Network Evolution Based on Centrality

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We study the evolution of networks when the creation and decay of links are based on the position of nodes in the network measured by their centrality. We show that the same network dynamics arises under various centrality measures, and solve analytically the network evolution. During the complete evolution, the network is characterized by nestedness: the neighborhood of a node is contained in the neighborhood of the nodes with larger degree. We find a discontinuous transition in the network density between hierarchical and homogeneous networks, depending on the rate of link decay. We also show that this evolution mechanism leads to double power-law degree distributions, with interrelated exponents.

I. INTRODUCTION

The underlying mechanisms of link formation governing the evolution of a network ultimately determine its emergent properties at the aggregate level [1, 2]. In particular, there exists ample empirical evidence that the network evolution can be driven by *centrality*, where nodes with higher centrality are more likely to form or receive links [3, 4]. The notion of centrality was recognized to play a fundamental role in the most despair fields, ranging from dynamical systems [5], synchronization [6], biology [7], and economics [3, 4]. In spite of its importance, a formal understanding of how networks evolve when the formation of links depends on the centrality of the nodes is still missing.

Depending on the context, several measures of centrality have been introduced to quantify the importance of the position of a node in a network: *degree*, *eigenvector*, *betweenness*, *closeness*, *PageRank* and *Bonacich centrality* are the most prominent ones [8, 9]. Due to this variety, few attempts have been made so far to elucidate the common features underlying the emergent properties of networks evolving by centrality [10].

At the macroscopic level, some real world networks exhibit a high degree of clustering while, coincidentally, their degree distributions show power-law tails. Taken together, these two characteristics indicate a hierarchical organization in the network [11]. In social and economic [12, 13], as well as biological systems [14], it has been found that the hierarchical organization of networks can further be characterized by *nestedness* [14, 15]: the neighborhood of a node is contained in the neighborhood of the nodes with higher degrees. In these examples, the extent of nestedness (defined as the fraction of links belonging to the nested structure) was shown to be above 93% [15]. A recent study [16] also finds nested core-periphery structures in over 100 large sparse real-world social and information networks.

In this paper, we study a model of network evolution where links are created or removed based on the centrality of the nodes incident to the links [17]. We show that in this model the network evolution is independent of the particular centrality measure used. Thus, for the first time, this model provides a general framework to study the evolution of networks under various measures of centrality. We show that there exist stationary networks which are highly hierarchical when the rate of link creation is low. Moreover, the networks are *nested* during the complete network evolution. As we show, both, a hierarchical organization as well as network nestedness can be the outcomes of a centrality based network formation process. We further show that in this framework, double power-law degree distributions [18–20] can be stationary solutions, and that each power-law exponent has a univocal relation to the other.

The paper is organized as follows. In Section II we introduce the basic network formation process and discuss the generality of its underlying assumptions, by showing the independence of the dynamics with respect to the particular centrality measure used. Next, in Section III we provide the characterization of the asymptotic network structures generated by our network formation process. In Section IV we then extend the basic network formation process by allowing for heterogeneous linking probabilities among the nodes in the network and study the effect this has on the emerging network structures. Finally, Section V concludes.

II. MODEL STUDIED

We consider a network composed of N nodes, initially connected by an arbitrary network. Each node has a centrality associated to it. At a constant rate (set arbitrarily to one and *a priori* equal for all nodes) a node is randomly selected and modifies its neighborhood: with probability

$\alpha \in [0,1]$, it creates a link to the node with the highest centrality it is not already connected to. With the complementary probability $1 - \alpha$, a link of the selected node decays. If this happens then the node removes the link to the neighbor with the lowest centrality. If the node is connected to all the other nodes in the network (resp. it is isolated), and it has to create (resp. remove) a link, nothing happens. One can show that the network formation process is ergodic, and starting from any initial network yields the same asymptotic results. Thus, and without any loss of generality, in the following we consider an empty network as initial condition.

Note that, when links are created, we could assume that a node has only local information of the network [21] and creates a link to the one with the highest centrality in its second-order neighborhood. It turns out that this leads to the same network evolution process. This makes sense in situations where centrality is known *ex ante*, for example, when centrality is a measure of performance in inter-organizational networks [22], or it indicates the fitness of biological species [7].

