Schwinger's Method for the Massive Casimir Effect

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Abstract

We apply to the massive scalar field a method recently proposed by Schwinger to calculate the Casimir Effect. The method is applied with two different regularization schemes: the Schwinger's original one by means of Poisson formula and another one by means of analytical continuation.

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The importance of the Casimir Effect[1] steems from its far reaching conceptual meaning in relativistic quantum field theory and for its appearance in rather simple physical conditions, which makes possible not only its precise calculation but also its experimental verification. The original setup proposed by Casimir consists in a pair of paralel conducting plates immersed in electrodynamic's vaccum and the predicted effect, the attraction between the plates, was indeed later observed[2]. The Casimir method of calculation[3] (summation of the zero-point energies) as well as other methods applied to several different setups, require the subtraction of infinities to arrive at the final correct result. In such circumstances it is highly desirable to have different methods of calculations, with the infinities controled by different regularization schemes, in order to check out the regularization independence of the result and further clarify the conceptual meaning of the effect. For example, for the massless scalar field in the original Casimir setup, an elegant method recently proposed by Schwinger[3] gives the force of atraction between the plates in the context of source theory. Schwinger uses a regularization by means of Poisson formula and subtracts infinities which are unrelated to the force of atraction. We have shown in a previous letter[4] that a slight modification in Schwinger's method, leads to the final result directly, in an exceedingly short and simple way, without requiring any subtraction of infinities. Our modification consists in the use of another regularization prescription, by means of analytical continuation of Schwinger's effective action. Here we further investigate Schwinger's method and its sensitivitness to the regularization procedure by applying it to the more complicated case of a massive scalar field in $(d+1)$-dimensional space-time. We use again the two different regularization prescriptions: Schwinger's original one and the other one by means of analytical continuation. The results presented below show that the calculations by Schwinger's original method are not much harder than in the massless case. On the other hand, in the modified Schwinger's method the calculations get a little bit more involved. However, we should notice that some infinities to be subtracted in the former case appear in the latter case as finite terms. Also, the modified Schwinger's method is very simply related to other methods of calculations, such as the zeta function method, as we will show elsewhere.

Schwinger's method consists essentially in using Schwinger's formula for the effective action $W^{(1)}$in proper time representation[5]:

$$W^{(1)}(s_o) = -\frac{i}{2}\int_{s_o}^{\infty} \frac{ds}{s}\, Tr\, e^{-isH} \; + constant, \tag{1}$$

where the exponential of $iW^{(1)}$ is defined as the vacuum persistency probability amplitude $\langle 0_+|0_-\rangle$ at the one-loop level, Tr means the total trace, H is the corresponding proper time Hamiltonian and s_o is a regularization cutoff that must be sent to zero only at the end of the calculations, after appropriate subtractions of divergent terms is made; the additional constant is used to subtract divergent terms, thereby establishing the physical normalization for the situation. The energy of the configuration under consideration is obtained from the prescription: $\mathcal{E} = -W^{(1)}/T$, where T is the duration of the measurement. Schwinger applied this method to calculate the vaccum energy of the massless scalar field between two perfectly conducting large parallel plates[4]; the force between them is then obtained as the derivative of the energy with respect to the plates separation. A crucial part of the calculation is the use of Poisson's formula[6],

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi\tau} = \tau^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi(1/\tau)}, \tag{2}$$

to exhibit two spurious terms in the energy, a uniform density vaccum energy and the self-energy associated to each individual plate, both diverging when $s_o \to 0$. To arrive at the final result these terms must be subtracted before taking the limit $s_o \to 0$. The same problem can be handled in a slightly modified way, by writing the effective action as:

$$W^{(1)}(\nu) = -\frac{i}{2}\int_{0}^{\infty} \frac{ds}{s} s^{\nu}\, Tr\, e^{-isH} \,, \tag{3}$$

where, differently from (1), the range of integration starts at zero and the factor s^{ν} in the integrand is taken large enough to regularize the integral and disappears at the end by analytical continuation to $\nu = 0$. With this new regularization scheme the final result for the massless case was obtained in a remarkable simple way[3], without the appearance of any divergent term, actually without the appearence of any spurious term.

