Generalized Impedance Boundary Conditions for Strongly Absorbing Obstacles: the full Wave Equations

Hoai-Minh Nguyen * and Linh V. Nguyen†

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Abstract

This paper is devoted to the study of the generalized impedance boundary conditions (GIBCs) for a strongly absorbing obstacle in the **time** regime in two and three dimensions. The GIBCs in the time domain are heuristically derived from the corresponding conditions in the time harmonic regime. The latters are frequency dependent except the one of order 0; hence the formers are non-local in time in general. The error estimates in the time regime can be derived from the ones in the time harmonic regime when the frequency dependence is well-controlled. This idea is originally due to Nguyen and Vogelius in [23] for the cloaking context. In this paper, we present the analysis to the GIBCs of orders 0 and 1. To implement the ideas in [23], we revise and extend the work of Haddar, Joly, and Nguyen in [9], where the GIBCs were investigated for a fixed frequency in three dimensions. Even though we heavily follow the strategy in [23], our analysis on the stability contains new ingredients and ideas. First, instead of considering the difference between solutions of the exact model and the approximate model, we consider the difference between their derivatives in time. This simple idea helps us to avoid the machinery used in [23] concerning the integrability with respect to frequency in the low frequency regime. Second, in the high frequency regime, the Morawetz multiplier technique used in [23] does not fit directly in our setting. Our proof makes use of a result by Hörmander in [12]. Another important part of the analysis in this paper is the well-posedness in the time domain for the approximate problems imposed with GIBCs on the boundary of the obstacle, which are non-local in time.

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*EPFL SB MATHAA CAMA, Station 8, CH-1015 Lausanne, Switzerland, E-mail: hoai-minh.nguyen@epfl.ch and School of Mathematics, University of Minnesota, MN, 55455, E-mail: hmnguyen@math.umn.edu.

†Department of Mathematics, University of Idaho, Moscow, Idaho 83843, USA, lnguyen@uidaho.edu.

1 Introduction and statement of the main results

The computation of electromagnetic scattering from an arbitrary obstacle has been an active research area for many decades. One technique is to replace the exact model inside the obstacle by appropriate boundary conditions on its surface; hence the problem of determining the external electromagnetic fields can be solved without considering the fields inside the obstacle (see, e.g., [11, 29]). These boundary conditions are called *Generalized Impedance Boundary Conditions* (GIBCs). The first GIBC for a highly absorbing obstacle (highly conducting obstacle) was proposed by Leontovich (see, e.g., [15]) and was extended later by Rytov in [28]. Antoine, Barucq, and Vernhet in [1], using the technique of pseudo-differential equations (following the ideas of Engquist and Majda in [7]), implemented a new derivation of such conditions. Recently, Haddar, Joly, and Nguyen in [9] revisited these GIBCs for the Helmholtz equation. More precisely, the authors first proposed a new construction of GIBCs which is based on an ansatz for the asymptotic expansion of exact solutions. They then developed mathematical tools, based on compactness arguments, to establish error estimates up to order 3. Related works are the edgy current problem studied by MacCamy and Stephan in [16] and the study of the GIBCs for highly conducting obstacle for the Maxwell system in the time harmonic regime by Haddar, Joly, and Nguyen in [10] (see also the work of Caloz, Dauge, Faou, and Péron in [4]) and references therein.

The study of GIBCs for highly conducting obstacle has been though studied extensively in the literature, the rigorous study of GIBCs for highly absorbing obstacle in the time regime is not known to our knowledge. The lack of the study in the time regime is not special for this context but a common problem in the study of acoustic and electromagnetic waves since problems in the time regime involve the interaction of all frequency and hence they are harder to analyze.

The goal of this paper is to provide the analysis of the GIBCs for highly conducting obstacle in the **time regime** in two and three dimensions. Heuristically, these are obtained by taking the inverse Fourier transform of the corresponding conditions in the time harmonic regime with respect to frequency. Since the GIBCs in the time harmonic regime are frequency dependent, the ones in the **time** regime are **non-local** with respect to time. The error estimates in the time regime can be derived from the ones in the time harmonic regime, when the frequency dependence is well-controlled. This idea is originally due to Nguyen and Vogelius in [23] used for the cloaking context. To implement this idea, we revise and extend the work of Haddar, Joly, and Nguyen in [9], where the GIBCs for the time harmonic regime were investigated for a fixed frequency in three dimensions. Even though, we follow the strategy in [23], our analysis on the stability contains new ingredients and ideas. First, instead of considering the difference between solutions of the exact model and the approximate model in the time harmonic regime, we consider the difference between their derivative in time. This simple idea helps us to avoid the machinery used in [23] concerning the integrability with respect to frequency in the low frequency regime. The proof of the stability in the low frequency regime is based on a compactness argument as in [9] and uses ideas in [19]. Second, the compactness argument is not appropriate in the high frequency regime; moreover, the well-known Morawetz multiplier technique used in [23] does not fit directly in our setting, mainly due to the structure of the GIBCs. This new technical challenge distinguishes our work from [23]. To tackle it, we use essentially a result by Hörmander in [12] (Lemma 14). Another important part of the proof is the well-posedness in the time domain of the approximate problems imposed with GIBCs on the boundary of the obstacle, which are non-local in time.

In this paper, we concentrate only on the GIBCs of order **0 and 1**. Even though our approach also works for the GIBCs of orders 2 and 3, we are still not able to obtain the optimal expected estimates as in the time harmonic regime due to the complexity of the structures of the GIBCs in these cases. We postpone the study of these conditions to our future work [21].

There are many other interesting situations in which the asymptotic expansions have been investigated in the time harmonic setting. For example, the thin coating effect in e.g., [2], the wave propagation in media with thin slots in e.g., [13], the wave propagation across thin periodic interfaces in e.g., [6] and references therein. We hope that our analysis also sheds light to the situations mentioned.

We next describe the problem in more details. Let Ω be a smooth bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\Gamma := \partial\Omega$ and let $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ be such that $\operatorname{supp} f \subset [0, T] \times (B_{R_0} \setminus \overline{\Omega})$ for some fixed $R_0 > 0$ and $T > 0$. Here and in what follows, B_r denotes the ball of radius r centered at the origin. Let $u^\varepsilon \in L^\infty_{loc}\big([0,\infty); H^1(\mathbb{R}^d)\big)$ with $\partial_t u^\varepsilon \in L^\infty_{loc}\big([0,\infty); L^2(\mathbb{R}^d)\big)$ be the unique weak solution to the problem

$$\begin{cases} \partial_{tt} u^\varepsilon(t,x) - \Delta u^\varepsilon(t,x) + \sigma_\varepsilon(x)\, \partial_t u^\varepsilon(t,x) = f(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ u^\varepsilon(0,x) = \partial_t u^\varepsilon(0,x) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where

$$\sigma_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^d \setminus \Omega, \\ \dfrac{1}{\varepsilon^2} & \text{if } x \in \Omega. \end{cases}$$

for some $\varepsilon > 0$ small. Roughly speaking, the absorption of Ω is of order $1/\varepsilon^2$. We consider here the case in which the initial conditions are zero. The general case could be treated similarly as discussed in the cloaking setting in [23].

Let $\hat{u}_\varepsilon(k,x)$ and $\hat{f}(k,x)$ be the Fourier transform of u_ε and f with respect to time respectively, i.e., [1]

$$\hat{u}_\varepsilon(k,x) = \mathcal{F}(u_\varepsilon)(k,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u_\varepsilon(t,x)\, e^{i\,k\,t}\, dt \tag{1.2}$$

and

$$\hat{f}(k,x) = \mathcal{F}(f)(k,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t,x)\, e^{i\,k\,t}\, dt. \tag{1.3}$$

Then for almost every $k > 0$, $\hat{u}_\varepsilon(k,x) \in H^1_{loc}(\mathbb{R}^d)$ be the unique solution to the equation

$$\Delta \hat{u}_\varepsilon(k,x) + k^2 \hat{u}_\varepsilon(k,x) + i\,k\,\sigma_\varepsilon\, \hat{u}_\varepsilon(k,x) = \hat{f}(k,x), \qquad \text{in } \mathbb{R}^d, \tag{1.4}$$

which satisfies the outgoing condition

$$\partial_r u_\varepsilon - ik\, u_\varepsilon = o\big(r^{-\frac{(d-1)}{2}}\big), \quad \text{as } r = |x| \to \infty. \tag{1.5}$$

This fact is formulated later in Lemma 4 whose proof has root in [23, Theorem A.1].

Set [2]

$$\hat{\varepsilon} = \frac{\varepsilon}{\sqrt{k}}, \qquad \alpha = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}, \tag{1.6}$$

and

$$\mathcal{D}_\ell^{\hat{\varepsilon}} = \begin{cases} 0 & \text{for } \ell = 0, \\ \dfrac{\hat{\varepsilon}}{\alpha} & \text{for } \ell = 1. \end{cases} \tag{1.7}$$

It is proved in [9] that the GIBCs of order 0 and 1 corresponding to (1.4) are

$$v + \mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v = 0, \quad \text{on } \Gamma. \tag{1.8}$$

Here and in what follows, $n = n(x)$ is the unit normal vector **directed into** Ω on Γ. More precisely, let $s \in H^1(\mathbb{R}^d)$ with support in $B_{R_0} \setminus \Omega$ and let $v^\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ and $v^a_\ell \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solutions to the problems

$$\Delta v^\varepsilon + k^2 v^\varepsilon + i\,k\,\sigma_\varepsilon\, v^\varepsilon = s, \qquad \text{in } \mathbb{R}^d, \tag{1.9}$$

and

$$\begin{cases} \Delta v^a_\ell + k^2\, v^a_\ell = s, & \text{in } \mathbb{R}^d \setminus \Omega, \\ v^a_\ell + \mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v^a_\ell = 0 & \text{on } \Gamma. \end{cases} \tag{1.10}$$

Haddar, Joly, and Nguyen [9, Theorem 3] proved that, for any $R > 0$, [3]

$$\|v^\varepsilon - v^a_\ell\|_{H^1(B_R \setminus \Omega)} \le C(k,R)\, \varepsilon^{\ell+1} \|s\|_{H^m(\mathbb{R}^d)}, \tag{1.11}$$

[1] We extend these function by 0 for $t < 0$.

[2] $\alpha^2 = -i$.

[3] In [9], the authors considered the bounded setting. However, their method also implies the results mentioned here.

for some positive constant $C(k, R)$ and for some $m > 0$ large enough. The dependence on k of $C(k, R)$ in [9] is not explicit.

We are now ready to **heuristically** derive the GIBCs for (1.1) by taking the inverse Fourier transform of the GIBCs in the time harmonic regime with respect to frequency. We have

GIBC of order 0:

$$G_0^\varepsilon(v) := v = 0, \quad \text{ on } \mathbb{R}_+ \times \Gamma. \tag{1.12}$$

This is clear from (1.7) and (1.8) with $\ell = 0$.

GIBC of order 1:

$$G_1^\varepsilon(v) := \partial_n v + B_1^\varepsilon v = 0 \text{ on } \mathbb{R}_+ \times \Gamma, \tag{1.13}$$

where

$$\big(B_1^\varepsilon v\big)(t, x) := \frac{1}{\sqrt{\pi}\,\varepsilon} \int_0^t \frac{\partial_t v(\tau, x)}{\sqrt{t-\tau}}\, d\tau. \tag{1.14}$$

The derivation goes as follows. For $\ell = 1$, condition (1.8) reads as:

$$\partial_n v + \frac{\alpha\sqrt{k}}{\varepsilon}\, v = 0, \quad \text{ on } \Gamma,$$

or equivalently

$$\partial_n v + \frac{1}{\varepsilon}\frac{1}{\alpha\sqrt{k}}\,(-ik)v = 0, \quad \text{ on } \Gamma.$$

The condition (1.13) is now a consequence of the fact (see, e.g., [8, p. 171])

$$\mathcal{F}(\varphi)(k) = \frac{1}{\sqrt{2\pi}}\,\frac{\sqrt{\pi}}{\alpha\sqrt{k}}, \quad \text{ where } \quad \varphi(t) = \begin{cases} \dfrac{1}{\sqrt{t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases} \tag{1.15}$$

We have heuristically derived the GIBCs of orders 0 (1.12) and 1 (1.13) for the full wave equation (1.1). Similarly, one can obtain the GIBCs of orders $2, 3$ for (1.1) from the corresponding ones in the time harmonic regime obtained in [9]. However, such conditions are more complicated. We have not been able yet to obtain the optimal expected estimates for them as in the ones in the time harmonic regime. We postpone the study of these conditions to our future work [21].