The general dynamics we have introduced above can be applied to different areas. In the following we provide a few illustrative examples.

A first example can be given for ecological networks. Consider a population of biological species in a catalytic network [7]. Let species $i = 1, \ldots, N$ have a fitness values $y_i \geq 0$ that evolve according to the dynamics

$$\dot{y}_i = \sum_{j=1}^{N} a_{ij} y_j - \phi y_i, \phi \geq 0,$$

where $a_{ij} \in \{0,1\}$ is the ij-th element of the symmetric adjacency matrix $\mathbf{A}$. In terms of relative fitness $x_i = y_i / \sum_{j=1}^{N} y_j$, we get

$$\dot{x}_i = \sum_{j=1}^{N} a_{ij} x_j - x_i \sum_{k,j=1}^{N} a_{kj} x_j.$$

This dynamics has a fixed point given by the eigenvector $\mathbf{v} \geq 0$ corresponding to the largest real (Perron-Frobenius) eigenvalue λ_{PF} of $\mathbf{A}$. Hence, in this model, the stationary fitness distribtion is directly given by the eigenvector centrality. Applying this centrality measure to our network formation process mimics an evolutionary process in which links to high fitness species are created while links to low fitness species decay.

A second example comes from a socio-economic context. Consider a population of agents whose payoffs are interdependent in a network. The agents choose a contribution level $x_i \geq 0$ and receive a payoff π_i given by

$$\pi_i = x_i - \frac{1}{2} x_i^2 + \lambda \sum_{j=1}^{N} a_{ij} x_i x_j,$$

where $\lambda < 1/\lambda_{\mathrm{PF}}$ [23]. Then, the unique Nash equilibrium satisfying the first-order condition $\partial \pi_i / \partial x_i = 0$ is given by the Bonacich centrality [8],

$$\mathbf{x}^* = (\mathbf{I} - \lambda \mathbf{A})^{-1} \mathbf{1}.$$

The linking dynamics introduced above corresponds to a game in which agents form links that maximize their Nash equilibrium payoffs $\pi_i(\mathbf{x}^*)$ in each period [17].

Further examples include degree centrality, closeness centrality [10], betweenness centrality [8], PageRank [9] and random walk centrality [24]. One can also show that the links created (removed) in our model are the ones which increase the most (decrease the least) the largest eigenvalue λ_{PF}. These links were shown to modify the most the dynamical properties of the system [25].

The distinctive characteristic of our network formation process that allows us to incorporate various centrality measures is the fact that, at every time step $t = 0, 1, 2, \ldots$, our dynamics yields a network whose adjacency matrix $\mathbf{A}$ is *stepwise*: the nodes can be ordered by their degree, such that the zero/one entries in the adjacency matrix are separated by a monotonic step-function $h(x)$ (see Fig. 1, right), where $x = 1 - r$, and r is the degree rank of a node. Networks with a stepwise adjacency matrix are also known as *threshold networks* [26, 27]. In such a network, if two nodes i and j have degrees such that $d_i < d_j$, then their neighborhoods satisfy $N_i \subset N_j$. Thus, these networks are characterized by nestedness. Moreover, the nodes can be partitioned into a *dominating set* and *independent sets*. In the dominating set S, every node not in S is linked to at least one member of S. Conversely, an independent set is one in which no two nodes are adjacent (see Fig. 1, left).

We now prove by induction that the adjacency matrix $\mathbf{A}$ representing the state of the network at every time step is stepwise for the case of eigenvector centrality. First, at time $t = 0$, the first link added generates a (trivial) stepwise matrix. Next, let us assume that this is true at time $t \geq 0$. Consider the creation of a link ij. Then

$$v_i = \frac{1}{\lambda_{\mathrm{PF}}} \sum_{k=1}^{n} a_{ik} v_k = \frac{1}{\lambda_{\mathrm{PF}}} \sum_{k \in N_i} v_k.$$

Thus, the larger is the degree of a node i, the higher is its eigenvector component v_i. In this way, the eigenvector centrality of the nodes is ranked in the same way as their degree. Therefore, for the model studied, a node has to establish a link to a node with the highest degree it is not connected to. This preserves the stepwise property of $\mathbf{A}$ (see Fig. 1, right). Similarly, for the removal of links, the node with the lowest degree among the neighbors is the least central one, and removing a link to it preserves the stepwise property of $\mathbf{A}$.

