Prepotential approach to solvable rational extensions of Harmonic Oscillator and Morse potentials

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We show how the recently discovered solvable rational extensions of Harmonic Oscillator and Morse potentials can be constructed in a direct and systematic way, without the need of supersymmetry, shape invariance, Darboux-Crum and Darboux-Bäcklund transformations.

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I. INTRODUCTION

It is fair to say that in the last three years some of the most interesting developments in mathematical physics have been the discoveries of new types of orthogonal polynomials, called the exceptional orthogonal polynomials, and the quantal systems related to them [1-18]. Unlike the classical orthogonal polynomials, these new polynomials have the remarkable properties that they still form complete sets with respect to some positive-definite measure, although they start with degree $\ell \geq 1$ polynomials instead of a constant.

Two families of such polynomials, namely, the Laguerre- and Jacobi-type X_1 polynomials, corresponding to $\ell = 1$, were first proposed by Gómez-Ullate et al. in [1], within the Sturm-Lioville theory, as solutions of second-order eigenvalue equations with rational coefficients. The results in [1] were reformulated in the framework of quantum mechanics in [2], and in supersymmetric quantum mechanics using superpotential in [3]. These quantal systems turn out to be rationally extended systems of the traditional ones which are related to the classical orthogonal polynomials. The most general X_ℓ exceptional polynomials, valid for all integral $\ell = 1, 2, \ldots$, were discovered by Odake and Sasaki [4] (the case of $\ell = 2$ was also discussed in [3]). Later, in [5] equivalent but much simpler looking forms of the Laguerre- and Jacobi-type X_ℓ polynomials were presented. Such forms facilitate an in-depth study of some important properties of the X_ℓ polynomials. Very recently, such systems have been generalized to multi-indexed cases [16, 17].

Other rational extensions of solvable systems, which are not related to the exceptional polynomials, are possible [19–23]. In these systems, the polynomial part of the wave functions start with degree zero. One of the simplest example of such systems was discussed in [20], which was later shown to be a certain supersymmetric partner of the harmonic oscillator in [21]. Extending the superpotential scheme of [21], Gandati and Bérard were able to generate an infinite set of solvable rational extension for translationally shape-invariant potentials of the second category [22]. More recently, rational extensions of the Morse and Kepler-Coulomb potentials have also been obtained by means of the Darboux-Bäcklund transformation in [23].

From the viewpoint of the generalized Crum's theorem, solvable rationally extended systems related to the exceptional polynomials are obtainable from the corresponding ordinary systems, which are related to the classical orthogonal polynomials, by deleting the lowest energy levels including the ground states [17]. The rational extensions of the harmonic and isotonic oscillators considered in [20–22] can be obtained in the same way, with the exception that the ground states were not deleted (see Appendix A of [24]).

So far most of the methods employed to generate solvable rational extensions of ordinary systems with or without the exceptional polynomials have invoked in one way or another the ideas of shape invariance and/or the related Darboux-Crum transformation (supersymmetry). This requires an exactly solvable ordinary system, such as the harmonic oscillator, to be known in the first place, and the superpotential associated with such system is modified for the extension.

In [18] we have proposed a simple constructive procedure to generate the exceptional orthogonal polynomials without the need of shape invariance and Darboux-Crum transformation. Thus in our work an exactly solvable ordinary system and its associated superpotential need not be assumed a priori as in the other works. The superpotential, as well as the potential, the eigenfunctions and eigenvalues of the new system are all derived from first principle in our method. To distinguish the different roles the superpotential play in our approach and in those employing Darboux-Crum transformation or supersymmetry, we prefer to call the superpotential "prepotential", and our procedure the "prepotential approach".

It is the purpose of this paper to demonstrate that the solvable rational extensions of the harmonic oscillator given in [20–22] and the Morse potential in [23] can also be generated very simply in the prepotential approach, without the need of supersymmetry, shape invariance, Darboux-Crum and Darboux-Bäcklund transformations. These two systems are in the same class as the rationally extended Jacobi system discussed in Sect. 4.4 of [18].

II. PREPOTENTIAL APPROACH

The main ideas of the prepotential approach are summarized here. We refer the reader to [18] for the details of the procedure. We adopt the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar = 2m = 1$.