The goal of this paper is to establish error estimates for (1.12) and (1.13). More precisely, we prove

Theorem 1. *Let $d = 2, 3$, $\ell = 0, 1$, $T > 0$, $R_0 > 0$, and let G_ℓ^ε be defined in* (1.12) *and* (1.13). *Assume that $f \in L^2\big([0,\infty) \times \mathbb{R}^d\big)$ with $\operatorname{supp} f \subset [0, T] \times (B_{R_0} \setminus \overline{\Omega})$. There exists a unique weak solution $u_\ell^a \in L^\infty_{loc}([0,\infty); H^1(\mathbb{R}^d \setminus \Omega))$ with $\partial_t u_\ell^a \in L^\infty_{loc}([0,\infty); L^2(\mathbb{R}^d \setminus \Omega))$ to*

$$\begin{cases} \partial^2_{tt} u_\ell^a - \Delta u_\ell^a = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \setminus \overline{\Omega}, \\ G_\ell^\varepsilon(u_\ell^a) = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ \partial_t u_\ell^a(0, \cdot) = u_\ell^a(0, \cdot) = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}; \end{cases} \tag{1.16}$$

moreover,

$$\int\limits_{\mathbb{R}^d\setminus\Omega} |\nabla u^a_\ell(t,x)|^2 + |\partial_t u^a_\ell(t,x)|^2\,dx \le C\,t\,\|f\|^2_{L^2([0,t]\times\mathbb{R}^d)} \quad \forall\, t\ge 0. \tag{1.17}$$

Assume in addition that Ω *is* **star-shaped** *and* $f\in C^\infty\big((0,\infty)\times\mathbb{R}^d\big)$ *with* $\operatorname{supp} f \subset\subset (0,+\infty)\times(B_{R_0}\setminus\overline{\Omega})$. *Then, for any* $t>0$ *and* $K\subset\subset\mathbb{R}^d\setminus\overline{\Omega}$ [4], *there is a positive constant* C *independent of* ε *and* f, *such that, for some integer* m [5].

$$\|u^\varepsilon - u^a_\ell\|_{L^\infty\big([0,t];H^1(K)\big)} \le C\,\varepsilon^{\ell+1}\|f\|_{H^m(\mathbb{R}_+\times\mathbb{R}^d)}. \tag{1.18}$$

The following definition of the weak solutions, which is motivated from the standard concept of weak solutions, is used in Theorem 1.

Definition 1. *Let* $d=2,3$ *and* $\ell=0,1$. *We say a function*

$$u^a_\ell\in L^\infty_{loc}([0,\infty);H^1(\mathbb{R}^d\setminus\Omega)) \text{ with } \partial_t u^a_\ell\in L^\infty_{loc}([0,\infty);L^2(\mathbb{R}^d\setminus\Omega))$$

is a weak solution to (1.16) *provided*

$$\frac{d^2}{dt^2}\int\limits_{\mathbb{R}^d\setminus\Omega} u^a_\ell(t,x)v(x)\,dx + \int\limits_{\mathbb{R}^d\setminus\Omega}\nabla u^a_\ell(t,x)\nabla v(x)\,dx$$

$$+\,\ell\int\limits_{\Gamma}\big(B^\varepsilon_\ell\,u^a_\ell\big)(t,x)\,v(x)\,dx = \int\limits_{\mathbb{R}^d\setminus\Omega} f(t,x)\,v(x)\,dx \quad \forall\, v\in H^1(\mathbb{R}^d\setminus\Omega), \tag{1.19}$$

for any $t>0$, *and*

$$u^a_\ell(0,x)=\partial_t u^a_\ell(0,x)=0 \text{ in } \mathbb{R}^d\setminus\Omega. \tag{1.20}$$

Remark 1. *In the definition, the last term of the LHS of* (1.19) *is 0 if* $\ell=0$. *The definition in the case* $\ell=0$ *is standard.*

The proof of Theorem 1 is presented in Section 2. The proof of well posedness of (1.16) for $\ell=1$ (non-local structure in time) is based on a nontrivial energy estimate (2.1), which is derived from the causality, see (1.14). Following the strategy in [22], we will derive (1.18) from estimates in the frequency domain. For this end, we establish estimates for $C(k,R)$ in (1.11) where the dependence on k is well-controlled. This is one of the main parts of the analysis and presented in the following three propositions which deal with different regimes of frequency.

[4] Roughly speaking, K is bounded and away from the boundary of the obstacle.

[5] The integer m can be chosen as follows $m=13$ if $\ell=0$ and $m=16$ if $\ell=1$. Assume in addition that $\operatorname{supp} f\cap\bar{\Omega}=\emptyset$. Then m can be chosen as follows $m=8$ if $\ell=0$ and $m=9$ if $\ell=1$; however the constant C in (2.1) now depends on the distance between $\operatorname{supp} f$ and $\bar{\Omega}$ (see Footnote 7).

Proposition 1. *Let $d = 2,3$, $\ell = 0,1$, $0 < \varepsilon < 1$, $0 < k < \varepsilon^2$, $R_0 > 0$, and $s \in L^2(\mathbb{R}^d \setminus \overline{\Omega})$ with* $\operatorname{supp} s \subset B_{R_0} \setminus \overline{\Omega}$. *Let $v^\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ and $v^a_\ell \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solutions to* (1.9) *and* (1.10) *respectively. We have, for any $r > 0$,*

$$\|v^\varepsilon - v^a_\ell\|_{H^1(B_r \setminus \Omega)} \leq C_r \, \|s\|_{L^2(\mathbb{R}^d)},$$

for some constant $C_r > 0$, independent of s, ε, and k.

Proposition 2. *Let $d = 2,3$, $\ell = 0,1$, $0 < \varepsilon < 1$, $k_0 > 0$, $\varepsilon^2 < k < k_0$, $R_0 > 0$, and $s \in H^{2\ell+5}(\mathbb{R}^d \setminus \overline{\Omega})$ with* $\operatorname{supp} s \subset B_{R_0} \setminus \overline{\Omega}$. *Let $v^\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ and $v^a_\ell \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solutions to* (1.9) *and* (1.10) *respectively. We have, for any $r > 0$,*

$$\|v^\varepsilon - v^a_\ell\|_{H^1(B_r \setminus \Omega)} \leq C_r \, \hat{\varepsilon}^{\ell+1} \, \|s\|_{H^{2\ell+5}(\mathbb{R}^d)}, \tag{1.21}$$

for some constant $C_r > 0$, independent of s, ε, and k.

Proposition 3. *Let $d = 2,3$, $\ell = 0,1$, $k_0 > 0$, $0 < \varepsilon < 1$, $k \geq k_0$, $R_0 > 0$, and $s \in H^{2\ell+5}(\mathbb{R}^d \setminus \overline{\Omega})$ with* $\operatorname{supp} s \subset B_{R_0} \setminus \overline{\Omega}$. *Let $v^\varepsilon \in H^1_{loc}(\mathbb{R}^d)$ and $v^a_\ell \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solutions to* (1.9) *and* (1.10) *respectively. Assume that Ω is* **star-shaped**. *Then, for any $K \subset\subset \mathbb{R}^d \setminus \overline{\Omega}$, we have*

$$\|\nabla(v^\varepsilon - v^a_\ell)\|_{L^2(K)} + k \, \|v^\varepsilon - v^a_\ell\|_{L^2(K)} \leq C_K \, k^{2\ell+7} \, \hat{\varepsilon}^{\ell+1} \, \|s\|_{H^{2\ell+5}(\mathbb{R}^d)}, \tag{1.22}$$

for some constant $C_K > 0$, independent of s, ε, and k.

The proofs of Propositions 1 and 2 are given in Section 4. They are based on a compactness argument as in [9] and use results on the Helmholtz equations in the low frequency regime in [19, 20]. The proof of Proposition 2 and 3 uses the asymptotic expansion introduced in [9]. To obtain explicit dependence on frequency of these estimates, we revise the asymptotic expansion given in [9] for all range of frequency with a focus on the dependence on frequency. The proof of the stability in Proposition 3 is given in Section 5. It is a heart matter of our paper. The compactness argument used in the proof of Propositions 1 and 2 is not appropriate in this regime. Moreover, the Morawetz's multiplier technique does not work directly in our settings. Due to the structure of the GIBCs, we are only able to obtain an estimate in $L^2(\Gamma)$-norm of the solution, not the $H^1(\Gamma)$-norm required for Morawetz's technique. To overcome this difficulty, we use a result due to Hörmander in [12] (see Lemma 14). The payoff for lacking of the control of the $H^1(\Gamma)$-norm is that we can only obtain estimates in regions away from Γ, see (1.22).

2 Proof of Theorem 1

This section containing two subsections is devoted to the proof of Theorem 1 assuming Propositions 1, 2, and 3 (their proofs are given in Section 4 and 5). In the first subsection, we establish the well-posedness and the stability for (1.16). We also show that the Fourier transform of the weak solutions satisfies the outgoing conditions for almost every positive frequency. The proof of Theorem 1 is given in the second subsection.

2.1 Preliminaries

In this section, we prepare some materials for the proof of Theorem 1. We first prove the well-posedness and the stability for (1.16).

Lemma 1. *Let $d = 2, 3$, $\ell = 0, 1$ and $f \in L^2\big([0,\infty), L^2(\mathbb{R}^d \setminus \Omega)\big)$ with compact support. There exists a unique weak solution $v_\ell^a \in L^\infty_{loc}\big([0,\infty), H^1(\mathbb{R}^d \setminus \Omega)\big)$ with $\partial_t v_\ell^a \in L^\infty_{loc}\big([0,\infty), L^2(\mathbb{R}^d \setminus \Omega)\big)$ to (1.16). Moreover,*

$$E(t, u_\ell^a) \le C\, t\, \|f\|^2_{L^2([0,t]\times\mathbb{R}^d)} \quad \forall\, t \ge 0. \tag{2.1}$$

Here,

$$E(t, \psi) := \frac{1}{2} \int\limits_{\mathbb{R}^d\setminus\Omega} \big(|\partial_t \psi(t,x)|^2 + |\nabla \psi(t,x)|^2\big)\, dx, \tag{2.2}$$

for $\psi \in L^\infty_{loc}\big([0,\infty), H^1(\mathbb{R}^d \setminus \Omega)\big)$ with $\partial_t \psi \in L^\infty_{loc}\big([0,\infty), L^2(\mathbb{R}^d \setminus \Omega)\big)$.

Proof. We need only prove the theorem for $\ell = 1$, since the case $\ell = 0$ is standard. We first establish the existence of a weak solution which satisfies (2.1). For this end, we use the Galerkin method. Let $(\varphi_j)_{j=1}^\infty \subset C_c^\infty(\mathbb{R}^d \setminus \Omega)$ be an orthonormal basis in $H^1(\mathbb{R}^d \setminus \Omega)$. For $m \in \mathbb{N}$, consider u_m of the form

$$u_m = \sum_{j=1}^m d_{m,j}(t)\ \varphi_j(x) \tag{2.3}$$

such that

$$\frac{d^2}{dt^2} \int\limits_{\mathbb{R}^d\setminus\Omega} u_m(t,x)\ \varphi_j(x)\, dx + \int\limits_{\mathbb{R}^d\setminus\Omega} \nabla u_m(t,x)\ \nabla \varphi_j(x)\, dx$$
$$+ \int\limits_{\Gamma} \big(B_1^\varepsilon\, u_m\big)(t,x)\ \varphi_j(x)\, dx = \int\limits_{\mathbb{R}^d\setminus\Omega} f(t,x)\ \varphi_j(x)\, dx, \qquad \text{for } j = 1, \dots, m \tag{2.4}$$

and

$$d_{m,j}(t) = d'_{m,j}(t) = 0 \qquad \text{for } j = 1, \dots, m. \tag{2.5}$$

Since $(\varphi_j)_j$ is linearly independent in $H^1(\mathbb{R}^d \setminus \Omega)$, it is also linearly independent in $L^2(\mathbb{R}^d \setminus \Omega)$. This implies the $(n \times n)$ matrix M given by $M_{i,j} = \langle \varphi_i, \varphi_j \rangle_{L^2(\mathbb{R}^d\setminus\Omega)}$ is invertible. The existence and uniqueness of u_m then follows; for example, one can use the theory of Volterra equation (see, e.g., [3, Theorem 2.1.1]).

We now derive an estimate for u_m. Let us multiply (2.4) by $d'_{m,j}(s)$ and sum it up with respect to j. Integrating the resulting equation over $[0,t]$ with respect to s and using (2.5), we obtain

$$E(t, u_m) + \int_0^t \int\limits_{\Gamma} \big(B_1^\varepsilon\, u_m\big)(s,x)\ \partial_t u_m(s,x)\, dx\, ds = \int_0^t \int\limits_{\mathbb{R}^d\setminus\Omega} f(s,x)\ \partial_t u_m(s,x)\, dx\, ds. \tag{2.6}$$

We claim that, for $t \geq 0$,

$$\int_0^t \int_\Gamma \big(B_1^\varepsilon u_m\big)(s,x)\, \partial_t u_m(s,x)\, dx\, ds \geq 0. \tag{2.7}$$

Indeed, recall

$$\big(B_1^\varepsilon u_m\big)(s,x) = \frac{1}{\sqrt{\pi}\,\varepsilon} \Big(\varphi * \partial_t u_m\Big)(s,x).$$

where φ is given in (1.15). Here and in what follows $*$ denotes the convolution with respect to time. Set

$$U(s,x) = \begin{cases} \partial_t u_m(s,x) & \text{if } s < t, \\ 0 & \text{if } s \geq t \text{ or } s < 0. \end{cases}$$

Then,

$$\int_0^t \int_\Gamma \big(B_1^\varepsilon u_m\big)(s,x)\, \partial_t u_m(s,x)\, dx\, ds = \frac{1}{\varepsilon\sqrt{\pi}} \int_\mathbb{R} \int_\Gamma \Big(\varphi * U\Big)(s,x)\, U(s,x)\, dx\, ds. \tag{2.8}$$

Using Parseval's identity and (1.15), we obtain

$$\begin{aligned} \int_\mathbb{R} \int_\Gamma \Big(\varphi * U\Big)(s,x)\, U(s,x)\, dx\, ds &= \int_\mathbb{R} \int_\Gamma \widehat{\Big(\varphi * U\Big)}(k,x)\, \overline{\widehat{U}}(k,x)\, dx\, dk \\ &= 2\Re \int_{\mathbb{R}_+} \int_\Gamma \frac{\sqrt{\pi}}{\alpha\sqrt{k}}\, |\widehat{U}(k,x)|^2\, dx\, dk \geq 0. \end{aligned} \tag{2.9}$$

A combination of (2.8) and (2.9) yields (2.7).