The nested neighborhood structure allows us to use similar arguments for other centrality measures. Consider two nodes i and j in a nested graph with $d_i > d_j$. All walks starting at node j are contained in the set of all walks starting from node i (after exchanging the starting node j with i). This implies that i has a higher centrality than j for any centrality measure that is based
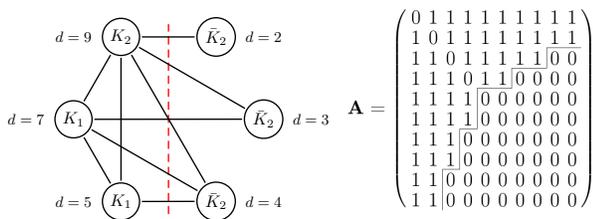
FIG. 1: Representation of a nested network (left) and the associated stepwise adjacency matrix (right) with $N = 10$ nodes. A nested network can be partitioned into subsets of nodes with the same degree (each subset is represented by circle, next to which the degree d of the nodes in the subset is indicated). A line connecting two subsets indicates that there exists a link between each node in one set to all nodes in the other set. The union of the sets represented by the circles to the left of the dashed line induce a dominating set, while to the right the circles indicate independent sets. In the matrix $\mathbf{A}$ to the right, the zero-entries are separated from the one-entries by a step-function, $h(r)$, of the rank degree of the nodes.

on walks or paths in the network. Hence, a proof by induction shows that the ranking of nodes by degree is equivalent to the ranking by centrality for this family of centrality measures. In general, this dynamics leads to a self-reinforcement of the nested structure.

III. RESULTS

Given the symmetry of the adjacency matrix $\mathbf{A}$, in order to solve the dynamic evolution of the network, it is enough to solve the dynamics for the nodes belonging to the independent sets (see Fig. 1). Let us denote by $n(d,t)$ the number of nodes in the independent sets with degree d at time t. The dynamic evolution of these populations can be written as a rate equation,

$$\partial_t n(d,t) = \omega[d+1 \to d]\, n(d+1,t) + \tag{1}$$
$$\omega[d-1 \to d)\, n(d-1,t) -$$
$$(\omega[d \to d-1] + \omega[d \to d+1])\, n(d,t),$$

where the transition rates are simply $\omega[d \to d+1] = \alpha/N$, $\omega[d \to d-1] = (1-\alpha)/N$. In Eq. (1) we have neglected the contributions of the nodes in the corresponding dominating set (which are selected with a probability $\sim \mathcal{O}(N^{-1})$) in the dynamics of the nodes in the independent sets. The dynamics studied is restricted to the profile separating (non-)existing edges, and is thus related to surface-growth models, such as those of polynuclear growth; because of this, it can also be linked to the one-dimensional Ising model with Kawasaki dynamics [28].

The dynamic evolution of the network can be written in terms of its degree distribution $P(k;t) = n(d,t)/N$, where $k = d/N$ denotes the normalized degree. For a finite population, the minimum increment possible in degree is $\delta k = 1/N$. At leading order in δk, the dynamic

evolution corresponding to Eq. (1) is given by

$$\partial_t P(k;t) = (1-2\alpha)\, \partial_k P(k;t)$$
$$+\, \delta k\, \partial^2_{kk} P(k;t) + \mathcal{O}(\delta k^2),$$
$$\partial_t P(0;t) = (1-2\alpha)\, (\delta k + \partial_k P(0;t)$$
$$-\alpha\, \delta k\, P(0;t)) + \mathcal{O}(\delta k^2), \tag{2}$$

with the additional boundary condition $P(1,t) = \mathcal{O}(\delta k^2)$ and an initial condition $P(k,0) = \delta(k)$.

When the terms of order δk can be neglected, Eq. (2) becomes a usual drift equation whose stationary solution is either a complete network for $\alpha > 1/2$ (when the link decay is low), or empty for $\alpha < 1/2$. The reason for this lays in the change of sign in the drift coefficient in such equation. Thus, there is a discontinuous phase transition in terms of the network density as a function of the decay rate α. If α is small (and the link decay is high), the network rapidly converges to a hierarchical structure, where only a few nodes immediately become central, and they remain in this central position during the network evolution. In this case it is the competition driven dynamics for centrality which leads to the spontaneous emergence of hubs [29].

