

NATURALLY REDUCTIVE PSEUDO RIEMANNIAN 2-STEP NILPOTENT LIE GROUPS

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ABSTRACT. This paper deals with naturally reductive pseudo Riemannian 2-step nilpotent Lie groups (N, \langle, \rangle_N) . In the cases under consideration they are related to bi-invariant metrics. On the one hand, whenever \langle, \rangle_N restricts to a metric in the center it is proved that the simply connected Lie group N arises from a Lie algebra \mathfrak{g} and a representation of it. The Lie algebra \mathfrak{g} carries an ad-invariant metric and its corresponding Lie group acts as a group of isometries of (N, \langle, \rangle) fixing the identity element. On the other hand, a bi-invariant metric \langle, \rangle on N provides another family of examples of naturally reductive spaces, namely those of the form $(N/\Gamma, \langle, \rangle)$ being $\Gamma \subset N$ a lattice, which are also investigated.

1. INTRODUCTION

The 2-step nilpotent groups are nonabelian and from the algebraic point of view as close as possible to be Abelian and they evidence a rich geometry when equipped with a metric tensor. While they have been extensively investigated in the Riemannian situation, in the case of indefinite metrics, there are significant advances as showed in [Bo, C-P1, C-P2, Ge, J-P-L, J-P, J-P-P, Pa] but there are still several open problems. A first obstacle appears when trying to traduce the left invariant metric to the Lie algebra level. So far all attempts in this direction take as starting point the Riemannian model. Among these pseudo Riemannian spaces, the *naturally reductive* ones are endowed with nice simple algebraic and geometric properties. Examples of them are provided by 2-step nilpotent Lie groups carrying a bi-invariant metric.

Important studies concerning the structure of a naturally reductive Riemannian Lie group G when G is compact and simple or when G is non compact and semisimple were given by D'Atri-Ziller [DA-Z] and Gordon [Go] respectively. Gordon showed that every naturally reductive Riemannian manifold may be realized as a homogeneous space G/H with Lie group G of the form $G = G_{nc}G_cN$ where G_{nc} is a non compact semisimple normal subgroup, G_c is compact semisimple and N is the nilradical of G . Furthermore $N \cap H = \{0\}$ and the induced metrics on each of $G_{nc}/(G_{nc} \cap H)$, $G_c/(G_c \cap H)$ and $N(= N/(N \cap H))$ are naturally reductive so that the study of naturally reductive metrics is partially reduced to the cases in which G is semisimple either of compact or non compact type or G is nilpotent. In the last case Gordon proved that G must be at most 2-step nilpotent. Lauret [La] exploited this result to afford a classification of naturally reductive Riemannian connected simply connected nilmanifolds. According to Wilson [Wi] such a manifold can be realized as a 2-step nilpotent Lie group equipped with a left invariant metric.

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Later Tricerri and Vanhecke [T-V2] proved that a Riemannian manifold is a naturally reductive homogeneous space if and only if there exists a homogeneous structure T satisfying $T_x x = 0$ for all tangent vectors x , offering in this way an infinitesimal description of these reductive manifolds. The notion of *homogeneous structure* was introduced by Ambrose and Singer [A-S] to characterize connected simply connected and complete homogeneous Riemannian manifolds. In the Riemannian case every homogeneous manifold is complete and reductive. More recently Gadea and Oubiña [G-O1] proved that a connected simply connected and complete pseudo Riemannian manifold admits a pseudo-Riemannian structure if and only if it is reductive homogeneous. While Tricerri and Vanhecke [T-V1] achieved the classification of homogeneous Riemannian structures, in the pseudo Riemannian case, a complete classification is still a pending item.

However Calvaruso and Marinosci [Ca, C-M1, C-M2] studied homogeneous structures in dimension three, obtaining with their results the naturally reductive Lie groups with a left invariant Lorentzian metric. In particular the Heisenberg Lie group admits two naturally reductive left invariant Lorentzian metrics (and for which the center is non degenerate).

In this paper we provide constructions for naturally reductive pseudo Riemannian 2-step nilpotent Lie groups. By following a similar approach to that one of Gordon, one gets necessary and sufficient conditions to have naturally reductive pseudo Riemannian 2-step nilpotent Lie groups with non degenerate center -Theorem (3.2)-. This enables to attach this kind of Lie groups to Lie algebras endowed with an ad-invariant metric and to certain kind of representations of them:

Theorem 3.3 *Let \mathfrak{g} denote a Lie algebra carrying an ad-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and let (π, \mathfrak{v}) be a real faithful representation of \mathfrak{g} without trivial subrepresentations and such that the metric on \mathfrak{v} , $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ is $\pi(\mathfrak{g})$ -invariant. Let \mathfrak{n} denote the Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ whose the Lie bracket is given by*

$$[\mathfrak{g}, \mathfrak{g}]_{\mathfrak{n}} = [\mathfrak{g}, \mathfrak{v}]_{\mathfrak{n}} = 0 \quad [\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{g}$$

$$\langle [u, v], x \rangle_{\mathfrak{g}} = \langle \pi(x)u, v \rangle_{\mathfrak{v}} \quad \text{for all } x \in \mathfrak{g}, \forall u, v \in \mathfrak{v},$$

equip \mathfrak{n} with the metric $\langle \cdot, \cdot \rangle$

$$\langle \cdot, \cdot \rangle_{\mathfrak{g} \times \mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{g}} \quad \langle \cdot, \cdot \rangle_{\mathfrak{v} \times \mathfrak{v}} = \langle \cdot, \cdot \rangle_{\mathfrak{v}} \quad \langle \mathfrak{g}, \mathfrak{v} \rangle = 0$$

then the corresponding simply connected 2-step nilpotent Lie group $(N, \langle \cdot, \cdot \rangle)$, being $\langle \cdot, \cdot \rangle$ the left invariant metric induced by $\langle \cdot, \cdot \rangle$ above, is a naturally reductive pseudo Riemannian space.

The converse holds whenever the center of \mathfrak{n} is non degenerate and j (defined as in (4)) is faithful.

The previous result empores the understanding of some geometrical features such as the isometry group -Proposition 3.5- and it sets up the construction of new examples, in particular by describing some naturally reductive metrics in the Heisenberg Lie group H_{2n+1} .

We also bring into consideration 2-step nilpotent Lie groups furnished with a bi-invariant metric in order to check geometrical and algebraic structure differences between metrics for which the center is either degenerate or either non degenerate. In

fact bi-invariant metrics offer examples of flat pseudo Riemannian metrics for which the e isometry group contains the group of orthogonal automorphisms as a proper subgroup. Another application of bi-invariant metrics promotes the construction of pseudo Riemannian naturally reductive compact spaces.

2. ON 2-STEP NILPOTENT LIE GROUPS WITH A LEFT INVARIANT PSEUDO RIEMANNIAN METRIC

In this section we show suitable decompositions of the Lie algebra corresponding to a 2-step nilpotent Lie group equipped with a left invariant pseudo Riemannian metric. We are mainly interested here in those metrics for which the center is non degenerate, a fact that determines unambiguously the decomposition.

A *metric* on a real vector space \mathfrak{v} is a non degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{v} \times \mathfrak{v} \rightarrow \mathbb{R}$. Whenever \mathfrak{v} is the Lie algebra of a given Lie group G , by identifying \mathfrak{v} with the set of left invariant vector fields on G , the metric induces by mean of the left translations, a pseudo Riemannian metric tensor on the corresponding Lie group. Conversely any left invariant pseudo Riemannian metric on G is completely determined by its value at the identity tangent space $T_e G$.

Let $(N, \langle \cdot, \cdot \rangle)$ denotes a 2-step nilpotent Lie group endowed with a left invariant pseudo Riemannian metric. There exist several ways to describe the structure of the corresponding Lie algebra \mathfrak{n} . The main difficult yields on the existence of degenerate subspaces as one can see below.

(a) If the center is degenerate, the null subspace is defined uniquely as

$$\mathfrak{u} = \{x \in \mathfrak{z} \text{ such that } \langle x, z \rangle = 0 \quad \forall z \in \mathfrak{z}\}$$

and therefore the center of \mathfrak{n} decomposes as a direct sum of vector subspaces

$$\mathfrak{z} = \mathfrak{u} \oplus \tilde{\mathfrak{z}}$$

where $\tilde{\mathfrak{z}}$ is a complementary subspace of \mathfrak{u} in \mathfrak{z} and it is easy to prove that the restriction of the metric to $\tilde{\mathfrak{z}}$ is non degenerate. Moreover it is possible to find a isotropic subspace $\mathfrak{v} \subset \mathfrak{n}$ such that $\mathfrak{v} \cap \mathfrak{z} = \{0\}$ and the metric on $\mathfrak{u} \oplus \mathfrak{v}$ is non degenerate. This subspace \mathfrak{v} is not well defined invariantly but once \mathfrak{v} is fixed, one can take $\tilde{\mathfrak{z}}$ as the portion of the center in $(\mathfrak{u} \oplus \mathfrak{v})^\perp$ and to complete the decomposition of \mathfrak{n} as a orthogonal direct sum

$$(1) \quad \mathfrak{n} = (\mathfrak{u} \oplus \mathfrak{v}) \oplus (\tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{v}})$$

in such a way that $(\mathfrak{u} \oplus \mathfrak{v})^\perp = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{v}}$ and (1) is a Witt decomposition. Note that $\tilde{\mathfrak{v}}$ is non degenerate. Moreover it is possible to define a lineal map $j : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v} \oplus \tilde{\mathfrak{v}})$ which play a similar role to that one in the Riemannian case (see [C-P1] for details).

