

# COX RINGS OF MODULI OF QUASI PARABOLIC PRINCIPAL BUNDLES AND THE $K$ -PIERI RULE

CHRISTOPHER MANON

ABSTRACT. We study a toric degeneration of the Cox ring of the moduli of principal  $SL_m(\mathbb{C})$  bundles on the projective line, with quasi parabolic data given by the the stabilizer of the highest weight vector in  $\mathbb{C}^m$  and its dual  $\bigwedge^{m-1}(\mathbb{C}^m)$ . The affine semigroup algebra resulting from this degeneration is described using the  $K$ -Pieri rule from Kac-Moody representation theory. Along the way we give a proof of the  $K$ -Pieri rule which utilizes the classical Pieri rule and elements of commutative algebra, and we describe a relationship between the Cox ring and a classical invariant ring studied by Weyl.

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## 1. INTRODUCTION

For a smooth, projective curve  $C$ , with a choice  $\vec{p} \subset C$  of marked points, and an assignment  $p_i \rightarrow P_i$  of parabolic subgroups  $P_i \subset SL_m(\mathbb{C})$  to these points, a quasi-parabolic principal bundle of type  $\vec{P}$  is a principal bundle  $E$  on  $C$  and a choice  $\rho_i \in [E \times_G G/P_i]_{p_i}$  from the fiber of the  $G/P_i$  associated bundle at the marked point  $p_i$ . The moduli stacks  $\mathcal{M}_{C, \vec{p}}(\vec{P})$  of these objects have been widely studied due to their place in algebraic geometry as a generalization to the Jacobian variety, and because of their relationship to conformal field theory and representation theory, which we will describe further below. In [M4] we began a program to study these stacks by describing the algebras formed by the global sections of their line bundles, sometimes called non-Abelian theta functions. Our primary object of interest is the algebra formed by all of the non-Abelian theta functions, obtained by endowing the direct sum of all global sections of line bundles on the stack with global section multiplication. This algebra is known as the Cox ring or total coordinate ring of the stack  $\mathcal{M}_{C, \vec{p}}(\vec{P})$ .

In this paper we describe this algebra in a special case  $\mathcal{M}_{\mathbb{P}^1, \vec{p}}(\vec{P}, \vec{P}^*)$ , where the curve  $C$  is a projective line, and the parabolic structure at each marked point is given by the stabilizers  $P, P^* \subset SL_m(\mathbb{C})$  of the highest weight vector in the representation  $\mathbb{C}^m$  or its dual  $\bigwedge^{m-1}(\mathbb{C}^m)$ . We let  $a$  be the number of marked points with parabolic data  $P$  and  $b$  be the number of marked points with parabolic data  $P^*$ . Line bundles  $\mathcal{L}(\vec{r}, \vec{s}, K)$  on this stack are indexed by a tuple  $\vec{r}$  of  $a$  non-negative integers, a tuple  $\vec{s}$  of  $b$  non-negative integers, and a non-negative integer  $K$  called the level.

$$(1) \quad \text{Pic}(\mathcal{M}_{\mathbb{P}^1, \vec{p}}(\vec{P}, \vec{P}^*)) = \mathbb{Z}^{a+b+1}$$

$$(2) \quad V_{\vec{p}}(a, b) = \bigoplus_{\vec{r}, \vec{s}, K} H^0(\mathcal{M}_{\mathbb{P}^1, \vec{p}}(\vec{P}, \vec{P}^*), \mathcal{L}(\vec{r}, \vec{s}, K))$$

By definition the algebra  $V_{\vec{p}}(a, b)$  contains each of the projective coordinate rings  $R_{\vec{p}}(\vec{r}, \vec{s}, K)$  defined by the line bundle  $\mathcal{L}(\vec{r}, \vec{s}, L) \in \text{Pic}(\mathcal{M}_{\mathbb{P}^1, \vec{p}}(\vec{P}, \vec{P}^*))$  as a graded subalgebra. We prove the following theorem.

**Theorem 1.1.** *For generic arrangements of points  $\vec{p}$ , the algebra  $V_{\vec{p}}(a, b)$  is minimally generated by the non-Abelian theta functions with  $K = 1$ . The relations which hold among these generators are generated by the quadratic relations.*

In the case  $m = 2$ , the parabolic subgroups  $P, P^*$  are equal, in this case we set  $n = a + b$ . This case is studied by Castravet and Tevelev in [CT], where the algebra  $V_{\vec{p}}(n)$  is shown to be a Cox-Nagata ring. This means that  $V_{\vec{p}}(n)$  is the algebra of invariants by a certain non-reductive (additive) group action on a polynomial ring, and can be identified with the Cox ring of a blow-up of a projective space. Sturmfels and Xu also study this case in [StXu], where they construct a SAGBI (Subalgebra Analogue of Gröbner Bases for Ideals) degeneration  $V_{\mathcal{T}}(n)$  of  $V_{\vec{p}}(n)$  for generic  $\vec{p}$ , for each trivalent tree  $\mathcal{T}$  with  $n$  leaves. The degenerations  $V_{\mathcal{T}}(n)$  constructed in [StXu] are notable as they are affine semigroup algebras which have also been studied by Buczyńska and Wiśniewski [BW] in the context of mathematical biology. The algebras  $V_{\mathcal{T}}(n)$  are the coordinate rings of the  $\mathbb{Z}/2\mathbb{Z}$  group-based phylogenetic statistical models. Generators for the ideal of relations presenting these coordinate rings are important for applications of these models, to that end Buczyńska and Wiśniewski prove results for  $V_{\mathcal{T}}(n)$  which imply Theorem 1.1 in the  $SL_2(\mathbb{C})$  case, via the degeneration techniques in [StXu].