There exists a first order phase transition in the network density that gives rise to nontrivial effects around the critical point $\alpha = 1/2$. If $|1 - 2\alpha|/\delta k \sim \mathcal{O}(1)$, then the diffusion term in Eq. (2) is not negligible anymore. Time scales must be rescaled to $\tau \equiv t\,\delta k$, and we get the Fokker-Planck equations

$$\partial_\tau P(k;\tau) = (1-2\alpha)\, \partial_k P(k;\tau) + \partial^2_{kk} P(k;\tau) \tag{3}$$
$$\partial_\tau P(0;\tau) = \frac{1-2\alpha}{\delta k}\partial_k P(0;\tau). \tag{4}$$

This prescription allows to relate the width of the transition from sparse to dense networks: it must be that $|1 - 2\alpha| \sim \mathcal{O}(1)$, or conversely, $\Delta\alpha \sim N^{-1}$.

We now study the stationary solutions for all values of $\alpha \in [0,1]$. First, notice that the network obtained for a value of $\alpha > 1/2$ is the complement of the network obtained for $1-\alpha < 1/2$. Thus, in the following we consider only values of $\alpha \le 1/2$. The step-function $h(x)$ can be decomposed in a part $h_u(x)$ below the diagonal and a part $h_l(x)$ above the diagonal of $\mathbf{A}$ (see Fig. 1, right panel). The point x^* is implicitly defined by $h_u(x^*) = h_l(x^*)$, where the step-function $h(x)$ intersects with the diagonal. Let $P(k)$ denote the stationary degree distribution. We have that $h_u(x) = \int_0^{1-x} P(k)dk$. From the stationary solution of Eq. (2) we find

$$h_u(x) = \mathcal{N}e^{-2(1-2\alpha)x},$$

with

$$\mathcal{N} = \frac{2(1-2\alpha)}{1 - e^{-2(1-2\alpha)N}}.$$

This result for the functional form of the step-function is valid for the elements below the diagonal, i.e. for the nodes with low degree.
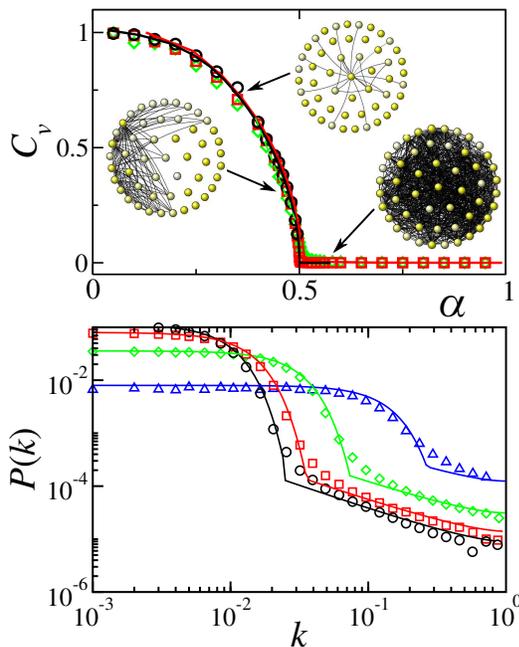
FIG. 2: (Upper panel) Eigenvector centralization $\mathcal{C}_v$ in stationary networks as a function of the link formation probability α for different system sizes $N = 100$ (○), $N = 1000$ (□) and $N = 5000$ (◇). Results of numerical simulations are superimposed with lines representing the analytical prediction. (Lower panel) Degree distributions of stationary networks for different values of $\alpha = 0.45$ (○), 0.48 (□), 0.49 (◇), 0.495 (△) and system size $N = 5000$. The figure reveals that the leading part of the distribution is exponential, while a logarithmic binning shows a power-law tail with exponent -1.