From (2.6) and (2.7), we have

$$E(t, u_m) \leq \int_0^t \int_{\mathbb{R}^d \setminus \Omega} f(s,x)\, \partial_t u_m(s,x)\, dx\, ds. \tag{2.10}$$

By the Gronwall inequality, it follows from (2.2) and (2.10) that

$$\int_0^t E(s, u_m)\, ds \leq C \left[\int_0^t \Big(\int_0^s \int_{\mathbb{R}^d \setminus \Omega} |f(\tau,x)|^2\, dx\, d\tau \Big)^{1/2} ds \right]^2,$$

which implies

$$\int_0^t E(s, u_m)\, ds \leq C\, t^2 \int_0^t \int_{\mathbb{R}^d \setminus \Omega} |f(s,x)|^2\, dx\, ds. \tag{2.11}$$

Due to (2.10),

$$E(t,u_m) \le \Big(\int_0^t \int_{\mathbb{R}^d\setminus\Omega} |\partial_t u_m(s,x)|^2\, dx\, ds \Big)^{1/2} \Big(\int_0^t \int_{\mathbb{R}^d\setminus\Omega} |f(s,x)|^2\, dx\, ds \Big)^{1/2}.$$

This implies

$$E(t,u_m) \le \Big(\int_0^t E(s,u_m)\, ds \Big)^{1/2} \Big(\int_0^t \int_{\mathbb{R}^d\setminus\Omega} |f(s,x)|^2\, dx\, ds \Big)^{1/2}.$$

It follows that from (2.11) that

$$E(t,u_m) \le Ct \int_0^t \int_{\mathbb{R}^d\setminus\Omega} |f(s,x)|^2\, dx\, ds. \tag{2.12}$$

Hence, for any fixed $T > 0$, there exists a subsequence of (u_m) (which is also denoted by u_m for notational ease) such that $u_m \to u$ weakly-star in $L^\infty\big([0,T], H^1(\mathbb{R}^d \setminus \Omega)\big)$ and $\partial_t u_m \to \partial_t u$ weakly star in $L^\infty\big([0,T], L^2(\mathbb{R}^d \setminus \Omega)\big)$. It is clear that u satisfies (1.19) for $t \in (0,T)$ and (1.20).

It remains to show that the limit is unique. It suffices to prove that if $u \in L^\infty\big([0,T], H^1(\mathbb{R}^d \setminus \Omega)\big)$ with $\partial_t u \in L^\infty\big([0,T], L^2(\mathbb{R}^d \setminus \Omega)\big)$ and u satisfies (1.19) with $f = 0$ and (1.20) then $u = 0$. Set $u^*(t,x) = \int_0^t u(\tau,x)\, d\tau$. We claim that u^* satisfies

$$\frac{d^2}{dt^2} \int_{\mathbb{R}^d\setminus\Omega} u^*(t,x)\; v(x)\, dx + \int_{\mathbb{R}^d\setminus\Omega} \nabla u^*(t,x)\; \nabla v(x)\, dx + \int_\Gamma \big(B_1^\varepsilon\, u^*\big)(t,x)\, v(x)\, dx = 0, \quad \forall\, v \in H^1(\mathbb{R}^d \setminus \Omega), \tag{2.13}$$

for any $t > 0$, and

$$u^*(0,x) = \partial_t u^*(0,x) = 0 \text{ in } \mathbb{R}^d \setminus \Omega. \tag{2.14}$$

The claim (more precisely, equation (2.13)) can be verified by integrating (1.19) with respect to t. We basically only need to verify the validity of the boundary term on the LHS. This follows from Lemma 2 below.

From (2.13), we derive

$$\int_{\mathbb{R}^d\setminus\Omega} \partial^2_{tt} u^*(t,x)\; v(x)\, dx + \int_{\mathbb{R}^d\setminus\Omega} \nabla u^*(t,x)\; \nabla v(x)\, dx + \int_\Gamma \big(B_1^\varepsilon\, u^*\big)(t,x)\, v(x)\, dx = 0, \quad \forall\, v \in H^1(\mathbb{R}^d \setminus \Omega), \tag{2.15}$$

Letting $v(x) = u_t^*$ in (2.15) and integrating in $[0,t]$, we obtain

$$\int_0^t \int_{\mathbb{R}^d\setminus\Omega} u_{tt}^*(t,x)u_t^*(t,x)\,dx + \int_0^t \int_{\mathbb{R}^d\setminus\Omega} \nabla u^*(t,x)\nabla u_t^*(t,x)\,dx + \int_0^t \int_\Gamma \big(B_1^\varepsilon\, u^*\big)(t,x)\, u_t^*(t,x)\,dx = 0. \tag{2.16}$$

By the same argument used to obtain (2.7), we have

$$\int_0^t \int_\Gamma \big(B_\ell^\varepsilon\, u^*\big)(t,x)\, u_t^*(t,x)\,dx \geq 0.$$

It follows from (2.16) that

$$E(t,u^*) \leq E(0,u^*) = 0.$$

Therefore, $u^* \equiv 0$ and, hence, $u \equiv 0$. The proof is complete. □

Remark 2. *Similar ideas are taken into account for the proof of the well-posedness of non-local the wave equations in [24] in which the Drude-Lorentz model is used to capture the dependence of the material on the frequency.*

The following lemma which reveals an interesting property of the integral kernel φ of B_1^ε is used in the proof of Lemma 1.

Lemma 2. *Assume that* $\psi \in L^\infty_{loc}[0,\infty)$. *Let*

$$\Psi(t) = \int_0^t \psi(s)\,ds.$$

Then, for any $t > 0$

$$\int_0^t (\varphi * \psi)(s)\;ds = (\varphi * \Psi)(t).$$

Here, φ *is defined in* (1.15).

Proof. The proof only involves a change of order of integration and an integration by parts. The details are left to the reader. □

Concerning the outgoing condition of $\hat{u}_\ell^a$ and $\hat{u}^\varepsilon$, we have the following two results whose proofs are similar to the one [23, Theorem A1]. The details are left to the reader.

Lemma 3. *Let* $d = 2,3$, $\ell = 0,1$ *and let* $\hat{u}_\ell^a(k,x)$ *be the Fourier transform of* $u_\ell^a(t,x)$ *with respect to* t. *Then, for almost every* $k > 0$, $\hat{u}_\ell^a(k,.) \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ *is the unique outgoing solution to*

$$\begin{cases} \Delta \hat{u}_\ell^a(k,x) + k^2\; \hat{u}_\ell^a(k,x) = -\hat{f}(k,x), & \text{in } \mathbb{R}^d \setminus \Omega, \\ \hat{u}_\ell^a(k,x) + \mathcal{D}_\ell^{\hat{\varepsilon}}\; \partial_n \hat{u}_\ell^a(k,x) = 0 & \text{on } \Gamma. \end{cases}$$

Moreover,

$$k\; \hat{u}_\ell^a(k,x) \in L^2_{loc}(\mathbb{R} \times (\mathbb{R}^d \setminus \Omega)).$$

Lemma 4. *Let $d = 2, 3$, $\ell = 0, 1$ and let $\hat{u}^\varepsilon(k, x)$ be the Fourier transform of $u^\varepsilon(t, x)$ with respect to t. Then, for almost every $k > 0$, $\hat{u}^\varepsilon(k, .) \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ is the unique outgoing solution to*

$$\Delta \hat{u}^\varepsilon(k, x) + k^2 \, \hat{u}^\varepsilon(k, x) + i \, k \, \sigma_\varepsilon \, \hat{u}^\varepsilon(k, x) = -\hat{f}(k, x), \quad \text{in } \mathbb{R}^d.$$

Moreover,

$$k \, \hat{u}^\varepsilon(k, x) \in L^2_{loc}(\mathbb{R} \times \mathbb{R}^d).$$

2.2 Proof of Theorem 1

The unique existence of u^a_ℓ and (2.1) follow from Lemma 1. It only remains to prove (1.18). To this end, we use Propositions 1, 2, and 3 (their proofs are given later in Sections 4 and 5). We follow the strategy in [23]. However, instead of estimating $\|u^\varepsilon - u^a_\ell\|_{L^2(\mathbb{R}_+, H^1(K))}$ as in [23], we estimate $\|\partial_t(u^\varepsilon - u^a_\ell)\|_{L^2(\mathbb{R}_+, H^1(K))}$. This simple idea helps us to avoid the technical issue of integrability in low frequency range in [23], which involves the theory of Gamma-convergence [6].

Applying Parseval's identity and using the fact that u^ε and u^a_ℓ are real, we have

$$\int_{\mathbb{R}_+} \|\partial_t(u^\varepsilon - u^a_\ell)(t, .)\|^2_{H^1(K)} \, dt = 2 \int_{\mathbb{R}_+} k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k, .)\|^2_{H^1(K)} \, dk. \tag{2.17}$$

For notational ease, we assume that the constant k_0 in Propositions 2 and 3 is 1. It is clear that

$$\int_{\mathbb{R}_+} k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k, .)\|^2_{H^1(K)} \, dk = \Big(\int_0^{\varepsilon^2} + \int_{\varepsilon^2}^1 + \int_1^\infty \Big) k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k, .)\|^2_{H^1(K)} \, dk. \tag{2.18}$$

We next estimate the RHS of (2.18). We begin with the first term. Applying Proposition 1, we have

$$\int_0^{\varepsilon^2} k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k, .)\|^2_{H^1(K)} \, dk \; \le C \int_0^{\varepsilon^2} k^2 \, \|\hat{f}(k, .)\|^2_{L^2(\mathbb{R}^d)} \, dk.$$

It follows that

$$\int_0^{\varepsilon^2} k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k, .)\|^2_{H^1(K)} dk \; \le C \, \varepsilon^6 \sup_{k>0} \, \|\hat{f}(k, .)\|^2_{L^2(\mathbb{R}^d)}. \tag{2.19}$$

Since $f(., x)$ is supported in $[0, T]$, it follows from the definition of the Fourier transform that

$$|\hat{f}(k, x)| \le \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(t, x)| \, dt \le C \left(\int_{\mathbb{R}} |f(t, x)|^2 \, dt \right)^{1/2};$$

[6] This simple idea is also very useful in the context of cloaking in [24].

which implies

$$\|\hat{f}(k,.)\|^2_{L^2(\mathbb{R}^d)} \leq C \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}} |f(t,x)|^2\,dt\,dx = C\,\|f\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}. \tag{2.20}$$

A combination of (2.19) and (2.20) yields

$$\int\limits_0^{\varepsilon^2} k^2\,\|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)}\,dk \leq C\,\varepsilon^6\,\|f\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}. \tag{2.21}$$

We next estimate the second term of the RHS in (2.18). Applying Proposition 2, we obtain

$$\int\limits_{\varepsilon^2}^{1} k^2\,\|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)}\,dk \leq C\,\sup_{k>0}\,\|\hat{f}(k,.)\|^2_{H^{2\ell+5}(\mathbb{R}^d)}\int\limits_{\varepsilon^2}^{1} k^2\,\hat{\varepsilon}^{2(\ell+1)}\,dk. \tag{2.22}$$

Similar to (2.20), we have

$$\sup_{k>0}\|\hat{f}(k,.)\|_{H^{2\ell+5}(\mathbb{R}^d)} \leq C\,\|f\|_{H^{2\ell+5}(\mathbb{R}_+\times\mathbb{R}^d)}. \tag{2.23}$$

Since $\hat{\varepsilon}^2 = \varepsilon^2/k$ and $\ell = 0, 1$, it follows that

$$\int\limits_{\varepsilon^2}^{1} k^2\,\hat{\varepsilon}^{2(\ell+1)}\,dk = \varepsilon^{2(\ell+1)}\int\limits_{\varepsilon^2}^{1} k^{1-\ell}\,dk \leq C\,\varepsilon^{2(\ell+1)}. \tag{2.24}$$

A combination of (2.22), (2.23), and (2.24) yields

$$\int\limits_{\varepsilon^2}^{1} k^2\,\|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)}\,dk \leq C\,\varepsilon^{2(\ell+1)}\,\|f\|^2_{H^{2\ell+5}(\mathbb{R}_+\times\mathbb{R}^d)}. \tag{2.25}$$

We now estimate the last term of the RHS in (2.18). Applying Proposition 3, we obtain

$$\begin{aligned}\int\limits_{1}^{\infty} k^2\,\|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)}\,dk &\leq \int\limits_{1}^{\infty} k^2\,\hat{\varepsilon}^{2(\ell+1)}\,k^{2(2\ell+7)}\,\|\hat{f}(k,.)\|^2_{H^{2\ell+5}(\mathbb{R}^d)}\,dk\\ &\leq C\,\varepsilon^{2(\ell+1)}\int\limits_{1}^{\infty} k^{3\ell+15}\,\|\hat{f}(k,.)\|^2_{H^{2\ell+5}(\mathbb{R}^d)}\,dk.\end{aligned}$$

It follows that

$$\int\limits_{1}^{\infty} k^2\,\|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)}\,dk \leq C\,\varepsilon^{2(\ell+1)}\|f\|^2_{H^{m_\ell}(\mathbb{R}_+\times\mathbb{R}^d)}, \tag{2.26}$$

where [7]

$$m_\ell = 13 \text{ if } \ell = 0 \quad \text{ and } \quad m_\ell = 16 \text{ if } \ell = 1.$$

Plugging (2.21), (2.25), and (2.26) into (2.18), we obtain:

$$\int_{\mathbb{R}} k^2 \, \|(\hat{u}^\varepsilon - \hat{u}^a_\ell)(k,.)\|^2_{H^1(K)} \, dk \leq C \, \varepsilon^{2(\ell+1)} \, \|f\|^2_{H^{m_\ell}(\mathbb{R}_+ \times \mathbb{R}^d)}. \tag{2.27}$$

A combination of (2.17) and (2.27) yields

$$\int_{\mathbb{R}_+} \|\partial_t(u^\varepsilon - u^a_\ell)(t,.)\|^2_{H^1(K)} \leq C \, \varepsilon^{2(\ell+1)} \, \|f\|^2_{H^{m_\ell}(\mathbb{R}_+ \times \mathbb{R}^d)}. \tag{2.28}$$

Since $u^\varepsilon - u^a_\ell \equiv 0$ at $t = 0$, the conclusion follows. □

3 Asymptotic expansion for highly conducting obstacle revisited

This section is on the asymptotic expansion of v^ε to (1.9) with respect to the small parameter $\hat{\varepsilon} := \varepsilon/\sqrt{k}$ and is essentially based on the work of [9]. Our goal is to keep track of the frequency dependence there. We recall the notations in [9] and state estimates which are used in the proof of Propositions 2 and 3. Their proofs are given in the appendix.

