \lambda_n^{s_n}, \quad \alpha_s, \alpha_t \in \mathbb{R},$$

to get $Z(p_i) = P_i$.

We say that a k -dimensional tropical variety is **generic** if it does not have two or more top-dimensional faces contained in some tropical primitive of dimension k . A tropical variety is called **reducible** if it is a weighted union of tropical varieties; otherwise is it called **irreducible**. In particular, when $H = Z(f)$ is a reducible hypersurface, then $H = Z(g) \overset{w}{\cup} Z(h)$ and $f = gh$ for some polynomials g and h in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ (cf. (6)). When $W = W' \overset{w}{\cup} P$ is a tropical variety and P is some k -dimensional tropical primitive, we say H is **primitively reducible**, otherwise W is called **primitively irreducible**.

Lemma 3.6. *Any non-primitive tropical hypersurface $H \subset \mathbb{R}^{(n)}$ which contains a tropical primitive P is primitively reducible.*

Proof. Assume P is of weight m . Since H contains P , then all of its top dimensional faces which lying over P have weight $\geq m$ (not necessarily all of the same weight). For any $(n-2)$ -dimensional face σ of H contained in P , there are $(n-1)$ -dimensional faces $\delta, \delta' \subset P \cap H$ whose intersection is σ , i.e. $\delta \cap \delta' = \sigma$, and for which the balancing condition (7) is satisfied. In particular, $n_\sigma(\delta) = -n_\sigma(\delta')$ and $m(\delta), m(\delta') \geq m$. Reducing $m(\delta)$ and $m(\delta')$ by m , Condition (7) is still satisfied for all $\sigma \subset P$, so we can erase P from H and denote the result as $H \setminus_w P$, which remains a tropical hypersurface. (Note that some faces of H which lie on P might exist also in $H \setminus_w P$, but with lower weights.) \square

We call the procedure described in the proof **extracting a primitive from H** and denote it $H \setminus_w P$. (When all the top-dimensional faces of H on P are of weight m , equal to the multiplicity of P , then $H \setminus_w P = H \setminus P$.) In view of Remark 3.5, assuming $H = Z(f)$, extracting a primitive from H is equivalent to cancelling a binomial factor from f .

Given a tropical hypersurface H , we define the procedure of **primitive reduction** by discarding sequentially all the possible primitives from H , and call the result, \tilde{H} , the **reduced tropical hypersurface** of H . (By construction, the primitive reduction procedure is independent of the order of extraction, and thus is canonically defined.) Accordingly, we say that two hypersurfaces H and H' are **equal modulo primitives** if their reductions are identical.

Remark 3.7. *From a more general algebraic point of view, we could define the semiring*

$$\tilde{\mathbb{T}}\langle\langle \lambda_1, \dots, \lambda_n \rangle\rangle,$$

whose elements are formal sums

$$\sum \alpha_i \lambda_1^{i_1} \cdots \lambda_n^{i_n}, \quad \alpha_i \in \mathbb{R}, \quad i_1, \dots, i_n \in \mathbb{Q},$$

where addition and multiplication are just as with polynomials. When we substitute $a \in \mathbb{R}$ tropically for λ in the monomial $\lambda^{m/n}$, using the standard notation, we just take $\frac{m}{n}a$.

A binomial $p = \alpha_s \lambda_1^{s_1} \cdots \lambda_n^{s_n} + \alpha_t \lambda_1^{t_1} \cdots \lambda_n^{t_n}$, where now the $s_i, t_i \in \mathbb{Q}$, can be rewritten (tropically) as

$$\left(\frac{\alpha_t}{\alpha_s} \lambda_1^{t_1 - s_1} \cdots \lambda_n^{t_n - s_n} + 0 \right) \alpha_s \lambda_1^{s_1} \cdots \lambda_n^{s_n}.$$

Cancelling out the monomial on the right, we obtain a binomial of the form $\alpha_s \lambda_1^{s_1} \cdots \lambda_n^{s_n} + 0$, which we say has **normal form**.

Given any binomial $p = \alpha_s \lambda_1^{s_1} \cdots \lambda_n^{s_n} + 0$ of normal form, we delete those λ_i for which $s_i = 0$, and thus rewrite p as $\alpha_s \lambda_1^{s_1} \cdots \lambda_m^{s_m} + 0$, where $s_m \neq 0$. The algebraic analog of extracting a primitive is to take the semiring obtained by identifying the two monomials of any binomial p . In order to do this, we replace λ_m by $(\alpha_s)^{-1/s_m} \lambda_1^{-s_1/s_m} \cdots \lambda_{m-1}^{-s_{m-1}/s_m}$. Performing this elimination process the same way that one reduces indeterminates in linear algebra via Gauss-Jordan elimination, effectively reduces the number of indeterminates.

We believe that this is the “correct” way to view tropical geometry in terms of polynomials.

Example 3.8. Let $f = \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_1 + \lambda_2 + 0 = (\lambda_1 + \lambda_2 + 0)(\lambda_1 + 0)$. Extracting a primitive from $H = Z(f)$, corresponds to cancelling the binomial factor $p = \lambda_1 + 0$ from f to get $\tilde{H} = Z(\lambda_1 + \lambda_2 + 0)$.

Similarly, the reduction \tilde{W} of a k -dimensional tropical variety W is obtained by discarding all possible primitive of dimension k , and

$$(8) \quad W \equiv W' \text{ modulo primitives} \iff \tilde{W} = \tilde{W}'.$$

Namely, each reduced tropical variety stands for a class of varieties. (Note that $\tilde{W} = \bigcap \tilde{H}_i$ is not the reduction of the tropical variety $W = \bigcap H_i$.)

Remark 3.9. In view of Lemma 3.6, $\tilde{W} = W$ for any irreducible, or primitively irreducible, tropical variety $W = \bigcap H_i$.