We now turn our attention to the high degree, central nodes. From the symmetry of the adjacency matrix, one finds that $h_l(x)$ for these nodes satisfies $x = \mathcal{N}e^{-2(1-2\alpha)h_l(x)}$. Thus, inverting this expression we get

$$h_l(x) = \frac{\ln(\mathcal{N}) - \ln(x)}{2(1 - 2\alpha)}.$$

Conversely, the degree distribution is given by $P(k) = -h'(1 - k)$, from which the following stationary degree distribution is found

$$P(k) = \begin{cases} \mathcal{N}e^{-2(1-2\alpha)k}, & \text{if } k < 1 - x^*, \\ \frac{1}{2(1-2\alpha)}k^{-1}, & \text{if } k > 1 - x^*. \end{cases} \quad (5)$$

In particular, for $\alpha = 1/2$, it results in a uniform distribution $P(k) = 1/N$. Degree distributions for different values of α in the stationary state can be seen in Fig. 2.

In these nested structures, the adjacency matrix is completely determined by the corresponding degree distribution from Eq. (5) or, conversely, from the profile function $h(x)$. Thus, it is possible to compute any network statistic of interest when the degree distribution is known. In doing so, one can show that the stationary

networks emerging in the link formation process are characterized by short path length, high clustering, negative degree-clustering correlations and dissortativity.

The emerging networks also show a clear core-periphery structure, which can be measured by their centralization. To quantify this, we compute the degree of centralization of the network $\mathcal{C}_v$, as [8]

$$\mathcal{C}_v = \frac{\sum_i \left(\mathcal{C}_v(i^*) - \mathcal{C}_v(i)\right)}{\sum_j \left(\mathcal{C}_v^*(j^*) - \mathcal{C}_v^*(j)\right)}, \quad (6)$$

where $\mathcal{C}_v(i)$ if the eigenvector centrality of node i, i^* is the node with the largest centrality in the network. The denominator normalizes the value between zero and one, by the computation of the centralization $\sum_j \left(\mathcal{C}_v^*(j^*) - \mathcal{C}_v^*(j)\right)$ of a star network the the same maximum degree as the considered one. In Fig. 2 (upper panel) we show the transition from hierarchical to decentralized networks measured in terms of the degree of centralization of the network, as a function of the parameter α. In the same plot, also exemplary stationary networks are depicted. It can be seen that there exists a transition at $\alpha = 1/2$ from highly centralized to highly decentralized networks. This means that for low arrival rates of linking opportunities α (and a strong link decay) the stationary network is strongly centralized, while for high arrival rates of linking opportunities, stationary networks are dense and largely homogeneous.

IV. GENERALIZED ATTACHMENT

The symmetry condition for the step-function $h(x)$ implies an important result when part of the degree distribution (for example around the head, i.e. $k \to 0$) shows a power-law decay: The tail of the distribution (i.e. $k \to \infty$) also follows a power-law distribution, but with a different exponent. To see this, let us assume that the head of the distribution has the functional dependence $P(k) = \beta k^{-\eta}$. If $\eta > 0$, this implies that the step-function $h_l(x)$ for low degree nodes is given by $h_l(x) = \beta k^{-\eta-1}/1 - \eta$. By inverting this function, we get

$$x = \frac{\beta}{1 - \eta}h_u(x)^{-\frac{1}{\eta-1}};$$

and the distribution in the tail yields

$$P_u(k) = \frac{1}{1 + \eta}\left(\frac{\beta}{1 - \eta}\right)^{\frac{1}{\eta+1}} k^{-\eta_u},$$

where $\eta_u = \eta/(\eta - 1)$. In the limit $\eta \to \infty$, (there is an exponential distribution for the head), it implies $\eta_u \to 1$, i.e. we recover the previous result of Eq. (5). The power-law distribution in the head and in the tail have the same exponent when $\eta = 2$.