Define

$$\Omega^\delta = \{x \in \Omega, \; d(x, \Gamma) \leq \delta\}. \tag{3.1}$$

In what follows, we fix $\delta > 0$ small enough such that any $x \in \Omega^\delta$ can be written uniquely in the form $x = x_\Gamma + \nu n$, where $(x_\Gamma, \nu) \in \Gamma \times \mathbb{R}_+$. Here, n is the unit normal vector of Γ at x_Γ pointing **toward** Ω.

Let $d = 3$, $\mathcal{H}$ and $\mathcal{G}$ be the mean and Gaussian curvatures of Γ and let $\mathcal{C} := \nabla_\Gamma n$ be the curvature tensor on Γ. Define the tangential operator $\mathcal{M}$ by the identity

$$\mathcal{C}\mathcal{M} = G I_\Gamma.$$

One has [9, (4.4)] for $x \in \Omega^\delta$,

$$J_\nu^3 \Delta = J_\nu \operatorname{div}_\Gamma (I_\Gamma + \nu \mathcal{M})^2 \nabla_\Gamma - J_\nu \cdot (I_\Gamma + \nu \mathcal{M})^2 \nabla_\Gamma + J_\nu^3 \partial^2_{\nu\nu} + 2 J_\nu^2 (\mathcal{H} + \nu G) \partial_n u, \tag{3.2}$$

where

$$J_\nu := \det(I + \nu C) = 1 + 2\nu \mathcal{H} + \nu^2 G. \tag{3.3}$$

[7] If $\operatorname{supp} f \cap \bar{\Omega} = \emptyset$, then $\|\hat{f}(k,.)\|^2_{H^{2\ell+5}(\mathbb{R}^d)}$ can be replaced by $\|\hat{f}(k,.)\|^2_{L^2(\mathbb{R}^d)}$. It follows that m_ℓ can be chosen as follows $m_\ell = 8$ if $\ell = 0$ and $m_\ell = 9$ if $\ell = 1$ and the constant C in (2.26) now depends on the distance between $\operatorname{supp} f$ and $\bar{\Omega}$.

The following differential operators $\mathcal{A}_m$ $(1 \le m \le 8)$ are defined in [9] [8]

$$\begin{aligned}
\mathcal{A}_1 =& 2\mathcal{H}\partial_\eta + 6\eta\mathcal{H}(\partial_\eta^2 + i),\\
\mathcal{A}_2 =& \Delta_\Gamma + k^2 + 2\eta(\mathcal{G}+4\mathcal{H}^2)\partial_\eta + 3\eta^2(\mathcal{G}+4\mathcal{H}^2)(\partial_\eta^2+i),\\
\mathcal{A}_3 =& 2\eta\Big[\mathcal{H}\Delta_\Gamma + \mathrm{div}_\Gamma(\mathcal{M}\nabla_\Gamma) - \nabla_\Gamma\mathcal{H}\nabla_\Gamma + 3k^2\mathcal{H}\Big] + 4\eta^2\mathcal{H}\Big[(3\mathcal{G}+2\mathcal{H}^2)\partial_\eta\Big]\\
&+ 4\eta^3\mathcal{H}(3\mathcal{G}+2\mathcal{H}^2)(\partial_\eta^2+i),\\
\mathcal{A}_4 =& \eta^2\Big[\mathcal{G}\Delta_\Gamma + 4\mathcal{H}\mathrm{div}_\Gamma(\mathcal{M}\nabla_\Gamma) + \mathrm{div}_\Gamma(\mathcal{M}^2\nabla_\Gamma)\Big] - \eta^2\Big[\nabla_\Gamma\mathcal{G}\nabla_\Gamma + 4\nabla_\Gamma\mathcal{H}(\mathcal{M}\nabla_\Gamma) - 3k^2(\mathcal{G}+4\mathcal{H}^2)\Big]\\
&+ 4\eta^3\mathcal{G}(\mathcal{G}+4\mathcal{H}^2)\partial_\eta + 3\eta^4\mathcal{G}(\mathcal{G}+4\mathcal{H}^2)(\partial_\eta^2+i),
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_5 =& 2\eta^3\Big[\mathcal{G}\mathrm{div}_\Gamma(\mathcal{M}\nabla_\Gamma) + \mathcal{H}\mathrm{div}_\Gamma\Big] - 2\eta^3\Big[\nabla_\Gamma\mathcal{G}(\mathcal{M}\nabla_\Gamma) + \nabla_\Gamma\mathcal{H}(\mathcal{M}^2\nabla_\Gamma) - 2k^2\mathcal{H}(3\mathcal{G}+2\mathcal{H}^2)\Big]\\
&+ 10\eta^4\mathcal{G}^2\mathcal{H}\partial_\eta + 6\eta^5\mathcal{G}^2\ \mathcal{H}(\partial_\eta^2+i),\\
\mathcal{A}_6 =& \eta^4\Big[\mathcal{G}\mathrm{div}_\Gamma(\mathcal{M}^2\nabla_\Gamma) - \nabla_\Gamma\mathcal{G}(\mathcal{M}^2\nabla_\Gamma) + 3k^2\mathcal{G}(\mathcal{G}+4\mathcal{H}^2)\Big] + 2\eta^5\mathcal{G}^3\partial_\eta + \eta^6\mathcal{G}^3(\partial_\eta^2+i),\\
\mathcal{A}_7 =& 6\eta^5k^2\mathcal{G}^3\mathcal{H}, \quad \text{and} \quad \mathcal{A}_8 = \eta^6k^2\mathcal{G}^3.
\end{aligned}$$

Using (3.2) and (3.3) as in [9, (5.22)], one has

$$\Delta + k^2 + \frac{i}{\hat{\varepsilon}^2} = \frac{1}{J_\nu^3\hat{\varepsilon}^2}\Big(-\partial_{\eta\eta}^2 - i - \sum_{m=1}^{8}\hat{\varepsilon}^m\mathcal{A}_m\Big). \tag{3.4}$$

Similarly, we also define the above operations in the case $d = 2$. In this case, the triple $(\mathcal{H},\mathcal{G},\mathcal{M})$ is replaced by $(\frac{\kappa}{2},0,0)$, where $\kappa = \kappa(x)$ is the (signed) curvature of Γ.

The following definitions by recurrence of w_e^ℓ in $\mathbb{R}^d\setminus\Omega$ and w_i^ℓ in $\Gamma\times\mathbb{R}_+$ are given in [9]. For $\ell = 0$, define

$$w_i^0(x) = 0 \text{ in } \Gamma\times\mathbb{R}_+, \tag{3.5}$$

and let $w_e^0 \in H^1_{loc}(\mathbb{R}^d\setminus\Omega)$ be the unique outgoing solution to

$$\begin{cases} \Delta w_e^0 + k^2\ w_e^0 = s & \text{in } \mathbb{R}^d\setminus\overline{\Omega},\\ w_e^0 = 0 & \text{on } \Gamma. \end{cases} \tag{3.6}$$

Let $\ell \ge 1$. Assume that w_e^j and w_i^j are defined for $j \le \ell-1$. Define w_i^ℓ to be the solution to

$$\begin{cases} (\partial_\eta^2 + i)\ w_i^\ell(x_\Gamma,\eta) = -\sum_{m=1}^{8}\mathcal{A}_m\ w_i^{\ell-m}(x_\Gamma,\eta) \text{ for } (x_\Gamma,\eta)\in\Gamma\times\mathbb{R}_+,\\ \partial_\eta w_i^\ell(x_\Gamma,0) = \partial_n w_e^{\ell-1}(x_\Gamma) \text{ and } \lim_{\eta\to\infty} w_i^\ell(x_\Gamma,\eta) = 0. \end{cases} \tag{3.7}$$

[8]The signs in front of i in our formulae are opposite to the ones in [9]. This is due to the difference between (1.9) and [9, (2.3)], because of different ways to take the Fourier transform in these papers. The reader should keep this fact in mind when comparing our calculations with those in [9].

(here we use the convention $w_i^\ell \equiv 0$ for $\ell < 0$) and let $w_e^\ell \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to

$$\begin{cases} \Delta w_e^\ell + k^2\, w_e^\ell = 0 & \text{for } x \in \mathbb{R}^d \setminus \overline{\Omega}, \\ w_e^\ell(x) = w_i^\ell(x,0) & \text{for } x \in \Gamma. \end{cases} \tag{3.8}$$

From (3.7), one has, [9, (4.27) and (4.28)],

$$w_i^1(x_\Gamma, \eta) = -\frac{1}{\alpha} \partial_n w_e^0\, e^{-\alpha\eta} \tag{3.9}$$

and

$$w_i^2(x_\Gamma, \eta) = \left\{ -\frac{1}{\alpha}\, \partial_n w_e^1 + \frac{\mathcal{H}}{\alpha^2}\, \partial_n w_e^0 + \frac{\eta\mathcal{H}}{\alpha}\, \partial_n w_e^0 \right\} e^{-\alpha\eta}. \tag{3.10}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ satisfy

$$\chi(\eta) = \begin{cases} 1 & \text{if } |\eta| \le \delta/2, \\ 0 & \text{if } |\eta| \ge \delta. \end{cases} \tag{3.11}$$

Following [9], we set

$$v_\ell^e(x) = w_e^0(x) + \hat\varepsilon\, w_e^1 + \dots + \hat\varepsilon^\ell\, w_e^\ell(x), \qquad x \in \mathbb{R}^d \setminus \overline{\Omega}$$

$$v_\ell^i(x) = \Big[w_i^0(x_\Gamma, \nu/\hat\varepsilon) + \hat\varepsilon\, w_i^1(x_\Gamma, \nu/\hat\varepsilon) + \dots + \hat\varepsilon^\ell\, w_i^\ell(x_\Gamma, \nu/\hat\varepsilon) \Big] \chi(x), \qquad x \in \Omega.$$

For $x \in \Omega^\delta$, define

$$\varphi_{\hat\varepsilon}(x) = \nu\, \chi(\nu/\hat\varepsilon)\, \partial_n w_e^\ell(x_\Gamma) \text{ where } x = x_\Gamma + \nu n. \tag{3.12}$$

It is clear that $\varphi_{\hat\varepsilon} \in C^\infty(\overline{\Omega})$,

$$\varphi_{\hat\varepsilon}(x) = 0 \text{ on } \Gamma \quad \text{and} \quad \partial_n \varphi_{\hat\varepsilon}(x) = \partial_n w_e^\ell \text{ on } \Gamma. \tag{3.13}$$

We define [9]

$$\mathbf{d}_\ell := \begin{cases} v^\varepsilon(x) - v_e^\ell(x) & \text{in } \mathbb{R}^d \setminus \Omega, \\ v^\varepsilon(x) - v_i^\ell(x) - \hat\varepsilon^\ell \varphi_{\hat\varepsilon}(x) & \text{in } \Omega. \end{cases} \tag{3.14}$$

and

$$q_\ell := \Delta \mathbf{d}_\ell + k^2\, \mathbf{d}_\ell + \frac{i}{\hat\varepsilon^2}\, \chi_\Omega\, \mathbf{d}_\ell \quad \text{in } \mathbb{R}^d. \tag{3.15}$$

We also set

$$\mathbf{e}_\ell := v_\ell^e - v_\ell^a \text{ in } \mathbb{R}^d \setminus \Omega \quad \text{and} \quad h_\ell := \mathbf{e}_\ell + \mathcal{D}_\ell^{\hat\varepsilon}\, \partial_n \mathbf{e}_\ell \text{ on } \Gamma. \tag{3.16}$$

Then $\mathbf{e}_\ell \in H^1_{loc}(\mathbb{R}^d)$ and $\mathbf{e}_\ell$ satisfies the outgoing condition and

$$\begin{cases} \Delta \mathbf{e}_\ell + k^2\, \mathbf{e}_\ell = 0, & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \mathbf{e}_\ell + \mathcal{D}_\ell^{\hat\varepsilon}\, \partial_n \mathbf{e}_\ell = h_\ell, & \text{on } \Gamma, \end{cases} \tag{3.17}$$

We now state estimates used later. We begin with two estimates on w_e^ℓ. The first one, whose proof is given in Section A.1, deals with the low and moderate frequency regimes.