Definition 3.10. A **primitive covering** of a tropical set $S \subset \mathbb{R}^{(n)}$ is a finite collection of k -dimensional tropical primitives $\mathcal{P}(S) = \{P_i : P_i \subset \mathbb{R}^{(n)}\}$ whose union contains S . Denoting the cardinality of $\mathcal{P}(S)$, counting multiplicities, by $|\mathcal{P}(S)|$, we say that $\mathcal{P}(S)$ is a **minimal covering** of S if $|\mathcal{P}(S)|$ is minimal over all the possible covers of S .

(A primitive cover may contain overlapping primitives; in this case the primitives are counted with their multiplicities.)

Clearly, any tropical set $S \subset \mathbb{R}^{(n)}$ has a primitive cover, where $|\mathcal{P}(S)| \leq$ the sum of all the multiplicities of faces of S . For a k -dimensional tropical variety W , this upper bound can be reduced to

$$|\mathcal{P}(W)| \leq \sum_{\delta} m(\delta),$$

(the operations here are the standard operations) where δ runs over all the k -dimensional faces of W . Yet, this naive upper bound often can be reduced much further.

3.2. Starred varieties. Among tropical varieties we identify a special family with a nice behavior which is much easier to analyze.

Definition 3.11. A tropical variety $W \subset \mathbb{R}^{(n)}$ is called **starred** if it has a single bottom-dimensional face.

Accordingly, a tropical variety that has a primitive cover, all of whose elements intersect at a single face, is starred. (This definition also includes cases in which varieties, or hypersurfaces, do not have a proper 0-dimensional face; for example $H = Z(f)$, with $f = \lambda_1 + 0$ in $\tilde{\mathbb{T}}[\lambda_1, \lambda_2]$, is starred of bottom-dimension 1.)

Example 3.12. The following are straightforward examples of starred varieties in $\mathbb{R}^{(n)}$:

- (1) A tropical hyperplane (thereby permitting one to use starred varieties to answer questions raised in Passare’s talk cited above),
- (2) A tropical primitive,

- (3) A tropical curve having a single vertex,
- (4) Example 3.24 below.

Locally, any tropical algebraic set $S \subset \mathbb{R}^{(n)}$ is a starred variety. When a local neighborhood contains points of only one face of S , then, locally, S is a tropical primitive.

Lemma 3.13. *Any tropical k -dimensional starred variety $W = \bigcap H_i$ is the intersection of tropical starred hypersurfaces.*

Proof. Let τ be the single bottom-dimension face of $W = \bigcap H_i$, where $H_i = Z(f_i)$. Then, $\tau \subset \bigcap_j \delta_{i,j}$, where $\delta_{i,j}$ are top-dimensional faces of H_i , each determined by a pair of monomials $f_{i,j}$ and $f_{i,k}$ of f_i . In case one of the H_i is not starred, one can replace it by the hypersurface determined by the pairs of monomials corresponding to the top-dimensional faces $\delta_{i,j}$ (and discarding all the other monomials of f_i). \square

3.3. Tropical hypersurfaces and subdivisions of their Newton polytopes. The convex hull Δ of the set Ω in the formula (2) (or, equivalently, in formula (3)) for a tropical polynomial f is called the **Newton polytope of f** . The Legendre dual to f is a convex piece-wise linear function $\nu_f : \Delta \rightarrow \mathbb{R}$, whose maximal linear domains form a subdivision

$$(9) \quad S(f) : \Delta = \Delta_1 \cup \dots \cup \Delta_N$$

into convex lattice polytopes of dimension $\dim \Delta_i = \dim \Delta$, $i = 1, \dots, N$. The vertices of the subdivision $S(f)$ bijectively correspond to the essential monomials of f ; in particular, the vertices of Δ always correspond to essential monomials of f . A subdivision $S(f)$ is called an **empty subdivision** if it has no interior vertices, i.e. vertices which are not vertices of Δ .

There is the following combinatorial duality, between the finite polyhedral complexes which inverts the incidence relation: Δ , covered by the faces of the subdivision $S(f)$, and $\mathbb{R}^{(n)}$, covered by the faces of the hypersurface $Z(f)$ and by the closures of the components of $\mathbb{R}^{(n)} \setminus Z(f)$.

Namely:

- (a) The vertices of $S(f)$ are in one-to-one correspondence with the components of $\mathbb{R}^{(n)} \setminus Z(f)$, so that the vertices of $S(f)$ on $\partial\Delta$ correspond to unbounded components, and the other vertices of $S(f)$ correspond to bounded components;
- (b) A k -dimensional face of $S(f)$, $k \geq 1$, corresponds to an $(n - k)$ -dimensional face of $Z(f)$, and they are orthogonal to each other.

A tropical hypersurface $Z(f)$ considered as a tropical variety (i.e., equipped with weights) determines the Newton polytope Δ and its subdivision $S(f)$ uniquely, up to translation in $\mathbb{R}^{(n)}$, and determines the essential part \tilde{f} (i.e., the sum of the essential monomials) of the tropical polynomial f up to multiplication by a monomial; therefore,

$$S(f) = S(\tilde{f}).$$

On the other hand, as a polynomial, f determines the Newton polytope uniquely. Without weights, $Z(f)$ determines the combinatorial type of Δ and of its subdivision, together with the slopes of all the faces of $S(f)$.

Note 3.14. *Accordingly:*

- (1) *Given a polynomial $f \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ whose tropical hypersurface $Z(f)$ is of bottom dimension k , then its Newton polytope Δ is of dimension $n - k$.*
- (2) *The Newton polytope Δ of an essential binomial $p \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ is simply a line segment in $\mathbb{R}^{(n)}$ with empty subdivision $S(p)$.*
- (3) *$Z(f)$ is starred iff the subdivision $S(f)$ of the Newton polytope Δ of f is empty.*
- (4) *If Δ has empty subdivision $S(f)$ and there exists $(n - 1)$ -plane cut π of Δ where all the 1-dimensional faces intersecting transversally with π are parallel, then $Z(f)$ contain a primitive.*

Abusing language slightly, for a tropical hypersurface $H = Z(f)$, we sometimes say that Δ is the Newton polytope of H , and their faces are said to be dual in the sense described above.

