Chart description for hyperelliptic Lefschetz fibrations and their stabilization

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Abstract

Chart descriptions are a graphic method to describe monodromy representations of various topological objects. Here we introduce a chart description for hyperelliptic Lefschetz fibrations, and show that any hyperelliptic Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations.

1 Introduction

Chart descriptions were originally introduced in order to describe 2-dimensional braids in [8, 9] (cf. [10]). In [13], a chart description for genus-one Lefschetz fibrations was introduced and an elementary proof of Matsumoto's classification theorem was given. At the third JAMEX meeting in Oaxaca, Mexico, 2004, the second author generalized it to a method describing any monodromy representation [11] and investigated genus-two Lefschetz fibrations as an application [12]. Here we introduce a chart description for hyperelliptic Lefschetz fibrations, and show that any hyperelliptic Lefschetz fibration can be stabilized by fiber-sum with certain basic Lefschetz fibrations.

2 Lefschetz fibrations

Let M and B be compact, connected, and oriented smooth 4-manifold and 2-manifold, respectively. Let $f : M \to B$ be a smooth map with $\partial M = f^{-1}(\partial B)$. A critical point p is called a *Lefschetz singular point* of *positive type* (or of *negative type*, respectively) if there exist local complex coordinates z_1, z_2 around p and a local complex coordinate ξ around $f(p)$ such that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$ (or $\overline{z_1} z_2$, resp.). We call f a (smooth or differentiable)

Lefschetz fibration if all critical points are Lefschetz singular points and if there exists exactly one critical point in the preimage of each critical value.

A *general fiber* is the preimage of a regular value of f. The *genus* of a Lefschetz fibration is defined to be the genus g of a general fiber. A *singular fiber* of *positive type* (or *negative type*, resp.) is the preimage of a critical value which contains a Lefschetz singular point of positive type (or negative type, resp.). A singular fiber is obtained by shrinking a simple loop, called a vanishing cycle, on a general fiber. In this paper we assume that a Lefschetz fibration is 'relatively minimal', i.e., all vanishing cycles are essential loops. We say that a singular fiber is of *type* I if the vanishing cycle is a non-separating loop. We say that a singular fiber is of *type* II_h for $h = 1, \ldots, [g/2]$ if the vanishing cycle is a separating loop which bounds a genus-h subsurface of the general fiber.

A singular fiber is of *type* I^+ if it is of type I and of positive type. Similarly *type* I^- and *type* II_h^+, *type* II_h^- for $h = 1, \ldots, [g/2]$ are defined. We denote by $n_0^+(f)$, $n_0^-(f)$, $n_h^+(f)$, and $n_h^-(f)$, the numbers of singular fibers of f of type I^+, I^-, II_h^+, and II_h^-, respectively. A Lefschetz fibration is called *irreducible* if every singular fiber is of type I, i.e., $n_h^+(f) = n_h^-(f) = 0$ for $h = 1, \ldots, [g/2]$. A Lefschetz fibration is called *chiral* or *symplectic* if every singular fiber is of positive type, i.e., $n_0^-(f) = n_h^-(f) = 0$ for $h = 1, \ldots, [g/2]$.

Let $f : M \to B$ be a Lefschetz fibration, and $\Delta = \{q_1, \ldots, q_n\}$ the set of critical values. Let $\rho : \pi_1(B \setminus \Delta, q_0) \to MC$ be the monodromy representation of f, where q_0 is a base point of $B \setminus \Delta$ and MC is the mapping class group of the fiber $f^{-1}(q_0)$. Consider a Hurwitz arc system for Δ, say $\mathcal{A} = (A_1, \ldots, A_n)$; each A_i is an embedded arc in B connecting q_0 and a point of Δ such that $A_i \cap A_j = \{q_0\}$ for $i \neq j$, and they appear in this order around q_0. When B is a 2-sphere or a 2-disk, the system $\mathcal{A}$ determines a system of generators of $\pi_1(B \setminus \Delta, q_0)$, say $(a_1, \ldots, a_n)$. We call $(\rho(a_1), \ldots, \rho(a_n))$ a *Hurwitz system* of f. For details on Hurwitz systems, refer to [1, 7, 15, 16, 17], etc.

Let ι be the mapping class of an involution of the fiber $f^{-1}(q_0)$ with $2g+2$ fixed points. The centralizer HMG of ι in MG is called the *hyperelliptic mapping class group* of $f^{-1}(q_0)$. A Lefschetz fibration is called *hyperelliptic* if the image of the monodromy representation ρ is included in HMG.

3 Main result

Let ζ_i $(i = 1, \ldots, 2g+1)$ be positive Dehn twists along the loops C_i $(i = 1, \ldots, 2g+1)$ illustrated in Figure 1. The hyperelliptic mapping class group HMC of a genus-g Riemann surface is generated by $\zeta_1, \ldots, \zeta_{2g+1}$, and the following relations are defining relations (cf. [4]).

$$\zeta_i \zeta_j = \zeta_j \zeta_i \quad \text{if } |i-j| \geq 2, \tag{1}$$

$$\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1} \quad \text{for } i = 1, \ldots, 2g, \tag{2}$$

$$\iota^2 = 1 \quad \text{where } \iota = \zeta_1 \cdots \zeta_{2g} \zeta_{2g+1}^2 \zeta_{2g} \cdots \zeta_1, \tag{3}$$

$$(\zeta_1 \cdots \zeta_{2g+1})^{2g+2} = 1, \tag{4}$$

$$\iota\,\zeta_i = \zeta_i\,\iota \quad \text{for } i = 1, \dots, 2g+1. \tag{5}$$

Let σ_h be a positive Dehn twist along the loop S_h illustrated in Figure 1. Then $\sigma_h = (\zeta_1 \cdots \zeta_{2h})^{4h+2}$ for $h = 1, \dots, [g/2]$.

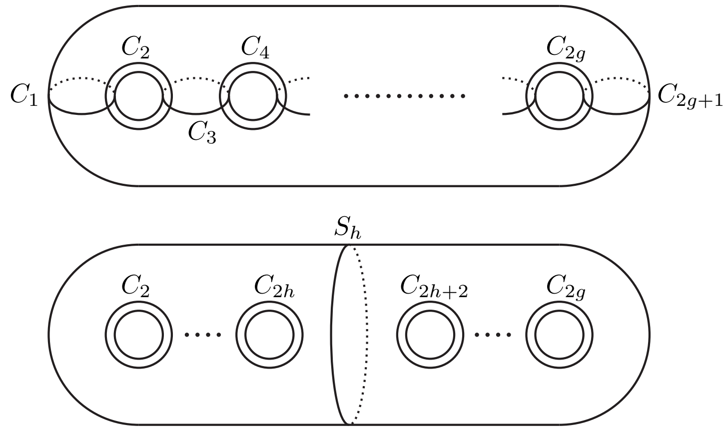

Figure 1: Curves on a general fiber

If $(g_1, \dots, g_n)$ is a Hurwitz system of a genus-g hyperelliptic Lefschetz fibration, then each g_j is a conjugate of ζ_i or ζ_i^{-1}, or a conjugate of σ_h or σ_h^{-1}.

Now we define basic Lefschetz fibrations.