In the last section of the present work, we show similar results for the case of bi-invariant metrics.

(b) Let e_1, \dots, e_p denote a basis of \mathfrak{z} . For any $u, v \in \mathfrak{n}$, the Lie bracket can be written

$$[u, v] = \sum_{i=1}^p \langle J_i u, v \rangle e_i,$$

where $J_i : \mathfrak{n} \rightarrow \mathfrak{n}$ are self adjoint endomorphisms with respect to $\langle \cdot, \cdot \rangle$ and $\mathfrak{z} = \bigcap_{i=1}^p \ker J_i$. In fact, $[u, v] = \sum \omega_i(u, v) e_i$ where $\omega_i : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathbb{R}$ for $i = 1, \dots, p$, is a family of skew symmetric bilinear 2-forms which represents the coordinates of $[u, v]$ with respect to

the fixed basis. Since the metric on \mathfrak{n} is non degenerate, for every i there exists a endomorphism $J_i : \mathfrak{n} \rightarrow \mathfrak{n}$ such that $\omega_i(u, v) = \langle J_i u, v \rangle$. The endomorphisms J_i are thus called the *structure endomorphisms* associated to e_1, \dots, e_p (see [Bo]).

Examples of pseudo Riemannian 2-step nilpotent Lie groups N arise by considering the simply connected Lie groups whose Lie algebra can be constructed as follows. Let $(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$ and $(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ denote vector spaces endowed with (non necessarily definite) metrics. Let \mathfrak{n} denote the direct sum as vector spaces

$$(2) \quad \mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} \quad \text{direct sum}$$

and let $\langle \cdot, \cdot \rangle$ denote the metric given by

$$(3) \quad \langle \cdot, \cdot \rangle_{|\mathfrak{z} \times \mathfrak{z}} = \langle \cdot, \cdot \rangle_{\mathfrak{z}} \quad \langle \cdot, \cdot \rangle_{|\mathfrak{v} \times \mathfrak{v}} = \langle \cdot, \cdot \rangle_{\mathfrak{v}} \quad \langle \mathfrak{z}, \mathfrak{v} \rangle = 0.$$

Let $j : \mathfrak{z} \rightarrow \text{End}(\mathfrak{v})$ be a linear map such that $j(z)$ is self adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ for all $z \in \mathfrak{z}$. Then \mathfrak{n} becomes a 2-step nilpotent Lie algebra if one defines a Lie bracket by

$$(4) \quad \begin{aligned} [x, y] &= 0 \quad \text{for all } x \in \mathfrak{z}, y \in \mathfrak{n} \\ \langle [u, v], x \rangle &= \langle j(x)u, v \rangle \quad \text{for } x \in \mathfrak{z}, u, v \in \mathfrak{v}. \end{aligned}$$

Conversely, let \mathfrak{n} denote a 2-step nilpotent Lie algebra furnished with a metric for which the center is non degenerate. Then \mathfrak{n} can be decomposed into a orthogonal direct sum as in (2) being $\mathfrak{v} := \mathfrak{z}^{\perp}$ and the Lie bracket on \mathfrak{n} induces self adjoint linear maps $j(x)$ for $x \in \mathfrak{z}$ given by (4).

Proposition 2.1. *Let $(N, \langle \cdot, \cdot \rangle)$ denote a simply connected 2-step nilpotent Lie group equipped with a left invariant pseudo Riemannian metric. If the center of N is non degenerate then its Lie algebra \mathfrak{n} admits a orthogonal decomposition as in (2) and the corresponding Lie bracket can be obtained by (4).*

This includes the Riemannian case, that is, when the metric $\langle \cdot, \cdot \rangle$ is positive definite. In this situation, the inner product $\langle \cdot, \cdot \rangle_+$ produces a decomposition of the center of the Lie algebra \mathfrak{n} as a orthogonal direct sum as vector spaces

$$\mathfrak{z} = \ker j \oplus C(\mathfrak{n})$$

and moreover j is injective if and only if there is no Euclidean factor in the De Rahm decomposition of the simply connected Lie group $(N, \langle \cdot, \cdot \rangle_+)$ (see [Go]). This does not necessarily hold in the pseudo Riemannian case. Below we show an example of a Lorentzian metric on a 2-step nilpotent Lie algebra \mathfrak{n} , where the center is non degenerate and such that $\ker(j) = [\mathfrak{n}, \mathfrak{n}]$, so that a splitting as above is not possible.

Example 2.2. Let $\mathbb{R} \times \mathfrak{h}_3$ be the 2-step nilpotent Lie algebra spanned by the vectors e_1, e_2, e_3, e_4 with the Lie bracket $[e_1, e_2] = e_3$. Define a metric where the non trivial relations are

$$\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_4 \rangle = 1.$$

After (4) one can verify that $j(e_3) \equiv 0$, while

$$j(e_4) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Notice that $e_4 \notin C(\mathbb{R} \times \mathfrak{h}_3)$ and $\ker j = \mathbb{R}e_3 = C(\mathbb{R} \times \mathfrak{h}_3)$, that is $\ker j = C(\mathfrak{n})$.

Let $O(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ denote the group of linear maps on \mathfrak{v} which are isometries for $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ and whose Lie algebra $\mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ is the set of linear maps on \mathfrak{v} that are self adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$. The next goal is to describe the group of isometries which plays an important role in the next section. Start with the next result proved in [C-P1].

Proposition 2.3. *Let N denote a 2-step nilpotent Lie group endowed with a left invariant pseudo Riemannian metric, with respect to which the center is non degenerate. Then the group of isometries fixing the identity coincides with the group of orthogonal automorphisms of N .*

Denote by H the group of orthogonal automorphisms and by N also the subgroup of isometries consisting of left translations by elements of N . Consider the isometries of the form hn where $h \in H$ and $n \in N$, and denote it by $I_a(N)$. Then N is a normal subgroup of $I_a(N)$, $N \cap H = \{e\}$ and therefore after (2.3) one has

$$I(N) = I_a(N) = H \times N.$$

Whenever $(N, \langle \cdot, \cdot \rangle)$ is simply connected, we do not distinguish between the group of automorphisms of N and of \mathfrak{n} . Thus one obtains that the group H is given by

$$(5) \quad H = \{(\phi, T) \in O(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : Tj(x)T^{-1} = j(\phi x), \quad x \in \mathfrak{z}\}$$

while its Lie algebra $\mathfrak{h} = \text{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is

$$(6) \quad \mathfrak{h} = \{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : [B, j(x)] = j(Ax), \quad x \in \mathfrak{z}\}.$$

In fact, let ψ denote an orthogonal automorphism of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$. As automorphism $\psi(\mathfrak{z}) \subseteq \mathfrak{z}$ and since the decomposition

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$$

is orthogonal then $\psi(\mathfrak{v}) \subseteq \mathfrak{v}$. Set $\phi := \psi|_{\mathfrak{z}}$ and $T := \psi|_{\mathfrak{v}}$, thus $(\phi, T) \in O(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times O(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ such that

$$\begin{aligned} \langle \phi^{-1}[u, v], x \rangle &= \langle [Tu, Tv], j(x) \rangle \quad \text{if and only if} \\ \langle j(\phi x)u, v \rangle &= \langle j(x)Tu, Tv \rangle \end{aligned}$$

which implies (5). By derivating (5) one gets (6).

Proposition 2.4. *Let N denote a simply connected 2-step nilpotent Lie group endowed with a left invariant pseudo Riemannian metric, with respect to which the center is non degenerate. Then the group of isometries is*

$$I(N) = H \times N.$$

where N denotes the set of left translations by elements of N and H the isotropy subgroup is given by (5) with Lie algebra as in (6).

Example 2.5. Let \mathfrak{n} be a 2-step nilpotent Lie algebra equipped with an inner product and denote it by $\langle \cdot, \cdot \rangle_+$. Let $J_z \in \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_+)$ denote the maps in (4) for the inner product.

We shall consider a non definite metric $\langle \cdot, \cdot \rangle$ on \mathfrak{n} by changing the sign of the metric on the center \mathfrak{z} ; thus the metric on \mathfrak{v} remains invariant and we take

$$\langle z_i, z_j \rangle = -\langle z_i, z_j \rangle_+ \quad \text{for } z_i, z_j \in \mathfrak{z} \quad \text{and} \quad \langle \mathfrak{z}, \mathfrak{v} \rangle = 0.$$

By (4) the maps $j(z)$ for the metric $\langle \cdot, \cdot \rangle$ on \mathfrak{n} are

$$-\langle z, [u, v] \rangle_+ = -\langle J(z)u, v \rangle_+ = \langle j(z)u, v \rangle = \langle z, [u, v] \rangle, \quad \text{for } z \in \mathfrak{z}$$

that is $j(z) = -J(z)$ for every $z \in \mathfrak{z}$.

We work out an example on the Heisenberg Lie group H_3 . This is the simply connected Lie group whose Lie algebra is \mathfrak{h}_3 which is spanned by the vectors e_1, e_2, e_3 , with the non trivial Lie bracket relation $[e_1, e_2] = e_3$. The canonical left invariant metric $(\langle \cdot, \cdot \rangle_+)$ is that one obtained by declaring the basis above to be orthogonal and the map $J(e_3)$ for $\langle \cdot, \cdot \rangle_+$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

A Lorentzian metric $\langle \cdot, \cdot \rangle$ is obtained on H_3 by changing the sign of the canonical metric on the center. Kaplan showed that $(H_3, \langle \cdot, \cdot \rangle_+)$ is naturally reductive ([Ka]). In the next sections we shall see that $(H_3, \langle \cdot, \cdot \rangle)$ and generalizations of it, are also naturally reductive.