One approach to define the weights $m(\delta)$ of the top-dimensional faces δ of a tropical hypersurface is by taking the integral lengths of their dual one-dimensional faces in the subdivision of the corresponding Newton polytope. For $(n - 1)$ -dimensional faces these integral lengths, which are equal to the number of lattice points on the dual faces minus 1, and satisfy the balancing condition (7).

Remark 3.15. *This setting, in which weights are defined using integral lengths, is canonical. Namely, the weights of the top-dimensional faces of a tropical hypersurface H are independent of its polynomial description; that is, even if $H = Z(f) = Z(g)$, where $f \neq g$, yet each top-dimensional face $\delta \subset H$ has the same weight. (Note that f and g need not to be e -equivalent.)*

For example, take a polynomial $f = gh$, where h is a monomial. Then, $f \approx g$. On the other hand, $Z(f) = Z(g)$, which implies that the Newton polytope Δ of f is an integral linear translation of Δ' , the Newton polytope of g ; thus, both Δ and Δ' determine the same weights for the top-dimensional faces of $Z(f)$.

Having the same weight setting as in Remark 3.15, let $P = Z(p)$ be a tropical primitive, where $p \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ is an essential binomial. Assume that p is rewritten as

$$p = \alpha_k \Lambda^k (\alpha_i \Lambda^i + \alpha_j \Lambda^j)^m$$

with maximal possible $m \in \mathbb{N}$, then the weight of P equals m .

Example 3.16. *Recall the Frobenius rule $f^m = \sum_i f_i^m$, for any polynomial $f = \sum_i f_i$ with monomials f_i in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, cf. [5, Theorem 2.40]. Let $p = \lambda_1^m \lambda_2^{m+j} + \lambda_2^j$, with $m, j \in \mathbb{N}$. Then*

$$p = \lambda_2^j (\lambda_1^m \lambda_2^m + 0) = \lambda_2^j (\lambda_1 \lambda_2 + 0)^m,$$

and thus p has multiplicity m .

Therefore, any primitive cover can be formed as a union of tropical primitives, each of multiplicity $m(P)$. Accordingly, we can write

$$|\mathcal{P}(W)| = \sum_P m(P),$$

where $P = Z(p)$ runs over all the primitives of $\mathcal{P}(W)$ and $m(P)$ are their multiplicities as defined above.

3.4. Supplements of tropical varieties.

Definition 3.17. *A **supplement** of a k -dimensional tropical variety W is a tropical variety W^{spl} of dimension k , whose weighted union with W produces a primitive cover of W , denoted by $\mathcal{P}(W)^{\text{spl}}$, i.e.*

$$(10) \quad W \bigcup^w W^{\text{spl}} \stackrel{w}{=} \mathcal{P}(W)^{\text{spl}},$$

*is called the **completion** of W . The supplement (resp. completion) is said to be a **pure supplement** (resp. completion) when the weighted union is a semidisjoint weighted union. We say that a supplement (resp. completion) is minimal when $|\mathcal{P}(W)^{\text{spl}}|$ is minimal possible.*

Since the union is a weighted union, a primitive cover by itself cannot be the supplement of a tropical variety unless it is a union of primitives. Conversely, the minimal supplement of a tropical primitive is the empty set.

Note that the supplement of a k -dimensional tropical variety $W \subset \mathbb{R}^{(n)}$ is not its set-theoretic complement in the primitive cover $\mathcal{P}(W)$, since the two sets are not disjoint. In fact, $W \cap W^{\text{spl}} \neq \emptyset$ is a collection of faces of dimension $\leq k$.

As will be seen later, the supplement of a tropical hypersurface H need not be of the same type as that of H . For example, the supplement of a tropical hyperplane is not a hyperplane. Moreover, the supplement of an irreducible hypersurface may be reducible, also they might have different combinatorial types.

Lemma 3.18. *The minimal supplement of a tropical variety W is unique.*

Proof. First assume that W is a tropical hypersurface H . Assume H_1^{spl} and H_2^{spl} are two different minimal supplements of a tropical hypersurface H , and let $\mathcal{P}_i(H)$ denote the corresponding primitive coverings $H \overset{w}{\cup} H_i^{\text{spl}}$, $i = 1, 2$. Then, without weights $\mathcal{P}_1(H) = \mathcal{P}_2(H)$; otherwise, one of the primitive coverings has a primitive which does not contain a face of H . So, $\mathcal{P}_1(H)$ and $\mathcal{P}_2(H)$ have a common primitive with unequal weight; say $m_1 > m_2$ respectively. But, then one can extract a primitive from $\mathcal{P}_1(H)$, thereby contradicting its minimality.

In general, the case of tropical variety $W = \bigcap H_i$ apply the same argument to possible k -dimensional primitive coverings of W . \square

We write $W^{\text{min-spl}}$ for the minimal supplement of a tropical variety W .

Lemma 3.19. *If $H^{\text{min-spl}}$ is the minimal supplement of a primitively irreducible undersurface H , then:*

- (1) H is the minimal supplement of $H^{\text{min-spl}}$.
- (2) $(H^{\text{min-spl}})^{\text{min-spl}} = H$.

Proof. Follows directly from the uniqueness of the minimal supplement. \square

Corollary 3.20. *Assume $W = \bigcap H_i$ is a tropical variety, where $H_i \subset \mathbb{R}^{(n)}$ are tropical primitively irreducible hypersurfaces. Then, $(W^{\text{min-spl}})^{\text{min-spl}} = W$.*

When $H^{\text{spl}} = Z(g)$, for some $g \in \mathbb{T}[\lambda_1, \dots, \lambda_n]$, is the supplement of a tropical hypersurface $H = Z(f)$, we also say that g is a supplement of f and denote it f^{spl} . (Note that f^{spl} need not to be unique.)