Definition 1 (cf. [1, 2, 12, 15, 17]) *Basic Lefschetz fibrations*, f_0, f_1, $f_{2,h}$, f'_1 and $f'_{2,h}$, are genus-g hyperelliptic Lefschetz fibrations over S^2 whose Hurwitz systems are

(1) $W_0 = (T)^2$ where $T = (\zeta_1, \zeta_2, \dots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \dots, \zeta_2, \zeta_1)$,

(2) $W_1 = (\zeta_1, \zeta_2, \dots, \zeta_{2g}, \zeta_{2g+1})^{2g+2}$,

(3) $W_{2,h} = (\zeta_{2g+1}, \dots, \zeta_1, (\zeta_{2g-2h+1}, \dots, \zeta_{2g+1}), \dots, (\zeta_2, \dots, \zeta_{2h+2}),$
$(\zeta_1, \dots, \zeta_{2h+1}), \sigma_h, (\zeta_{2h+1}, \dots, \zeta_1),$
$(\zeta_{2h+2}, \dots, \zeta_2), \dots, (\zeta_{2g+1}, \dots, \zeta_{2g-2h+1}), \zeta_1, \dots, \zeta_{2g+1})$,

(4) $W'_1 = (\zeta_1, \zeta_1^{-1})$,

(5) $W'_{2,h} = (\sigma_h, \sigma_h^{-1})$,

respectively.

For example, f_0 has $4(2g+1)$ singular fibers, which are of type I^+. Thus f_0 is chiral and irreducible.

LF	number of singular fibers				chiral	irreducible
	n_0^+	n_0^-	n_k^+	n_k^-		
f_0	$4(2g+1)$	0	0	0	◯	◯
f_1	$2(g+1)(2g+1)$	0	0	0	◯	◯
$f_{2,h}$	$8h(g-h)+4(2g+1)$	0	δ_{hk}	0	◯	×
f_1'	1	1	0	0	×	◯
$f_{2,h}'$	0	0	δ_{hk}	δ_{hk}	×	×

For two Lefschetz fibrations f and f' over S^2, we denote by $f\#f'$ a fiber-sum of f and f'. By $\#mf$ for a positive integer m, we mean the fiber-sum of m copies of f. If both f and f' are hyperelliptic, we assume that $f\#f'$ is also hyperelliptic.

Theorem 2 *Let f be a genus-g hyperelliptic Lefschetz fibration over S^2. Suppose that $n_h^+(f) \geq n_h^-(f)$ for $h = 1, \ldots, [g/2]$. Then*

(1) $\mathcal{E}(f) := n_0^+(f) - n_0^-(f) - 4\sum_{h=1}^{[g/2]}(n_h^+(f) - n_h^-(f))(2h(g-h)+2g+1)$ *is a multiple of $2(2g+1)$ if g is even, and that of $4(2g+1)$ if g is odd.*

(2) *There exists a positive integer m_0 such that for any integer $m \geq m_0$,*

$$f\,\#\,m\,f_0 \cong \#\,(a+m)\,f_0\,\#\,b\,f_1\,\#\,(\#_{h=1}^{[g/2]} c_h\,f_{2,h})\,\#\,d\,f_1'\,\#\,(\#_{h=1}^{[g/2]} e_h\,f_{2,h}')$$

for some non-negative integers $a, b, c_1, \ldots, c_{[g/2]}, d, e_1, \ldots, e_{[g/2]}$.

(3) *In (2), it holds that $c_h = n_h^+(f) - n_h^-(f)$, $d = n_0^-(f)$ and $e_h = n_h^-(f)$. Although a and b are not determined uniquely, we have $a = (\mathcal{E}(f) - 2(g+1)(2g+1)b)/4(2g+1)$ and we can take $b \in \{0, 1\}$.*

Remark 3 If f is chiral, then $n_0^-(f) = n_1^-(f) \cdots = n_{[g/2]}^-(f) = 0$. By Theorem 2, we have

$$f\,\#\,m\,f_0 \cong \,\#\,(a+m)\,f_0\,\#\,b\,f_1\,\#\,c_1\,f_{2,1}\cdots\#\,c_{[g/2]}\,f_{2,[g/2]}$$

for a sufficiently large integer m. Auroux and Smith [3] pointed out that a similar result follows from a work of Kharlamov and Kulikov [14].

Remark 4 If g is even, then $b \equiv \mathcal{E}(f)/2(2g+1) \pmod 2$. If g is odd, the parity of b is not determined by $\mathcal{E}(f)$.

4 Chart description

In this section we introduce a chart description for genus-g hyperelliptic Lefschetz fibrations. We use the terminologies on chart description in [11]. For simplicity's sake, we only consider genus-g hyperelliptic Lefschetz fibrations over B such that ∂B is empty or connected, and if ∂B is not empty, we assume that the monodromy along ∂B is trivial. Unless otherwise stated, genus-g hyperelliptic Lefschetz fibrations over B are assumed to be so.

Definition 5 (cf. [10, 11, 12, 13]) A *chart* in B is a finite graph Γ in B (possibly being empty or having *hoops* that are closed edges without vertices) whose edges are labeled with an element of $\{1, \ldots, 2g+1, \sigma_1, \ldots, \sigma_{[g/2]}\}$, and oriented so that the following conditions are satisfied (see Figure 2, 3, 4):

(1) The degree of each vertex is $1, 4, 6, 4(2g+1), 2(g+1)(2g+1), 2(4g+3)$ or $4h(2h+1)+1$.

(2) For a degree-1 vertex, the adjacent edge is oriented outward or inward.

(3) For a degree-4 vertex, two edges in each diagonal position have the same label and are oriented coherently; and the labels i and j of the diagonals are in $\{1, \ldots, 2g+1\}$ with $|i-j| > 1$.

(4) For a degree-6 vertex, the six edges are alternately labeled i and j in $\{1, \ldots, 2g+1\}$ with $|i-j| = 1$; and three consecutive edges are oriented outward while the other three are oriented inward.

(5) For a vertex of degree $4(2g+1)$, the edges are labeled with $(1, \ldots, 2g+1, 2g+1, \ldots, 1)^2$; and all edges are oriented outward or all edges are oriented inward.

(6) For a vertex of degree $2(g+1)(2g+1)$, the edges are labeled with $(1, \ldots, 2g+1)^{2g+2}$ in a counterclockwise direction (or clockwise direction, resp.); and all edges are oriented outward (or inward, resp.).

(7) For a vertex of degree $2(4g+3)$, the edges are labeled with $(1, \ldots, 2g+1, 2g+1, \ldots, 1, i)^2$ in a counterclockwise direction where $i \in \{1, \ldots, 2g+1\}$; and the first $4g+3$ edges are oriented outward and the latter ones are oriented inward.

(8) For a vertex of degree $4h(2h+1)+1$, the edges are labeled with $((1, \ldots, 2h)^{4h+2}, \sigma_h)$ in a counterclockwise direction (or clockwise direction, resp.); and the edges with labels $1, \ldots, 2h$ are oriented outward (or inward, resp.), and the edge with label σ_h is oriented inward (or outward, resp.).

(9) $\Gamma \cap \partial B = \emptyset$.

(10) Γ misses the base point $q_0 \in B$.