By (5) the group of isometries for any of these both metrics is $(\mathbb{R} \times O(2)) \ltimes H_3$, where the action of the isotropy group is given by $(\lambda, A) \cdot (z + v) = \lambda z + Av$ for $z \in \mathfrak{z}$ and $v \in \mathfrak{v} = \text{span}\{e_1, e_2\}$, $\lambda \in \mathbb{R}$ and $A \in O(2)$.

Definition 2.6. A homogeneous manifold M is said to be *naturally reductive* if there is a transitive Lie group of isometries G with Lie algebra \mathfrak{g} and there exists a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ complementary to \mathfrak{h} , the Lie algebra of the isotropy group H , in \mathfrak{g} such that

$$\text{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m} \quad \text{and} \quad \langle [x, y]_{\mathfrak{m}}, z \rangle + \langle y, [x, z]_{\mathfrak{m}} \rangle = 0 \quad \text{for all } x, y, z \in \mathfrak{m}.$$

Frequently we will say that a metric on a homogeneous space M is naturally reductive even though it is not naturally with respect to a particular transitive group of isometries (see Lemma 2.3 in [Go]).

For naturally reductive metrics the geodesics passing through $m \in M$ are of the form

$$\gamma(t) = \exp(tx) \cdot m \quad \text{for some } x \in \mathfrak{m}.$$

A point p of a pseudo Riemannian manifold is called a *pole* provided the exponential map \exp_p is a diffeomorphism. Furthermore if o is a pole of the naturally reductive pseudo Riemannian manifold G/H , then the map $(x, h) \rightarrow \exp(x)h$ is a diffeomorphism of $\mathfrak{m} \times H \rightarrow G$ [O'N] Ch. 11.

Indeed pseudo Riemannian symmetric spaces are naturally reductive. Examples of naturally reductive spaces arise from Lie groups equipped with a bi-invariant metric, which could exist for nilpotent ones. In the Riemannian case, if a nilmanifold N admits a naturally reductive metric, then N is at most 2-step nilpotent [Go].

3. NATURALLY REDUCTIVE METRICS WITH NON DEGENERATE CENTER: A CHARACTERIZATION

In this section we achieve a characterization of naturally reductive pseudo Riemannian simply connected 2-step nilpotent Lie groups with non degenerate center by studying the set of maps $j(z)$ $z \in \mathfrak{z}$ defined in (4), showing that they build a subalgebra of the Lie algebra of the isotropy group H .

Lemma 3.1. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie algebra equipped with a metric for which its center \mathfrak{z} is non degenerate and assume j is injective. Let $\mathfrak{h} = \mathfrak{so}(\mathfrak{n}, \langle \cdot, \cdot \rangle) \cap \text{Der}(\mathfrak{n})$ denote the Lie subalgebra of the group of isometries fixing the identity element in the corresponding simply connected Lie group N . Then*

- i) \mathfrak{h} leaves each of \mathfrak{z} and \mathfrak{v} invariant,
- ii) For $\phi \in \mathfrak{h}$,

$$\phi|_{\mathfrak{z}} = j^{-1} \circ \text{ad}_{\mathfrak{so}(\mathfrak{v})} \phi|_{\mathfrak{v}} \circ j.$$

In particular $\phi \rightarrow \phi|_{\mathfrak{v}}$ is an isomorphism of \mathfrak{h} onto a subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$.

iii) Let $\phi \in \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$. Then ϕ extends to an element of \mathfrak{h} if and only if $[\phi, j(\mathfrak{z})] \subseteq J(\mathfrak{z})$ and $j^{-1} \circ \text{ad}_{\mathfrak{so}(\mathfrak{v})} \phi|_{\mathfrak{v}} \circ j \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$.

Proof. i) is easy to prove. We shall show (ii) and (iii). Let $A \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$ and $B \in \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$, the linear map ϕ which agrees with $(A, B) \in \mathfrak{z} \oplus \mathfrak{v}$ lies in \mathfrak{h} if and only if

$$\langle j(Ax)u, v \rangle = \langle (Bj(x) - j(x)B)u, v \rangle \quad \text{for } x \in \mathfrak{z}, u, v \in \mathfrak{v}$$

which is equivalent to $j(A(x)) = [B, j(x)]$ the last one denotes the Lie bracket in $\mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ and since j was assumed injective one gets $A = j^{-1} \circ \text{ad}_{\mathfrak{so}(\mathfrak{v})}(B) \circ j$. \square

The proof of the next theorem coincides with that one given by C. Gordon in [Go]. For the sake of completeness we include it here. However the consequences are quite different from the Riemannian situation.

Theorem 3.2. *Let $(N, \langle \cdot, \cdot \rangle)$ denote a 2-step simply connected Lie group equipped with a left invariant pseudo Riemannian metric such that the center is non degenerate and assume j is injective. Then the metric is naturally reductive with respect to $G = H \ltimes N$ being H the group of orthogonal automorphisms, if and only if*

- (i) $j(\mathfrak{z})$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ and
- (ii) $[j(x), j(y)] = j(\tau_x y)$ where $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$ for any $x \in \mathfrak{z}$.

Proof. Let $\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{n}$ be the Lie algebra of $G = H \ltimes N$ and assume N is naturally reductive with respect to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Set $\pi : \mathfrak{n} \rightarrow \mathfrak{h}$ so that

$$\mathfrak{m} = \{x + \pi(x) : x \in \mathfrak{n}\}.$$

The condition for natural reductivity says

$$\langle [x + \pi(x), y + \pi(y)]_{\mathfrak{m}}, z + \pi(z) \rangle_{\mathfrak{m}} = -\langle y + \pi(y), [x + \pi(x), z + \pi(z)]_{\mathfrak{m}} \rangle_{\mathfrak{m}}$$

where $\langle \cdot, \cdot \rangle$ is the pseudo Riemannian metric on \mathfrak{m} , so that the previous equality can be interpreted on \mathfrak{n} as

$$(7) \quad \langle [x, y] + \pi(x)y - \pi(y)x, z \rangle = -\langle y, [x, z] + \pi(x)z - \pi(z)x \rangle.$$

where $\pi(x)$ is view as a linear operator on \mathfrak{n} and one writes $\pi(x)y = [x, y]$ when $x, y \in \mathfrak{n}$. Since $\pi(x) \in \mathfrak{so}(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ the terms involving $\pi(x)$ cancel and (7) yields

$$(8) \quad \text{ad}(y)^* z + \text{ad}(z)^* y = \pi(y)z + \pi(z)y \quad \text{for all } y, z \in \mathfrak{n}.$$

Since $[\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{n}$ and $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, one has

$$[\pi(x), y + \pi(y)] = \pi(x)y + [\pi(x), \pi(y)] \in \mathfrak{m}$$

and therefore

$$(9) \quad \pi(\pi(x)y) = [\pi(x), \pi(y)] \quad \text{for all } x, y \in \mathfrak{n}.$$

If $z \in \mathfrak{z}$ and $y \in \mathfrak{v}$, $\text{ad}(z)^*y = 0$ and (8) says

$$(10) \quad j(z)y = \pi(y)z + \pi(z)y.$$

But $\pi(y)z \in \mathfrak{z}$ and $\pi(z)y \in \mathfrak{v}$, so (10) implies

$$\pi(z)|_{\mathfrak{v}} = j(z) \in \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) \quad \text{for every } z \in \mathfrak{z}.$$

It then follows that

$$[j(x), j(\mathfrak{z})] \subset j(\mathfrak{z}) \quad \text{and} \quad [j(x), j(y)] = j(\tau_x y) \quad \text{for } \tau_x \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}), \quad x, y \in \mathfrak{z}.$$

Conversely if (i) and (ii) hold, extend $j(x)$ to an element $\pi(x)$ of \mathfrak{h} such that the restriction of $\pi(x)$ to \mathfrak{z} is given by the left hand side of (ii). Extend ρ as a linear map of \mathfrak{n} by declaring $\pi|_{\mathfrak{v}} \equiv 0$. We claim (9) hold for all $x, y \in \mathfrak{n}$. In fact it is easy to verify it if at least one of $x, y \in \mathfrak{v}$. Assume $x, y \in \mathfrak{z}$, then

$$\pi(\pi(x)y)|_{\mathfrak{v}} = j(j^{-1}[j(x), j(y)]) = [j(x), j(y)]$$

and therefore (9) is true after (3.1) ii). Define

$$\mathfrak{l} = \pi(\mathfrak{n}), \quad \mathfrak{m} = \{x + \pi(x) : x \in \mathfrak{n}\}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}.$$

By (9) \mathfrak{l} is a Lie subalgebra of \mathfrak{h} and $[\mathfrak{l}, \mathfrak{m}] \subseteq \mathfrak{m}$ and since $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{n}$, \mathfrak{k} is a Lie subalgebra of \mathfrak{g} .

We assert that (8) is valid. This can be easily checked whenever at least one of $x, y \in \mathfrak{v}$. If both $x, y \in \mathfrak{z}$ the left-hand side of (8) is zero. The right-hand side lies in $\mathfrak{z} \cap \ker(\pi)$, but $\ker(\pi) = \ker(j)$ and since j is injective one has $\mathfrak{z} \cap \ker(\pi) = \{0\}$, which proves (8). By following the argument preceding (8) backwards, one can see that M is naturally reductive with respect to \mathfrak{k} . \square

If \mathfrak{h} is a Lie subalgebra of $\text{End}(\mathfrak{v})$ such that $\mathfrak{h} \subseteq \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ then we call $\langle \cdot, \cdot \rangle$ an \mathfrak{h} -invariant metric.