Theorem 3.21. *Any tropical hypersurface $H \subset \mathbb{R}^{(n)}$ has a supplement, $H^{\text{spl}} \subset \mathbb{R}^{(n)}$, which is also a tropical hypersurface; when H is generic, then its supplement is pure.*

Proof. Write $H = Z(f)$ for some $f = \sum f_i$ in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ and apply Corollary 1.9. Denoting $\bar{f} = \prod_{i < j} (f_i + f_j)$, it is clear that $Z(\bar{f}) = \mathcal{P}(H)$ is a primitive cover of H , explicitly, $P_{i,j} = Z(f_i + f_j)$, $\mathcal{P}(H) = \bigcup P_{i,j}$. Let

$$(11) \quad g = \left(\sum_{i < j} f_i f_j \right) \cdots \left(\sum_i \prod_{j \neq i} f_j \right),$$

and take $G = Z(g)$, clearly a tropical hypersurface. Using Equation (5), we have $fg = h$ and thus $Z(f) \overset{w}{\cup} Z(g) \overset{w}{=} Z(h)$. Namely $H^{\text{spl}} = G$ is a supplement of H .

Assume that H is generic. Thus, on each primitive $P_{i,j}$ of the primitive cover $\mathcal{P}(H)$, H has at most one top-dimension face δ , i.e. $P_k \setminus \delta \subset H^{\text{spl}}$. So, all the intersections of H and H^{spl} are of dimension $< (n - 1)$. \square

Example 3.22. *The supplement (11) of a tropical hypersurface $H = Z(f) \subset \mathbb{R}^{(n)}$ whose Newton polytope Δ is a simplex (and thus has empty subdivision) is a minimal pure supplement.*

Indeed, Δ has $n + 1$ vertices, each corresponding to a monomial of f , and exactly $\binom{n+1}{2}$ 1-dimensional faces, dual to the top-dimensional faces of H . On the other hand, the primitive cover of H consists of $\binom{n+1}{2}$ primitives (not counting multiplicities) which are determined by the pairs of different monomials of f , cf. Theorem 0.1 (1). Thus, $H^{\text{spl}} = Z(f^{\text{spl}})$ is a pure supplement.

To see that $\mathcal{P}(H) = H \overset{w}{\cup} H^{\text{spl}}$ is the minimal cover, just note, by construction, that each primitive has the same multiplicity as the top-dimensional face it covers.

Corollary 3.23. *Assume $W = \bigcap H_i$ is a tropical variety, where $H_i \subset \mathbb{R}^{(n)}$ are tropical hypersurfaces. Then, $W^{\text{spl}} = \bigcap H_i^{\text{spl}}$.*

Proof. Each top-dimensional face δ of W is the intersection of top-dimensional faces δ_i of H_i contained in some k -dimensional primitive $P = \bigcap_i P_i$ with $\delta_i \subset P_i$. The supplement of each δ_i is also in P_i and thus their intersection is contained in P . \square

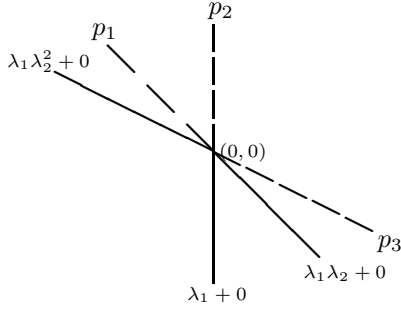


FIGURE 1. Illustration for Example 3.24.

3.5. **Examples.** In this subsection we present a few examples of typical planar supplements.

Example 3.24. A tropical planar curve with a single node. Let $C = Z(f)$, where $f = \lambda_1^2 \lambda_2 + \lambda_1 + 0$. Take $f^{\text{sp1}} = \lambda_1^3 \lambda_2 + \lambda_1^2 \lambda_2 + \lambda_1$. Then $C^{\text{sp1}} = Z(f^{\text{sp1}})$ is a supplement (and also a point symmetry, as explained below) of C along $(0, 0)$. The primitive cover is determined by the binomials $p_1 = \lambda_1^2 \lambda_2 + 2$, $p_2 = \lambda_1^2 \lambda_2 + 0$, $p_3 = \lambda_1 + 0$, yielding the equality

$$f f^{\text{sp1}} = p_1 p_2 p_3 = \lambda_1 + \lambda_1^2 + \lambda_1^2 \lambda_2 + \lambda_1^3 \lambda_2 + \lambda_1^4 \lambda_2 + \lambda_1^4 \lambda_2^2 + \lambda_1^5 \lambda_2^2.$$

See Fig. 1, where the dashed lines correspond to C^{sp1} and the solid lines correspond to C . This is a pure supplement which is the minimal supplement.

Example 3.25. (see Fig. 2). A tropical conic with two vertices.

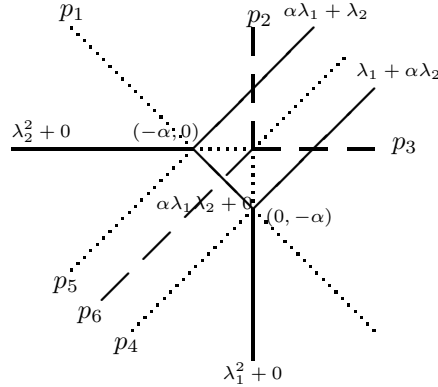


FIGURE 2. Illustration for Example 3.25.

Let $C = Z(f)$, where $f = \lambda_1^2 + \lambda_2^2 + \alpha \lambda_1 \lambda_2 + 0$, with $\alpha > 0$. The supplement of f consists of two components, drawn in dashed and dotted lines for $Z(f^{(2)})$ and $Z(f^{(2)})$, respectively. The primitive cover is determined by the following binomials (which are obtained by taking the sums of all pairs of monomials of f): $p_1 = \alpha \lambda_1 \lambda_2 + 0$, $p_2 = \lambda_1^2 + 0$, $p_3 = \lambda_2^2 + 0$, $p_4 = \lambda_1 + \alpha \lambda_2$, $p_5 = \alpha \lambda_1 + \lambda_2$, and $p_6 = \lambda_1 + \lambda_2$. The supplement here is pure but not minimal.

Example 3.26. Let $C = Z(f)$, $f = \lambda_1^2 \lambda_2^2 + \alpha \lambda_1 \lambda_2 + \lambda_1 + \lambda_2$, with $\alpha > 0$, be a (generic) tropical curve of genus 1 (see Fig. 3). The supplement of f consists of two components (drawn in dashed and dotted lines). The supplement again consists of six primitives; thus the supplement is minimal (and also pure).