In this paper we also employ degeneration methods, developed in [M4] from elements of conformal field theory (see below). The tree combinatorics is still present, however it stems from a relationship to the moduli of marked genus 0 curves  $\bar{\mathcal{M}}_{0,n}$ , rather than a SAGBI construction. It would be interesting to find a SAGBI construction for  $V_{\vec{p}}(a, b)$  in the general case, and also an interpretation of this algebra as the invariants of an additive group action on a polynomial ring.

**1.1. Degeneration and the  $K$ -Pieri rule.** We choose the standard set of positive roots  $\alpha_{ij}, i < j$  for  $SL_m(\mathbb{C})$ , with corresponding Weyl chamber  $\Delta$ . Dominant weights  $\lambda$  of  $SL_m(\mathbb{C})$  and positive dominant weights of  $GL_m(\mathbb{C})$  in this Weyl chamber are weakly decreasing lists of  $m - 1$  and  $m$  numbers, respectively.

$$(3) \quad \lambda = [\lambda_1 \geq \lambda_2 \geq \dots]$$

We let  $V(\lambda)$  denote the irreducible representation corresponding to a dominant weight  $\lambda$ . The fundamental weight  $\omega_k$  is the list with  $k$  1's followed by  $m - k$  0's, and its corresponding irreducible representation is the exterior product  $V(\omega_k) = \bigwedge^k(\mathbb{C}^m)$ . Note that  $V(\omega_m)$  is the determinant representation of  $GL_m(\mathbb{C})$  and therefore the trivial representation of  $SL_m(\mathbb{C})$ .

In order to prove Theorem 1.1, we first degenerate  $V_{\vec{p}}(a, b)$  to an affine semigroup algebra, using Theorem 1.1 in [M4]. When specialized to the case we consider here, this theorem says that for any trivalent tree  $\mathcal{T}$  with  $a + b$  labelled leaves, there is a degeneration of  $V_{\vec{p}}(a, b)$  to an algebra  $V_{\mathcal{T}}(a, b)$ . The algebra  $V_{\mathcal{T}}(a, b)$  is built out of the total coordinate rings of moduli of  $SL_m(\mathbb{C})$  quasi-parabolic principal bundles on a triple marked projective line,  $\mathbb{P}^1, 0, 1, \infty$  according to a specific recipe dictated by  $\mathcal{T}$ . The structure of  $V_{\mathcal{T}}(a, b)$  depends on the tree  $\mathcal{T}$ , so it is critical to choose a tree topology which gives an algebra with advantageous properties. From now on, we consider the "caterpillar" tree  $\mathcal{T}_0$ , depicted in Figure 1. This tree has the property that each vertex of  $\mathcal{T}_0$  is connected to a leaf by a single edge.

Each leaf-edge of this tree corresponds to a marked point, and is assigned a  $P$  or  $P^*$ , the internal edges are each assigned a copy of the Borel subgroup  $B \subset P, P^* \subset SL_m(\mathbb{C})$ . Each internal vertex of  $\mathcal{T}_0$  now has three parabolic subgroups assigned to its incident edges, so following [M4], we think of this vertex as corresponding to the total coordinate ring of the moduli of quasi-parabolic principal bundles on  $\mathbb{P}^1, 0, 1, \infty$  with parabolic structure coming from the corresponding parabolic subgroups. Four types of total coordinate rings appear,  $V_{0,3}(P, P, B)$ ,  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ , and  $V_{0,3}(B, P^*, P^*)$ , we call these the  $K$ -Pieri algebras for reasons that will be made clear. The following theorem expresses the degeneration of  $V_{\vec{p}}(a, b)$  in terms of these algebras, it follows from Theorem 1.1 from [M4].

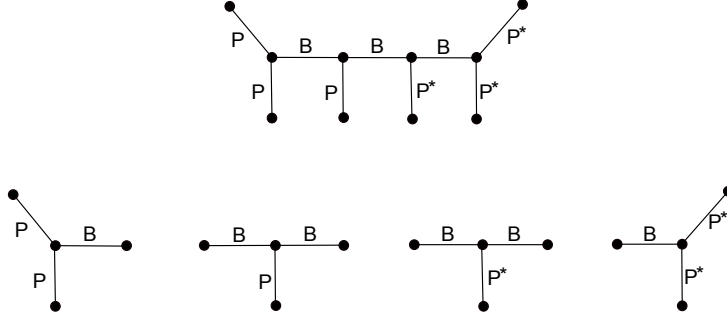


FIGURE 1.