[9] Our definition of $\mathbf{d}_\ell$ is slightly different from the one in [9] in Ω. We include the term $-\hat\varepsilon^\ell \varphi_{\hat\varepsilon}(x)$ to make $\partial_n \mathbf{d}_\ell$ continuous across Γ while maintaining the continuity of $\mathbf{d}_\ell$. This modification is convenient for the use of Morawetz's technique in the high frequency regime later. The scaling for the variable of function χ in the definition of φ reflects the skin effect.

Lemma 5. *Let $d = 2, 3$, $\ell = 0, 1, 2$, $m \geq 0$, $k_0 > 0$, and $R > 0$. There exist two positive constants $C_{R,\ell}$ and $C_{\ell,m}$ independent of ε and k such that, for $0 < k \leq k_0$,*

$$\|w_e^\ell\|_{H^{m+1}(B_R\setminus\Omega)} \leq C_{R,\ell,m}\, \|s\|_{H^{2\ell+m}(\mathbb{R}^d)} \tag{3.18}$$

and, for $m \geq 1$,

$$\|\partial_n w_e^\ell\|_{H^{m-1/2}(\Gamma)} \leq C_{\ell,m}\, \|s\|_{H^{2\ell+m}(\mathbb{R}^d)}. \tag{3.19}$$

Here is the second estimate of w_e^ℓ in the high frequency regime whose proof is given in Section A.2.

Lemma 6. *Let $d = 2, 3$, $\ell = 0, 1, 2$, $m \geq 0$, $k_0 > 0$, and $R > 0$. Assume that Ω is* **star-shaped***. There exist two positive constants $C_{R,\ell}$ and $C_{\ell,m}$ independent of ε and k such that, for $k \geq k_0$,*

$$\|w_e^\ell\|_{H^{m+1}(B_R\setminus\Omega)} + k\, \|w_e^\ell\|_{H^m(B_R\setminus\Omega)} \leq C_{R,\ell,m}\, k^{2\ell+m}\, \|s\|_{H^{2\ell+m}(\mathbb{R}^d)} \tag{3.20}$$

and, for $m \geq 1$,

$$\|\partial_n w_e^\ell\|_{H^{m-1/2}(\Gamma)} \leq C_{\ell,m}\, k^{2\ell+m} \|s\|_{H^{2\ell+m}(\mathbb{R}^d)}. \tag{3.21}$$

The following two lemmas give us the essential estimates for q_ℓ and h_ℓ. Their proofs are given in Sections A.3 and A.4, respectively.

Lemma 7. *We have $\operatorname{supp} q_\ell \subset \overline{\Omega}$. Moreover, let $0 < \varepsilon < 1$, $k \geq 0$, $k_0 > 0$, $\ell = 0, 1, 2$. Assume that $m \geq 1$ and let $s \in H^{\ell+m}(\mathbb{R}^d)$. Then, there is a positive constant C independent of ε and k, such that*

i) for $\varepsilon^2 < k \leq k_0$, we have

$$\|q_\ell\|_{L^2(\mathbb{R}^d)} \leq C\, \hat{\varepsilon}^{\ell-1}\, \|s\|_{H^{2\ell+3}(\mathbb{R}^d)}. \tag{3.22}$$

ii) assuming in addition that Ω is **star-shaped***, for $k \geq k_0$, we have*

$$\|q_\ell\|_{L^2(\mathbb{R}^d)} \leq C\, k^{2\ell+3}\, \hat{\varepsilon}^{\ell-1}\, \|s\|_{H^{2\ell+3}(\mathbb{R}^d)}, \tag{3.23}$$

Lemma 8. *Let $0 < \varepsilon < 1$, $k \geq 0$, $k_0 > 0$, $\ell = 0, 1, 2$. Assume that $m \geq 1$ and let $s \in H^{\ell+m}(\mathbb{R}^d)$. Then, there is a positive constant $C_{\ell,m}$ independent of ε and k such that*

i) We have, for $\varepsilon^2 < k \leq k_0$,

$$\|h_\ell\|_{H^{m-1/2}(\Gamma)} \leq C_{\ell,m}\, \hat{\varepsilon}^{\ell+1}\, \|s\|_{H^{2\ell+m}(\mathbb{R}^d)}.$$

ii) Assume in addition that Ω is **star-shaped***. We have, for $k \geq k_0$,*

$$\|h_\ell\|_{H^{m-1/2}(\Gamma)} \leq C_{\ell,m}\, k^{2\ell+m}\, \hat{\varepsilon}^{\ell+1}\, \|s\|_{H^{2\ell+m}(\mathbb{R}^d)}.$$

4 Proofs of Propositions 1 and 2

This section containing three subsections is devoted to Propositions 1 and 2. In the first subsection, we present several useful lemmas. The proofs of Propositions 1 and 2 are given in the last two subsections.

4.1 Preliminaries

In this section, we present useful lemmas used in the proof of Propositions 1 and 2. We first recall the following results established in [20, Lemmas 2.2] (see also [19, Lemma 2.2]).

Lemma 9. *Let $d = 2, 3$, $0 < k < k_0$, and Ω be a smooth bounded connected subset of $\mathbb{R}^d$. Let $g_k \in H^{\frac{1}{2}}(\partial\Omega)$ and $v_k \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v_k + k^2\, v_k = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v_k = g_k & \text{on } \Gamma. \end{cases}$$

Assume that $g_k \rightharpoonup g$ weakly in $H^{\frac{1}{2}}(\partial\Omega)$ as $k \to 0$. Then $v_k \rightharpoonup v$ weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ where $v \in W^1(\mathbb{R}^d \setminus \bar{\Omega})$ is the unique solution to

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v = g & \text{on } \Gamma. \end{cases}$$

Here, for an open unbounded subset U of $\mathbb{R}^d$, the space $W^1(U)$ is defined as follows

$$W^1(U) = \begin{cases} \Big\{\psi \in L^1_{loc}(U);\ \dfrac{\psi(x)}{\ln(2+|x|)\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U)\Big\}, & \text{if } d = 2, \\ \Big\{\psi \in L^1_{loc}(U);\ \dfrac{\psi(x)}{\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U)\Big\}, & \text{if } d = 3. \end{cases} \tag{4.1}$$

Using Lemma 9, we can prove

Lemma 10. *Let $d = 2, 3$, $\ell = 0, 1$, $0 < \varepsilon < 1$, $k_0 > 0$, $r_0 > 0$, $0 < k < k_0$, $q \in L^2(\mathbb{R}^d \setminus \Omega)$ with $\operatorname{supp} q \subset B_{r_0} \setminus \Omega$, and $g \in H^{1/2}(\Gamma)$. Let $v \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v + k^2\, v = q & \text{in } \mathbb{R}^d \setminus \Omega, \\ v + \mathcal{D}^{\hat{\varepsilon}}_{\ell}\, \partial_n v = g & \text{on } \Gamma. \end{cases} \tag{4.2}$$

We have

$$\|v\|_{H^1(B_r\setminus\Omega)} \le C_r\big(\|q\|_{L^2(\mathbb{R}^d\setminus\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\big), \tag{4.3}$$

for some positive constant C_r independent of g, q, ε, and k.

Proof. We only derive the estimate for small enough ε and k. The other case follows in the same spirit.

Set $r_1 := r_0 + 1$. We first prove that, for small enough k,

$$\|v\|_{L^2(B_{r_1}\setminus\Omega)} \le C\big(\|q\|_{L^2(\mathbb{R}^d\setminus\Omega)} + \|g\|_{H^{1/2}(\Gamma)}\big), \tag{4.4}$$

for some positive constant C, independent of ε, k, q and g, by contradiction. Suppose this is not true. Then there exist $\varepsilon_n \to 0^+$, $k_n \to 0^+$, $q_n \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} q_n \subset B_{r_0}\setminus\Omega$, and $g_n \in H^{1/2}(\Gamma)$ such that

$$\|q_n\|_{L^2(\mathbb{R}^d\setminus\Omega)} + \|g_n\|_{H^{1/2}(\Gamma)} \to 0 \text{ and } \|v_n\|_{L^2(B_{r_1}\setminus\Omega)} = 1. \tag{4.5}$$

Here, $v_n \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ is the unique outgoing solution to the problem

$$\begin{cases} \Delta v_n + k_n^2\, v_n = q_n & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v_n + \mathcal{D}_\ell^{\hat{\varepsilon}_n} \partial_n v_n = g_n & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Using the standard regularity theory of elliptic equations and the representation formula for the equation $\Delta v_n + k_n^2\, v_n = 0$ in $\mathbb{R}^d \setminus B_{r_0}$, we have

$$\|v_n\|_{H^1(B_r \setminus B_{r_0+1/2})} \le C_r \text{ for all } r > 0. \tag{4.7}$$

Multiplying the first equation of (4.6) by $\overline{v}_n$ (the conjugate of v_n), integrating over $B_{r_1} \setminus \Omega$, and using the second equation of (4.6), we obtain

$$\int_{B_{r_1}\setminus\Omega} \nabla v_n\, \nabla \overline{v}_n + \int_\Gamma \partial_n v_n\, \overline{\mathcal{D}_\ell^{\hat{\varepsilon}_n} \partial_n v_n} = \int_{\partial B_{r_1}} \partial_n v_n\, \overline{v}_n + k_n^2 \int_{B_{r_1}\setminus\Omega} |v_n|^2 - \int_{B_{r_1}\setminus\Omega} q_n\, \overline{v}_n + \int_\Gamma \partial_n v_n\, \overline{g_n}. \tag{4.8}$$

Since, by (1.6) and (1.7),

$$\Re \left[\partial_n v_n\, \overline{\mathcal{D}_\ell^{\hat{\varepsilon}_n}\, \partial_n v_n} \right] \ge 0, \tag{4.9}$$

it follows from (4.7) and (4.8) that

$$\int_{B_{r_1}\setminus\Omega} |\nabla v_n|^2 \le C, \tag{4.10}$$

for a constant C independent of n. From (4.5), (4.7), and (4.10), w.l.o.g. one may assume that $v_n \to v$ weakly in $H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ and, by (4.6),

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v = 0 & \text{on } \Gamma. \end{cases} \tag{4.11}$$

Applying Lemma 9, we have

$$v \in W^1(\mathbb{R}^d \setminus \bar{\Omega}). \tag{4.12}$$

This implies

$$v = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

On the other hand, we derive from (4.5) that

$$\|v\|_{L^2(B_{r_1}\setminus\Omega)} = \lim_{n\to\infty} \|v_n\|_{L^2(B_{r_1}\setminus\Omega)} = 1.$$

We have a contradiction. Thus (4.4) holds.

From (4.4), as in the proof of (4.7) and (4.10), we obtain (4.3). The proof is complete. □

We now state the last results in this section dealing with (1.9) in the low and moderate frequency regimes.

Lemma 11. *Let $d = 2, 3$, $0 < \varepsilon < 1$, $k_0 > 0$, $r_0 > 0$, $0 < k < k_0$, and let $q \in L^2(\mathbb{R}^d)$ with* $\operatorname{supp} q \subset B_{r_0}$. *Let $v \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to*

$$\Delta v + k^2\, v + ik\, \sigma_\varepsilon\, v = q \text{ in } \mathbb{R}^d.$$

We have, for $r > 0$,

$$\|v\|_{H^1(B_r)} \le C_r \big(\|q\|_{L^2(\Omega^c)} + \hat{\varepsilon} \|q\|_{L^2(\Omega)} \big). \tag{4.13}$$

where C_r is a positive constant independent of ε, k, and q.

Proof. The proof of Lemma 11 is similar to one of Lemma 10. The details are left to the reader. □

4.2 Proof of Proposition 1

Proposition 1 is a direct consequence of Lemmas 10 and 11. Indeed, applying Lemma 10 for $v = v^a_\ell$, $q = s$, we obtain, for $\ell = 0, 1$,

$$\|v^a_\ell\|_{H^1(B_r \setminus \Omega)} \le C\, \|s\|_{L^2(\mathbb{R}^d)}. \tag{4.14}$$

and, applying Lemma 11 for $v = v^\varepsilon$ and $q = s$, we have

$$\|v^\varepsilon\|_{H^1(B_r)} \le C\, \|s\|_{L^2(\mathbb{R}^d)}. \tag{4.15}$$

A combination of (4.14) and (4.15) yields the conclusion. □

4.3 Proof of Proposition 2

It is from the definition of $\mathbf{d}_\ell$ (3.14) and $\mathbf{e}_\ell$ (3.16) that

$$\|v^\varepsilon - v^a_\ell\|_{H^1(B_r \setminus \Omega)} \le \|\mathbf{d}_\ell\|_{H^1(B_r \setminus \Omega)} + \|\mathbf{e}_\ell\|_{H^1(B_r \setminus \Omega)}. \tag{4.16}$$

Applying Lemmas 10 and 8 (with $m = 1$), from (3.17), we have for $\ell = 0, 1$,

$$\|\mathbf{e}_\ell\|_{H^1(B_r \setminus \Omega)} \le C\, \|h_\ell\|_{H^{1/2}(\Gamma)} \le C\, \hat{\varepsilon}^{\ell+1}\, \|s\|_{H^{2\ell+1}(\mathbb{R}^d)}. \tag{4.17}$$

Using Lemmas 11 and 7, from (3.15), we obtain, for $\ell = 0, 1, 2$,

$$\|\mathbf{d}_\ell\|_{H^1(B_r \setminus \Omega)} \le C\, \hat{\varepsilon}\, \|q_\ell\|_{L^2(\Omega)} \le C\, \hat{\varepsilon}^{\ell}\, \|s\|_{H^{2\ell+3}(\mathbb{R}^d)}.$$

By Lemma 5, it follows that, for $\ell = 0, 1$,

$$\begin{aligned} \|\mathbf{d}_\ell\|_{H^1(B_r \setminus \Omega)} \le & \hat{\varepsilon}^{\ell+1}\, \|w_e^{\ell+1}\|_{H^1(B_r \setminus \Omega)} + C\, \hat{\varepsilon}^{\ell+1}\, \|s\|_{H^{2\ell+5}(\mathbb{R}^d)} \\ \le & C\, \hat{\varepsilon}^{\ell+1}\, \|s\|_{H^{2\ell+5}(\mathbb{R}^d)}. \end{aligned} \tag{4.18}$$

A combination of (4.16), (4.17), and (4.18) yields the conclution. □

5 Proof of Proposition 3

This section is devoted to the proof of Proposition 3. In order to obtain the desired estimate for $u_a^\ell - u^\varepsilon$, we will derive separate estimates for the functions $\mathbf{d}_\ell$ and $\mathbf{e}_\ell$, introduced in Section 3. This goal is achieved by Corollaries 1 and 2 below. Our presentation is divided into two subsections. In the first one, we present some useful lemmas. The proof of Proposition 3 is given in the second subsection.