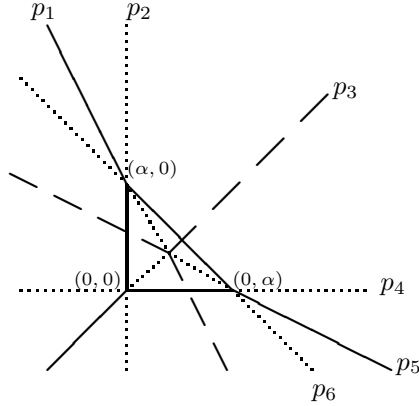


FIGURE 3. Illustration for Example 3.26.

3.6. The reduced completion. Once we have specified a supplement H^{spl} of hypersurfaces, with an explicit algebraic description, cf. Theorem 3.21 we define the **reduced supplement** $\widetilde{H}^{\text{spl}}$ by taking the primitive reduction of H^{spl} . In this sense, for a tropical variety W , we minimize $|\mathcal{P}(W)^{\text{spl}}|$ as much as possible. The weighted $H \cup^w \widetilde{H}^{\text{spl}}$ union is called **reduced completion** of H .

Remark 3.27. *In the case of a hypersurface (the algebraic set of a polynomial f), its reduced completion is the algebraic set of the reduced closure of f .*

Claim 3.28. $\widetilde{H}^{\text{spl}}$ is the minimal supplement of the tropical surface H .

Proof. The primitive reduction procedure discards only primitives, whose set-theoretic completions are always the empty set, so $\widetilde{H}^{\text{spl}}$ is a completion as well. To see that it is minimal, apply Lemma 3.18. \square

Corollary 3.29. *The minimal supplement of a tropical variety $W = \bigcap_i H_i$ is $\widetilde{W}^{\text{spl}} = \bigcap \widetilde{H}^{\text{spl}}$.*

Proof. Immediate from Corollary 3.23. \square

Remark 3.30. *Suppose H is a primitively irreducible surface, i.e., $H = \widetilde{H}$. Denote the minimal primitive cover correspond to $\widetilde{H}^{\text{spl}}$ by $\widetilde{\mathcal{P}}(H)^{\text{spl}}$. Since $\widetilde{H}^{\text{spl}}$ is the minimal supplement of H , the multiplicity of each primitive P in $\widetilde{\mathcal{P}}(H)^{\text{spl}}$ is equal to the sum of the weights of the top-dimensional overlapping faces of H and $\widetilde{H}^{\text{spl}}$ that P covers.*

Locally, any tropical variety $W \subset \mathbb{R}^{(n)}$ can be viewed as a starred variety. Given a point $\mathbf{a} \in W$, taking a small enough neighborhood $B(\mathbf{a}) \subset \mathbb{R}^{(n)}$ of \mathbf{a} and restricting W to $B(\mathbf{a})$, one can see that locally W is either a starred or a primitive variety. The latter situation is trivial, and we are mostly interested in the starred varieties.

Claim 3.31. *The reduced completion of a tropical starred variety $W \subset \mathbb{T}^{(n)}$ is a pure completion.*

Proof. Clear from the fact that H is starred. \square

Let $\tau \subset H$ be a bottom-dimensional face, $\mathbf{a} \in \tau$ a point, and $B(\mathbf{a})$ a small neighborhood. Denoting the restriction of H to $B(\mathbf{a})$ by H_τ ; then H_τ is a starred hypersurface of the same bottom-dimension as H . Let H_τ^{spl} be its completion. Constructing the supplement locally and viewing it in $\mathbb{R}^{(n)}$, we have the following identification:

Theorem 3.32. *The global reduced supplement of a generic tropical hypersurface H is equal to the primitive reduction of the weighted union of the local supplements along its bottom-dimension faces, i.e.*

$$\widetilde{H}^{\text{spl}} \stackrel{w}{=} \bigcup_{\tau}^w H_\tau^{\text{spl}}$$

where τ runs over all the bottom-dimensional faces of H .

Proof. Each of the faces of H contains at least one of the bottom-dimensional faces τ ; thus, it enough to take the supplement along these faces to get a completion of H . Taking the primitive reduction of $\bigcup_{\tau}^w H_{\tau}^{\text{spl}}$ we get a minimal supplement of H , which is unique by Lemma 3.18 and thus equal to $\widetilde{H}^{\text{spl}}$; cf. Claim 3.28. \square

Given a bottom-dimensional face τ of a tropical variety $W = \bigcap_i H_i$, locally $W_{\tau} = \bigcap_{i,\sigma} H_{i,\sigma}$, where σ is a top-dimensional face of H_i that contains τ . Combining Corollary 3.23 and Theorem 3.32 we conclude:

Corollary 3.33. *Given a generic tropical variety $W = \bigcap_i H_i$, then*

$$\widetilde{W}^{\text{spl}} \stackrel{w}{=} \bigcup_{\tau}^w W_{\tau}^{\text{spl}},$$

where τ runs over all the bottom-dimensional faces of W .

3.7. Supplemental duality. A dual correspondence for tropical hypersurfaces is established by taking the reduced supplement.

Theorem 3.34. *The reduced supplement of a tropical hypersurface admits a duality; i.e., for any $H \subset \mathbb{R}^{(n)}$*

$$(12) \quad \left(\widetilde{H}^{\text{spl}} \right)^{\text{spl}} \stackrel{w}{=} \widetilde{H}.$$

Proof. We may assume that H is primitively irreducible. $\widetilde{H}^{\text{spl}}$ is the minimal supplement of H ; cf. Claim 3.28.