**Theorem 1.2.** *The algebra  $V_{\vec{P}}(a, b)$  has a flat degeneration to the following algebra.*

$$(4) \quad V_{\mathcal{T}_0}(a, b) = [V_{0,3}(P, P, B) \otimes [V_{0,3}(B, P, B)]^{\otimes a-2} \otimes [V_{0,3}(B, P^*, B)]^{\otimes b-2} \otimes V_{0,3}(B, P^*, P^*)]^T$$

This is the algebra of invariants in a tensor product of  $K$ -Pieri algebras by a torus  $T$ . This torus is a product of  $a + b - 3$  copies of  $\mathbb{T} \times \mathbb{C}^*$ , one for each internal edge of  $\mathcal{T}_0$ , where  $\mathbb{T} \subset SL_m(\mathbb{C})$  is the torus of diagonal matrices, we describe the action of this torus below.

Equation 1 is a special case of a more general theorem due to Laszlo and Sorger, [LS], which expresses the Picard group of the moduli  $\mathcal{M}_{C, \vec{P}}(\vec{P})$  of quasi-parabolic principal bundles on a curve  $C$  with parabolic structure  $P_i$  at  $p_i$  in terms of the character groups  $\mathcal{X}(P_i)$  of the associated parabolic subgroups.

$$(5) \quad \text{Pic}(\mathcal{M}_{C, \vec{P}}(\vec{P})) = \mathcal{X}(P_1) \times \dots \times \mathcal{X}(P_n) \times \mathbb{Z}$$

When each  $P_i$  is the Borel subgroup  $B$ , we let  $V_{C, \vec{P}}(SL_m(\mathbb{C}))$  denote the corresponding Cox ring. This algebra contains the Cox rings of all other moduli of principal bundles as a multigraded subalgebra, in particular  $V_{0,3}(SL_m(\mathbb{C}))$  contains each of the  $K$ -Pieri algebras as multigraded subalgebras. By Laszlo and Sorger's theorem, the Cox rings  $V_{0,3}(P, P, B)$ ,  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ ,  $V_{0,3}(B, P^*, P^*)$  are multigraded by triples of dominant  $SL_m(\mathbb{C})$  weights. The algebra  $V_{0,3}(B, P, B)$  is given as an example below.

$$(6) \quad V_{0,3}(B, P, B) = \bigoplus_{\lambda, \eta \in \Delta, r, K \in \mathbb{Z}_{\geq 0}} V_{0,3}(\lambda, r\omega_1, \eta, K)$$

Here  $V_{0,3}(\lambda, r\omega_1, \eta, K) = H^0(\mathcal{M}_{0,3}(B, P, B), \mathcal{L}(\lambda, r\omega_1, \eta, K))$ . The weights  $\lambda, \eta$  are allowed to vary over all dominant weights in the Weyl chamber  $\Delta$ , whereas  $r\omega_1$  is always taken from the ray through  $\omega_1 \in \Delta$ , corresponding to the parabolic subgroup  $P \subset SL_m(\mathbb{C})$ . The analogous expression holds for  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ ,  $V_{0,3}(B, P^*, P^*)$ , this implies that the tensor product

$$(7) \quad V_{0,3}(P, P, B) \otimes [V_{0,3}(B, P, B)]^{\otimes a-2} \otimes [V_{0,3}(B, P^*, B)]^{\otimes b-2} \otimes V_{0,3}(B, P^*, P^*)$$

is a multigraded sum of spaces

$$(8) \quad V_{0,3}(r_1\omega_1, r_2\omega_1, \eta_1, K_1) \otimes V_{0,3}(\lambda_2, r_3\omega_1, \eta_2, K_2) \otimes \dots$$

$$\dots \otimes V_{0,3}(\lambda_{a-1}, r_a, \eta_{a-1}, K_{a-1}) \otimes V_{0,3}(\lambda_a, s_1\omega_{m-1}, \eta_a, K_a) \otimes \dots$$

$$\dots \otimes V_{0,3}(\lambda_{a+b-3}, s_{b-2}\omega_{m-1}, \eta_{a+b-3}, K_{a+b-3}) \otimes V_{0,3}(\lambda_{a+b-2}, s_{b-1}\omega_{m-1}, s_b\omega_{m-1}, K_{a+b-2})$$

The torus  $T$  action on these spaces is by the characters determined by the weight information  $(\eta_i - \lambda_{i+1}^*, K_i - K_{i+1})$ . A graded component of the invariant algebra  $V_{\mathcal{T}_0}(a, b)$  of the torus  $T$  therefore has  $\eta_i = \lambda_{i+1}^*$  and  $K_i = K_{i+1}$ .