Consider a wave function $\phi(x)$ which is written in terms of a function $W(x)$ as $\phi(x) \equiv \exp(W(x))$. The function $W(x)$ is assumed to have the form

$$W(x,\eta) = W_0(x) - \ln\xi(\eta) + \ln p(\eta). \tag{1}$$

Here $\eta(x)$ is a function of x which we shall choose to be one of the sinusoidal coordinates, i.e., coordinates such that $\dot{\eta}(x)^2$, where the dot denotes derivative with respect to x, is at most quadratic in η.

The functions $W_0(x)$, $\xi(\eta)$ and $p(\eta)$ are functions to be determined later. We shall assume $\xi(\eta)$ to be a polynomial in η. The wave function is

$$\phi(x) = \frac{e^{W_0(x)}}{\xi(\eta)}\, p(\eta). \tag{2}$$

Operating on $\phi(x)$ by the operator $-d^2/dx^2$ results in a Schrödinger equation $\mathcal{H}\phi = 0$, where $\mathcal{H} = -d^2/dx^2 + \bar{V}$, $\bar{V} \equiv \dot{W}^2 + \ddot{W}$. For simplicity of presentation, we shall often leave out the independent variable of a function if no confusion arises.

Since $W(x)$ determines the potential $\bar{V}$, it is therefore called the prepotential. To make $\bar{V}$ exactly solvable, we demand that: (1) W_0 is a regular function of x, (2) the function $\xi(\eta)$ has no zeros in the the ordinary (or physical) domain of $\eta(x)$, and (3) the function $p(\eta)$ does not appear in $\bar{V}$.

For $\xi(\eta) = 1$, the prepotential approach can generate exactly and quasi-exactly solvable systems associated with the classical orthogonal polynomials [25]. The presence of ξ in the denominators of $\phi(x)$ and $V(x)$ thus gives a rational extension, or deformation, of the traditional system. We therefore call $\xi(\eta)$ the deforming function.

If the factor $\exp(W_0(x))/\xi(\eta)$ in Eq. (2) is normalizable, then $p(\eta) =$ constant (in this case we shall take $p(\eta) = 1$ for simplicity) is admissible. This gives the ground state

$$\phi_0(x) = \frac{e^{W_0(x)}}{\xi(\eta)}. \tag{3}$$

However, if $\exp(W_0(x))/\xi(\eta)$ is non-normalizable, then $\phi_0(x)$ cannot be the ground state. In this case, the ground state, like all the excited states, must involve non-trivial $p(\eta) \neq 1$. Typically it is in such situation that the exceptional orthogonal polynomials arise [18]. The cases considered in [23], which we shall rederived by means of the prepotential approach in this paper, are such that $\exp(W_0(x))/\xi(\eta)$ is normalizable and thus $\phi_0(x)$ is the ground state.

Following the procedure in [18], we assume $\xi(\eta)$ to satisfy the equation

$$c_2(\eta)\xi'' + c_1(\eta)\xi' + \widetilde{\mathcal{E}}(\eta)\xi = 0. \tag{4}$$

Here the prime denotes derivative with respective to η. We choose $c_2(\eta) = \pm\dot{\eta}^2$, and c_1 is determined by

$$c_1(\eta) = \pm\left[\frac{1}{2}\frac{d}{d\eta}\left(\dot{\eta}^2\right) - 2Q(\eta)\right], \tag{5}$$

where $Q(\eta) \equiv \dot{W}_0\dot{\eta}$. The function $\widetilde{\mathcal{E}}(\eta)$ was taken to be a real constant in [18], but here we allow the possibility that $\widetilde{\mathcal{E}}(\eta)$ may be a function of η. Nonetheless, it turns out that all formulae presented in [18] remain intact.

By matching Eq. (4) with the (confluent) hypergeometric equation, one determines $\tilde{\mathcal{E}}$, $Q(\eta)$ and $\xi(\eta)$. Integrating $Q(x) = \dot{W}_0\dot{\eta}$ then gives the prepotential $W_0(x)$:

$$\begin{aligned} W_0(x) &= \int^x dx\, \frac{Q(\eta(x))}{\dot{\eta}(x)} \\ &= \int^{\eta(x)} d\eta\, \frac{Q(\eta)}{\dot{\eta}^2(\eta)}; \end{aligned} \tag{6}$$