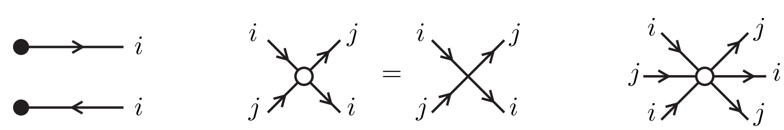

Figure 2: Vertices of degree 1, 4, 6

Remark 6 When we would treat genus-g hyperelliptic Lefschetz fibrations over B with $\partial B \neq \emptyset$ such that the monodromies along ∂B are not trivial, the condition (9) should be removed. See [11].

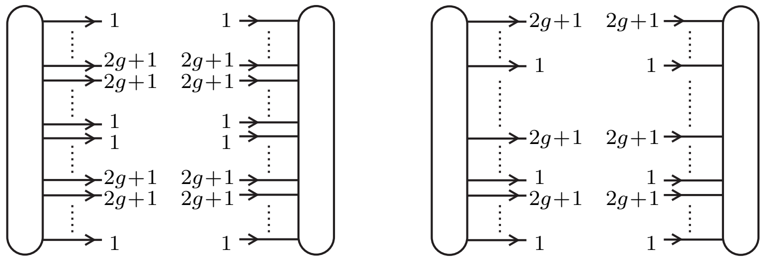

Figure 3: Vertices of degree $4(2g+1)$, $2(g+1)(2g+1)$

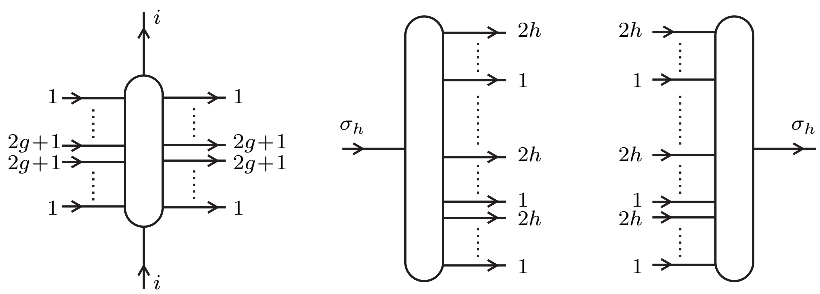

Figure 4: Vertices of degree $2(4g+3)$, $4h(2h+1)+1$

We call a degree-1 vertex a *black vertex*. We say that a chart is *chiral* if every black vertex has an adjacent edge oriented outward. We say that a chart is *irreducible* if there exist no edges with label σ_h.

For a chart Γ, let Δ_Γ be the set of black vertices. A chart Γ determines a homomorphism $\pi_1(B \setminus \Delta_\Gamma, q_0) \to HMC$ as in [11]. By Theorem 5 of [11], we have the following theorem.

Theorem 7 *Let f be a genus-g hyperelliptic Lefschetz fibration over B, and let ρ be the monodromy representation. Then there is a chart Γ in B such that the monodromy representation ρ equals the homomorphism ρ_Γ determined by Γ.*

A chart Γ as in Theorem 7 is called a *chart description* of f or a *chart describing* f. Moreover, for a Hurwitz system $(g_1, \ldots, g_n)$ of f over $B = D^2$, we call Γ a chart description of $(g_1, \ldots, g_n)$. A chart Γ in D^2 is also regarded as a chart in S^2 in the trivial way.

We introduce some local moves on chart descriptions.

(C1) For a chart Γ, suppose that there exists a chart Γ' and an embedded 2-disk, say E, in B such that (i) ∂E intersects with Γ and Γ' transversely (or does not intersect with them) avoiding their vertices, (ii) Γ and Γ' have no black vertices in E, and (iii) Γ and Γ' are identical outside of E. Then we say that Γ' is obtained from Γ by a C1-move.

(C2) For a chart, suppose that there is an edge e joining a degree-4 vertex and a black vertex. Remove the edge e as in Figure 5(1). We call this local move a C2-move.

(C3) For a chart, suppose that there is an edge e joining a degree-6 vertex and a black vertex. Suppose that e is neither the middle of three edges oriented outward nor the middle of the three edges oriented inward. Then, remove the edge as in Figure 5(2). We call this local move a C3-move.

(C4) In a chart, suppose that there is an edge e joining a degree-$2(4g+3)$ vertex and a black vertex. Suppose that e is one of the two edges labeled i in Figure 4. Then, remove the edge as in Figure 5(3). We call this local move a C4-move.

When $\partial B \neq \emptyset$ and the base point q_0 is in ∂B, we introduce another move.

(C5) Suppose that $\partial B \neq \emptyset$ and $q_0 \in \partial B$. Let Γ' be a chart that is the union of a chart Γ and some hoops which are parallel to and sufficiently near ∂B. Then we say that Γ' is obtained from Γ by a C5-move.

Definition 8 (1) *Chart moves* are C1-moves, C2-moves, C3-moves, C4-moves and their inverse moves.

(2) Two charts in B are said to be *chart move equivalent* (with respect to the base point q_0) if they are related by a finite sequence of chart moves and ambient isotopies of B rel q_0, where we assume that chart moves are applied in embedded 2-disks in B missing q_0.

(3) Two charts in B are said to be *chart move equivalent up to conjugation* (with respect to the base point q_0) if they are related by a finite sequence of chart moves, C5-moves and ambient isotopies of B rel q_0. (It is not necessary to assume that chart moves are applied in embedded 2-disks in B missing q_0.)

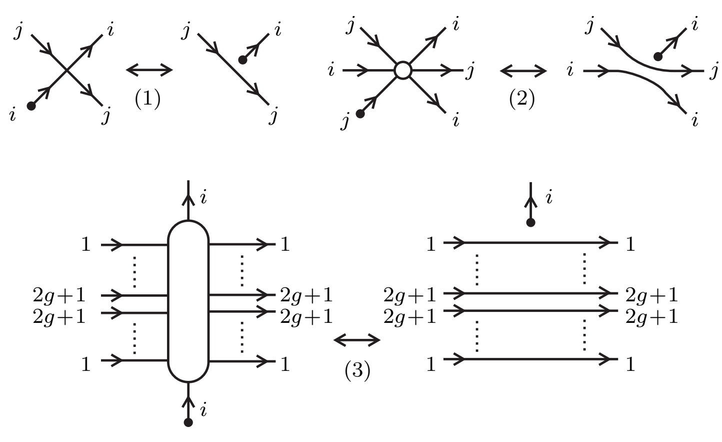

Figure 5: Some chart moves

We say that two monodromy representations $\rho : \pi_1(B \setminus \Delta, q_0) \to HMC$ and $\rho' : \pi_1(B \setminus \Delta', q_0) \to HMC$ are *equivalent* if there is a diffeomorphism $h : (B, q_0) \to (B, q_0)$ which is isotopic to the identity map rel q_0 such that

$h(\Delta) = \Delta'$ and $\rho = \rho' \circ h_*$, where $h_* : \pi_1(B \setminus \Delta, q_0) \to \pi_1(B \setminus \Delta', q_0)$ is the induced isomorphism.

We say that two monodromy representations $\rho : \pi_1(B \setminus \Delta, q_0) \to HMC$ and $\rho' : \pi_1(B \setminus \Delta', q_0) \to HMC$ are *equivalent up to conjugation* if there is an inner-automorphism of HMC, say t, and there is a diffeomorphism $h : (B, q_0) \to (B, q_0)$ which is isotopic to the identity map rel q_0 such that $h(\Delta) = \Delta'$ and $\rho = t \circ \rho' \circ h_*$.