This allows us to understand properties of the degeneration  $V_{\mathcal{T}_0}(a, b)$  by studying the  $K$ -Pieri algebras. In order to understand the structure of these algebras, in particular their multigraded components, we exploit a connection between moduli of principal bundles on curves and mathematical physics. The moduli of quasi-parabolic principal bundles on a smooth marked curve have a close connection with conformal field theory, in particular the non-Abelian theta functions are identified with the partition functions from the Wess-Zumino-Witten model of conformal field theory, and so-called conformal blocks. This theory associates a space of conformal blocks  $V_{C, \vec{p}}(\vec{\lambda}, K)$  to each stable, marked curve  $C, \vec{p}$ , assignment of dominant weights  $p_i \rightarrow \lambda_i$ , and non-negative integer  $K$ . The theorem establishing this relationship in various cases is due to Narasimhan, Ramanathan, Kumar; Beauville, Laszlo, Sorger; Faltings; and Pauly.

**Theorem 1.3.** *For  $C, \vec{p}$  a smooth marked curve,  $\mathcal{L}(\vec{\lambda}, K) \in \text{Pic}(\mathcal{M}_{C, \vec{p}}(\vec{P}))$ ,*

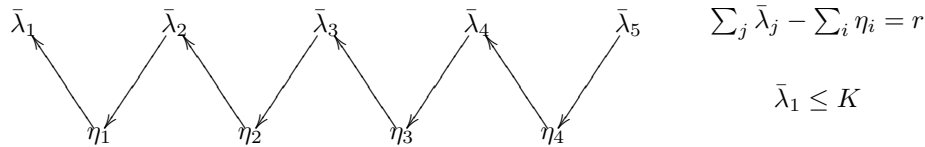
$$(9) \quad H^0(\mathcal{M}_{C, \vec{p}}(\vec{P}), \mathcal{L}(\vec{\lambda}, K)) = V_{C, \vec{p}}(\vec{\lambda}, K)$$

Conformal blocks are also the structure spaces of a monoidal operation called the fusion product on integrable highest-weight representations of the affine Kac-Moody algebra  $sl_m(\mathbb{C})$ . As a consequence, we may bring in the associated combinatorial representation theory to analyze the graded pieces of  $V_{\mathcal{T}_0}(a, b)$ . In particular, we use a result called the  $K$ -Pieri rule, that each graded component of  $V_{0,3}(P, P, B)$ ,  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ , and  $V_{0,3}(B, P^*, P^*)$  is multiplicity free. For a description of the  $K$ -Pieri rule in the context of fusion coefficients see Section 3 of [MS]. This result implies that each  $K$ -Pieri algebra has a basis labelled by the weight data  $(\lambda, r, \eta)$  of their graded components  $V_{0,3}(\lambda, r\omega_1, \eta, K)$ , and multiplication of basis members is simple addition on this weight data. Formally then we say that the  $K$ -Pieri algebras are affine semigroup algebras for semigroups of weight data  $P_3(P, P, B)$ ,  $P_3(B, P, B)$ ,  $P_3(B, P^*, B)$ ,  $P_3(B, P^*, P^*)$ . We must first determine the structure of these semigroups.

**Proposition 1.4.** *[The  $K$ -Pieri rule (reformulated)] The space  $V_{0,3}(\lambda, r\omega_1, \eta, K)$  has dimension 0 or 1. This space has dimension 1 if and only if there is a positive  $GL_m(\mathbb{C})$  dominant weight  $\bar{\lambda}$  such that the following hold.*

- (1)  $\lambda^* = \bar{\lambda} - \bar{\lambda}_m \omega_m$
- (2)  $\sum_j \bar{\lambda}_j - \sum_i \eta_i = r$
- (3)  $r, \bar{\lambda}_1 \leq K$
- (4)  $\bar{\lambda}_i - \eta_i, \eta_i - \bar{\lambda}_{i+1} \geq 0$

The entries of the dominant weights  $\eta, \bar{\lambda}$  from Proposition 1.4 can be placed in an interlacing diagram, shown in Figure 1.1 for  $m = 5$ . Notice that  $\bar{\lambda}$  can be recovered from the data  $\lambda, \eta, r$ , namely  $\bar{\lambda} = \lambda^* + \bar{\lambda}_m \omega_m$ ,  $\bar{\lambda}_m = \frac{1}{m}(r + \sum_i \eta_i - \sum_j \bar{\lambda}_j - \bar{\lambda}_j)$ .



For such a pattern  $\mathbf{b}$  we define its boundary weights as follows.

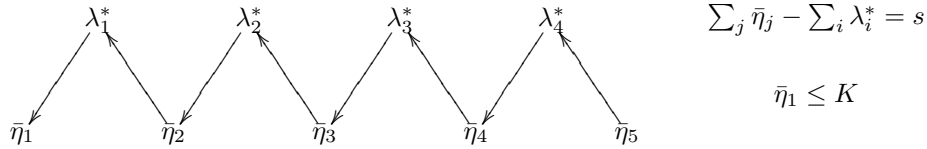
$$(10) \quad \partial_1(\mathbf{b}) = \eta$$

$$(11) \quad \partial_2(\mathbf{b}) = (\bar{\lambda}_1 - \bar{\lambda}_m, \dots, \bar{\lambda}_1 - \bar{\lambda}_2) = \lambda$$

By duality considerations, the space  $V_{0,3}(\lambda, r\omega_1, \eta, K)$  is non-zero with dimension 1 if and only if the same is true for  $V_{0,3}(\lambda^*, r\omega_{m-1}, \eta^*, K)$ . For this reason, interlacing patterns also give a mechanism to label the (multiplicity free) graded components of the algebra  $V_{0,3}(B, P^*, B)$ . The space  $V_{0,3}(\lambda, s\omega_{m-1}, \eta, K)$  is labelled by  $GL_n(\mathbb{C})$  weights  $\lambda^*$  and  $\bar{\eta}$ , which satisfy the following.