The function $p(\eta)$ is then given by a linear combination of ξ and ξ':

$$p(\eta) = \xi'(\eta)F(\eta) + \xi(\eta)G(\eta). \tag{7}$$

Here the two functions $F(\eta)$ and $G(\eta)$ are determined by

$$F(\eta) = c_2(\eta)\mathcal{V}(\eta), \tag{8}$$
$$G(\eta) = (c_1 - c_2')\,\mathcal{V} - c_2\mathcal{V}', \tag{9}$$

with the function $\mathcal{V}(\eta)$ satisfying

$$c_2\mathcal{V}'' + (2c_2' - c_1)\,\mathcal{V}' + \left[c_2'' - c_1' + \widetilde{\mathcal{E}} \pm \mathcal{E}\right]\mathcal{V} = 0. \tag{10}$$

By matching Eq. (10) with the (confluent) hypergeometric equation, one determines $\mathcal{V}$, and thus $F(\eta),\ G(\eta),\ p(\eta)$ and $\mathcal{E}$.

Once all the relevant functions and parameters are determined, we would have constructed an exactly solvable quantal system $H\phi = \mathcal{E}\phi$ defined by $H = -d^2/dx^2 + V(x)$, with the wave function (2) and the potential

$$V(x) \equiv \dot{W}_0^2 + \ddot{W}_0 + \frac{\xi'}{\xi}\left[2\dot{\eta}^2\left(\frac{\xi'}{\xi}\right) - \left(2\dot{W}_0\dot{\eta} + \ddot{\eta}\right) \pm c_1\right] \pm \tilde{\mathcal{E}}. \tag{11}$$

Lastly, we note that the functions $p_{\mathcal{E}}$ (here we add a subscript to distinguish p corresponding to a particular eigenvalue $\mathcal{E}$) are orthogonal , i.e.,

$$\int d\eta\ p_{\mathcal{E}}(\eta)p_{\mathcal{E}'}(\eta)\frac{\mathcal{W}^2(x(\eta))}{\dot{\eta}} \propto \delta_{\mathcal{E},\mathcal{E}'} \tag{12}$$

in the η-space with the weight function

$$\mathcal{W}(x) \equiv \exp\left(\int^x dx\left(\dot{W}_0 - \frac{\dot{\xi}}{\xi}\right)\right) = \frac{e^{W_0(x)}}{\xi_\ell(\eta(x))}. \tag{13}$$

III. HARMONIC OSCILLATOR

Let us choose $\eta(x) = x \in (-\infty, \infty)$. Then $\dot{\eta}^2 = 1$. For c_2 and c_1, we take the upper signs in c_1 and c_2 (it turns out that the lower signs give the same model) . Thus $c_2(\eta) = 1$ and $c_1 = -2Q(\eta)$.

Eq. (4) becomes

$$\xi'' - 2Q(\eta)\xi' + \tilde{\mathcal{E}}\xi = 0. \tag{14}$$

Comparing Eq. (14) with the Hermite equation

$$H_\ell''(\eta) - 2\eta H_\ell'(\eta) + 2\ell H_\ell(\eta) = 0, \quad \ell = 0, 1, 2, \ldots, \tag{15}$$

where $H_\ell(\eta)$ is the Hermite polynomial, we would have

$$\xi(\eta) \equiv \xi_\ell(\eta;\alpha) = H_\ell(\eta), \quad \tilde{\mathcal{E}} = 2\ell, \quad Q(\eta) = \eta. \tag{16}$$

But this choice is not viable, as $\xi_\ell(\eta;\alpha) = H_\ell(\eta)$ has zeros in the ordinary domain $(-\infty, \infty)$, which we want to avoid. A simple way to solve this is to make the zeros of Hermite polynomials lie on the imaginary axis. This is achieved if we set $\eta \to i\eta$ in Eq. (15), giving

$$H_\ell''(i\eta) + 2\eta H_\ell'(i\eta) - 2\ell H_\ell(i\eta) = 0, \quad \ell = 0, 1, 2, \ldots. \tag{17}$$

Matching Eq. (14) with (17) gives

$$\xi(\eta) \equiv \xi_\ell(\eta;\alpha) = H_\ell(i\eta), \quad \tilde{\mathcal{E}} = -2\ell, \quad Q(\eta) = -\eta. \tag{18}$$

One notes that the Hermite polynomials are odd functions in η for odd ℓ. So in this case ξ_ℓ has a zero at $\eta = 0$. Further study of this case reveals that the wave functions are not normalizable. So here we shall only consider the case with even $\ell = 2m (m = 1, 2, \ldots)$. By Eq. (6), the form of $Q(\eta)$ leads to

$$W_0(x) = -\frac{x^2}{2}. \tag{19}$$

We shall ignore the constant of integration as it can be absorbed into the normalization constant. As noted in Sect. II, in this case $p(\eta) = 1$ is admissible, as $\exp(W_0(x))/\xi(\eta)$ is normalizable. So the energy and eigenfunction of the ground state of this system are $\mathcal{E}_0 = 0$ and $\phi_0(x) = \exp(W_0(x))/\xi(\eta)$. Below we determine the energies and eigenfunctions of the excited states.