Conversely, H is minimal supplement of $\widetilde{H}^{\text{spl}}$, cf. Lemma 3.19. On the other hand $\widetilde{H}^{\text{spl}}$ is also minimal supplement of $\widetilde{H}^{\text{spl}}$, which is unique, cf. Lemma 3.18. \square

Example 3.35. *The dual curves for the Examples 3.24 and 3.26 are precisely their supplements, which we recall are minimal; see Figs 1 and 3 respectively. The dual curve of Example 3.25 is obtained by extracting $Z(p_6)$ from the supplement of $Z(f)$, drawn in dotted and dashed lines, see Fig 2.*

Corollary 3.36. *Given a tropical variety $W = \bigcap_i H_i$ then $\left(\widetilde{W}^{\text{spl}} \right)^{\text{spl}} \stackrel{w}{=} \widetilde{W}$.*

We call the relation in Corollary 3.36, the **supplemental duality** of tropical varieties. This duality is quite general; note that although the dual of a variety has the same dimension, a variety need not to be of the same type as its dual. For example:

- The dual of an irreducible variety might be irreducible, or vice versa, cf. Example 3.25;
- The dual of a curve of genus 1 (which, in tropical sense, is not a rational variety) can be a rational curve, cf. Example 3.26.
- a tropical variety and its dual might be of different combinatorial types, cf. Example 3.26.

4. THE REVERSAL ISOMORPHISM AND ITS GEOMETRIC INTERPRETATION

The supplement can be understood better from the decomposition of Formula (5), by means of another algebraic tool. Before introducing this tool, we pass to a more convenient semiring than the polynomial ring. The motivation is that the monomial $\alpha_i \Lambda^i$ has no tropical roots other than $-\infty$. Thus, when considering nonzero roots (or when studying projective tropical geometry) one could multiply or divide the polynomials defining the variety by powers of the λ_i without affecting the variety. This observation often enables us to “clean up” some of the computations, by erasing powers of the λ_i that arise for example in Remark 1.10(ii),(iii).

Accordingly, it is just as natural to work with the semiring $\mathbb{T}[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}]$ of **Laurent polynomials**, defined just as polynomials except that powers of the λ_i may be taken to be negative integers as well. Given any Laurent polynomial, denoted $f(\lambda_1, \dots, \lambda_n)$, one can define the natural substitution $f(\mathbf{a})$, for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{T}^{(n)}$. (Indeed, $a_k^{-i_k}$ can be viewed as the inverse of $a_k^{i_k}$.)

Remark 4.1. *There is a natural isomorphism, which we denote as*

$$(13) \quad * : \mathbb{T}[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}] \longrightarrow \mathbb{T}^*[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}],$$

given by $\alpha \mapsto \alpha^{-1}$ for each $\alpha \in \mathbb{T}$ and $\lambda_i \mapsto \lambda_i^{-1}$ for each i . (Thus, for any monomial h , $h^* = \frac{1}{h}$.) We call f^* the **reversal** of f . Clearly, by definition, $(f^*)^* = f$, so we have a duality, which also induces a duality of the geometry.

To understand the connection between the algebra and the geometry here, we note that $*$ reverses the order of values in the monomials, and thus switches (max, plus) with (min, plus).

Given a finitely generated ideal $I = \langle f_1, \dots, f_m \rangle$, we write I^* for $\langle f_1^*, \dots, f_m^* \rangle$ and call it the **reversal ideal** of I .

When $f = \sum f_i$, the product of the f_i 's is denoted

$$(14) \quad N(f) = \prod_i f_i .$$

Example 4.2. *If $f = f_i + f_j$ for monomials f_i, f_j , we have $(f_i + f_j)^* = f_i^* + f_j^* = \frac{1}{f_i} + \frac{1}{f_j} = \frac{f_i + f_j}{f_i f_j}$, which has the same variety as f . Thus, the reversal of a binomial has the same variety as the binomial.*

More generally, writing $f = \sum_{i=1}^m f_i$ as a sum of monomials f_i , recalling the notation

$$f^{\text{trn}} = f^{(m-1)} = \sum_i \prod_{j \neq i} f_j, \quad \bar{f} = \text{per}(V_f) = \prod_{j < i} (f_i + f_j),$$

we have

$$\begin{aligned} \bar{f}^* &= \prod_{i \neq j} (f_i^* + f_j^*) = \prod_{i \neq j} \frac{f_i + f_j}{f_i f_j} = \frac{\bar{f}}{N(f)^{m-1}} ; \\ f^{(m-u)} &= \prod \frac{N(f)}{f_{i_1} \cdots f_{i_u}} = N(f)^{\binom{m}{u}} (f^*)^{(u)} \quad \forall u ; \\ f^{\text{trn}} &= f^{(m-1)} = N(f) f^* ; \\ \overline{f^{\text{trn}}} &= \prod_{i \neq j} \left(\sum \frac{N(f)}{f_i} \right) = N(f)^{\frac{n(n-1)}{2}} \bar{f}^* . \end{aligned}$$

To understand the connection between the algebra and the geometry here, we note that $*$ reverses the order of values in the monomials; thus, $Z(f^*) = Z(f^{(m-1)})$.

Definition 4.3. *The **reversal** of a tropical hypersurface $H = Z(f)$ is defined as $H^{\text{rvl}} = Z(f^*)$, with f^* as defined above. The reversal W^{rvl} of a tropical variety $W = \bigcap H_i$ is then $W^{\text{rvl}} = \bigcap H_i^{\text{rvl}}$.*

Note that reversals are defined for any tropical variety W , not necessarily for starred or irreducible varieties. In particular, if W is a tropical primitive, then $W = W^{\text{rvl}}$ and W is self dual. From Remark 4.1 we can conclude immediately:

Corollary 4.4. *If H is a tropical surface, then $(H^{\text{rvl}})^{\text{rvl}} = H$. When $W = \bigcap H_i$ is a tropical variety, $(W^{\text{rvl}})^{\text{rvl}} = W$.*

We call the relation in Corollary 4.4, the **reversal duality** of tropical varieties. Clearly, for W and W^{rvl} we have the following properties:

- W and W^{rvl} are isomorphic of the same dimension,
- they are of a same combinatorial type,
- the weights of top-dimensional reversal faces of W and W^{rvl} are equal.

Remark 4.5. *In Remark 1.6, we considered $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n] \times \tilde{\mathbb{T}}^*[\lambda_1, \dots, \lambda_n]$. In light of Remark 4.1, to see the entire picture, one should view this in $\tilde{\mathbb{T}}[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}] \times \tilde{\mathbb{T}}^*[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}]$, in which the isomorphism $*$ becomes an automorphism of degree 2.*

For deformations of surfaces, a geometric approach to duality has been suggested recently by Nisse [12].