In the conditions of Theorem (3.2) it follows that if $(N, \langle \cdot, \cdot \rangle)$ is naturally reductive then the bilinear map τ defines a Lie algebra structure on \mathfrak{z} and the map $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ becomes a real representation of the Lie algebra (\mathfrak{z}, τ) . Furthermore the metric on \mathfrak{v} is $j(\mathfrak{z})$ -invariant and since $\tau_x \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}})$ the metric on \mathfrak{z} is $\text{ad}(\mathfrak{z})$ -invariant, where ad denotes the adjoint representation of (\mathfrak{z}, τ) .

Conversely let \mathfrak{g} be a real Lie algebra endowed with an $\text{ad}(\mathfrak{g})$ -invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and let (π, \mathfrak{v}) be a faithful representation of \mathfrak{g} endowed with a $\pi(\mathfrak{g})$ -invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$ and without trivial subrepresentations, that is, $\bigcap_{x \in \mathfrak{g}} \ker \pi(x) = \{0\}$. Define a 2-step nilpotent Lie algebra structure on the vector space underlying $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ by the following bracket

$$(11) \quad \begin{aligned} [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{n}} &= [\mathfrak{g}, \mathfrak{v}]_{\mathfrak{n}} = 0 & [\mathfrak{v}, \mathfrak{v}] &\subseteq \mathfrak{g} \\ \langle [u, v], x \rangle_{\mathfrak{g}} &= \langle \pi(x)u, v \rangle_{\mathfrak{v}} & \forall x \in \mathfrak{g}, u, v \in \mathfrak{v}. \end{aligned}$$

and equip \mathfrak{n} with the metric obtained as the product metric

$$(12) \quad \langle \cdot, \cdot \rangle_{|\mathfrak{g} \times \mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{g}} \quad \langle \cdot, \cdot \rangle_{|\mathfrak{v} \times \mathfrak{v}} = \langle \cdot, \cdot \rangle_{\mathfrak{v}} \quad \langle \mathfrak{g}, \mathfrak{v} \rangle = 0.$$

Take N the simply connected 2-step nilpotent Lie group with Lie algebra \mathfrak{n} and endow it with the left invariant metric determined by $\langle \cdot, \cdot \rangle$.

Since (π, \mathfrak{v}) has no trivial subrepresentations, the center of \mathfrak{n} coincides with \mathfrak{g} . Moreover \mathfrak{v} is its orthogonal complement and the transformation $j(x)$ defined as in (4) is precisely $\pi(x)$ for all $x \in \mathfrak{g}$. Since (π, \mathfrak{v}) is faithful, the commutator of \mathfrak{n} is \mathfrak{g} : $C(\mathfrak{n}) = \mathfrak{g}$. Since the set $\{\pi(x)\}_{x \in \mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ we conclude that $(N, \langle \cdot, \cdot \rangle)$ is naturally reductive.

Theorem 3.3. *Let \mathfrak{g} denote a Lie algebra equipped with an ad-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ and let (π, \mathfrak{v}) be a real faithful representation of \mathfrak{g} without trivial subrepresentations and endowed with a $\pi(\mathfrak{g})$ -invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{v}}$. Let \mathfrak{n} be the Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ direct sum of vector spaces, together with the Lie bracket given by (4) and furnished with the metric $\langle \cdot, \cdot \rangle$ as in (12). Then the corresponding simply connected 2-step nilpotent Lie group $(N, \langle \cdot, \cdot \rangle)$, being $\langle \cdot, \cdot \rangle$ the induced left invariant metric, is a naturally reductive pseudo Riemannian space.*

The converse holds whenever the center of N is non degenerate and j is faithful.

Remark. Suppose the representation (π, \mathfrak{v}) of \mathfrak{g} is not faithful. Thus

$$z \in \ker \pi \iff \langle z, [u, v] \rangle = 0 \quad \forall u, v \in \mathfrak{v}$$

$\implies z \in C(\mathfrak{n})^{\perp}$. Since the metric on the center \mathfrak{g} is non definite, $\ker \pi \cap C(\mathfrak{n})$ could be non trivial, so that the sum as vector spaces $\ker \pi + C(\mathfrak{n})$ is not necessarily direct.

When π has some trivial subrepresentation,

$$u \in \cap_{x \in \mathfrak{g}} \pi(x) \iff \langle \pi(x)u, v \rangle = 0 \quad \forall v \in \mathfrak{v},$$

$\implies \langle x, [u, v] \rangle = 0$ for all $x \in \mathfrak{g}$, thus $[u, v] = 0$ for all $v \in \mathfrak{v}$ which says $u \in \mathfrak{z}(\mathfrak{n})$. Hence $\mathfrak{g} \subsetneq \mathfrak{z}(\mathfrak{n})$.

Remark. While in the Riemannian case, the condition of the metric to be positive definite says that \mathfrak{g} must be compact, in the pseudo Riemannian case the statement above imposes the restriction on \mathfrak{g} to carry an ad-invariant metric. See the next example.

Example 3.4. The Killing form on any semisimple Lie algebra is an ad-invariant metric.

Any Lie algebra \mathfrak{g} can be embedded into a Lie algebra which admits an ad-invariant metric. In fact, the cotangent $\mathbb{T}^* \mathfrak{g} = \mathfrak{g} \times_{\text{coad}} \mathfrak{g}^*$, being *coad* the coadjoint representation, admits a neutral ad-invariant metric which is given by:

$$\langle (x_1, \varphi_1), (x_2, \varphi_2) \rangle = \varphi_1(x_2) + \varphi_2(x_1) \quad x_1, x_2 \in \mathfrak{g}, \quad \varphi_1, \varphi_2 \in \mathfrak{g}^*.$$

Notice that both \mathfrak{g} and \mathfrak{g}^* are isotropic subspaces.

A data set $(\mathfrak{g}, \mathfrak{v}, \langle \cdot, \cdot \rangle)$ consists of

- (i) a Lie algebra \mathfrak{g} equipped with a ad-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$,
- (ii) a real faithful representation of \mathfrak{g} : (π, \mathfrak{v}) , without trivial subrepresentations,
- (iii) $\langle \cdot, \cdot \rangle$ is a \mathfrak{g} -invariant metric on $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$, i.e. $\langle \cdot, \cdot \rangle|_{\mathfrak{g} \times \mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is ad(\mathfrak{g})-invariant and $\langle \cdot, \cdot \rangle|_{\mathfrak{v} \times \mathfrak{v}}$ is $\pi(\mathfrak{g})$ -invariant and $\langle \mathfrak{g}, \mathfrak{v} \rangle = 0$.

A data set $(\mathfrak{g}, \mathfrak{v}, \langle \cdot, \cdot \rangle)$ determines a 2-step nilpotent Lie group denoted by $N(\mathfrak{g}, \mathfrak{v})$ whose Lie algebra is the underlying vector space $\mathfrak{n} = \mathfrak{g} \oplus \mathfrak{v}$ with the Lie bracket defined by (11). Extend the metric on \mathfrak{g} by left translations after identifying $\mathfrak{n} \simeq T_e N(\mathfrak{g}, \mathfrak{v})$,

so that $N(\mathfrak{g}, \mathfrak{v})$ becomes a naturally reductive pseudo Riemannian 2-step nilpotent Lie group (3.3).

We study the isometry group in this case. Let \mathfrak{h} denote the Lie algebra of the isometries fixing the identity element; by (6) an element $D \in \mathfrak{h}$ is a self adjoint derivation which can be written as $D = (A, B) \in \mathfrak{so}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$ such that

$$B\pi(x) - \pi(x)B = \pi(Ax), \quad \forall x \in \mathfrak{g}.$$

Denote by $[\cdot, \cdot]_{\mathfrak{n}}$ the Lie bracket on \mathfrak{n} and by $[\cdot, \cdot]$ the Lie brackets on \mathfrak{g} and $\text{End}(\mathfrak{v})$. Then

$$\begin{aligned} \pi(A[x, y]) &= B\pi([x, y]) - \pi([x, y])B = B[\pi(x), \pi(y)] - [\pi(x), \pi(y)]B \\ &= [B, [\pi(x), \pi(y)]] = [[B, \pi(x)] + [\pi(x), B], \pi(y)] \\ &= [\pi(Ax), \pi(y)] + [\pi(x), \pi(Ay)] = \pi([Ax, y] + [x, Ay]). \end{aligned}$$

Since π is faithful then

$$A[x, y] = [Ax, y] + [x, Ay] \quad \text{for all } x, y \in \mathfrak{g},$$

that is, $A \in \text{Der}(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$.

Proposition 3.5. *The group of isometries fixing the identity on a naturally reductive pseudo Riemannian 2-step nilpotent Lie group $N(\mathfrak{g}, \mathfrak{v})$ as in (3.3) has Lie algebra*

$$\mathfrak{h} = \{(A, B) \in (\text{Der}(\mathfrak{g}) \cap \mathfrak{so}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : [\pi(x), B] = \pi(Ax) \quad \forall x \in \mathfrak{g}\}.$$

Whenever \mathfrak{g} is semisimple, the ad-invariant metric on \mathfrak{g} is the Killing form; therefore any self adjoint derivation of \mathfrak{g} is of the form $\text{ad}(x)$ for some $x \in \mathfrak{g}$. In this case one can consider $\mathfrak{g} \subset \mathfrak{h}$ where the action is given as

$$x \cdot (z + v) = \text{ad}(x)z + \pi(x)v \quad x \in \mathfrak{g}, z + v \in \mathfrak{n}$$

being $\text{ad}(x)$ the adjoint map on the semisimple Lie algebra \mathfrak{g} . Thus an element $D = (A, B) \in \mathfrak{h}$ is of the form

$$(A, B) = (\text{ad}(x), \pi(x)) + (0, B') \quad x \in \mathfrak{g}$$

with $B' = B - \pi(x) \in \text{End}_{\mathfrak{g}}(\mathfrak{v}) \cap \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) = \mathfrak{e}_{\mathfrak{g}}$, where $\text{End}_{\mathfrak{g}}(\mathfrak{v})$ denotes the set of intertwining operators of the representation (π, \mathfrak{v}) of \mathfrak{g} . Since \mathfrak{g} and $\mathfrak{e}_{\mathfrak{g}}$ commute, then $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{e}_{\mathfrak{g}}$ is a direct sum of Lie algebras, here we identify \mathfrak{g} with the set $\{(\text{ad}(x), \pi(x)) : x \in \mathfrak{g}\} \subseteq \mathfrak{h}$. This argues the following result.