5.1 Preliminaries

In this section, we present useful lemmas used in the proof of Proposition 3. We start this section with the following lemma.

Lemma 12. *Let $d = 2, 3$, D be a* **star-shaped** *domain of $\mathbb{R}^d$, and $r_* > 0$ such that $D \subset B_{r_*}$. Define*

$$P(r) = \begin{cases} \dfrac{2r_*}{d-1} & \text{if } \; r > r_* \; , \\ \dfrac{2r}{d-1} & \text{if } \; 0 < r < r_* \; , \end{cases} \qquad \text{and} \quad Q(r) = \begin{cases} \dfrac{r_*}{r} & \text{if } \; r > r_* \; , \\ 1 & \text{if } \; 0 < r < r_* \; , \end{cases}$$

and let $v \in H^1_{loc}(\mathbb{R}^d)$ be such that $\Delta v + k^2 v \in L^2_{loc}(\mathbb{R}^d)$. For any $R > r_*$ and $k > 0$, we have

$$\Re \int\limits_{B_R \setminus \Omega} \big(\Delta v + k^2 \, v\big) \big[P(r) \, \overline{v}_r + Q(r) \, \overline{v}\big] \leq$$

$$- \frac{1}{d-1} \int\limits_{B_{r_*} \setminus \Omega} \Big(|\nabla v|^2 + k^2 \, |v|^2\Big) + \frac{r_* \, (3-d)}{2} \int\limits_{B_R \setminus B_{r_*}} \frac{u^2}{r^3} + F_0(v) - F(R, v). \quad (5.1)$$

Here, n denotes the **inward** *unit normal vector of ∂D,*

$$F_0(v) = \Re \Big(\int\limits_{\partial D} \frac{2}{d-1} \partial_n v \; (x \cdot \nabla \overline{v}) - \frac{1}{d-1} (x \cdot n) \; |\nabla v|^2 + \partial_n v \; \overline{v} + \frac{k^2}{d-1} (x \cdot n) \; |v|^2 \Big),$$

and

$$F(r, v) = \Re \Big(- \int\limits_{\partial B_r} \frac{r_*}{d-1} |v_r|^2 - \frac{r_*}{d-1} |\nabla_{\partial B_r} v|^2 + \frac{r_*}{2r^2} |v|^2 + \frac{r_*}{r} \; v_r \, \bar{v} + \frac{k^2 r_*}{d-1} |v|^2 \Big).$$

Remark 3. *This lemma has been stated and proved in [22, Lemma 1] for the spherical domains. The proof presented here follows heavily from the one of [22, Lemma 1]. In the proof, we also use the "Rellich" identity* (5.2) *which has root from [27, 25, 17]. Estimate* (5.1) *is in the spirit of Morawetz-Ludwig [17]. The choice of the weight functions $P(r), Q(r)$ appeared in the work of Perthame and Vega [26].*

Proof. We will prove the lemma for $v \in C^\infty(\mathbb{R}^d)$. The general case follows by a standard regularizing argument. We have

$$\Re \int\limits_{B_R\setminus D} \left(\Delta v + k^2 v\right) \left[P(r)\,\overline{v}_r + Q(r)\,\overline{v}\right] = A_1 + A_2,$$

where

$$A_1 = \Re \int\limits_{B_{r_*}\setminus D} \left(\Delta v + k^2 v\right) \left[P(r)\,\overline{v}_r + Q(r)\,\overline{v}\right]$$

and

$$A_2 = \Re \int\limits_{B_R\setminus B_{r_*}} \left(\Delta v + k^2 v\right)\left[P(r)\,\overline{v}_r + Q(r)\,\overline{v}\right].$$

<u>Calculate A_1</u>: Since $P(r) = \frac{2r}{d-1}$ and $Q(r) = 1$ for $0 < r < r_*$,

$$A_1 = \Re \int\limits_{B_{r_*}\setminus D} \left(\Delta v + k^2 v\right)\Big[\frac{2r}{d-1}\overline{v}_r + \overline{v}\Big] = \int\limits_{B_{r_*}\setminus D} \Re\Big[\left(\Delta v + k^2 v\right)\Big(\frac{2}{d-1}\, x\cdot\nabla\overline{v} + \overline{v}\Big)\Big].$$

We have [10]

$$\Re\left[\,\left(\Delta v + k^2 v\right)\left(\frac{2}{d-1}x\cdot\nabla\overline{v} + \overline{v}\right)\right] = -\frac{1}{d-1}\left(|\nabla v|^2 + k^2|v|^2\right)$$
$$+ \Re\,\nabla\left[\frac{2}{d-1}\nabla v\,(x\cdot\nabla\overline{v}) - \frac{1}{d-1}x\,|\nabla v|^2 + \nabla v\,\overline{v} + \frac{k^2}{d-1}x\,|v|^2\right]. \quad (5.2)$$

Integrating over the domain $B_{r_*}\setminus D$, we obtain:

$$A_1 = -\frac{1}{d-1}\int\limits_{B_{r_*}\setminus D}\left(|\nabla v|^2 + k^2|v|^2\right) + \Re\int\limits_{\partial B_{r_*}} \frac{2r_*}{d-1}\,|v_r|^2 - \frac{r_*}{d-1}\,\left|\nabla v\right|^2 + v_r\,\overline{v} + \frac{k^2\,r_*}{d-1}\,|v|^2$$
$$+ \Re\int\limits_{\partial D} \frac{2}{d-1}\partial_n v\,(x\cdot\nabla\overline{v}) - \frac{1}{d-1}\,(x\cdot n)\,|\nabla v|^2 + \partial_n v\,\overline{v} + \frac{k^2}{d-1}\,(x\cdot n)\,|v|^2.$$

It follows that

$$A_1 = -\frac{1}{d-1}\int\limits_{B_{r_*}\setminus D}\left(|\nabla v|^2 + k^2|v|^2\right) + F_0(v) + \Re\int\limits_{\partial B_{r_*}} \frac{2r_*}{d-1}|v_r|^2 - \frac{r_*}{d-1}\,|\nabla v|^2 + v_r\,\overline{v} + \frac{k^2\,r_*}{d-1}\,|v|^2.$$

Since $|\nabla v|^2 = |v_r|^2 + |\nabla_{\partial B_{r_*}} v|^2$, we have

$$\Re\int\limits_{\partial B_{r_*}} \frac{2r_*}{d-1}|v_r|^2 - \frac{r_*}{d-1}\,|\nabla v|^2 + v_r\,\overline{v} + \frac{k^2\,r_*}{d-1}\,|v|^2$$

$$= \Re\int\limits_{\partial B_{r_*}} \frac{r_*}{d-1}\,|v_r|^2 - \frac{r_*}{d-1}\,|\nabla_{\partial B_{r_*}} v|^2 + v_r\,\overline{v} + \frac{r_* k^2}{d-1}\,|v|^2 \le -F(r_*, v).$$

[10] This is the "Rellich" identity (5.2) which has root from [27, 25, 17].

It follows that

$$A_1 \le -\frac{1}{d-1} \int\limits_{B_{r_*}\setminus D} \left(|\nabla v|^2 + k^2 |v|^2\right) + F_0(v) - F(r_*, v). \tag{5.3}$$

Estimate A_2: Applying [22, Lemma 2], we have

$$A_2 \le \frac{r_*(3-d)}{2} \int\limits_{B_R\setminus B_{r_*}} \frac{u^2}{r^3} + F(r_*, v) - F(R, v). \tag{5.4}$$

The conclusion now follows from (5.3) and (5.4). □

The following lemma, in spirit of Morawetz-Ludwig [17] (see also [26]), is important for our analysis.

Lemma 13. *Let $d = 2, 3$, $k_0 > 0$, $r_0 > 0$, $q \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} q \subset B_{r_0} \setminus \Omega$, and $g \in H^1(\Gamma)$. Let $k \ge k_0$ and $v \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v + k^2\, v = q & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v = g & \text{on } \Gamma. \end{cases} \tag{5.5}$$

Assume that Ω is **star-shaped**. *Given $r_* > 0$, there exists a positive constant $C = C(r_*, r_0, k_0, \Omega)$ independent of k such that*

$$\|\nabla v\|_{L^2(B_{r_*}\setminus\Omega)} + k\,\|v\|_{L^2(B_{r_*}\setminus\Omega)} + \|\partial_n v\|_{L^2(\Gamma)} \le C\left(\|q\|_{L^2(\mathbb{R}^d\setminus\Omega)} + \|\nabla_\Gamma g\|_{L^2(\Gamma)} + k\,\|g\|_{L^2(\Gamma)}\right). \tag{5.6}$$

Proof of Lemma 13. The idea is to apply Lemma 12 for $D = \Omega$. We have

$$F_0(v) = \Re \int\limits_{\partial\Omega} \Big(\frac{1}{d-1}(x\cdot n)|\partial_n v|^2 + \frac{2}{d-1}\partial_n v\,\nabla_\Gamma \bar v - \frac{1}{d-1}(x\cdot n)|\nabla_\Gamma v|^2 + \partial_n v\,\bar v + \frac{k^2}{d-1}(x\cdot n)\,|v|^2\Big),$$

where n denotes the **inward** unit normal vector of $\partial\Omega$. Since Ω is star-shaped, it follows that

$$F_0(v) \le -\frac{1}{C}\,\|\partial_n v\|^2 + C\,\left(\|\nabla_\Gamma v\|_{L^2(\Gamma)} + k^2\|v\|_{L^2(\Gamma)}\right). \tag{5.7}$$

Applying Lemma 12 and using (5.7), we obtain

$$\|\nabla v\|^2_{L^2(B_{r_*}\setminus\Omega)} + k^2\,\|v\|^2_{L^2(B_{r_*}\setminus\Omega)} + \|\partial_n v\|^2_{L^2(\Gamma)} \le C\Big(\frac{r_*(3-d)}{2} \int\limits_{B_R\setminus B_{r_*}} \frac{u^2}{r^3} + \|q\|^2_{L^2(\mathbb{R}^d\setminus\Omega)} + \|\nabla_\Gamma g\|^2_{L^2(\Gamma)} + k\,\|g\|^2_{L^2(\Gamma)}\Big),$$

This implies the conclusion in the case $d = 3$. For $d = 2$, it remains to absorb the first term in the RHS into the LHS. Without loss of generality, we may assume that r_* is big enough. The absorption then can be done as in [22, p. 11-12]. The details are left to the reader. □

When the control is only available on $L^2(\Gamma)$ (not $H^1(\Gamma)$), one has the following result by Hörmander [12, Theorem 3.1] (see also [12, page 65]).

Lemma 14 (Hörmander)**.** *Let D be a bounded smooth domain of $\mathbb{R}^d$ $(d \geq 2)$ and $g \in H^{1/2}(\partial D)$. Assume $v \in H^1(D)$ is the unique solution to the system*

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ v = g & \text{on } \partial D. \end{cases} \tag{5.8}$$

Then

$$\|v\|_{L^2(D)} \leq C \|g\|_{L^2(\partial D)}, \tag{5.9}$$

for some positive constant C independent of g [11]*.*

Here is a result related to equation (3.17) of $\mathbf{e}_\ell$:

Lemma 15. *Let $d = 2, 3$, $\ell = 0, 1$, $0 < \varepsilon < 1$, $k_0 > 0$, $k \geq k_0$, $h \in L^2(\Gamma)$, and let $v \in H^1_{_{loc}}(\mathbb{R}^d \backslash \Omega)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v + k^2\, v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v + \mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v = h & \text{on } \Gamma. \end{cases} \tag{5.10}$$

We have

$$\|v\|_{L^2(\Gamma)} \leq C\, \|h\|_{L^2(\Gamma)},$$

where C is a positive constant, independent of ε, k, q, and h.