- (1)  $\bar{\eta} = \eta + \bar{\eta}_m \omega_m$
- (2)  $\sum_j \bar{\eta}_j - \sum_i \lambda_i^* = s$
- (3)  $s, \bar{\eta}_1 \leq K$
- (4)  $\bar{\eta}_i - \lambda_i^*, \lambda_i^* - \bar{\eta}_{i+1} \geq 0$

We illustrate interlacing patterns  $\mathbf{b}^*$  for the spaces  $V_{0,3}(\lambda, s\omega_{m-1}, \eta)$  with the opposite orientation.



We apply the boundary maps  $\partial_1, \partial_2$  to interlacing patterns  $\mathbf{b}^*$  as follows.

$$(12) \quad \partial_1(\mathbf{b}^*) = (\bar{\eta}_1 - \bar{\eta}_m, \dots, \bar{\eta}_{m-1} - \bar{\eta}_m) = \eta$$

$$(13) \quad \partial_2(\mathbf{b}^*) = (\lambda_1^*, \lambda_1^* - \lambda_{m-1}^*, \dots, \lambda_1^* - \lambda_2^*) = \lambda$$

Each such interlacing pattern labels one of the multiplicity-free spaces in the multigraded decomposition of  $V_{0,3}(P, P, B)$  and  $V_{0,3}(B, P, B)$  and likewise  $V_{0,3}(B, P^*, B)$  and  $V_{0,3}(B, P^*, P^*)$ . The algebra  $V_{\mathcal{T}_0}(a, b)$  is also graded by multiplicity-free components, and is therefore an affine semigroup algebra for an affine semigroup  $P(a, b)$ . The elements of  $P(a, b)$  are lists of elements  $(\mathbf{b}_1, \dots, \mathbf{b}_{a-1}, \mathbf{b}_1^*, \dots, \mathbf{b}_{b-1})$  from the semigroups above, which can be glued along their boundary data,  $\partial_1(\mathbf{b}_i) = \partial_2(\mathbf{b}_{i+1})^*$ ,  $\partial_1(\mathbf{b}_i^*) = \partial_2(\mathbf{b}_{i+1})$ ,  $\partial_1(\mathbf{b}_{a-1}) = \partial_2(\mathbf{b}_1^*)$ . We will present this semigroup explicitly, by first analyzing each of the four algebras  $V_{0,3}(P, P, B)$ ,  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ ,  $V_{0,3}(B, P^*, P^*)$ , and then by controlling the "gluing" procedure. The following proposition, proved in Section 4, says that the four algebras we consider as our building components are not very complicated.

**Proposition 1.5.** *The algebras  $V_{0,3}(B, P, B), V_{0,3}(B, P^*, B)$  are polynomial rings on  $2m$  variables. The algebras  $V_{0,3}(P, P, B), V_{0,3}(B, P^*, P^*)$  are polynomial rings on 4 variables.*

We will prove this proposition by carefully reproving the  $K$ -Pieri rule using elements of commutative algebra. This continues a theme first established in [M4] and continued in [M10], that the combinatorics of the conormal blocks are related to toric degenerations of the moduli of quasi-parabolic principal bundles.

The gluing operation described above should be a familiar operation for researchers who study affine semigroup algebras. The following proposition is a consequence of the description of  $V_{\mathcal{T}_0}(a, b)$  as an algebra of torus invariants in a tensor product of the  $K$ -Pieri algebras.

**Proposition 1.6.** *The affine semigroup  $P(a, b)$  is a toric fiber product of the affine semigroups associated to the  $K$ -Pieri algebras.*

Toric fiber products of affine semigroups are defined in [S], and further explored in [M6]. Informally we may think of the toric fiber product as a way to glue the  $K$ -Pieri semigroups corresponding to the above interlacing diagrams together to produce a composite semigroup  $P(a, b)$ , and indeed we describe

the defining inequalities and corresponding diagram of this semigroup in Section 5. In Section 5 we also describe a polytope  $P(\vec{r}, \vec{s}, K)$  whose graded affine semigroup algebra is induced the degeneration of the projective coordinate ring  $R_{\vec{p}}(\vec{r}, \vec{s}, K) \subset V_{\vec{p}}(a, b)$ . We make use of the fiber product structure on  $P(a, b)$ , and general theorems on toric fiber products from [S] and [M6], to prove an explicit presentation of  $V_{\mathcal{T}_0}(a, b)$ .