With the solutions in Eq. (18), Eq. (10) becomes

$$\mathcal{V}'' - 2\eta\mathcal{V}' + (\mathcal{E} - 2\ell - 2)\,\mathcal{V} = 0. \tag{20}$$

Comparing Eqs. (20) and (15) (with ℓ replaced by $n = 0, 1, 2, \ldots$), we get

$$\mathcal{V}(\eta) = H_n(\eta), \quad \mathcal{E} \equiv \mathcal{E}_{\ell,n} = 2(n + \ell + 1), \quad \ell = 2m. \tag{21}$$

From Eqs. (8) and (9), one eventually obtains

$$\begin{aligned} p(\eta) &\equiv p_{\ell,n}(\eta) = \xi' F + \xi G \\ &= H_n(\eta)\xi'_\ell(\eta) + [2\eta H_n(\eta) - H'_n(\eta)]\,\xi_\ell(\eta) \\ &= H_n(\eta)H'_\ell(i\eta) + H_{n+1}(\eta)H_\ell(i\eta). \end{aligned} \tag{22}$$

Use has been made of the identity $H'_n = 2\eta H_n - H_{n+1}$ in obtaining the last line in Eq. (22). We note that $p_{\ell,n}(\eta)$ is a polynomial of degree $\ell + n + 1$.

By Eq. (12), one finds that $p_{\ell,n}(\eta;\alpha)$'s are orthogonal in the sense

$$\int_{-\infty}^{\infty} d\eta\, \frac{e^{-\eta^2}}{\xi_\ell^2}\, p_{\ell,n}(\eta;\alpha) p_{\ell,k}(\eta;\alpha) \propto \delta_{nk}. \tag{23}$$

The exactly solvable potential is given by Eq. (11) with $W_0(x)$ and $\xi_\ell(\eta;\alpha)$ given by Eqs. (19) and (18), respectively. Explicitly, the potential is

$$V(x) = x^2 - 1 + 2\frac{\xi'_\ell}{\xi_\ell}\left[\left(\frac{\xi'_\ell}{\xi_\ell}\right) + 2\eta\right] + 2\ell. \tag{24}$$

The complete eigenfunctions and energies are

$$\phi_0(x;\alpha) \;\propto\; \frac{e^{-\frac{x^2}{2}}}{\xi_\ell}, \qquad\qquad \mathcal{E}_0 = 0, \tag{25}$$

$$\phi_{\ell,n}(x;\alpha) \;\propto\; \frac{e^{-\frac{x^2}{2}}}{\xi_\ell} p_{\ell,n}(\eta(x);\alpha), \quad \mathcal{E}_{\ell,n} = 2(n + \ell + 1), \quad n = 0, 1, 2 \ldots. \tag{26}$$

Using the identity for $\ell = 2m$ [26, 27], i.e.,

$$H_{2m}(i\eta) = (-1)^m 2^{2m} m! L_m^{(-\frac{1}{2})}(-\eta^2), \tag{27}$$

where $L_\ell^{(\alpha)}(\eta)$ is the Laguerre polynomial, and the identity

$$\frac{d}{d\eta} L_\ell^{(\alpha)}(\eta) = -L_{\ell-1}^{(\alpha+1)}(\eta), \tag{28}$$

we can reduce Eq. (26) to

$$\phi_{2m,n} \sim \frac{e^{-\frac{x^2}{2}}}{L_m^{(-\frac{1}{2})}(-\eta^2)} \left[\frac{1}{2} L_m^{(-\frac{1}{2})}(-\eta^2) H_{n+1}(\eta) + \eta L_{m-1}^{(\frac{1}{2})}(-\eta^2) H_n(\eta)\right]. \tag{29}$$

This result is identical with that given in [23].

IV. MORSE POTENTIAL

Now we consider rational extension of the Morse potential. It turns out that in this case $\tilde{\mathcal{E}}$ cannot be a constant.