Let us now apply Schwinger's method to the proper-time Hamiltonian:

$$H = \sum_{j=1}^{d} p_j^2 - \omega^2 + m^2, \tag{4}$$

where $p_j = -i\partial/\partial x^j$, $\omega = i\partial/\partial t$ and m is the mass. The boundary conditions are dictated by two perfectly conducting hyperplanes, perpendicular to the d-direction and separated by a distance a. The total trace is for this case:

$$Tr\, e^{-i\,s\,H} = T\,\frac{i\,L^{d-1}}{(4\pi i)^{d/2}}\frac{1}{s^{d/2}}\,e^{-ism^2}\sum_{n=1}^{\infty}e^{-i\,s\,(\pi n/a)^2}. \tag{5}$$

By substituing this trace in (1) we get for the energy $\mathcal{E}$ the expression:

$$\frac{\mathcal{E}(s_o)}{L^{d-1}} = -\frac{1}{2}\frac{1}{(4\pi i)^{d/2}}\int_{s_o}^{\infty}\frac{ds}{s^{1+(d/2)}}\,e^{-ism^2}\sum_{n=1}^{\infty}e^{-is(\pi n/a)^2}, \tag{6}$$

which by using Poisson's formula can be recasted in the following form:

$$\begin{aligned}\mathcal{E}(s_o) = {} & L^{d-1}\frac{1}{4(4\pi i)^{d/2}}\int_{s_o}^{\infty}\frac{ds}{s^{1+d/2}}\,e^{-ism^2}\\ & - aL^{d-1}\frac{1}{2(4\pi i)^{(d+1)/2}}\int_{s_o}^{\infty}\frac{ds}{s^{1+(d+1)/2}}\,e^{-ism^2}\\ & - \frac{1}{(4\pi i)^{(d+1)/2}}\int_{s_o}^{\infty}\frac{ds}{s^{1+(d+1)/2}}\,e^{-ism^2}\,aL^{d-1}\sum_{n=1}^{\infty}e^{in^2a^2/s}.\end{aligned} \tag{7}$$

The first term, on the right-hand side of this equation, proportional to the conductor area L^{d-1}, comes from the self-energy of each conductor and must be normalized to zero because one is concerned only with the energy shift produced by varying the distance a between the conductors. The second term, proportional to the spatial volume aL^{d-1}, comes from a uniform spatial density of vaccum energy and must also be eliminated by a term in the *constant* of equation (1), in order to normalize the vaccum energy density to zero in the limit $a \to \infty$. Hence, we take the last term as the physical energy of interaction; by taking the limit $s_o \to 0$ and changing the integration variable to $\sigma = a^2/is$ we arrive at:

$$\frac{\mathcal{E}(0)}{L^{d-1}} = -\frac{1}{(4\pi)^{(d+1)/2}}\frac{1}{a^d}\sum_{n=1}^{\infty}\int_0^{\infty}d\sigma\,\sigma^{\frac{1}{2}(d+1)-1}\,e^{-n^2\sigma - m^2a^2/\sigma}. \tag{8}$$

Since both m^2a^2 and n^2 are positive the above integral is well defined and can be written as a series in modified Bessel functions[(7)]. The final expression for the energy $\mathcal{E}(0)$ is then obtained as[(8)]:

$$\frac{\mathcal{E}(0)}{L^{d-1}} = -2\left(\frac{m}{4\pi}\right)^{(d+1)/2}\frac{1}{a^{(d-1)/2}}\sum_{n=1}^{\infty}\frac{1}{n^{(d+1)/2}}K_{(d+1)/2}(2amn). \tag{9}$$

Let us now turn to the modified Schwinger method, in which we start with expression (3) and use the total trace (5) to arrive at the expression:

$$\frac{\mathcal{E}(\nu)}{L^{d-1}} = -\frac{1}{2}\frac{1}{(4\pi i)^{d/2}}\sum_{n=1}^{\infty}\int_0^{\infty} ds\, s^{(\nu-d/2)-1}\, e^{-s[i\pi^2(n^2+a^2m^2/\pi^2)/a^2]}. \tag{10}$$