Corollary 3.6. *In the conditions of (3.5) with data set $(\mathfrak{g}, \mathfrak{v}, \langle \cdot, \cdot \rangle)$ for \mathfrak{g} semisimple the group of isometries fixing the identity element is*

$$H = G \times U \quad U = \text{End}_{\mathfrak{g}}(\mathfrak{v}) \cap \text{O}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}).$$

Proof. By (5) we have that

$$H = \{(\phi, T) \in \text{O}(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}) \times \text{O}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) : T\pi(x)T^{-1} = \pi(\phi x), \quad x \in \mathfrak{g}\}.$$

Hence $\phi = \pi^{-1} \circ \text{Ad}(T) \circ \pi \in \text{Aut}(\mathfrak{g})$. Since \mathfrak{g} is semisimple any automorphism of \mathfrak{g} is an inner automorphism, thus there exist $g \in G$ such that $\phi = \text{Ad}(g)$. By the paragraph above, $(\text{Ad}(g), \pi(g)) \in H$ and therefore $\pi(g)^{-1}T \in U$. Hence

$$(\phi, T) = (\text{Ad}(g), \pi(g)) \cdot (I, \pi(g)^{-1}T),$$

which says $H = G \times U$. □

Remark. Compare with [La].

4. GEOMETRY AND EXAMPLES OF NATURALLY REDUCTIVE 2-STEP NILMANIFOLDS WITH NON DEGENERATE CENTER

The aim of this section is twofold. In the first part we write explicitly some geometric features to bring into the proof of (2.3), while in the second part we show examples of naturally reductive pseudo Riemannian 2-step nilpotent Lie groups with non degenerate center.

Recall that a 2-step nilpotent Lie algebra \mathfrak{n} is said to be *non singular* if $\text{ad}(x)$ maps \mathfrak{n} onto \mathfrak{z} for every $x \in \mathfrak{n} - \mathfrak{z}$. Suppose \mathfrak{n} is equipped with a metric as in (2) then \mathfrak{n} is non singular if and only if $j(x)$ is non singular for every $x \in \mathfrak{z}$. We shall say that a Lie group is non singular if its corresponding Lie algebra is non singular.

Whenever N is simply connected 2-step nilpotent the exponential map $\exp : \mathfrak{n} \rightarrow N$ produces global coordinates. In terms of this map the product on N can be obtained by

$$\exp(z_1 + v_1) \exp(z_2 + v_2) = \exp(z_1 + z_2 + \frac{1}{2}[v_1, v_2] + v_1 + v_2) \quad \text{for } z_1, z_2 \in \mathfrak{z}, v_1, v_2 \in \mathfrak{v}.$$

We shall study the geometry of 2-step nilpotent Lie groups when they are endowed with a left invariant (pseudo Riemannian) metric $\langle \cdot, \cdot \rangle$ with respect to which the center is non degenerate. In the Riemannian case a deep study of the geometry can be found in the works of P. Eberlein [Eb1, Eb2].

The covariant derivative ∇ is left invariant, hence one can see ∇ as a bilinear form on \mathfrak{n} getting the formula

$$(13) \quad \nabla_x y = \frac{1}{2}([x, y] - \text{ad}(x)^* y - \text{ad}(y)^* x) \quad \text{for } x, y \in \mathfrak{n},$$

where $\text{ad}(x)^*$ denotes the adjoint of $\text{ad}(x)$. By writing this explicitly one obtains

$$(14) \quad \begin{aligned} \nabla_x y &= \frac{1}{2}[x, y] && \text{for all } x, y \in \mathfrak{v} \\ \nabla_x y = \nabla_y x &= -\frac{1}{2}j(y)x && \text{for all } x \in \mathfrak{v}, y \in \mathfrak{z} \\ \nabla_x y &= 0 && \text{for all } x, y \in \mathfrak{z} \end{aligned}$$

Since translations on the left are isometries, to describe the geodesics of $(N, \langle \cdot, \cdot \rangle)$ it suffices to describe those geodesics that begin at e the identity of N . Let $\gamma(t)$ be a curve with $\gamma(0) = e$, and let $\gamma'(0) = z_0 + v_0 \in \mathfrak{n}$, where $z_0 \in \mathfrak{z}$ and $v_0 \in \mathfrak{v}$. In exponential coordinates we write

$$\gamma(t) = \exp(z(t) + v(t)), \quad \text{where } z(t) \in \mathfrak{z}, v(t) \in \mathfrak{v} \quad \text{for all } t \text{ and } z'(0) = z_0, v'(0) = v_0.$$

The curve $\gamma(t)$ is a geodesic if and only if the following equations are satisfied:

$$(15) \quad v''(t) = j(z_0)v'(t) \text{ for all } t \in \mathbb{R}$$

$$(16) \quad z_0 \equiv z'(t) + \frac{1}{2}[v'(t), v(t)] \text{ for all } t \in \mathbb{R}$$

These equations were derived by A. Kaplan in [Ka] to study 2-step nilpotent groups N of Heisenberg type, but the proof is valid in general for 2-step nilpotent Lie groups equipped with a left invariant pseudo Riemannian metric where the center is non degenerate as noted in [Ge] and [Bo].

Let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$. Write $\gamma'(0) = z_0 + v_0$, where $z_0 \in \mathfrak{z}$ and $v_0 \in \mathfrak{v}$ and identify $\mathfrak{n} = T_e N$. Then

$$(17) \quad \gamma'(t) = dL_{\gamma(t)}(e^{tj(z_0)}v_0 + z_0) \quad \text{for all } t \in \mathbb{R}$$

where $e^{tj(z_0)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} j(z_0)^n$. In fact, write $\gamma(t) = \exp(z(t) + v(t))$, where $z(t)$ and $v(t)$ lie in \mathfrak{z} and \mathfrak{v} respectively for all $t \in \mathbb{R}$. By using the previous equations (15) one has

$$\begin{aligned} \gamma'(t) &= d\exp_{z(t)+v(t)}(z'(t) + v'(t))_{z(t)+v(t)} \\ &= dL_{\gamma(t)}(z'(t) + \frac{1}{2}[v'(t), v(t)] + v') \\ &= dL_{\gamma(t)}(z_0 + v'). \end{aligned}$$

Now by integrating the first equation of (15) one gets $v'(t) = e^{tj(z_0)}v_0$ which proves (17).

For x, y elements in \mathfrak{n} the curvature tensor is defined by

$$R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x, y]}.$$

Using (14) one gets

$$(18) \quad R(x, y)z = \begin{cases} \frac{1}{2}j([x, y])z - \frac{1}{4}j([y, z])x + \frac{1}{4}j([x, z])y & \text{for } x, y, z \in \mathfrak{v}, \\ -\frac{1}{4}[x, j(y)z] & \text{for } x, y \in \mathfrak{v}, z \in \mathfrak{z}, \\ -\frac{1}{4}[x, j(z)y] + \frac{1}{4}[y, j(z)x] & \text{for } x, z \in \mathfrak{v}, y \in \mathfrak{z}, \\ -\frac{1}{4}j(y)j(z)x & \text{for } x \in \mathfrak{v}, y, z \in \mathfrak{z}, \\ \frac{1}{4}[j(x), j(y)]z & \text{for } x, y \in \mathfrak{z}, z \in \mathfrak{v}, \\ 0 & \text{for } x, y, z \in \mathfrak{z}. \end{cases}$$

Let $\Pi \subseteq \mathfrak{n}$ denote a non degenerate plane and let Q be given by

$$Q(x, y) = \langle x, x \rangle \langle y, y \rangle - \langle x, y \rangle^2.$$

The non degeneracy property is equivalent to ask $Q(v, w) \neq 0$ for one -hence every- basis $v, w \in \Pi$ [O'N]. The sectional curvature of Π is the number $K(x, y) := \langle R(x, y)y, x \rangle / Q(x, y)$, which is independent of the choice of the basis. Now take a orthonormal basis for Π , that is a linearly independent set $\{x, y\}$ such that $\langle x, y \rangle = 0$ and $\langle x, x \rangle = \pm 1$ and $\langle y, y \rangle = \pm 1$.

After (18) one obtains

$$(19) \quad K(x, y) = \begin{cases} -\frac{3\varepsilon_1\varepsilon_2}{4}\langle [x, y], [x, y] \rangle & \text{for } x, y \in \mathfrak{v} \\ \frac{\varepsilon_1\varepsilon_2}{4}\langle j(y)x, j(y)x \rangle & \text{for } x \in \mathfrak{v}, y \in \mathfrak{z}, \\ 0 & \text{for } x, y \in \mathfrak{z} \end{cases}$$

being $\varepsilon_1 := \langle x, x \rangle$ and $\varepsilon_2 := \langle y, y \rangle$.