C1-moves in this paper are called *chart moves of type W* in Definition 7 of [11]. C2-moves, C3-moves, C4-moves, C5-moves are not given explicitly in [11]. However, as shown in Fig. 22 and 23 of [11], C2-moves and C3-moves are equivalent to some local moves called *chart moves of transition* in Definition 14 of [11]. C4-moves are also equivalent to chart moves of transition in the sense of [11]. Thus, as stated in Section 8 of [11], we see that if two charts are chart move equivalent in our sense (Definition 8 (2)) then the monodromy representations determined by them are equivalent. C5-moves are equivalent to *chart moves of conjugacy* in (3) and (4) of Fig. 17 of [11]. Again as in Section 7 of [11], we see that if two charts are chart move equivalent up to conjugation (Definition 8 (3)) then the monodromy representations determined by them are equivalent up to conjugation.

Thus we have the following.

Theorem 9 *For two charts in B, if they are chart move equivalent (or chart move equivalent up to conjugation, resp.) then the monodromy representations determined by them are equivalent (or equivalent up to conjugation, resp.), and hence the hyperelliptic Lefschetz fibrations described by them are isomorphic.*

Remark 10 By Theorem 16 of [11], we see that two charts determine equivalent monodromy representations if and only if they are related by C1-moves (chart move of type W), chart moves of transition, and ambient isotopies of B rel q_0. It is unknown to the authors whether all chart moves of transition are consequence of our chart moves.

We say that a black vertex of a chart Γ is of *type* I^+, *type* I^-, *type* II_h^+ or *type* II_h^- if the adjacent edge is labeled in $\{1, \ldots, 2g+1\}$ and oriented outward, if the adjacent edge is labeled in $\{1, \ldots, 2g+1\}$ and oriented inward, if the adjacent edge is labeled σ_h and oriented outward, or if the adjacent edge is labeled σ_h and oriented inward, respectively.

When Γ is a chart description of a genus-g hyperelliptic Lefschetz fibration $f : M \to B$, black vertices correspond to critical values of f, and the types of the vertices are the same with the types of the singular fibers over the corresponding critical values. For a chart Γ, we denote by $n_0^+(\Gamma)$, $n_0^-(\Gamma)$, $n_h^+(\Gamma)$, and $n_h^-(\Gamma)$, the numbers of black vertices of type I^+, type I^-, type II_h^+ and type II_h^-, respectively. They are equal to $n_0^+(f)$, $n_0^-(f)$, $n_h^+(f)$, and $n_h^-(f)$, respectively.

If a chart Γ is irreducible, then it is obvious that $n_h^+(\Gamma) = n_h^-(\Gamma) = 0$ for $h = 1, \ldots, [g/2]$. The converse is not true. However we have the following.

Lemma 11 *Every chart Γ with $n_h^+(\Gamma) = n_h^-(\Gamma) = 0$ for $h = 1, \dots, [g/2]$ is chart move equivalent to an irreducible chart.*

Proof. We can replace every hoop labeled σ_h into $4h(2h+1)$ parallel hoops with labels $1, \dots, 2h$ by a chart move depicted in Figure 6 (1) followed by one in Figure 6 (2). Every edge labeled σ_h whose endpoints are degree-$4h(2h+1)+1$ vertices is also removed by the latter move. $\square$

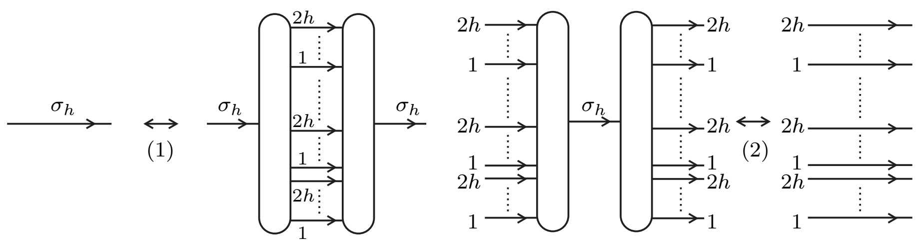

Figure 6: Chart moves

Proposition 12 *A chiral (or irreducible, resp.) genus-g hyperelliptic Lefschetz fibration has a chart description which is chiral (or irreducible, resp.).*

Proof. If f is chiral, local monodromies around the critical values are all positive Dehn twists. By the definition of a chart description, the adjacent edges of the black vertices are oriented outward. Thus any chart description of f is chiral. If f is irreducible, local monodromies around the critical values are Dehn twists along non-separating simple loops, which are conjugates of $\zeta_1, \dots, \zeta_{2g+1}$ and their inverses. Thus any chart description Γ of f satisfies $n_h^+(\Gamma) = n_h^-(\Gamma) = 0$ for $h = 1, \dots, [g/2]$. By Lemma 11, it changes to an irreducible one. $\square$

In Figures 7 and 8, we show charts N_0, N_1, $N_{2,h}$, F_1 and $F_{2,h}$ describing f_0, f_1, $f_{2,h}$, f_1' and $f_{2,h}'$. We call N_0 a (positive) *nucleon of degree*-$4(2g+1)$ and N_1 a (positive) *nucleon of degree*-$2(g+1)(2g+1)$. The region named $M_{2,h}$ is a chart with some special properties (Lemma 15). A *free edge* means a chart consisting two black vertices and a single edge connecting them. F_1 and $F_{2,h}$ are free edges.

Let Γ and Γ' be charts in $B = D^2$. Divide D^2 into 2-disks D_1^2 and D_2^2 by a properly embedded arc in D^2. Put a small copy of Γ in D_1^2 and a small copy of Γ' in D_2^2. We have a new chart in $D^2 = D_1^2 \cup D_2^2$. We call it the *product* of Γ and Γ' and denote it by $\Gamma \oplus \Gamma'$. We say that Γ is a *factor* of $\Gamma \oplus \Gamma'$. The chart $\Gamma \oplus \Gamma'$ is a chart description of the fiber sum $f \# f'$ of the Lefschetz fibrations f and f' described by Γ and Γ'. We denote by $n\Gamma$ the product $\Gamma \oplus \cdots \oplus \Gamma$ of n copies of Γ. (When $B = D^2$, the fiber sum $f \# f'$ of f and f' over B is defined by using the boundary connected sum of the base spaces.)