So far we have assumed that all nodes are selected at the same rate, regardless of their position in the network. Depending on the context, this assumption may
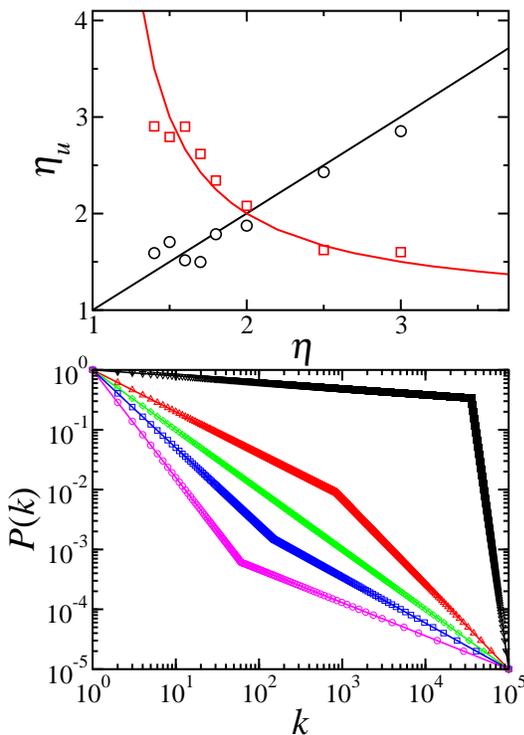
FIG. 3: (Upper panel) Degree distribution for different exponents in the head of the distribution $\eta = 1.2$ (∇), 1.5 ($\triangle$), 2 ($\diamond$), 2.5 ($\square$), 3 ($\circ$). If a nested network exhibits a power-law in the head (tail) of the degree distribution then the distribution will also exhibit a power-law behavior in the tail (head), with an exponent that can be completely determined by the head (tail). (Lower panel) Power-law exponents for the tail of the degree distribution, i.e. $k \to \infty$ ($\square$), and the head of the distribution, i.e. $k \to 0$ ($\circ$), as a function of the power-law exponent of the head. The symbols correspond to networks of $N = 10^5$ nodes, and the lines represent the numerical simulations.

not apply. In order to overcome this limitation, we assume that nodes are selected at a rate which depends on their position in the network. Note that the rate at which nodes are selected affects only the frequency but not the way in which they create or remove links. Therefore, the nestedness of the network is preserved. Moreover, in these nested structures, the nodes with the same degree are indistinguishable, as only their degree rank in the network is important. We therefore assume that the node selection rate F is a function of the degree of the node. As a simple example, we set $F(k) = k^\eta + A$, where $A > 0$ denotes the idiosyncratic activity of every node, and $\eta > 0$ a parameter governing nonlinearly the preferential selection of nodes with higher degree. Using similar arguments as in the derivation of Eq. (2), we can

write the evolution of the degree distribution as follows,

$$
\begin{aligned}
\partial_t P(k;t) = {} & \frac{(1-2\alpha)\eta}{\mathcal{N}} k^{\eta-1} P(k;t) \\
& + \frac{1-2\alpha}{2\mathcal{N}} \left[k^\eta + A \right] \partial_k P(k;t) \delta k + \mathcal{O}(\delta k^2).
\end{aligned}
$$

In the continuous limit, the stationary solution is given by

$$
P(k) = \frac{D}{A + k^\eta},
$$

where D is a normalization constant such that $\int_0^1 P(k) \, dk = 1$. The solution reduces to the exponential one when $\eta \to \infty$ and $A \ll N$. In the general case, the degree distribution exhibits two different power-law behaviors and an inflection point. These two power-laws have the functional form $P(k) \sim k^{-\eta}$ for the head of the distribution, and consequently

$$
P(k) \sim k^{-\frac{\eta}{\eta-1}}
$$

for the tail.

V. CONCLUSIONS

In this paper we have introduced a network formation process in which link creation and removal is based on the position of the nodes in the network measured by their centrality. We have shown that the network evolution is independent of the exact measure of centrality, and our results hold irrespective of whether degree centrality or any more general centrality measure that is based on walks or paths in the network is used. Thus, our model provides a general framework to study the evolution of networks under various measures of centrality. Moreover, we can show that the link formation decision of nodes does not require global information of the complete network structure. A further characteristic property of our model is that the emerging network structures are nested with a tunable degree of centralization, depending on the likelihood with which links are formed. This illustrates that both, a hierarchical organization as well as network nestedness can be the outcomes of a centrality based network formation process. Finally, we extend the model to allow for heterogeneous activity levels in the linking process of the nodes. We show that this generalization keeps the basic properties of the model unaltered, although the degree distribution is modified, and a restricted set of double power-law degree distributions is found.

We have also discussed the broad range of applications of this kind of dynamics. In this context, it is worth mentioning the recent empirical analysis of the European interbank payment network, which shows that our dynamic model matches closely the observed network pattern [30].

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