The Ricci tensor is given by $Ric(x, y) = \text{trace}(z \rightarrow R(z, x)y), z \in \mathfrak{n}$ for arbitrary elements $x, y \in \mathfrak{n}$.

Proposition 4.1. *Let $\{z_i\}$ denote a orthonormal basis of \mathfrak{z} and $\{v_j\}$ a orthonormal basis of \mathfrak{v} . It holds*

$$Ric(x, y) = \begin{cases} 0 & \text{for } x \in \mathfrak{v}, y \in \mathfrak{z} \\ \frac{1}{2} \sum_i \varepsilon_i \langle j(z_i)^2 x, y \rangle & \text{for } x, y \in \mathfrak{v}, \varepsilon_i = \langle z_i, z_i \rangle \\ -\frac{1}{4} \sum_j \varepsilon_j \langle j(x)j(y)v_j, v_j \rangle & \text{for } x, y \in \mathfrak{z}, \varepsilon_j = \langle v_j, v_j \rangle. \end{cases}$$

Due to symmetries of the curvature tensor, the Ricci tensor is a symmetric bilinear form on \mathfrak{n} and hence there exists a symmetric linear transformation $T : \mathfrak{n} \rightarrow \mathfrak{n}$ such that $Ric(x, y) = \langle Tx, y \rangle$ for all $x, y \in \mathfrak{n}$. T is called the Ricci transformation. Let $\{e_k\}$ denote a orthonormal basis of \mathfrak{n} ; it holds

$$Ric(x, y) = \sum_k \varepsilon_k \langle R(e_k, x)y, e_k \rangle = \langle -\sum_k \varepsilon_k R(e_k, x)e_k, y \rangle$$

which implies

$$(20) \quad T(x) = -\sum_k \varepsilon_k R(e_k, x)e_k, \quad \text{being } \varepsilon_k = \langle e_k, e_k \rangle.$$

According to the results in (4.1) we have that \mathfrak{z} and \mathfrak{v} are T -invariant subspaces and

$$T(x) = \begin{cases} \frac{1}{2} j(\sum_i \varepsilon_i z_i)^2 x & x \in \mathfrak{v}, \quad \varepsilon_i = \langle z_i, z_i \rangle \\ \frac{1}{4} \sum_j \varepsilon_j [v_j, j(x)v_j] & x \in \mathfrak{z} \quad \varepsilon_j = \langle v_j, v_j \rangle. \end{cases}$$

where $\{z_i\}$ and $\{v_j\}$ are orthonormal basis of \mathfrak{z} and \mathfrak{v} respectively.

Remark. The formulas above were used in [C-P1] to prove (2.3).

Remark. For naturally reductive metrics in the formulas above, replace the maps j be the corresponding representation $\pi : \mathfrak{g} \rightarrow \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}})$.

Below we expose examples of naturally reductive metrics on 2-step nilpotent Lie groups. This is achieved by translating the data at the Lie algebra level to the corresponding simply connected Lie group by following the key results provided in (3.3). We shall make use of euclidean and semisimple Lie algebras in order to obtained ad-invariant metrics. For further details on Lie algebras with ad-invariant metrics see for instance [F-S, M-R]. Concerning isometries between pseudo Riemannian 2-step nilpotent Lie groups notice that orthogonal isomorphisms gives rise to isometries between the corresponding Lie groups (*).

(i) *Riemannian examples.* Naturally reductive Riemannian nilmanifolds arise by considering a data set with \mathfrak{g} compact. Recall that if \mathfrak{g} is compact then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{c}$ where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is a compact semisimple Lie algebra and \mathfrak{c} is the center (see [Wa]). In [La] they were extended studied.

In the Riemannian case the converse of (*) above holds [Wi].

(ii) *Modified Riemannian.* Take any of those data sets corresponding to the positive definite case and follow the ideas in (2.5). Clearly all requirements in (3.3) apply and so one can produce naturally reductive pseudo Riemannian metrics of signature $(\dim \mathfrak{g}, \dim \mathfrak{v})$.

Let $N(\mathfrak{g}, \mathfrak{v})$ denote a Riemannian naturally reductive nilmanifold obtained from a data set $(\mathfrak{g}, \mathfrak{v}, \langle \cdot, \cdot \rangle)$. Let $\tilde{N}(\mathfrak{g}, \mathfrak{v})$ denote the pseudo Riemannian 2-step nilpotent Lie group obtained by changing the sign of the metric on \mathfrak{g} . Therefore by [Wi]

$$N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}') \iff \mathfrak{n}(\mathfrak{g}, \mathfrak{v}) \simeq \mathfrak{n}(\mathfrak{g}, \mathfrak{v}')$$

and this occurs if and only if there an isometric isomorphism $\phi : (\mathfrak{g}, \langle \cdot, \cdot \rangle_+) \rightarrow (\mathfrak{g}', \langle \cdot, \cdot \rangle'_+)$ and a isometry $T : (\mathfrak{v}, \langle \cdot, \cdot \rangle_+) \rightarrow (\mathfrak{v}', \langle \cdot, \cdot \rangle_+)$ such that

$$T\pi(x)T^{-1} = \pi'(\phi x) \quad \text{for all } x \in \mathfrak{g}.$$

Clearly $\phi : (\mathfrak{g}, -\langle \cdot, \cdot \rangle_+) \rightarrow (\mathfrak{g}', -\langle \cdot, \cdot \rangle'_+)$ is also a isometric isomorphism, so that the corresponding simply connected Lie groups are isometric. Thus one has what follows.

Proposition 4.2. *If $N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}')$ then $\tilde{N}(\mathfrak{g}, \mathfrak{v}) \simeq \tilde{N}'(\mathfrak{g}', \mathfrak{v}')$.*

In [La] detailed conditions to get the isometries $N(\mathfrak{g}, \mathfrak{v}) \simeq N'(\mathfrak{g}', \mathfrak{v}')$ were obtained.

(iii) *Abelian center.* Let \mathbb{R}^{2n} be equipped with a metric B , that is, B is determined by a non singular symmetric linear map such that

$$B(x, y) = \langle bx, y \rangle \quad \langle \cdot, \cdot \rangle \text{ the canonical inner product on } \mathbb{R}^{2n}.$$

Let $t \in \mathfrak{so}(\mathbb{R}^{2n}, B)$, that is t may satisfy $t^* = -btb$ where t^* denotes adjoint with respect to the canonical inner product on \mathbb{R}^{2n} .

Any non singular $t \in \mathfrak{so}(\mathbb{R}^{2n}, B)$ gives rise to a faithful representation of \mathbb{R} to (\mathbb{R}^{2n}, B) without trivial subrepresentations. Let \mathfrak{n} be the vector space direct sum $\mathbb{R}z \oplus \mathbb{R}^{2n}$ equipped with a metric $\langle \cdot, \cdot \rangle$ such that

$$\langle z, \mathbb{R}^{2n} \rangle = 0 \quad \langle z, z \rangle = \lambda \in \mathbb{R} - \{0\} \quad \langle \cdot, \cdot \rangle_{\mathbb{R}^{2n}} = B.$$

Define a Lie bracket on \mathfrak{n} by

$$[z, y] = 0 \quad \forall y \in \mathfrak{n} \quad \text{and} \quad \langle [u, v], z \rangle = B(tu, v) \quad u, v \in \mathbb{R}^{2n}.$$

According to (3.3) this Lie bracket makes of \mathfrak{n} a 2-step nilpotent Lie algebra and the given metric is naturally reductive whenever the center is non degenerate. This Lie algebra is isomorphic to the Heisenberg Lie algebra.

Furthermore, the group of isometries fixing the identity element has Lie algebra

$$(21) \quad \mathfrak{h} = \mathcal{Z}_{\mathfrak{so}(\mathbb{R}^{2n}, B)}(t)$$

where $\mathcal{Z}_{\mathfrak{so}(\mathbb{R}^{2n}, B)}(t)$ denotes the centralizer of t in $\mathfrak{so}(\mathbb{R}^{2n}, B)$, which can be verified by applying Proposition (3.5).

In this way one gets naturally reductive metrics on the Heisenberg Lie group of dimension $2n+1$. The converse also holds.

Proposition 4.3. *Any left invariant pseudo Riemannian metric on the Heisenberg Lie group H_{2n+1} for which the center is non degenerate is naturally reductive.*

The isotropy group has Lie algebra \mathfrak{h} as in (21).

Proof. Let \mathfrak{h}_{2n+1} denote the Lie algebra of H_{2n+1} and decompose it as a orthogonal direct sum $\mathfrak{h}_{2n+1} = \mathbb{R}z \oplus \mathfrak{v}$. Then the restriction of the metric to \mathfrak{v} defines a metric B of signature (k, m) . The map j defined in (4) is indeed self adjoint with respect to

$B := \langle \cdot, \cdot \rangle_{\mathfrak{v} \times \mathfrak{v}}$ and it generates a subalgebra of $\mathfrak{so}(\mathfrak{v}, B)$. Thus $z \rightarrow j(z)$ defines a faithful representation without trivial subrepresentations since by $t := j(z)$ one has

$$tu = 0 \iff B(tu, v) = 0 \quad \forall v \in \mathfrak{v} \iff B(z, [u, v]) = 0 \quad \forall v \in \mathfrak{v}.$$

But since the center is non degenerate then $[u, v] = 0$ for all $v \in \mathfrak{v}$ which implies $u = 0$. Indeed any non degenerate metric on $\mathbb{R}z$ is ad-invariant. Hence the statements of (3.3) are satisfied and the metric on \mathfrak{h}_{2n+1} is naturally reductive. \square

Example 4.4. Let \mathfrak{h}_3 denote the Heisenberg Lie algebra of dimension three with basis e_1, e_2, e_3 satisfying the Lie brackets $[e_1, e_2] = e_3$. Lorentzian metrics on \mathfrak{h}_3 with non degenerate center can be defined by

$$\begin{aligned} (1) \quad & -\langle e_3, e_3 \rangle = 1 = \langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle \\ (2) \quad & \langle e_3, e_3 \rangle = 1 = -\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle \end{aligned}$$

Thus in the basis e_1, e_2 the map $j_1(e_3)$ for the metric in (1) is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(compare with (2.5)) while $j_2(e_3)$ for the metric (2) one has

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

See [Ge] for more results concerning Lorentzian metrics.