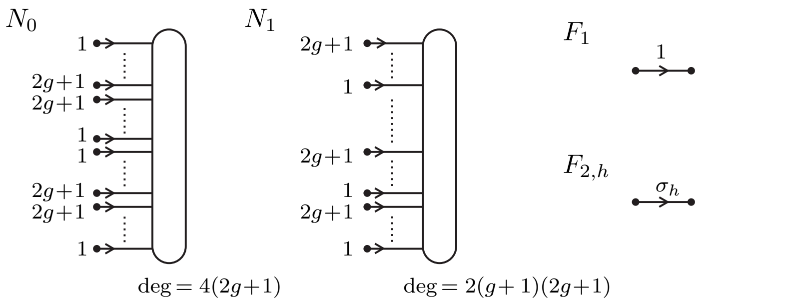

Figure 7: Charts N_0, N_1, F_1 and $F_{2,h}$ describing f_0, f_1, f_1' and $f_{2,h}'$

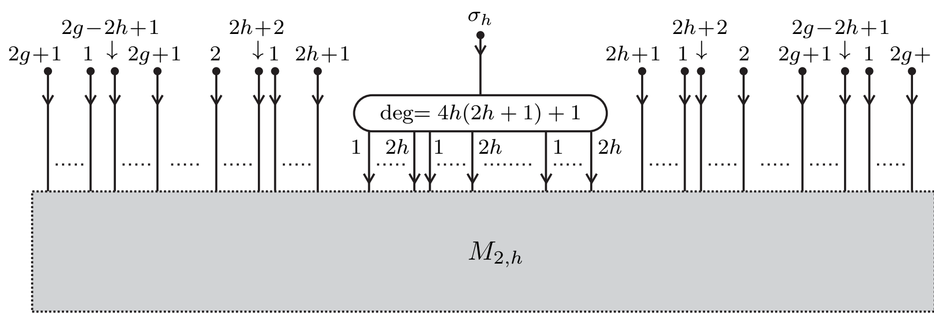

Figure 8: Chart $N_{2,h}$ describing $f_{2,h}$

Theorem 13 *Let Γ be a chart in $B = D^2$. Suppose that $n_h^+(\Gamma) \geq n_h^-(\Gamma)$ for $h = 1, \ldots, [g/2]$. Then there exists a positive integer m_0 such that for any integer $m \geq m_0$, the chart $\Gamma \oplus m\, N_0$ is chart move equivalent to*

$$\Gamma' \oplus \left(\oplus_{h=1}^{[g/2]} (n_h^+(\Gamma) - n_h^-(\Gamma)) N_{2,h}\right) \oplus n_0^-(\Gamma)\, F_1 \oplus \left(\oplus_{h=1}^{[g/2]} n_h^-(\Gamma) F_{2,h}\right)$$

for some chart Γ' with $n_0^-(\Gamma') = n_h^+(\Gamma') = n_h^-(\Gamma') = 0$ for $h = 1, \ldots, [g/2]$ such that Γ' has N_0 as a factor. Moreover if $n_h^-(\Gamma) = 0$ for $h = 1, \ldots, [g/2]$, we may take m_0 to be $n_0^-(\Gamma) + \sum_{h=1}^{[g/2]} (h+1) n_h^+(\Gamma) + 1$.

We prove Theorem 13 in Section 5.

Corollary 14 *Let f be a genus-g hyperelliptic Lefschetz fibration over $B = D^2$ (or S^2) with $n_h^+(f) \geq n_h^-(f)$ for $h = 1, \ldots, [g/2]$. Then there exists a positive integer m_0 such that for any integer $m \geq m_0$, the fiber sum $f \# m f_0$ is equivalent to*

$$f' \# \left(\#_{h=1}^{[g/2]} (n_h^+(f) - n_h^-(f)) f_{2,h}\right) \# n_0^-(f)\, f_1' \# \left(\#_{h=1}^{[g/2]} n_h^-(f) f_{2,h}'\right)$$

for some chiral and irreducible genus-g hyperelliptic Lefschetz fibration f' over $B = D^2$ (or S^2) such that the monodromy representation of f' is transitive. Moreover if $n_h^-(f) = 0$ for $h = 1, \ldots, [g/2]$, we may take m_0 to be $n_0^-(f) + \sum_{h=1}^{[g/2]} (h+1) n_h^+(f) + 1$.

Lemma 15 *There is a chart $M_{2,h}$ satisfying the following conditions.*

(1) *$M_{2,h}$ consists of edges with labels in $\{1, \ldots, 2g+1\}$ and vertices whose degrees are in $\{4, 6, 4(2g+1), 2(4g+3)\}$.*

(2) *The chart $P_{2,h}$ depicted in Figure 9 is chart move equivalent to $(h+1)N_0$.*

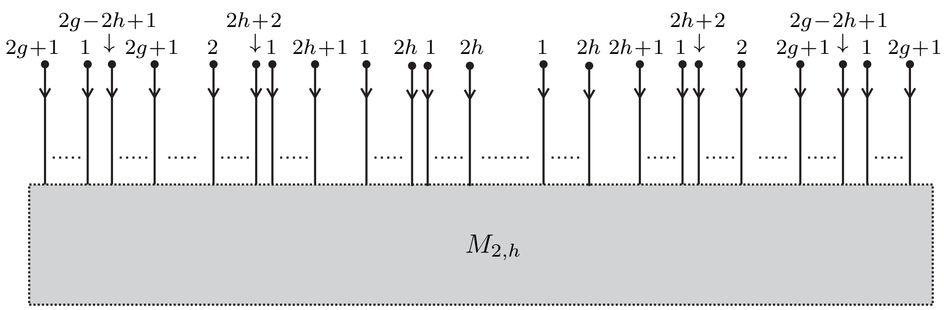

Figure 9: Chart $P_{2,h}$, which is equivalent to $(h+1)N_0$

We prove Lemma 15 in Section 7.

5 Proof of Theorem 13

Definition 16 A chart Γ in a 2-disk is *nomadic with respect to* a chart Γ_0 in B if for any two regions of the complement $B \setminus \Gamma_0$, say R_1 and R_2, the chart Γ_0 together with a small copy of Γ in R_1 is chart move equivalent to the chart Γ_0 together with a small copy of Γ in R_2. A chart Γ in a 2-disk is *nomadic* if it is nomadic with respect to every chart.

Lemma 17 *Let D be a 2-disk and B a compact, connected and oriented surface.*

(1) *Let Γ be a chart in D. If there is a 2-disk U in D such that $\Gamma \cap U$ is as in Figure 10, then Γ is nomadic.*

(2) *Let Γ_0 be a chart in B. If there is a 2-disk U in B such that $\Gamma_0 \cap U$ is as in Figure 10, then any chart Γ in a 2-disk is nomadic with respect to Γ_0.*

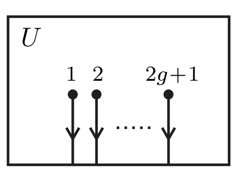

Figure 10: Nomadic chart

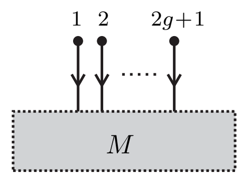

Figure 11: Nomadic chart

Proof. (1) First we consider a special case where Γ is as in Figure 11. Let Γ_0 be any chart in B, and put a small copy of Γ in a region of $B \setminus \Gamma_0$. As shown in Figure 12, it can pass through any edge of Γ_0 which is labeled in $\{1, \ldots, 2g+1\}$. For an edge labeled σ_h, apply a chart move as in Figure 6(1), let Γ pass through the $4(2h+1)$ edges with labels $1, \ldots, 2h$, and recover the edge labeled σ_h by the move in Figure 6. Thus we see that Γ is nomadic. Now we consider a general case. Take a point y_0 in the region U and a point y_1 in the boundary ∂D. Consider a simple path $\eta : [0,1] \to D$ connecting y_0 and y_1 such that η intersects Γ transversely. Let w be the intersection word of η with respect to Γ (see [10, 11]). Let Γ' be a chart obtained from Γ by adding some hoops surrounding Γ such that the intersection word w' of η with respect to Γ' is $w \cdot w^{-1}$. Applying a chart move in a neighborhood of η as in Figure 13, we have a chart Γ'' such that it coincides with Γ' outside of the neighborhood of η and the path η misses Γ''. So Γ'' is as in Figure 11. Note that Γ'' is chart move equivalent to Γ, since one can add or remove any hoop surrounding it by chart moves as in Figure 14. Since Γ'' is nomadic as shown in the previous case, we see that Γ is nomadic.