Let us choose $\eta(x) = e^{-x} \in (0, \infty)$, with $\dot{\eta}^2 = \eta^2$. For definiteness we shall take the upper signs for c_2 and c_1, as the lower signs lead to the same results. So we have $c_2(\eta) = \eta^2$ and $c_1 = (\eta - 2Q(\eta))$.

Equation determining ξ is

$$\eta^2 \xi''(\eta) + (\eta - 2Q(\eta))\, \xi'(\eta) + \tilde{\mathcal{E}}(\eta)\xi(\eta) = 0. \tag{30}$$

In order to link Eq. (30) with the Laguerre equation

$$\eta L_\ell^{\prime\prime(\alpha)} + (\alpha + 1 - \eta)\, L_\ell^{\prime(\alpha)} + \ell L_\ell^{(\alpha)} = 0, \quad \ell = 0, 1, 2, \ldots, \tag{31}$$

we rewrite Eq. (30) as

$$\eta \xi''(\eta) + \left(1 - 2\frac{Q(\eta)}{\eta}\right) \xi'(\eta) + \frac{\tilde{\mathcal{E}}(\eta)}{\eta}\xi(\eta) = 0. \tag{32}$$

Directly matching this equation with Eq. (31) will leads to unnormalizable wave functions. So instead we set $\eta \to -\eta$ in Eq. (32). This gives

$$\eta \xi''(-\eta) + \left(1 + 2\frac{Q(-\eta)}{\eta}\right) \xi'(-\eta) + \frac{\tilde{\mathcal{E}}(-\eta)}{\eta}\xi(-\eta) = 0. \tag{33}$$

Comparing this equation with Eq. (31) leads to

$$\xi(-\eta) \equiv \xi_\ell(-\eta; \alpha) = L_\ell^{(\alpha)}(\eta), \quad \tilde{\mathcal{E}}(-\eta) = \ell\eta, \quad Q(-\eta) = \frac{\alpha}{2}\eta - \frac{1}{2}\eta, \tag{34}$$

or equivalently,

$$\xi_\ell(\eta; \alpha) = L_\ell^{(\alpha)}(-\eta), \quad \tilde{\mathcal{E}}(\eta) = -\ell\eta, \quad Q(\eta) = -\frac{\alpha}{2}\eta - \frac{1}{2}\eta, \tag{35}$$

The form of $Q(\eta)$ leads to

$$W_0(x) = -\frac{\alpha}{2}\ln\eta - \frac{\eta}{2}. \tag{36}$$

According to the Kienast-Lawton-Hahn's Theorem [26, 27], the deforming function $\xi_\ell(\eta)$ will have no positive zeros in $(0, \infty)$ if: (i) $-2k - 1 < \alpha < -2k$ with $-\ell < \alpha < -1$, or (ii) ℓ is even with $\alpha < -\ell$. Again, in this case, $p(\eta) = 1$ is admissible. Thus the energy and eigenfunction of the ground state of this system are $\mathcal{E}_0 = 0$ and $\phi_0(x) = \exp(W_0(x))/\xi(\eta)$. We now determine the energies and eigenfunctions of the excited states.

With the solutions in Eq. (35), Eq. (10) becomes

$$\mathcal{V}'' + (-\alpha + 3 - \eta)\, \mathcal{V}' + \left[\frac{\mathcal{E} - \alpha + 1}{\eta} - (\ell + 2)\right] \mathcal{V} = 0. \tag{37}$$

In order that $\mathcal{E}$ be dependent on n, we try $\mathcal{V} = \eta^\gamma U(\eta)$ where γ is a real parameter and $U(\eta)$ a function of η. From Eq. (10) we get

$$\eta U'' + (2\gamma - \alpha + 3 - \eta)\, U' + \left[\frac{\mathcal{E} - \alpha + 1 + \gamma(\gamma - \alpha + 2)}{\eta} - (\gamma + \ell + 2)\right] U = 0. \tag{38}$$

Matching this equation with Eq. (31), we have $(n = 0, 1, 2, \ldots)$

$$\begin{aligned} \gamma &= -(n + \ell + 2), \\ \mathcal{E}_{\ell,n} &= \alpha - 1 - \gamma(\gamma - \alpha + 2) = \alpha - 1 - (n + \ell + 2)(n + \ell + \alpha), \\ \beta &= 2\gamma - \alpha + 2 = -\alpha - 2(n + \ell + 1), \\ U_n(\eta) &= L_n^\beta(\eta), \quad \beta > -1. \end{aligned} \tag{39}$$