The integration is now given by the mere definition of the gamma function,

$$\alpha^{-\xi}\Gamma(\xi) = \int_0^{\infty} ds\, s^{\xi-1}\, e^{-\alpha s},$$

and leads after some elementary manipulations to:

$$\frac{\mathcal{E}(\nu)}{L^{d-1}} = -\frac{1}{2}\left(\frac{\pi}{4a^2}\right)^{d/2}\left(\frac{a^2}{i\pi^2}\right)^{\nu}\Gamma(\nu-d/2)\sum_{n=1}^{\infty}\frac{1}{(n^2+\mu^2)^{\nu-d/2}}, \tag{11}$$

where we have defined $\mu = am/\pi$ and assumed $\nu > (d+1)/2$, to guarantee the convergence of the series; its sum can then be analytically continued by means of the following equality[9,8]:

$$\sum_{n=1}^{\infty}\frac{1}{(n^2+\mu^2)^z} = -\frac{1}{2\mu^{2z}} + \frac{\sqrt{\pi}}{2\mu^{2z-1}\Gamma(z)}\left[\Gamma(z-1/2) + 4\sum_{n=1}^{\infty}(\pi n\mu)^{z-1/2}K_{z-1/2}(2\pi n\mu)\right], \tag{12}$$

where $\Re z > 1/2$. Using this equality in (11) we arrive at:

$$\begin{aligned}\frac{\mathcal{E}(\nu)}{L^{d-1}} &= \frac{1}{4}\left(\frac{m}{2\sqrt{\pi}}\right)^{d}\frac{1}{(im^2)^{\nu}}\Gamma(\nu-d/2)\\ &\quad - a\frac{1}{2}\left(\frac{m}{2\sqrt{\pi}}\right)^{d+1}\frac{1}{(im^2)^{\nu}}\Gamma(\nu-(d+1)/2)\\ &\quad - 2\left(\frac{m}{4\pi}\right)^{(d+1)/2}\frac{1}{a^{(d-1)/2}}\left(\frac{a}{im}\right)^{\nu}\sum_{n=1}^{\infty}\frac{1}{n^{(d+1)/2-\nu}}K_{(d+1)/2-\nu}(2amn).\end{aligned} \tag{13}$$

Now we have to use the analytical continuation to take the limit $\nu \to 0$ and this is all that we have to do in the massless case to get the final result[3]. However, in the present case of a massive field we identify in (13) two spurious terms in the limit $\nu \to 0$, not necessarily divergent as in (7), but still spurious: the first term in (13), independent of a, gives the self-energies associated with the individual plates, and the second term, proportional to a, gives a uniform energy density of the vaccum. After discarding these two terms and then taking the limit we are left with an expression for the Casimir energy which is identical to (9).

After these calculations a short comment is in order. Schwinger's method, which showed itself as a very powerful method for QED, is also a very economical way of computing the Casimir energy in other situations. Depending on the regularization scheme adopted, the spurious terms can be made finite or divergent. In the regularization by Poisson formula they are both divergent, while in the analytical continuation scheme they will never be simultaneously divergent. It is interesting to note that in the latter case the two terms exchange between themselves the divergent character each time the dimension of space changes by a unit.

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References

1. H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51** (1948), 793; for a review see G. Plunien, B. Muller and W. Greiner, Phys. Rep. **134** (1986), 89.
2. M. J. Sparnay, Physica **24** (1958), 751.
3. J. Schwinger, Lett. Math. Phys. **24**, 59 (1992).
4. M. V. Cougo-Pinto, C. Farina, and A. J. Seguí-Santoja, Lett. Math. Phys., to be published.
5. J. Schwinger, Phys. Rev. **82** (1951), 664.
6. S. D. Poisson, Journal de l'Ecole Polytechnique, **XII** (cahier **XIX**), (1823), 420.
7. Formula **3.471**,9 in Gradshteyn, I. S. and Ryshik, I. M., *Table of Integrals, Series and Products* (Academic Press, New York and London, 1965).
8. J. Ambjorn and S. Wolfram, Annals of Phys. **147** (1983), 1.
9. E. Elizalde and A. Romeo, J. Math. Phys. **30** (1989), 1133; erratum, **31**, (1990), 771.