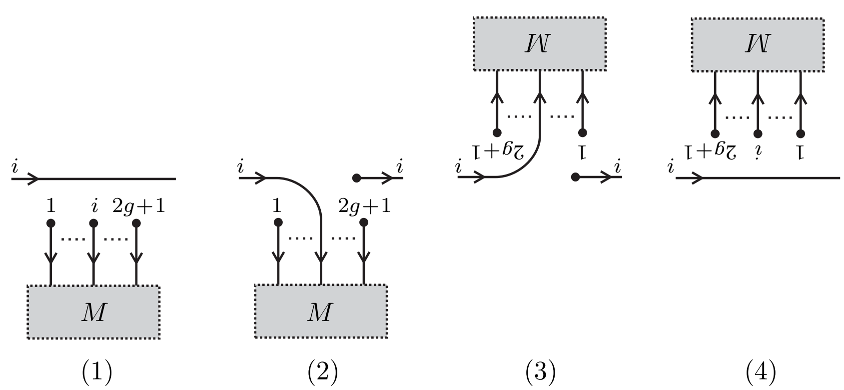

Figure 12: Chart moves

Now we prove (2). Let U be a region such that $\Gamma_0 \cap U$ is as in Figure 10. It is sufficient to show that any chart Γ put in a region of $B \setminus \Gamma_0$ can be moved into U. As shown in Figure 15, Γ can pass through any edge of Γ_0 by getting a surrounding hoop. When Γ arrives in U, it is surrounded some hoops, which can be removed by use of the edges of Γ_0 in U as in Figure 14. $\square$

Now we prove Theorem 13.

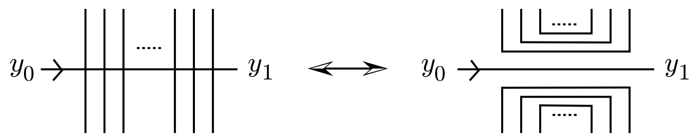

Figure 13: Chart moves

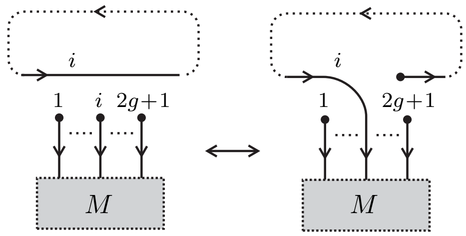

Figure 14: Chart moves

Proof of Theorem 13. First we consider a case where Γ is a chart with $n_h^-(\Gamma)=0$ for $h=1,\dots,[g/2]$. It suffices to show that $\Gamma \oplus (n_0^-(\Gamma)+\sum_{h=1}^{[g/2]}(h+1)n_h^+(\Gamma)+1)\,N_0$ is chart move equivalent to

$$\Gamma' \oplus \left(\oplus_{h=1}^{[g/2]} n_h^+(\Gamma)N_{2,h}\right) \oplus n_0^-(\Gamma)\,F_1$$

for some chart Γ' with $n_0^-(\Gamma')=n_h^+(\Gamma')=n_h^-(\Gamma')=0$ such that Γ' has N_0 as a factor. By Lemma 17, N_0 is nomadic. Thus we can move N_0 freely up to chart move equivalence. For each black vertex of type I^-, move a chart N_0 near the vertex and apply a chart move as in Figure 16 to make a free edge. Move the free edge toward the boundary of B by the chart move as in Figure 15. Since there is at least one N_0 near ∂B, the hoops surrounding the free edge can be removed (Figure 14), and we may also assume that the label of the free edge is 1 (Lemma 18.24 of [10]). Thus we can change $\Gamma \oplus (n_0^-(\Gamma)+\sum_{h=1}^{[g/2]}(h+1)n_h^+(\Gamma)+1)\,N_0$ so that all black vertices of type I^- are endpoints of F_1's near ∂B. We still have $\sum_{h=1}^{[g/2]}(h+1)n_h^+(\Gamma)+1$ N_0's near ∂B. For each black vertex of type II_h^+, move $h+1$ copies of N_0 near the vertex. Change the copies of N_0 to a chart $P_{2,h}$ in Figure 9 (Lemma 15). The edge adjacent to the vertex of type II_h^+ is oriented outward and is labeled σ_h. Apply a chart move as in Figure 17, and then apply a chart move between the $4(2h+1)$ edges there and the $4(2h+1)$ edges of $P_{2,h}$ to get one $N_{2,h}$. Move the chart $N_{2,h}$ toward ∂B. (Note that $N_{2,h}$ is nomadic by Lemma 17.) Now all black vertices of type II_h^+ belong to $N_{2,h}$'s near ∂B. We still have one N_0 near ∂B. Thus the chart is $\Gamma' \oplus (\oplus_{h=1}^{[g/2]} n_h^+(\Gamma)N_{2,h}) \oplus n_0^-(\Gamma)\,F_1$ for a chart Γ' with $n_0^-(\Gamma')=n_h^+(\Gamma')=n_h^-(\Gamma')=0$ such that Γ' has N_0 as a factor.

We consider a case where Γ is a chart with $n_h^+(\Gamma)\geq n_h^-(\Gamma)>0$ for $h=1,\dots,[g/2]$. Let v be a black vertex of type II_h^-. Choose a black vertex v' of type II_h^+ and consider a simple path η from v to v'. If η intersects an edge

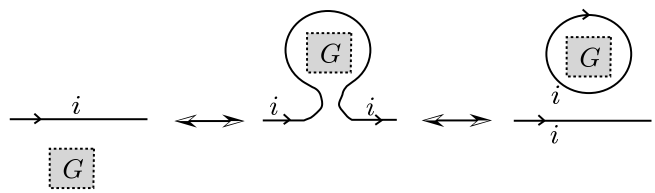

Figure 15: Chart moves

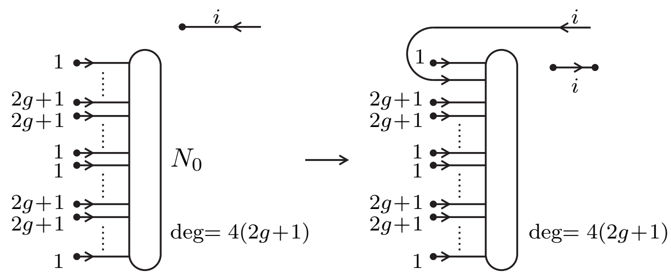

Figure 16: Chart moves

labeled σ_h, then apply a chart move depicted in Figure 6(1) and we assume that η intersects only edges with labels in $\{1, \dots, 2g+1\}$. For each intersection of η and the chart, we assert one N_0 and apply a chart move as in Figure 12(2) so that η does not intersect the chart. Now move v along η toward v' and by a chart move we can make a free edge with label σ_h, that is $F_{2,h}$. Move this $F_{2,h}$ toward ∂B by moves as in Figure 15. The hoops surrounding the free edge can be removed by adding one N_0 near ∂B as before. By this procedure, we can move all black vertices of type II_h^- near ∂B as endpoints of $F_{2,h}$'s. The number of $F_{2,h}$'s is $n_h^-(\Gamma)$. There are $n_h^+(\Gamma) - n_h^-(\Gamma)$ black vertices of type II_h^+ in the chart, besides the endpoints of $F_{2,h}$'s. For each black vertex of type II_h^+, that is not an endpoint of $F_{2,h}$, add $h+1$ copies of N_0 to make $P_{2,h}$. As in the previous case, we can move the black vertex of type II_h^+ as an endpoint of $N_{2,h}$ near ∂B. The number of $N_{2,h}$'s is $n_h^+(\Gamma) - n_h^-(\Gamma)$. As in the previous case, we move black vertices of type I^- as endpoints of F_1'as near ∂B. The number of F_1's is $n_0^-(\Gamma)$. Thus we have a chart written as