Putting all these results into $F(\eta)$ and $G(\eta)$ gives

$$\begin{aligned} p(\eta) &\equiv p_{\ell,n}(\eta;\alpha) = \eta^{-n-\ell-1} P_{\ell,n}(\eta;\alpha) \\ P_{\ell,n}(\eta;\alpha) &\equiv \eta L_n^{(\beta)} \xi_\ell' - \left(\ell L_n^{(\beta)} + (n+1) L_{n+1}^{(\beta)}\right) \xi_\ell. \end{aligned} \tag{40}$$

$P_{\ell,n}(\eta;\alpha)$ is a polynomial of degree $\ell+n+1$. It is also easy to check that $p_{\ell,n}(\eta;\alpha)$'s are orthogonal with respect to the weight function

$$\frac{e^{-\eta}\eta^{-\alpha}}{\xi_\ell^2}. \tag{41}$$

The exactly solvable potential is given by

$$V(x) = \frac{1}{4}e^{-2x} + \frac{1}{2}(\alpha - 4\ell - 1)e^{-x} + \frac{\alpha^2}{4} + 2\frac{\xi_\ell'}{\xi_\ell}\left[e^{-2x}\left(\frac{\xi_\ell'}{\xi_\ell} + 1\right) + \alpha e^{-x}\right]. \tag{42}$$

The complete eigenfunctions are

$$\phi_0(x,\alpha) \propto \frac{e^{-\frac{\eta}{2}}\eta^{-\frac{\alpha}{2}}}{\xi_\ell}, \tag{43}$$

$$\phi_{\ell,n}(x;\alpha) \propto \frac{e^{-\frac{\eta}{2}}\eta^{-\frac{\alpha}{2}}}{\xi_\ell} p_{\ell,n}(\eta;\alpha), \tag{44}$$

where $p_{\ell,n}(\eta;\alpha)$'s are given in (40). The corresponding eigen-energies are $\mathcal{E}_0 = 0$ and $\mathcal{E}_{\ell,n}$ in (39), respectively. For the wave functions to be regular at $x = 0$, one must have $n < -\alpha/2 - \ell - 1$. This means the system admits only finite number of bound states.

With the help of the identities (28) and

$$L_\ell^{(\alpha)}(\eta) - L_\ell^{(\alpha-1)}(\eta) = L_{\ell-1}^{(\alpha)}(\eta), \tag{45}$$

$$\eta L_{\ell-1}^{(\alpha+1)}(\eta) - \alpha L_{\ell-1}^{(\alpha)}(\eta) = -\ell L_\ell^{(\alpha-1)}(\eta), \tag{46}$$

one can recast $P_{\ell,n}(\eta;\alpha)$ in (40) into

$$P_{\ell,n}(\eta;\alpha) = -\left[(\alpha+\ell)\, L_{\ell-1}^{(\alpha)}(-\eta) L_n^{(\beta)}(\eta) + (n+1) L_\ell^{(\alpha)}(-\eta) L_{n+1}^{(\beta)}(\eta)\right]. \tag{47}$$

This expression is exactly the same as that given in [23] in the case $\ell = 2m$, with the identification

$$\begin{aligned} \alpha &= -2(a+\ell+1), \\ n &\to k, \end{aligned} \tag{48}$$

$$\beta = -\alpha - 2(n+\ell+1) = 2(a-k). \tag{49}$$

V. SUMMARY

We have shown how the recently discovered solvable rational extensions of Harmonic Oscillator and Morse potentials can be constructed in a direct and systematic way, without the need of supersymmetry, shape invariance, Darboux-Crum and Darboux-Bäcklund transformations. In our approach, the prepotential, the deforming function, the potential, the eigenfunctions and eigenvalues are all derived within the same framework.

With the results given here and in [18], rational extensions of all well-known one-dimensional solvable quantal systems based on sinusoidal coordinates have been generated by the prepotential approach. One would like to apply the same approach to find rational extensions of the other solvable models based on non-sinusoidal coordinates, following the work of the third paper in [25]. Unfortunately, such way of rational extensions only lead to quasi-exactly solvable systems, because the energy quantum number n will appear in the x-dependent terms in $V(x)$. A non-trivial generalization of the present approach may be in order, which we hope to report in the near future.

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