5. SYMMETRY OF VARIETIES

We call a k -dimensional plane (in the classical sense) in $\mathbb{R}^{(n)}$, for short, k -plane. Let us state the definition of symmetry which is used in this paper:

Definition 5.1. A set $S \subset \mathbb{R}^{(n)}$ is said to be **point symmetric** if there exists a point $\mathbf{o} \in \mathbb{R}^{(n)}$ for which, in the standard notation,

$$(15) \quad \mathbf{a} \in S \implies 2\mathbf{o} - \mathbf{a} \in S,$$

for any $\mathbf{a} \in S$.

A set is **k -plane symmetric**, $k < (n - 1)$, if all of its restrictions to $(n - k)$ -planes orthogonal to a fixed k -plane π are point symmetric (with respect to a point $\mathbf{o} \in \pi$). We say that a set S^{sym} is a k -plane **symmetry** of S if their union is k -plane symmetric.

Let us describe explicitly the action of the isomorphism (13) on a monomial $f_i = \alpha_i \Lambda^{\mathbf{i}}$:

$$(16) \quad f_i^{-1}(\mathbf{a}) = \frac{1}{\alpha_i \mathbf{a}^{\mathbf{i}}} = \frac{f_i(\mathbf{a}^{-1})}{\alpha_i^2}.$$

Tropically, we also write \mathbf{a}^2 for (a_1^2, \dots, a_n^2) and have the relation

$$(17) \quad f_i(\mathbf{a}^2) = \alpha_i \mathbf{a}^{2\mathbf{i}} = \frac{f_i(\mathbf{a})^2}{\alpha_i},$$

for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{(n)}$.

Lemma 5.2. Let $H = Z(f) \subset \mathbb{R}^{(n)}$, $f = \sum_i \alpha_i \Lambda_i$, be a primitively irreducible starred hypersurface of bottom dimension 0. Then $H^{\text{trn}} = Z(f^{\text{trn}})$ is the point symmetry of H , and vice versa.

Proof. Let $\mathbf{o} \in Z(f)$, $f = \sum f_i$, be the face of bottom dimension 0, in particular

$$(18) \quad f(\mathbf{o}) = f_1(\mathbf{o}) = \dots = f_m(\mathbf{o}) = c.$$

Assume $\mathbf{a} \in Z(f)$, then $f(\mathbf{a}) = f_i(\mathbf{a}) = f_j(\mathbf{a})$ for some i and j . Since $Z(f^{\text{trn}}) = Z(\sum_i f_i^{-1})$, we can rewrite condition (15) tropically, and need to prove $\frac{\mathbf{o}^2}{\mathbf{a}} \in Z(\sum_i f_i^{-1})$. Indeed, using Equations (16) and (17) we have,

$$\sum_i f_i^{-1}\left(\frac{\mathbf{o}^2}{\mathbf{a}}\right) = \sum_i \alpha_i^{-2} f_i\left(\frac{\mathbf{a}}{\mathbf{o}^2}\right) = \sum_i \alpha_i^{-1} \frac{f_i(\mathbf{a})}{f_i(\mathbf{o}^2)} = \sum_i \frac{f_i(\mathbf{a})}{f_i(\mathbf{o})^2}.$$

But, $f_i(\mathbf{o})^2 = c^2$ for each i , cf. Equation (18), and this completes the proof. \square

Theorem 5.3. The primitively irreducible tropical hypersurface $H^{\text{trn}} = Z(f^{\text{trn}})$ is the k -plane symmetry of a tropical starred hypersurface $H = Z(f)$, i.e. $H^{\text{trn}} = H^{\text{sym}}$.

Proof. H is starred, thus has a single face $\tau \subset H$ of bottom dimension, which is a k -plane. Consider its orthogonal $(n - k)$ -planes, and apply Lemma 5.2 to the restrictions of H to these $(n - k)$ -planes. \square

Note 5.4. In the case of $\mathbb{R}^{(2)}$, for starred curves, the minimal supplement and the point symmetry coincide, and thus by Theorem 5.3 we provide the explicit polynomial that determines the minimal pure supplement for this class of curves, i.e. $C^{\text{spl}} = Z(f^{\text{trn}})$. But, in dimension 3 or higher, H^{sym} is purely contained in H^{spl} even for the simplest case of a non-degenerate tropical hyperplane (i.e. non-primitive hyperplane).

Corollary 5.5. If $W = \bigcap H_i$ is a tropical starred variety, where H_i are tropical hypersurfaces, then $W^{\text{sym}} = \bigcap H_i^{\text{sym}}$.

Proof. The bottom-dimensional face of W is contained in the intersection of the bottom-dimensional faces of H_i . Apply Theorem 5.3 to each H_i , and consider the intersection of their symmetries. \square

Corollary 5.6. Suppose W is a tropical starred variety, then $(W^{\text{sym}})^{\text{sym}} = W$.

We call the identification in Corollary 5.6 the **symmetry duality** of starred tropical varieties; for this duality we have the following properties:

- W and W^{sym} are isomorphic of the same dimension,
- W and W^{sym} are of the same combinatorial type,
- the weights of top-dimensional symmetric faces of W and W^{sym} are equal.

(The symmetry duality need coincide with the supplemental duality only in the case of starred varieties in $\mathbb{R}^{(2)}$.)

By proving Theorem 5.3 we have also proved the following identity of polynomials:

Theorem 5.7. *Suppose $f = \sum_{i=1}^m f_i$ is a polynomial in $\tilde{\mathbb{T}}[\lambda_1, \lambda_2]$ with monomials f_i whose Newton polytope Δ has empty subdivision. Then*

$$(19) \quad f f^{\text{trn}} = \left(\sum f_i \right) \left(\sum_i \prod_{j \neq i} f_j \right) = (f_1 + f_2)(f_2 + f_3) \cdots (f_{m-1} + f_m)(f_m + f_1),$$

where the f_i 's are ordered according the order of the corresponding vertices on the Newton polytope of f .