The construction on the Heisenberg Lie algebra, can be extended in the following way. Set B a non degenerate symmetric bilinear form on \mathbb{R}^k and let t_1, \dots, t_l be commuting linear maps in $\mathfrak{so}(\mathbb{R}^k, B)$ and such that $\bigcap_i \ker(t_i) = \{0\}$.

Set $\mathfrak{n} = \mathbb{R}^l \oplus \mathbb{R}^k$ direct sum of vector spaces, equipped \mathbb{R}^l with any metric and \mathfrak{n} with the product metric such that $\langle \mathbb{R}^l, \mathbb{R}^k \rangle = 0$.

The triple $(\mathbb{R}^l, \mathbb{R}^k, \langle \cdot, \cdot \rangle)$ is a data set which induces a naturally reductive metric on the corresponding simply connected 2-step nilpotent Lie group with Lie algebra \mathfrak{n} .

Semisimple center. Let $\mathbb{R}^{p,q}$ denote the real vector space \mathbb{R}^{p+q} endowed with a metric $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q) . Let $\mathfrak{so}(p, q)$ denote the set of self adjoint transformations for $\langle \cdot, \cdot \rangle_{p,q}$. This a semisimple Lie algebra and the Killing form K a natural ad-invariant metric on $\mathfrak{so}(p, q)$. Indeed $\mathfrak{so}(p, q)$ acts on $\mathbb{R}^{p,q}$ just by evaluation. Take the direct sum as vector spaces $\mathfrak{n} = \mathfrak{so}(p, q) \oplus \mathbb{R}^{p,q}$ and equipped with the product metric $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$ such that $\langle \cdot, \cdot \rangle_{\mathfrak{so}(p,q) \times \mathfrak{so}(p,q)} = K$, $\langle \cdot, \cdot \rangle_{\mathbb{R}^{p,q} \times \mathbb{R}^{p,q}} = \langle \cdot, \cdot \rangle_{p,q}$ and $\langle \mathfrak{so}(p, q), \mathbb{R}^{p,q} \rangle = 0$. Thus a Lie bracket can be defined on \mathfrak{n} by

$$K([u, v], A) = \langle Au, v \rangle_{p,q} \quad \text{for all } u, v \in \mathbb{R}^{p,q}, A \in \mathfrak{so}(p, q).$$

The corresponding 2-step nilpotent Lie group equipped with the left invariant metric induced by the metric above, makes of N a naturally reductive pseudo Riemannian space (Theorem (3.3)).

A similar construction can be done by restriction of the evaluating action to a non degenerate subalgebra of $\mathfrak{so}(p, q)$.

(v) *Modified tangent semisimple.* The Killing form K is an ad-invariant metric on any semisimple Lie algebra \mathfrak{g} . As usual the tangent Lie algebra $T\mathfrak{g}$ is the semidirect product $\mathfrak{g} \ltimes \mathfrak{g}$ via the adjoint representation. We shall modify the algebraic structure

on $T\mathfrak{g}$ in order to get a naturally reductive pseudo Riemannian 2-step nilpotent Lie group.

Take the Lie algebra \mathfrak{g} together with the Killing form and let \mathfrak{v} denote the underlying vector space to \mathfrak{g} endowed also with the Killing form metric. To this pair $(\mathfrak{g}, \mathfrak{v})$ attach

- the metric given by $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = \langle \cdot, \cdot \rangle_{\mathfrak{v}} = K$ and $\langle \mathfrak{g}, \mathfrak{v} \rangle = 0$;
- the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{so}(\mathfrak{v}, K)$.

The adjoint representation is faithful and there is no trivial subrepresentations, so that $(\mathfrak{g}, \mathfrak{v}, K + K)$ constitutes a data set for a 2-step nilpotent Lie group $N(\mathfrak{g}, \mathfrak{v})$ and by (3.3) it is naturally reductive pseudo Riemannian. Clearly the signature of this metric is twice as much the signature of B and the isometry group can be computed with (3.6).

Notice that whenever \mathfrak{g} is compact the procedure above is a case of the construction for naturally reductive Riemannian nilmanifolds (see (i)).

In the next section we shall see that the 2-step nilpotent Lie algebra above, together with another metric gives rise to a Lie algebra carrying an ad-invariant metric.

5. OTHER EXAMPLES OF NATURALLY REDUCTIVE METRICS

In this section we study 2-step nilpotent Lie algebras with ad-invariant metrics. The corresponding Lie group carries a bi-invariant metric for which the center is degenerate.

An *ad-invariant metric* on a Lie algebra \mathfrak{g} is a non degenerate symmetric bilinear map $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ such that

$$(22) \quad \langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0 \quad \text{for all } x, y, z \in \mathfrak{n}.$$

Recall that on a connected Lie group G furnished with a left invariant pseudo Riemannian metric $\langle \cdot, \cdot \rangle$, the following statements are equivalent (see [O’N] Ch. 11):

- (1) $\langle \cdot, \cdot \rangle$ is right invariant, hence bi-invariant;
- (2) $\langle \cdot, \cdot \rangle$ is $\text{Ad}(G)$ -invariant;
- (3) the inversion map $g \rightarrow g^{-1}$ is an isometry of G ;
- (4) $\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0$ for all $x, y, z \in \mathfrak{g}$;
- (5) $\nabla_x y = \frac{1}{2}[x, y]$ for all $x, y \in \mathfrak{g}$, where ∇ denotes the Levi Civita connection;
- (6) the geodesics of G starting at e are the one parameter subgroups of G .

Clearly $(G, \langle \cdot, \cdot \rangle)$ is naturally reductive, which by (3) is a symmetric space. Furthermore one has

- the Levi-Civita connection is given by

$$\nabla_x y = \frac{1}{2}[x, y] \quad \text{for all } x, y \in \mathfrak{g},$$

- the curvature tensor is

$$R(x, y) = \frac{1}{4} \text{ad}([x, y]) \quad \text{for } x, y \in \mathfrak{g}.$$

Hence any *simply connected 2-step nilpotent Lie group equipped with a bi-invariant metric is flat*.

The set of nilpotent Lie groups carrying a bi-invariant pseudo Riemannian metric is non empty. An element of this set is for instance the simply connected Lie group whose

Lie algebra is the free 3-step nilpotent Lie algebra in two generators: in fact, \mathfrak{n} the Lie algebra spanned as vector space by e_1, e_2, e_3, e_4, e_5 with the non zero Lie brackets

$$[e_1, e_2] = e_3 \quad [e_1, e_3] = e_4 \quad [e_2, e_3] = e_5,$$

carries the ad-invariant metric defined by the non vanishing symmetric relations

$$\langle e_3, e_3 \rangle = 1 = \langle e_1, e_5 \rangle = -\langle e_2, e_4 \rangle.$$

Otherwise in the Riemannian case, a naturally reductive nilpotent Lie group may be at most 2-step nilpotent [Go].

Let \mathfrak{n} denote a 2-step nilpotent Lie algebra with an ad-invariant metric $\langle \cdot, \cdot \rangle$. It is not hard to prove that $\mathfrak{z}^\perp = C(\mathfrak{n})$ and therefore the center is always an isotropic ideal. Moreover \mathfrak{n} decomposes as a orthogonal product

$$(23) \quad \mathfrak{n} = \tilde{\mathfrak{z}} \times \tilde{\mathfrak{n}}$$

where $\tilde{\mathfrak{z}}$ is a non degenerate central ideal and $\tilde{\mathfrak{n}}$ is a 2-step nilpotent ideal of corank zero, being the corank of \mathfrak{n} uniquely defined by the scalar $k := \dim \mathfrak{z} - \dim C(\mathfrak{n})$. This follows essentially from the fact that the ad-invariant metric is non degenerate on any complementary subspace of $C(\mathfrak{n})$ in \mathfrak{z} . Thus by choosing such a complement $\tilde{\mathfrak{z}}$, $\mathfrak{z} = \tilde{\mathfrak{z}} \oplus C(\mathfrak{n})$ and its orthogonal complement in \mathfrak{n} , $\mathfrak{n} = \tilde{\mathfrak{z}} \oplus \tilde{\mathfrak{z}}^\perp$ one gets a decomposition as above (23) with $\tilde{\mathfrak{n}} := \tilde{\mathfrak{z}}^\perp$.