**Definition 1.7.** *The set  $X_{a,b}$  is defined to be the set of all tuples  $[\vec{i}]$  of  $a + b - 1$  elements from  $\mathbb{Z}/m\mathbb{Z}$  such that the first and last entries are  $m - 1$  or  $0$ , and the difference  $i_k - i_{k+1}$  is in  $\{0, 1\}$  if  $1 \leq k \leq a - 1$  and in  $\{0, -1\}$  if  $a \leq k \leq a + b - 1$ .*

**Definition 1.8.** *On the set of tuples above, a swap is a quadratic polynomial of the following form,*

$$(14) \quad [\vec{i}_1, i, \vec{i}_2][\vec{i}_1^*, i, \vec{i}_2^*] - [\vec{i}_1, i, \vec{i}_2^*][\vec{i}_1^*, i, \vec{i}_2]$$

**Theorem 1.9.** *The algebra  $V_{\mathcal{T}_0}(a, b)$  is presented by generators  $X_{a,b}$ , subject to all possible swap relations.*

This is a presentation of  $V_{\mathcal{T}_0}(a, b)$  by generators and relations. The fiber product structure of  $P(a, b)$  also allows us to prove that the algebra  $V_{\mathcal{T}_0}(a, b)$  is Gorenstein, and that it possesses a quadratic square-free Gröbner basis given by the swap relations above, in Section 4. These properties then lift to the algebra  $V_{\vec{p}}(a, b)$  for generic  $\vec{p}$  by general properties satisfied by algebras in flat families, see the proof of Theorem 1.11 in [M6]. In order to prove these results, in particular Proposition 1.5, we study an auxiliary semigroup  $Q(a, b)$ , which comes from a classical algebra from invariant theory.

**1.2. Relation to Weyl's invariant ring.** One of our motivations in studying the algebra  $V_{\mathbb{P}^1, \vec{p}}(a, b)$  is that it bears a resemblance to the object of study in a classical theorem in invariant theory due to Weyl. We let  $R(a, b)$  be the algebra of  $SL_m(\mathbb{C})$  invariants in the coordinate ring of the affine space  $[\bigwedge^{m-1}(\mathbb{C}^m)]^a \times [\mathbb{C}^m]^b$ .

$$(15) \quad R(a, b) = \mathbb{C}[[\bigwedge^{m-1}(\mathbb{C}^m)]^a \times [\mathbb{C}^m]^b]^{SL_m(\mathbb{C})}$$

We view the algebra  $\mathbb{C}[[\bigwedge^{m-1}(\mathbb{C}^m)]^a \times [\mathbb{C}^m]^b]$  as a polynomial ring on  $m \times (a + b)$  variables, arranged in a matrix, with the columns labelled by elements of  $[a + b] = [a] \amalg [b]$ .

$$\begin{array}{cccccc} x_{11} & \dots & x_{a1} & y_{11} & \dots & y_{b1} \\ x_{12} & \dots & x_{a2} & y_{12} & \dots & y_{b2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{1m} & \dots & x_{am} & y_{1m} & \dots & y_{bm} \end{array}$$

We let  $\Delta_I$  be the determinant form on the variables determined by a subset  $I \subset [a]$  in  $[\bigwedge^{m-1}(\mathbb{C}^m)]^a$  of size  $m$ , and  $\Delta_J$  be the dual determinant in  $[\mathbb{C}^m]^b$ , for  $J \subset [b]$ . We let  $P_{ij}$  be the column-wise inner product of the variables on the indices  $i \in [a]$  with those in  $j \in [b]$ . Each of the elements  $\Delta_I, \Delta_J, P_{ij}$  is an  $SL_m(\mathbb{C})$  invariant, and therefore a member of  $R(a, b)$ . Weyl described a collection of quadratic relations on these elements, known as the Plücker relations.

**Theorem 1.10** (Weyl's first fundamental theorem of invariant theory). *The algebra  $R(a, b)$  is generated by the  $\Delta_I, \Delta_J, P_{ij}$ , and all relations among these generators are generated by the Plücker relations.*

The algebra  $R(a, b)$  is multigraded by an action of  $a + b$  copies of the maximal diagonal torus  $\mathbb{T} \subset SL_m(\mathbb{C})$ . The components of this multigrading are the  $SL_m(\mathbb{C})$  invariant spaces in the tensor products,

$$(16) \quad V(\vec{r}\omega_1, \vec{s}\omega_{m-1}) = V(r_1\omega_1) \otimes \dots \otimes V(r_a\omega_1) \otimes V(s_1\omega_{m-1}) \otimes \dots \otimes V(s_b\omega_{m-1})$$

In [M4] a relationship is established between  $R(a, b)$  and  $V_{\vec{p}}(a, b)$ .