Here, the Newton polytope Δ is a polygon whose vertices all lie on the boundary $\partial\Delta$ of Δ , corresponding to monomials f_i of f . These vertices, and respectively the f_i 's, can be labelled according to their order on $\partial\Delta$. So, in the theorem, any pair f_i and f_{i+1} correspond to adjacent vertices on $\partial\Delta$.

6. SYMMETRY OF LATTICE POLYTOPES

A lattice polytope Δ is a polytope whose vertices are lattice points of a lattice Σ . Assume $\Sigma = \mathbb{Z}^{(n)}$ is a lattice embedded in $\mathbb{R}^{(n)}$, and Δ is a lattice polytope on Σ ; often an integral linear translation, we may assume that Δ is a lattice polytope on $\mathbb{N}^{(n)} \hookrightarrow \mathbb{R}_+^{(n)}$. So we may assume that $\Sigma = \mathbb{N}^{(n)}$.

Given a lattice polytope $\Delta \in \mathbb{N}^{(n)}$, one can assign to Δ a tropical polynomial $f = \sum_i f_i$ in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, whose Newton polytope is Δ . Indeed, to any vertex $\mathbf{v} = (v_1, \dots, v_n)$ of Δ assign the monomial $f_i = \lambda_1^{v_1} \cdots \lambda_n^{v_n}$. Thus, a lattice polytope can be regraded as a Newton polytope (with empty subdivision).

Assume $Z(f) = H \subset \mathbb{R}^{(n)}$ is a primitively irreducible tropical starred hypersurface. When $n = 2$, the Newton polytope Δ of f does not contain any parallel edges; otherwise by the duality between Δ and H , the latter must have a primitive factor. In the general case of $\mathbb{R}^{(n)}$, the edges of Δ that intersect transversally with a hyperplane cut of Δ are not all parallel.

Due to duality between tropical hypersurfaces and their Newton polytope, the point symmetry of a primitively irreducible tropical starred hypersurface also induces a symmetry of the corresponding Newton polytope. Moreover, in the case of Newton polytopes, the symmetry is always a point symmetry.

Theorem 6.1. *Let Δ be the Newton polytope of a primitively irreducible polynomial $f \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, assume Δ has empty subdivision, and let Δ^{trn} be the Newton polytope of f^{trn} . Then Δ^{trn} is a point symmetry of Δ , and vice versa.*

Note that the point-symmetry need not be along a lattice point, and might be along any point $\mathbf{o} \in \mathbb{R}^{(n)}$.

Proof. We prove the theorem for the boundary $\partial\Delta$, which is enough since Δ is convex. f and f^{trn} , in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, determine the Newton polytopes Δ , and Δ^{trn} uniquely. $H = Z(f)$ is dual to Δ and H^{trn} is dual to Δ^{trn} . The symmetry from H to H^{trn} , cf. Theorem 5.3, completes the proof. \square

Corollary 6.2. $(\Delta^{\text{trn}})^{\text{trn}} = \Delta$ and thus the relation in Theorem 6.1 induces a duality of lattice polytopes.

Remark 6.3. *Theorem 6.1 can be extended to polytopes of the same type on $\mathbb{Z}^{(n)}$. In this case the assigned polynomials are Laurent polynomials, i.e. tropical polynomials over $\tilde{\mathbb{T}}[\lambda_1, \lambda_1^{-1}, \dots, \lambda_n, \lambda_n^{-1}]$, whose Newton polytopes are lattice polytopes over $\Sigma = \mathbb{Z}^{(n)}$.*

Lemma 6.4. *Any integral translation of Δ^{sym} on Σ is also a symmetry of Δ .*

Proof. Assume Δ is the Newton polytope of $f \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, and take $Z(f^{\text{trn}})$ which determines the Newton polytope Δ^{trn} uniquely up to integral translation on Σ ; that is, for any monomial $h \in \tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$, $Z(f^{\text{trn}}) = Z(hf^{\text{trn}})$. But the Newton polytope of hf^{trn} is just an integral translation of Δ^{trn} on Σ . \square

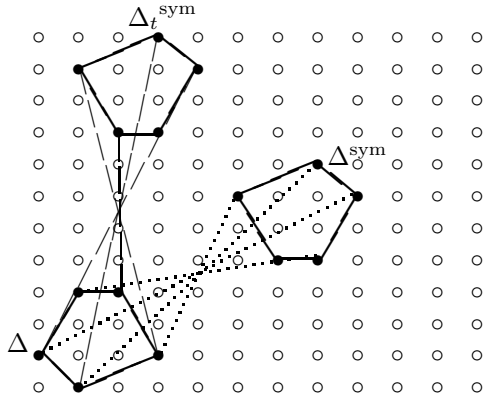


FIGURE 4. Illustration for Example 6.5.

Example 6.5. Let Δ be the lattice polytope in $\mathbb{N}^{(n)}$ with vertices $(1, 0), (0, 1), (3, 2), (1, 3),$ and $(2, 3)$, which has no parallel edges. Assign to Δ the primitively irreducible polynomial

$$f = \lambda_1 + \lambda_2 + \lambda_1^3 \lambda_2 + \lambda_1 \lambda_2^3 + \lambda_1^2 \lambda_2^3$$

in $\tilde{\mathbb{T}}[\lambda_1, \dots, \lambda_n]$ (with Newton polytope Δ), and compute

$$f^{\text{sym}} = \lambda_1^8 \lambda_2^8 + \lambda_1^9 \lambda_2^7 + \lambda_1^6 \lambda_2^7 + \lambda_1^8 \lambda_2^5 + \lambda_1^7 \lambda_2^5,$$

whose Newton polytope Δ^{sym} has the vertices $(8, 8), (9, 7), (6, 7), (8, 5)$ and $(7, 5)$. Then Δ^{sym} is a point symmetry of Δ and vice versa. See Fig 4, the dotted lines show the symmetry between Δ and Δ^{sym} . The dashed lines show the symmetry between Δ and Δ_t^{sym} , an integral translation of Δ^{sym} on Σ .

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