$$\Gamma' \oplus \left(\oplus_{h=1}^{[g/2]}(n_h^+(\Gamma) - n_h^-(\Gamma))N_{2,h}\right) \oplus n_0^-(\Gamma)\, F_1 \oplus \left(\oplus_{h=1}^{[g/2]} n_h^-(\Gamma) F_{2,h}\right)$$

for some chart Γ' with $n_0^-(\Gamma') = n_h^+(\Gamma') = n_h^-(\Gamma') = 0$ such that Γ' has N_0 as a factor. $\square$

6 Proof of Theorem 2

Proposition 18 *Let Γ be a chart description of a chiral and irreducible genus-g hyperelliptic Lefschetz fibration over $B = D^2$ (or S^2). There exists a positive*

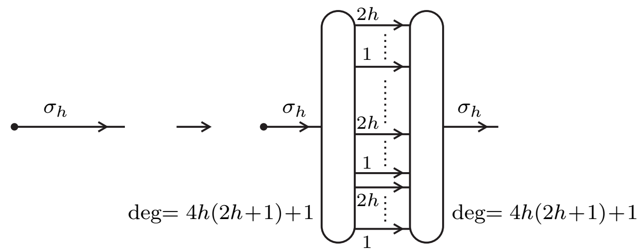

Figure 17: Chart moves

integer m such that $\Gamma \oplus mN_0$ is chart move equivalent to $(a+m)N_0 \oplus bN_1$ for some integers a and b.

Proof. Since f is chiral and irreducible, we may assume that Γ is chiral and irreducible by Proposition 12. Adding some N_0's to the chart and applying chart moves shown in Figures 18–20, we can remove all degree-6 vertices, degree-$2(4g+3)$ vertices, degree-$4(2g+1)$ vertices whose adjacent edges are oriented outward, and degree-$2(g+1)(2g+1)$ vertices whose adjacent edges are oriented outward. (Since it is easily seen that $(g+1)N_0$ is chart move equivalent to $2N_1$ (cf. [6]), we may add N_1's too.) Remove all hoops using an N_0 (Figure 14). Now every edge is adjacent to a black vertex, a degree-4 vertex, a degree-$4(2g+1)$ vertex whose adjacent edges are oriented inward or a degree-$2(g+1)(2g+1)$ vertex whose adjacent edges are oriented inward. Note that for a degree-4 vertex, the two incoming adjacent edges have black vertices at the other end. Thus by a C2-move (Figure 5(1)), we can remove the degree-4 vertex. Remove all degree-4 vertices this way. Now the chart is a union of some N_0's and N_1's. □

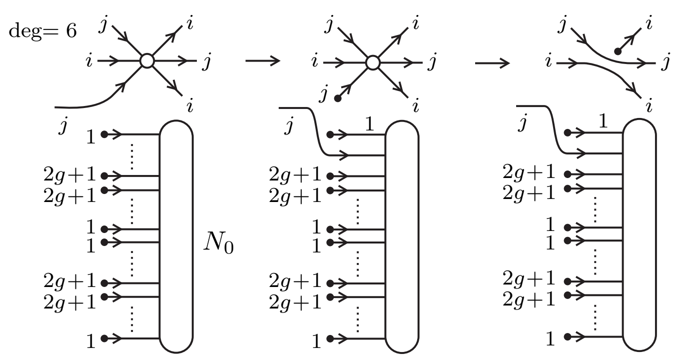

Figure 18: Chart moves

Now we have a corollary to Proposition 18.

Corollary 19 *Let f be a chiral and irreducible genus-g hyperelliptic Lefschetz fibration over S^2. There exists a positive number m such that $f \# m f_0 \cong (a+m)f_0 \# b f_1$ for some integers a and b.*

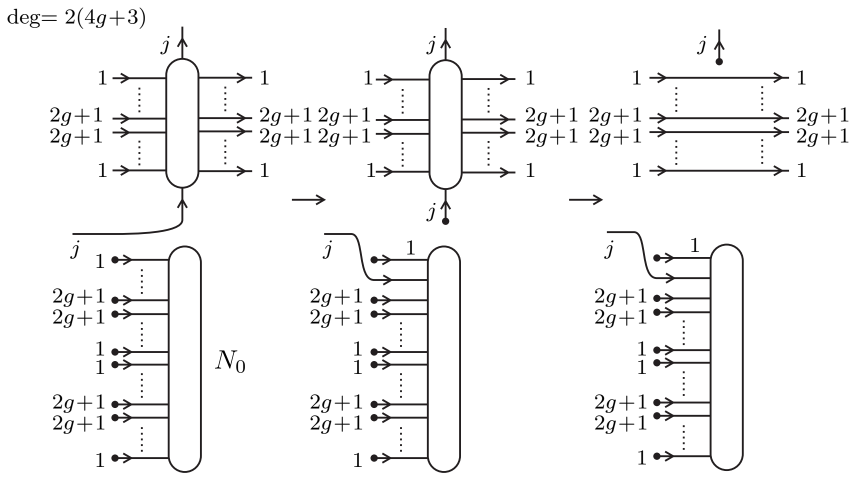

Figure 19: Chart moves

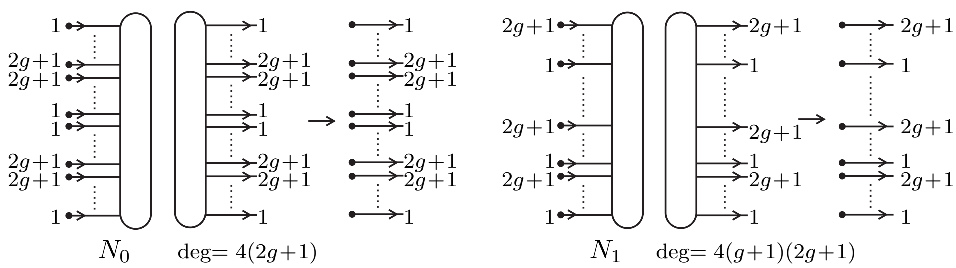

Figure 20: Chart moves

Using this corollary, we have a proof of Theorem 2.