**Theorem 1.11.** *For any collection of  $a + b$  points  $\vec{p} \subset \mathbb{P}^1$ , the algebra  $V_{\vec{p}}(a, b)$  is a  $\mathbb{Z}^{a+b+1}$ -multigraded subalgebra of the polynomial ring over Weyl's invariant ring,  $R(a, b) \otimes \mathbb{C}[t]$ .*

$$(17) \quad V_{\mathbb{P}^1, \vec{p}}(\vec{r}\omega_1, \vec{s}\omega_{m-1}, L) \subset V(\vec{r}\omega_1, \vec{s}\omega_{m-1})^{SL_m(\mathbb{C})} t^L$$

Theorem 1.11 in [M4] implies that the degeneration of  $V_{\vec{p}}(a, b)$  corresponding to  $\mathcal{T}_0$  we use here also applies to the algebra  $R(a, b)$ . This connection allows us to use the algebra  $R(a, b)$  to study  $V_{\vec{p}}(a, b)$ . We will show how a similar analysis of  $R(a, b)$  using the classical Pieri rule yields the following presentation of this degeneration,  $R_{\mathcal{T}_0}(a, b)$ . Let  $Y_{a,b} \subset X_{a,b}$  be the subset of tuples which are unbroken strings of non-zero entries.

**Theorem 1.12.** *The algebra  $R_{\mathcal{T}_0}(a, b)$  is presented by  $Y_{a,b}$  with respect to the swap relations on these generators.*

This gives a presentation of  $R_{\mathcal{T}_0}(a, b)$ , in Section 3 we will describe an explicit affine semigroup  $Q(a, b)$  such that  $\mathbb{C}[Q(a, b)] = R_{\mathcal{T}_0}(a, b)$ .

The elements  $\Delta_I$ ,  $\Delta_J$ , and  $P_{ij}$  from Weyl's presentation correspond to minimal generators in the degeneration  $R_{\mathcal{T}_0}(a, b)$ , as follows. For the set  $I \subset [a]$ , we let  $t_I$  be the tuple with  $t_{i+1} - t_i = 1$  exactly for  $i \in I$ , and we define  $t_J$  for  $J \subset [b]$  similarly. We let  $t_{ij}$  be the tuple with  $t_{i+1} - t_i = 1$ ,  $t_{j+1} - t_j = -1$ , and all other differences equal to 0. These elements constitute a proper subset of  $Y_{a,b}$ , so the presentation emerging from the degeneration  $R_{\mathcal{T}_0}(a, b)$  is perhaps not as efficient as Weyl's presentation, however it is better suited to aid in our description of  $V_{\mathcal{T}_0}(a, b)$ .

**1.3. Organization.** The paper is organized as follows. In Section 2, we review the toric fiber product construction and some of its properties. In Section 3, we review the classical Pieri rule from the representation theory of  $SL_m(\mathbb{C})$ . In Section 4, we study a valuation on the algebra of tensor product invariants which will help us to understand the semigroup algebra  $P(a, b)$ , we prove the  $K$ -Pieri rule from the classical Pieri rule, we prove Theorem 1.1. In Section 5, we construct  $P(a, b)$  as a set of weightings on an interlacing pattern.

## 2. TORIC FIBER PRODUCTS

In this section we review the toric fiber product operation on affine semigroup algebras. This is the "gluing" operation we will need to build the affine semigroup algebras  $R_{\mathcal{T}_0}(a, b)$ ,  $V_{\mathcal{T}_0}(a, b)$  from their component semigroup algebras. To this end, we will pay special attention to toric fiber products of simplicial cones over a simplicial cone base. We will review results from [S] and [M6] which control the algebraic behavior of these semigroups.

**Definition 2.1.** *For  $P_1 \subset \mathbb{R}^n$ ,  $P_2 \subset \mathbb{R}^k$ ,  $D \subset \mathbb{R}^m$  polyhedral cones, and  $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear maps with  $\pi_i(P_i) \subset D$ , the toric fiber product is the set,*

$$(18) \quad P_1 \times_D P_2 = \{(x, y) | \pi_1(x) = \pi_2(y)\} \subset \mathbb{R}^n \times \mathbb{R}^k$$

The fiber product  $P_1 \times_D P_2$  is also a polyhedral cone. If  $\mathcal{L}_1 \subset \mathbb{R}^n$ ,  $\mathcal{L}_2 \subset \mathbb{R}^k$ , and  $\mathcal{L} \subset \mathbb{R}^m$  are lattices, with  $\pi_i(\mathcal{L}_i) \subset \mathcal{L}$ , then the set  $\mathcal{L}_1 \times \mathcal{L}_2 \cap P_1 \times_D P_2$  is called the fiber product semigroup of the affine semigroups  $\mathcal{L}_1 \cap P_1$  and  $\mathcal{L}_2 \cap P_2$ . From now on we abuse notation and refer to the semigroup by its defining cone.

If the generators of the affine semigroups of these bodies behave well with respect to the maps  $\pi_1, \pi_2$ , then the affine semigroup of the associated toric fiber product also behaves well. The following can be found in [S], and a variant appears in [M6].