Proof. We only consider the case $\ell = 1$ since the lemma is trivial for $\ell = 0$. Let $\ell = 1$. Multiplying the first equation of (5.10) by $\bar{v}$, integrating in $B_r \setminus \Omega$, and using the boundary condition, we obtain

$$\int\limits_{B_r \backslash \overline{\Omega}} -|\nabla v|^2 + k^2\, |v|^2 \; + \int\limits_{\partial B_r} \partial_r v\, \overline{v} + \int\limits_{\Gamma} \partial_n v\, \overline{h} - \int\limits_{\Gamma} \partial_n v\, \overline{\mathcal{D}_1^{\hat{\varepsilon}}\, \partial_n v} = 0. \tag{5.11}$$

Recall that

$$\mathcal{D}_1^{\hat{\varepsilon}} = \frac{\hat{\varepsilon}}{\alpha} = \frac{\sqrt{2}}{2}\hat{\varepsilon} + i\frac{\sqrt{2}}{2}\hat{\varepsilon}, \tag{5.12}$$

which implies

$$-\Im\left[\partial_n v\, \overline{\mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v}\right] = \frac{\sqrt{2}}{2}\hat{\varepsilon}\, |\partial_n v|^2. \tag{5.13}$$

Since v satisfies the outgoing condition (1.5), i.e.,

$$\partial_r v - ikv = o(r^{-(d-1)/2}),$$

it follows that

$$\liminf_{r \to \infty} \Im\Big(\int\limits_{\partial B_r} \partial_r v\, \overline{v}\Big) = \liminf_{r \to \infty} \int\limits_{\partial B_r} k|v|^2 \geq 0. \tag{5.14}$$

[11] In (5.9), $\|g\|_{L^2(\partial D)}$ is used not $\|g\|_{H^{1/2}(\partial D)}$

Considering the imaginary part of (5.11) and letting $r \to \infty$, we derive from (5.13) and (5.14) that

$$\int_\Gamma \hat{\varepsilon}\, |\partial_n v|^2 \leq C \int_\Gamma |h||\partial_n v|.$$

This implies

$$\hat{\varepsilon}\, \|\partial_n v\|_{L^2(\Gamma)} \leq \|h\|_{L^2(\Gamma)}. \tag{5.15}$$

Since $v = h - \mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v$, the conclusion follows from (5.12) and (5.15). □

Here is an important consequence of Lemmas 13, 14, and 15, which will be applied to obtain the estimate for $\mathbf{e}_\ell$:

Corollary 1. *Let $\ell = 0, 1$, $k_0 > 0$, $0 < \varepsilon < 1$, $k \geq k_0$, $h \in H^{1/2}(\Gamma)$, and let $v \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ be the unique outgoing solution to*

$$\begin{cases} \Delta v + k^2\, v = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ v + \mathcal{D}_\ell^{\hat{\varepsilon}}\, \partial_n v = h & \text{on } \Gamma. \end{cases} \tag{5.16}$$

Assume that Ω is **star-shaped**. *Then, for all $K \subset\subset \mathbb{R}^d \setminus \overline{\Omega}$,*

$$\|\nabla v\|_{L^2(K)} + k\, \|v\|_{L^2(K)} \leq C\, k^2\, \|h\|_{L^2(\Gamma)},$$

for some positive constant $C = C(K)$ independent of ε, k, and h.

Proof. Let $r > 0$ such that $K \subset B_r \setminus \Omega$. Set $r_1 = r + 1$ and let $\phi \in H^1(B_{r_1} \setminus \Omega)$ be the solution to

$$\begin{cases} \Delta \phi = 0 \text{ in } B_{r_1} \setminus \bar{\Omega}, \\ \phi = v \text{ on } \Gamma \quad \text{and} \quad \phi = 0 \text{ on } \partial B_{r_1}. \end{cases} \tag{5.17}$$

Applying Lemma 15 we have

$$\|v\|_{L^2(\Gamma)} \leq C \|h\|_{L^2(\Gamma)}.$$

It follows from Lemma 14 that

$$\|\phi\|_{L^2(B_{r_1} \setminus \Omega)} \leq C \|h\|_{L^2(\Gamma)}. \tag{5.18}$$

Fix $\chi \in C^\infty(\mathbb{R}^d)$ such that $\chi = 1$ in B_r and $\operatorname{supp} \chi \subset B_{r+1/2}$. Set

$$V = v - \chi \phi \text{ in } \mathbb{R}^d \setminus \Omega. \tag{5.19}$$

It is clear that $V \in H^1_{loc}(\mathbb{R}^d \setminus \Omega)$ is the unique outgoing solution to the problem

$$\begin{cases} \Delta V + k^2\, V = -\Delta(\chi\phi) - k^2\, \chi\phi & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ V = 0 & \text{on } \Gamma. \end{cases} \tag{5.20}$$

Since $\Delta \phi = 0$ in $B_{r_1} \setminus \Omega$, $\chi = 1$ in B_r, and $\chi = 0$ in $\mathbb{R}^d \setminus B_{r+1/2}$,

$$\|\Delta(\chi\phi)\|_{L^2(\mathbb{R}^d \setminus \Omega)} + k^2\, \|\chi\phi\|_{L^2(\mathbb{R}^d \setminus \Omega)} \leq C\, \big(\|\phi\|_{H^2(B_{r+1/2} \setminus B_r)} + k^2\, \|\phi\|_{L^2(B_{r_1} \setminus \Omega)} \big).$$

Using the standard regularity of elliptic equations, we derive from (5.17) that

$$\|\Delta(\chi\phi)\|_{L^2(\mathbb{R}^d\setminus\Omega)} + k^2\,\|\chi\phi\|_{L^2(\mathbb{R}^d\setminus\Omega)} \le C\,k^2\,\|\phi\|_{L^2(B_{r_1}\setminus\Omega)}. \tag{5.21}$$

Applying Lemma 13 for (5.20) and using (5.21), we arrive to:

$$\|\nabla V\|_{L^2(B_r\setminus\Omega)} + k\,\|V\|_{L^2(B_r\setminus\Omega)} \le C\,k^2\,\|h\|_{L^2(\Gamma)}. \tag{5.22}$$

The conclusion now follows from (5.17), (5.19), (5.22), and the standard regularity theory for elliptic equations. □

The following lemma, which is a variant of [23, Proposition 1], plays an important role in analyzing $\mathbf{d}_\ell$.

Lemma 16. *Let $d = 2, 3$, $k_0 > 0$, $r_0 > 0$, $0 < \varepsilon < 1$, $k \ge k_0$, and $q \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} q \subset \Omega$. Let $v \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to*

$$\Delta v + k^2\,v + i\,k\,\sigma_\varepsilon\,v = q \text{ in } \mathbb{R}^d. \tag{5.23}$$

We have, for $r_* > 0$,

$$\|\nabla v\|_{L^2(B_{r_*})} + k\|v\|_{L^2(B_{r_*})} \le C_{r_*}\,\|q\|_{L^2(\Omega)} \tag{5.24}$$

and

$$\|v\|_{L^2(\Omega)} \le C\,\varepsilon^2\,\|q\|_{L^2(\Omega)} \tag{5.25}$$

where C_{r_*} and C are positive constants independent of k, ε, and q. As a consequence,

$$\|v\|_{L^2(\Gamma)} \le C\,\varepsilon\,\|q\|_{L^2(\Omega)}. \tag{5.26}$$

Proof. We follow the strategy used in the proof of [23, Proposition 1]. Multiplying equation (5.23) by $\overline{v}$ and integrating over B_R, we have

$$-\int_{B_R} |\nabla v|^2 + k^2 \int_{B_R} |v|^2 + \int_{\partial B_R} \partial_r v\;\bar{v} + ik\int_{\Omega} \sigma_\varepsilon\;|v|^2 = \int_{\Omega} q\;\bar{v}.$$

Letting $R \to \infty$, using the outgoing condition, and considering the imaginary part, we obtain

$$k\;\limsup_{R\to\infty} \int_{\partial B_R} |v|^2 + \frac{1}{\hat{\varepsilon}^2}\int_{\Omega} |v|^2 \le \|v\|_{L^2(\Omega)}\;\|q\|_{L^2(\Omega)}. \tag{5.27}$$

This implies

$$\int_{\Omega} |v|^2 \le \hat{\varepsilon}^4 \int_{\Omega} |q|^2 \quad \text{ and } \quad k\;\limsup_{R\to\infty} \int_{\partial B_R} |v|^2 \le \hat{\varepsilon}^2 \int_{\Omega} |q|^2. \tag{5.28}$$

Hence (5.25) is proved. Let $\gamma_0 > 0$ such that $B_{2\gamma_0} \subset \Omega$. It only remains to prove (5.24).

Multiplying (5.23) by $\phi^2\bar{v}$, with $\phi \in C^\infty_c(B_{2\gamma_0})$ and $\phi = 1$ in B_{γ_0}, and integrating over Ω. We obtain, by Caccioppoli's inequality,

$$\int_{B_{\gamma_0}} |\nabla v|^2 \le C\,\Big(k^2 \int_{\Omega} |v|^2 + \int_{\Omega} |q|^2\Big).$$

In this proof, C denotes a positive constant independent of ε, k, and q. It follows from (5.28) that

$$\int_{B_{\gamma_0}} |\nabla v|^2 + k^2 \int_{B_{\gamma_0}} |v|^2 \le C(\hat{\varepsilon}^4 k^2 + 1) \int_{\Omega} |q|^2 = C(\varepsilon^4 + 1) \int_{\Omega} |q|^2 \le C \int_{\Omega} |q|^2.$$

Hence, there exists $\gamma_0/2 < \tau \le \gamma_0$ such that

$$\int_{\partial B_\tau} |\nabla v|^2 + k^2 \int_{\partial B_\tau} |v|^2 \le C \int_{\Omega} |q|^2. \tag{5.29}$$

Applying Lemma 12 with $D = B_\tau$, we obtain, for any $R > r_* > R_0$,

$$\Re \int_{B_R \setminus B_\tau} \left[r\overline{v}_r + \overline{v} \right] \left[\Delta v + k^2 \, v \right] \le -\frac{1}{d-1} \int_{B_{r_*} \setminus B_\tau} \Big(|\nabla v|^2 + k^2 \, |v|^2 \Big)$$
$$+ \frac{r(3-d)}{2} \int_{B_R \setminus B_{r_*}} \frac{u^2}{r^3} + F_0(v) - F(R, v). \tag{5.30}$$

Here $F_0(v)$ and $F(R, v)$ are defined in Lemma 12. Using (5.23), we derive from (5.30) that

$$\frac{1}{d-1} \int_{B_{r_*} \setminus B_\tau} \Big(|\nabla v|^2 + k^2 \, |v|^2 \Big) \le -\Re \int_{\Omega \setminus B_\tau} \left[r\overline{v}_r + \overline{v} \right] \left[-\frac{i}{\hat{\varepsilon}^2} v + q \right]$$
$$+ \frac{r_*(3-d)}{2} \int_{B_R \setminus B_{r_*}} \frac{u^2}{r^3} + F_0(v) - F(R, v). \tag{5.31}$$

We have

$$\left| \Re \int_{\Omega \setminus B_\tau} \left[r\overline{v}_r + \overline{v} \right] \left[-\frac{i}{\hat{\varepsilon}^2} v + q \right] \right| \le \frac{C}{\hat{\varepsilon}^2} \int_{\Omega \setminus B_\tau} |v_r| \, |v| + C \int_{\Omega \setminus B_\tau} (|v_r| + |v|)|q|.$$

We derive from Young's inequality that

$$\left| \Re \int_{\Omega \setminus B_\tau} \left[r\overline{v}_r + \overline{v} \right] \left[-\frac{i}{\hat{\varepsilon}^2} v + q \right] \right| \le \frac{1}{2(d-1)} \int_{\Omega \setminus B_\tau} |v_r|^2 + C \Big(\frac{1}{\hat{\varepsilon}^4} \int_{\Omega \setminus B_\tau} |v|^2 + \int_{\Omega \setminus B_\tau} |q|^2 \Big). \tag{5.32}$$

A combination of (5.28), (5.31) and (5.32) yields

$$\frac{1}{2(d-1)} \int_{B_{r_*} \setminus B_\tau} \Big(|\nabla v|^2 + k^2 \, |v|^2 \Big) \le C \int_{\Omega} |q|^2 + \frac{r_*(3-d)}{2} \int_{B_R \setminus B_{r_*}} \frac{u^2}{r^3} + F_0(v) - F(R, v). \tag{5.33}$$

Recall that

$$-F(R,v) = \Re\Big(\int\limits_{\partial B_R} \frac{r_*}{d-1}|v_r|^2 - \frac{r_*}{d-1}|\nabla_{\partial B_R} v|^2 + \frac{r_*}{2R^2}\,|v|^2 + \frac{r_*}{R}\,v_r\,\bar v + \frac{k^2\,r_*}{d-1}|v|^2\Big).$$

Since v satisfies the outgoing condition, it follows that [12]

$$\limsup_{R\to\infty}\; -F(R,v) \le \frac{2\,k^2\,r_*}{d-1}\,\limsup_{R\to\infty}\int\limits_{\partial B_R} |v|^2. \tag{5.34}$$

A combination of (5.28) and (5.34) yields

$$\limsup_{R\to\infty}\; -F(R,v) \le \frac{2r\,\hat\varepsilon^2\,k}{d-1}\int\limits_{\Omega} |q|^2 = C\varepsilon^2\int\limits_{\Omega} |q|^2. \tag{5.35}$$

We have

$$F_0(v) = \Re\int\limits_{\partial B_\tau} -\frac{2\tau}{d-1}\,|\partial_n v|^2 + \frac{\tau}{d-1}\,|\nabla v|^2 + \partial_n v\,\bar v - \frac{k^2\,\tau}{d-1}\,|v|^2. \tag{5.36}$$

This implies

$$F_0(v) \le C\Big(\int\limits_{\partial B_\tau} |\nabla v|^2 + k^2\int\limits_{\partial B_\tau} |v|^2\Big). \tag{5.37}$$

We derive from (5.29) that

$$F_0(v) \;\le\; C\int\limits_{\Omega} |q|^2. \tag{5.38}$$

Combining (5.33), (5.35), and (5.38), we obtain

$$\frac{1}{2(d-1)}\int\limits_{B_{r_*}\setminus B_\tau} |\nabla v|^2 + k^2\,|v|^2 \le C\int\limits_{\Omega} |q|^2 + \frac{r_*(3-d)}{2}\int\limits_{B_R\setminus B_{r_*}} \frac{u^2}{r^3}.$$

The proof for $d = 3$ is complete. For $d = 2$, it remains to absorb the second term on the RHS to the LHS. Without of generality we may assume that r_* is big enough. Then, the absorption can be done as in [22, pp. 11-12]. The details are left to the reader. □

The following result will be used to obtain the estimate for $\mathbf{d}_\ell$.