Proof of Theorem 2. Corollary 14 and Corollary 19 imply the assertions (2) of Theorem 2. We shall compare the number of singular fibers of each type of $f \# m\, f_0$ with that of $\# \,(a{+}m)\, f_0 \# b\, f_1 \# (\#_{h=1}^{[g/2]} c_h\, f_{2,h}) \# d\, f_1' \# (\#_{h=1}^{[g/2]} e_h\, f_{2,h}')$. We have already used the information on the numbers of singular fibers of type I^-, II_h^+ and type II_h^- to determine c_h, d and e_h; $c = n_h^+(f) - n_h^-(f)$, $d = n_0^-(f)$ and $e = n_h^-(f)$. The number of singular fibers of type I^+ of $f \# m\, f_0$ is $n_0^+(f) + 4(2g{+}1)m$, and that of $\# \,(a{+}m)\, f_0 \# b\, f_1 \# (\#_{h=1}^{[g/2]} c_h\, f_{2,h}) \# d\, f_1' \# (\#_{h=1}^{[g/2]} e_h\, f_{2,h}')$ is $4(2g+1)(a+m) + 2(g+1)(2g+1)b + \sum_{h=1}^{[g/2]} (8h(g-h) + 4(2g+1))c_h + d$. From this equality, we have

$$\begin{aligned} & 4(2g+1)a + 2(g+1)(2g+1)b \\ = {} & n_0^+(f) - n_0^-(f) - 4\sum_{h=1}^{[g/2]} (n_h^+(f) - n_h^-(f))(2h(g-h) + 2g + 1). \end{aligned}$$

Thus the right hand side, which is $\mathcal{E}(f)$, is a multiple of $2(2g+1)$ if g is even, and that of $4(2g+1)$ if g is odd. It is well-known that $(g+1)f_0 \cong 2f_1$ (cf. [6]).

Therefore we have $a = (\mathcal{E}(f) - 2(g+1)(2g+1)b)/4(2g+1)$ and we can take b to be 0 or 1. $\square$

7 Proof of Lemma 15

We introduce some local moves on Hurwitz systems.

(H1) For a Hurwitz system, suppose that there are two consecutive components (ζ_i, ζ_j) with $|i-j| > 1$. Replace them by (ζ_j, ζ_i). We call this local move an H1-move.

(H2) For a Hurwitz system, suppose that there are three consecutive components $(\zeta_i, \zeta_j, \zeta_i)$ with $|i-j| = 1$. Replace them by $(\zeta_j, \zeta_i, \zeta_j)$. We call this local move an H2-move.

(H3) For a Hurwitz system, suppose that there are $4g+3$ consecutive components $(\zeta_1, \ldots, \zeta_{2g+1}, \zeta_{2g+1}, \ldots, \zeta_1, \zeta_i)$ where $i \in \{1, \ldots, 2g+1\}$. Replace them by $(\zeta_i, \zeta_1, \ldots, \zeta_{2g+1}, \zeta_{2g+1}, \ldots, \zeta_1)$. We call this local move an H3-move.

Lemma 20 *Let Γ be a chart description of a Hurwitz system $(g_1, \ldots, g_n)$. If a Hurwitz system $(g'_1, \ldots, g'_n)$ is obtained from $(g_1, \ldots, g_n)$ by a H1-move (or H2-move, H3-move, resp.) or its inverse, a chart description Γ' is obtained from Γ by a C2-move (or C3-move, C4-move, resp.) or its inverse.*

Proof. C2-move, C3-move, C4-move and their inverses (Figure 5) realize H1-move, H2-move, H3-move and their inverses. See Figure 21. $\square$

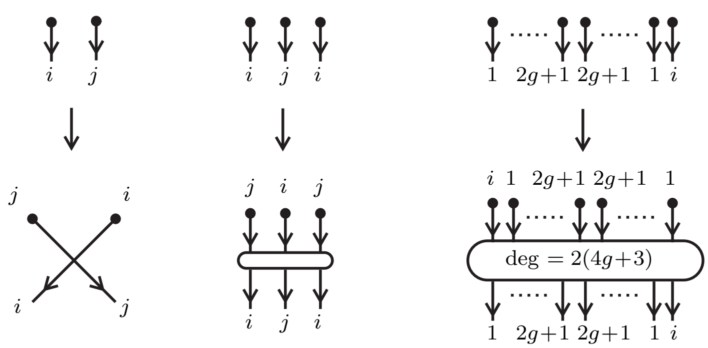

Figure 21: Chart moves

Now we prove Lemma 15.

Proof of Lemma 15. We shall construct a chart $P_{2,h}$ by applying C2-moves, C3-moves, C4-moves and their inverses to $(h+1)N_0$. Such $P_{2,h}$ obviously satisfies the conditions (1) and (2). By Lemma 20, it suffices to show that

$$W'_{2,h} = (\zeta_{2g+1}, \ldots, \zeta_1, (\zeta_{2g-2h+1}, \ldots, \zeta_{2g+1}), \ldots, (\zeta_1, \ldots, \zeta_{2h+1}), \\ (\zeta_1, \ldots, \zeta_{2h})^{4h+2}, (\zeta_{2h+1}, \ldots, \zeta_1), \ldots, (\zeta_{2g+1}, \ldots, \zeta_{2g-2h+1}), \zeta_1, \ldots, \zeta_{2g+1})$$

is obtained from W_0^{h+1} by H1-moves, H2-moves, H3-moves, their inverses, and cyclic permutations of components. Note that a cyclic permutation of components of a Hurwitz system does not affect a chart description. Applying H3-moves to $W_0^{h+1} = (\zeta_1, \zeta_2, \ldots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \ldots, \zeta_2, \zeta_1)^{2(h+1)}$, we have a Hurwitz system

$$((\zeta_1, \zeta_2, \ldots, \zeta_{2g}, \zeta_{2g+1})^{2(h+1)}, (\zeta_{2g+1}, \zeta_{2g}, \ldots, \zeta_2, \zeta_1)^{2(h+1)}).$$

Permuting the components of this system cyclically, we obtain

$$((\zeta_{2g+1}, \zeta_{2g}, \ldots, \zeta_2, \zeta_1)^{2(h+1)}, (\zeta_1, \zeta_2, \ldots, \zeta_{2g}, \zeta_{2g+1})^{2(h+1)}).$$

Applying H1-moves and H2-moves to this, we have

$$\begin{aligned}&(\zeta_{2g+1}, \ldots, \zeta_1, (\zeta_{2g-2h+1}, \ldots, \zeta_{2g+1}), \ldots, (\zeta_1, \ldots, \zeta_{2h+1}), (\zeta_{2h}, \ldots, \zeta_1)^{2h+1},\\ &(\zeta_1, \ldots, \zeta_{2h})^{2h+1}, (\zeta_{2h+1}, \ldots, \zeta_1), \ldots, (\zeta_{2g+1}, \ldots, \zeta_{2g-2h+1}), \zeta_1, \ldots, \zeta_{2g+1})\end{aligned}$$

by virtue of Lemma 4.6 and Lemma 4.10 of [5]. It follows from Lemma A.1 of [6] that $(\zeta_1, \ldots, \zeta_{2h})^{2h+1}$ is obtained from $(\zeta_{2h}, \ldots, \zeta_1)^{2h+1}$ by H1-moves and H2-moves. Thus we have

$$\begin{aligned}&(\zeta_{2g+1}, \ldots, \zeta_1, (\zeta_{2g-2h+1}, \ldots, \zeta_{2g+1}), \ldots, (\zeta_1, \ldots, \zeta_{2h+1}),\\ &(\zeta_1, \ldots, \zeta_{2h})^{4h+2}, (\zeta_{2h+1}, \ldots, \zeta_1), \ldots, (\zeta_{2g+1}, \ldots, \zeta_{2g-2h+1}), \zeta_1, \ldots, \zeta_{2g+1}),\end{aligned}$$

which is nothing but the Hurwitz system $W'_{2,h}$. □

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