Assume now the corank of $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ vanishes, so that $\mathfrak{z}^\perp = C(\mathfrak{n}) = \mathfrak{z}$. One can produce a isotropic subspace \mathfrak{v}_1 such that the ad-invariant metric on $\mathfrak{z} \oplus \mathfrak{v}_1$ is non degenerate. Hence one obtains a orthogonal decomposition as vector spaces

$$\mathfrak{n} = (\mathfrak{z} \oplus \mathfrak{v}_1) \oplus \mathfrak{v}_2, \quad \text{where } \mathfrak{v}_2 = (\mathfrak{z} \oplus \mathfrak{v}_1)^\perp.$$

We claim $\mathfrak{v}_2 = 0$. In fact for all $x \in \mathfrak{v}_2$ one has $\langle x, [u, v] \rangle = 0$ for all $u, v \in \mathfrak{n}$ implying that $x \in C(\mathfrak{n})^\perp \cap \mathfrak{v}_2 = \{0\}$. Hence there is a splitting of \mathfrak{n} as a direct sum of the isotropic subspaces \mathfrak{z} and \mathfrak{v} so that the metric on \mathfrak{n} is neutral:

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}.$$

Among other possible constructions, 2-step nilpotent Lie algebras admitting an ad-invariant metric can be obtained as follows. Let $(\mathfrak{v}, \langle \cdot, \cdot \rangle_+)$ denote a real vector space equipped with an inner product and let $\rho : \mathfrak{v} \rightarrow \mathfrak{so}(\mathfrak{v})$ an injective linear map satisfying

$$(24) \quad \rho(u)u = 0 \quad \text{for all } u \in \mathfrak{v}.$$

Consider the vector space $\mathfrak{n} := \mathfrak{v}^* \oplus \mathfrak{v}$ furnished with the canonical neutral metric $\langle \cdot, \cdot \rangle$ and define a Lie bracket on \mathfrak{n} by

$$(25) \quad \begin{aligned} [x, y] &= 0 \quad \text{for } x \in \mathfrak{v}^*, y \in \mathfrak{n} \quad \text{and} \quad [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{v}^* \\ \langle [u, v], w \rangle &= \langle \rho(w)u, v \rangle_+ \quad \text{for all } u, v, w \in \mathfrak{v}. \end{aligned}$$

Then \mathfrak{n} becomes a 2-step nilpotent Lie algebra of corank zero for which the metric $\langle \cdot, \cdot \rangle$ is ad-invariant. This construction was called the *modified cotangent*, since \mathfrak{n} is linear isomorphic to the cotangent of \mathfrak{v} . Notice that the commutator coincides with the center and it equals \mathfrak{v}^* . This allows to construct 2-step nilpotent Lie algebras of null corank which carry an ad-invariant metric. Furthermore this is basically the way to obtain such Lie algebras, see [Ov]:

Theorem 5.1. *Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie algebra of corank m endowed with an ad-invariant metric. Then $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is isometric isomorphic to an orthogonal direct product of the Lie algebras \mathbb{R}^m and a modified cotangent.*

One can get 2-step nilpotent examples by proceeding as follows. Let (\mathfrak{g}, B) denote a compact semisimple Lie algebra and B its Killing form. Since B is negative definite on \mathfrak{g} , $-B$ determines an inner product on \mathfrak{g} . The adjoint map, $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{so}(\mathfrak{g}, B)$ satisfies (24), therefore the vector space $\mathfrak{g}^* \oplus \mathfrak{g}$ equipped with the Lie bracket defined in (25) makes of $\mathfrak{g}^* \oplus \mathfrak{g}$ a 2-step nilpotent Lie algebra which carries an ad-invariant metric, the usual neutral metric on $\mathfrak{g}^* \oplus \mathfrak{g}$.

From (24) it is clear that the non singular Lie algebras cannot carry an ad-invariant metric. Note that if a 2-step nilpotent Lie algebra admits an ad-invariant metric, then $\dim \mathfrak{n} - \dim \mathfrak{z} = \dim C(\mathfrak{n})$. This condition is however not sufficient.

A self adjoint derivation ϕ in such a Lie algebra of zero corank has the form

$$\begin{aligned} \phi(z) &= -A^*z \in \mathfrak{v}^* & \text{for } z \in \mathfrak{v}^* \\ \phi(v) &= Bv + Av & \text{where } Bv \in \mathfrak{v}^*, Av \in \mathfrak{v}, \text{ for } v \in \mathfrak{v} \end{aligned}$$

and such that A^* denotes the dual map of A : $A^*\varphi = \varphi \circ A$ for $\varphi \in \mathfrak{v}^*$. On the other hand according to the results in [Mu] the isotropy group of isometries fixing the identity element on the corresponding 2-step nilpotent Lie group consists of the self adjoint transformations with respect to $\langle \cdot, \cdot \rangle$. Thus

$$I_a(N) \subseteq I(N).$$

Examples of 2-step nilpotent Lie algebras with ad-invariant metrics arise by taking $T^*\mathfrak{n}$, the cotangent of any 2-step nilpotent Lie algebra \mathfrak{n} together with the canonical neutral metric (see (3.4)). Let $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ denote a 2-step nilpotent Lie algebra, where \mathfrak{v} is any complementary subspace of \mathfrak{z} in \mathfrak{g} . Let z_1, \dots, z_m be a basis of the center \mathfrak{z} and let v_1, \dots, v_n be a basis of the vector space \mathfrak{v} . Thus

$$[v_i, v_j] = \sum_{s=1}^m c_{ij}^s z_s \quad i, j = 1, \dots, n.$$

Let $T^*\mathfrak{n} = \mathfrak{n} \times \mathfrak{n}^*$ denote the cotangent Lie algebra obtained via the coadjoint representation. Indeed $z^1, \dots, z^m, v^1, \dots, v^n$ becomes the dual basis of the basis above adapted to the decomposition $\mathfrak{n}^* = \mathfrak{z}^* \oplus \mathfrak{v}^*$. The non trivial Lie bracket relations concerning the coadjoint action follow

$$[v_i, z^j] = \sum_{s=1}^n d_{ij}^s v^s \quad \text{for } i = 1, \dots, n, j = 1, \dots, m.$$

Thus $[v_i, z^j](v_k) = d_{ij}^k$ and by the definition

$$[v_i, z^j](v_k) = -z^j \left(\sum_{s=1}^m c_{ik}^s z^s \right) = -c_{ik}^j \quad i, k = 1, \dots, n, j = 1, \dots, m.$$

Therefore $d_{ij}^k = -c_{ik}^j$ for $i, k = 1, \dots, n, j = 1, \dots, m$.

It is clear that if for some basis of \mathfrak{n} the structure constants are rational numbers then by choosing the union of this basis and its dual on $T^*\mathfrak{n}$ one gets rational structure constants for $T^*\mathfrak{n}$. Thus by the Mal'cev criterium N and its cotangent T^*N , the simply

connected Lie group with Lie algebra $T^* \mathfrak{n}$, admits a lattice which induces a compact quotient (see [O-V, Ra] for instance).

Let $\Gamma \subset T^* N$ denote a cocompact lattice of $T^* N$. Indeed $T^* N$ acts on the compact nilmanifold $(T^* N)/\Gamma$ by left translation isometries if we induce to the quotient the bi-invariant metric corresponding to the neutral canonical one on $T^* \mathfrak{n}$. The tangent space at the representative e can be identified with $T^* \mathfrak{n} \simeq T_e((T^* N)/\Gamma)$ so that $T^* \mathfrak{n} = \{0\} \oplus T^* \mathfrak{n}$ and clearly $Ad(\Gamma) T^* \mathfrak{n} \subseteq T^* \mathfrak{n}$ which says that $(T^* N)/\Gamma$ is homogeneous reductive. Moreover the induced metric on the quotient satisfies

$$\langle [x, y], z \rangle + \langle [x, z], y \rangle = 0 \quad \forall x, y, z \in T^* \mathfrak{n}.$$

Proposition 5.2. *Let N denote a 2-step nilpotent Lie group. If it admits a cocompact lattice then the cotangent Lie group $T^* N$ admits a cocompact lattice Γ such that $(T^* N)/\Gamma$ is pseudo Riemannian naturally reductive.*

Example 5.3. The low dimensional 2-step nilpotent Lie group N admitting an ad-invariant metric occurs in dimension six. This Lie algebra can be also be described as the cotangent of the Heisenberg Lie algebra $T^* \mathfrak{h}_3$. Explicitly let $e_1, e_2, e_3, e_4, e_5, e_6$ be a basis of \mathfrak{n} ; the Lie brackets are

$$[e_4, e_5] = e_1 \quad [e_4, e_6] = e_2 \quad [e_5, e_6] = e_3$$

and the ad-invariant metric is defined by the non zero symmetric relations

$$1 = \langle e_1, e_6 \rangle = \langle e_2, e_5 \rangle = \langle e_3, e_4 \rangle.$$

The corresponding simply connected six dimensional Lie group N can be modelled on \mathbb{R}^6 together with the multiplication group given by

$$(x_1, x_2, x_3, x_4, x_5, x_6) \cdot (y_1, y_2, y_3, y_4, y_5, y_6) = \begin{pmatrix} x_1 + y_1 + \frac{1}{2}(x_4 y_5 - x_5 y_4), \\ x_2 + y_2 + \frac{1}{2}(x_4 y_6 - x_6 y_4), \\ x_3 + y_3 + \frac{1}{2}(x_5 y_6 - x_6 y_5), \\ x_4 + y_4, x_5 + y_5, x_6 + y_6. \end{pmatrix}$$

By the Malcev criterium N admits a cocompact lattice Γ . By inducing the bi-invariant metric of N to N/Γ one gets a invariant metric on N/Γ , and in this way N/Γ is a pseudo Riemannian naturally reductive compact nilmanifold.

For instance the subgroup of N given by

$$\Gamma = \{(k_1, k_2, k_3, 2k_4, k_5, 2k_6) / \text{for } k_i \in \mathbb{Z} \forall i = 1, 2, 3, 4, 5, 6\}$$

is a co-compact lattice of N , so that N/Γ is a compact homogeneous manifold.

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