Corollary 2. *Let $d = 2, 3$, $k_0 > 0$, $r_0 > 0$, $0 < \varepsilon < 1$, $k \ge k_0$, and $q \in L^2(\mathbb{R}^d)$ with $\operatorname{supp} q \subset \overline{\Omega}$. Let $v \in H^1_{loc}(\mathbb{R}^d)$ be the unique outgoing solution to*

$$\Delta v + k^2\,v + i\,k\,\sigma_\varepsilon\,v = q \text{ in } \mathbb{R}^d.$$

Then, for any $K \subset\subset \mathbb{R}^d \setminus \bar\Omega$, there is a positive constant C_K independent of k, ε, and q such that

$$\|\nabla v\|_{L^2(K)} + k\|v\|_{L^2(K)} \le C_K\,\hat\varepsilon\,k^2\,\|q\|_{L^2(\Omega)}.$$

Proof. The proof of this corollary is similar to that of Corollary 1. One only needs to use Lemma 16 (more precisely, the estimate (5.26)) in place of Lemma 15. The details are left to the reader. □

[12] For the details of the argument, see the one used to obtain [22, (2.19)].

5.2 Proof of Proposition 3

Proposition 3 can now be proved in a similar way to Proposition 2. We only need to use Corollary 1 in place of Lemma 10 and Corollary 2 in place of Lemma 11. We present the proof here for the convenience of the reader. From the definition of $\mathbf{d}_\ell$ (3.14) and $\mathbf{e}_\ell$ (3.16), we have

$$\|v^\varepsilon - v_\ell^a\|_{H^1(B_r\setminus\Omega)} \leq \|\mathbf{d}_\ell\|_{H^1(B_r\setminus\Omega)} + \|\mathbf{e}_\ell\|_{H^1(B_r\setminus\Omega)}. \tag{5.39}$$

Applying Corollary 1 and Lemma 8 (with $m = 1$), we have, for $\ell = 0, 1$,

$$\|\nabla \mathbf{e}_\ell\|_{L^2(K)} + k\,\|\mathbf{e}_\ell\|_{L^2(K)} \leq C\,k^2\,\|h_\ell\|_{L^2(\mathbb{R}^d)} \leq C\,k^{2\ell+3}\,\hat{\varepsilon}^{\ell+1}\,\|s\|_{H^{2\ell+1}(\mathbb{R}^d)}. \tag{5.40}$$

Applying Corollary 2 and Lemma 7, we obtain, for $\ell = 0, 1, 2$,

$$\|\nabla \mathbf{d}_\ell\|_{L^2(K)} + k\,\|\mathbf{d}_\ell\|_{L^2(K)} \leq C\,\hat{\varepsilon}\,k^2\,\|q_\ell\|_{L^2(\Omega)} \leq C\,k^{2\ell+5}\,\hat{\varepsilon}^\ell\,\|s\|_{H^{2\ell+3}(\mathbb{R}^d)}.$$

This and Lemma 6 imply, for $\ell = 0, 1$,

$$\begin{aligned}\|\nabla \mathbf{d}_\ell\|_{L^2(K)} + k\,\|\mathbf{d}_\ell\|_{L^2(K)} \leq& \hat{\varepsilon}^{\ell+1}\left(\|\nabla w_e^\ell\|_{L^2(K)} + k\,\|w_e^\ell\|_{L^2(K)}\right) + C\,k^{2\ell+7}\,\hat{\varepsilon}^{\ell+1}\,\|s\|_{H^{2\ell+5}(\mathbb{R}^d)}\\ \leq& C\,k^{2\ell+7}\,\hat{\varepsilon}^{\ell+1}\,\|s\|_{H^{2\ell+5}(\mathbb{R}^d)}.\end{aligned} \tag{5.41}$$

A combination of (5.39), (5.40), and (5.41) yields the conclution. □

A Appendix: The Proof of the Estimates for the Asymptotic Expansions

A.1 Proof of Lemma 5

The conclusion of Lemma 5 follows from the definition of w_e^ℓ for $\ell = 0, 1, 2$, and the standard regularity theory of elliptic equations. The details are left to the reader. □

A.2 Proof of Lemma 6

Using the definition of w_e^0 and applying Lemma 13 for $r_* = R$, we have

$$\|\nabla w_e^0\|_{L^2(B_R\setminus\Omega)} + k\,\|w_e^0\|_{L^2(B_R\setminus\Omega)} \leq C\|s\|_{L^2(\mathbb{R}^d\setminus\Omega)}.$$

Using the regularity theory of elliptic equations and applying Lemma 13, we have, for $m \geq 1$,

$$\|\nabla w_e^0\|_{H^m(B_R\setminus\Omega)} + k\,\|w_e^0\|_{H^m(B_R\setminus\Omega)} \leq Ck^m\|s\|_{H^m(\mathbb{R}^d\setminus\Omega)}.$$

and hence by the trace theory, we obtain

$$\|\partial_n w_e^0\|_{H^{m-1/2}(\Gamma)} \leq C\|w_e^0\|_{H^{m+1}(\mathbb{R}^d\setminus\Omega)} \leq Ck^{m+1}\|s\|_{H^{m+1}(\mathbb{R}^d\setminus\Omega)}.$$

By (3.9) and (3.10), the conclusion follows from the definition of w_e^1, w_e^2 in (3.8), Lemma 13, and the standard regularity theory of elliptic equations. □

A.3 Proof of Lemma 7

We only consider $k \geq k_0$. The other case follows similarly. From the definition of q_ℓ, we have

$$q_\ell(x) = \begin{cases} -[\Delta v_i^\ell(x) + k^2\, v_i^\ell(x) + \frac{i}{\hat{\varepsilon}^2}\, v_i^\ell(x)] - \hat{\varepsilon}^\ell[\Delta \varphi_{\hat{\varepsilon}}(x) + k^2\, \varphi_{\hat{\varepsilon}}(x) + \frac{i}{\hat{\varepsilon}^2}\, \varphi_{\hat{\varepsilon}}(x)] & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

It follows that

$$\|q_\ell\|_{L^2(\mathbb{R}^d)} \leq \|\Delta v_i^\ell + k^2\, v_i^\ell + \frac{i}{\hat{\varepsilon}^2}\, v_i^\ell\|_{L^2(\Omega)} + \hat{\varepsilon}^\ell \Big\|\Delta \varphi_{\hat{\varepsilon}} + k^2 \varphi_{\hat{\varepsilon}} + \frac{i}{\hat{\varepsilon}^2}\, \varphi_{\hat{\varepsilon}}\Big\|_{L^2(\Omega)} \tag{A1}$$

Since $\varphi_{\hat{\varepsilon}}(x) = \nu\, \chi(\nu/\hat{\varepsilon})\, \partial_n w_e^\ell(x_\Gamma)$, we obtain

$$\hat{\varepsilon}^\ell \Big\|\Delta \varphi + k^2 \varphi + \frac{i}{\hat{\varepsilon}^2}\, \varphi\Big\|_{L^2(\Omega)} \leq C \hat{\varepsilon}^\ell \Big(\|\partial_n w_e^\ell\|_{H^2(\Gamma)} + k^2\, \|\partial_n w_e^\ell\|_{L^2(\Gamma)}\Big) \times \big(\|\nu \chi(\nu/\hat{\varepsilon})\|_{H^2(0,\infty)} + \frac{1}{\hat{\varepsilon}^2} \|\nu \chi(\nu/\hat{\varepsilon})\|_{L^2(0,\infty)}\big).$$

We have [13]

$$\|\nu \chi(\nu/\hat{\varepsilon})\|_{H^2(0,\infty)} + \frac{1}{\hat{\varepsilon}^2} \|\nu \chi(\nu/\hat{\varepsilon})\|_{L^2(0,\infty)} \leq C \hat{\varepsilon}^{-1/2}. \tag{A2}$$

Therefore,

$$\hat{\varepsilon}^\ell \Big\|\Delta \varphi_{\hat{\varepsilon}} + k^2 \varphi_{\hat{\varepsilon}} + \frac{i}{\hat{\varepsilon}^2}\, \varphi_{\hat{\varepsilon}}\Big\|_{L^2(\Omega)} \leq C\, \hat{\varepsilon}^{\ell - 1/2} \Big(\|\partial_n w_e^\ell\|_{H^2(\Gamma)} + k^2\, \|\partial_n w_e^\ell\|_{L^2(\Gamma)}\Big).$$

We derive from Lemma 6 with $m = 3$ that, for $\ell = 0, 1, 2$,

$$\hat{\varepsilon}^\ell \Big\|\Delta \varphi_{\hat{\varepsilon}} + k^2 \varphi_{\hat{\varepsilon}} + \frac{i}{\hat{\varepsilon}^2}\, \varphi_{\hat{\varepsilon}}\Big\|_{L^2(\Omega)} \leq C\, \hat{\varepsilon}^{\ell - 1/2} k^{2\ell + 3}\, \|s\|_{H^{2\ell+3}(\mathbb{R}^d)}. \tag{A3}$$

Using (3.4) and the definitions of v_i^1 and v_i^2, as in [9], we have

$$\Big\|\Delta v_i^1 + k^2 v_i^1 + \frac{i}{\hat{\varepsilon}^2} v_i^1\Big\|_{L^2(\Omega)} \leq C\, \big(\|\mathcal{A}_1 w_i^1\|_{L^2(\Gamma \times \mathbb{R}_+)} + \hat{\varepsilon}^{-1} \|v_i^1\|_{H^1(\Gamma \times \mathbb{R}_+)}\big) \tag{A4}$$

and

$$\Big\|\Delta v_i^2 + k^2 v_i^2 + \frac{i}{\hat{\varepsilon}^2} v_i^2\Big\|_{L^2(\Omega)} \leq C\, \big(\hat{\varepsilon} \|\mathcal{A}_1 w_i^2 + \mathcal{A}_2 w_i^1 + \mathcal{A}_1 w_i^2\|_{L^2(\Gamma \times \mathbb{R}_+)} + \|v_i^2\|_{H^1(\Gamma \times \mathbb{R}_+)}\big). \tag{A5}$$

Recall that

$$\begin{aligned} \mathcal{A}_1 =& 2\mathcal{H} \partial_\eta + 6\eta \mathcal{H}(\partial_\eta^2 + i), \\ \mathcal{A}_2 =& \Delta_\Gamma + k^2 + 2\eta(\mathcal{G} + 4\mathcal{H}^2)\partial_\eta + 3\eta^2(\mathcal{G} + 4\mathcal{H}^2)(\partial_\eta^2 + i). \end{aligned}$$

[13] The scaling for the variable of χ gives us the optimal estimate in term of $\hat{\varepsilon}$.

Applying Lemma 6 with ($m = 2$ and $\ell = 0$) and using (3.9), we obtain:

$$\|w_i^1\|_{L^2(\Gamma\times\mathbb{R}_+)} + \hat{\varepsilon}^{-1}\,\|v_i^1\|_{H^1(\Gamma\times\mathbb{R}_+)} \le Ck^2\|s\|_{H^2(\mathbb{R}^d)}. \tag{A6}$$

Applying Lemma 6 with ($m = 3$ and $\ell = 0$) and ($m = 1$ and $\ell = 1$) and using (3.10), we obtain

$$\|w_i^1\|_{H^2(\Gamma\times\mathbb{R}_+)} + \|w_i^2\|_{L^2(\Gamma\times\mathbb{R}_+)} \le Ck^3\|s\|_{H^3(\mathbb{R}^d)}. \tag{A7}$$

Moreover, applying Lemma 6 with ($m = 2$ and $\ell = 0, 1$)

$$\hat{\varepsilon}^{-1}\,\|v_i^2\|_{H^1(\Gamma\times\mathbb{R}_+)} = \|w_i^1 + \hat{\varepsilon}\,w_i^2\|_{H^1(\Gamma\times\mathbb{R}_+)} \le Ck^4\|s\|_{H^4(\mathbb{R}^d)}. \tag{A8}$$

The conclusion now follows from (A1), (A3), (A4), (A5), (A6), (A7), (A8). We note here that for the case $\ell = 0$, we use the fact that $v_i^0 \equiv 0$. □

A.4 Proof of Lemma 8

A calculation (see e.g., [9, (4.34)]) shows that $h_0 = 0$, $h_1 = \frac{\hat{\varepsilon}^2}{\alpha}\,\partial_n w_e^1$. The conclusion follows from Lemmas 5 and 6. □

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