**Proposition 2.2.** *Let  $S_1 \subset P_1$ ,  $S_2 \subset P_2$ , and  $T \subset D$  denote the generating sets of the affine semigroups associated to polyhedral cones  $P_1, P_2, D$  with respect to lattices  $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}$ . Suppose that  $\pi_i(S_i) \subset T$ , and that  $D$  is a simplicial cone, then the set  $S_1 \times_T S_2$  generates the product semigroup  $P_1 \times_D P_2$ .*

*Proof.* For any semigroup element  $(x, y) \in P_1 \times_D P_2$ , we can factor  $x = s_1 + \dots + s_i$ ,  $y = t_1 + \dots + t_j$ . We then get factorizations,

$$(19) \quad \pi_1(x) = \pi_2(y) = \pi_1(s_1) + \dots + \pi_1(s_i) = \pi_2(t_1) + \dots + \pi_2(t_j).$$

The affine semigroup  $D$  has unique factorization, because it is a simplicial cone. It follows then that  $i = j$  and the sets  $\{\pi_1(s_1), \dots, \pi_1(s_i)\}$ ,  $\{\pi_2(t_1), \dots, \pi_2(t_j)\}$  are equal. The elements of the sets  $\{s_1, \dots, s_i\}$  and  $\{t_1, \dots, t_i\}$  can be matched in some way  $s_k \rightarrow t'_k$ , so that the resulting pairs have the same image under  $\pi_1, \pi_2$ . We therefore obtain a factorization,

$$(20) \quad (x, y) = (s_1, t'_1) + \dots + (s_i, t'_i).$$

□

Several of the elements  $\{s_1, \dots, s_i\}$  may have the same image under  $\pi_1$ , this could lead to distinct assignments  $s_k \rightarrow t'_k$  which produce different factorizations of the element  $(x, y)$ . These factorizations are related by swap relations. Let  $a, b \in P_1$ ,  $c, d \in P_2$  with  $\pi_1(a) = \pi_2(c) = \pi_1(b) = \pi_2(d)$ , then the following is a swap relation in  $P_1 \times_D P_2$ .

$$(21) \quad (a, c) + (b, d) = (b, c) + (a, d)$$

**Proposition 2.3.** *With the assumptions of Proposition 2.2, the relations of  $P_1 \times_D P_2$  are generated by those which generate relations for the  $P_1, P_2$  and the swap relations.*

*Proof.* We consider a relation in  $P_1 \times_D P_2$ ,

$$(22) \quad (x_1, y_1) + \dots + (x_k, y_k) = (X_1, Y_1) + \dots + (X_l, Y_l)$$

Once again, the sets images of these two sets of elements in  $D$  must agree, so  $k = l$ . The word  $x_1 + \dots + x_k$  can be transformed to  $X_1 + \dots + X_k$  by a series of generating relations in  $P_1$ . We claim that each such relation can be lifted to a relation on generators in  $P_1 \times_D P_2$ . If  $s_1 + \dots + s_m = S_1 + \dots + S_m$  is such a relation, then the image sets under  $\pi_1$  must agree, so each relation can be performed without changing the set  $\{\pi_1(x_1), \dots, \pi_1(x_k)\}$ . This implies that at each stage, these sets can be assigned to elements in the set  $\{y_1, \dots, y_m\}$  to form elements from the fiber product. It follows that  $(x_1, y_1) + \dots + (x_k, y_k)$  can be transformed to  $(X'_1, Y'_1) + \dots + (X'_k, Y'_k)$  by relations lifted this way from  $P_1$  and  $P_2$ , where the  $X'_i$  and  $Y'_i$  are the  $X_i$  and  $Y_i$  matched in a different way. Now this word can be rearranged into  $(X_1, Y_1) + \dots + (X_k, Y_k)$  using swap relations. □

In [S] and [M6] it is shown that if the ideals defining  $\mathbb{C}[P_1], \mathbb{C}[P_2]$  under presentation by  $S_1, S_2$  have quadratic, square-free Gröbner bases, then so does the ideal defining  $\mathbb{C}[P_1 \times_D P_2]$ . In particular, if  $P_1, P_2, D$  are simplicial cones with  $\pi_i(S_i) \subset T$ , the ideal defining  $\mathbb{C}[P_1 \times_D P_2]$  has a quadratic, square-free Gröbner basis made of swap relations. From this we can deduce the following proposition.

**Proposition 2.4.** *Let  $P = \Delta_{k_1} \times_{\Delta_{m_1}} \dots \times_{\Delta_{m_\ell}} \Delta_{k_{\ell+1}}$  be a fiber product of simplicial cones over simplicial cones, by maps which take generators to generators. Then the swap relations define a quadratic, square-free Gröbner basis on the defining ideal of  $\mathbb{C}[P]$  under the presentation by the fiber product generating set.*

Affine semigroup algebras are appealing because their inherent combinatorial nature allows many of their commutative algebra properties to be expressed in terms of polyhedral geometry. The following is Corollary 6.3.8 of [BH].

**Theorem 2.5.** *Let  $C$  be an affine semigroup algebra, with  $\text{int}(C) \subset C$  the set of interior lattice points. The algebra  $\mathbb{C}[C]$  is Gorenstein if and only if  $\text{int}(C) = w + C$  for some  $w \in \text{int}(C)$ .*

In [M10], we gave the following condition for a fiber product semigroup algebra of Gorenstein semigroups to be Gorenstein. The conditions of this proposition are satisfied for the semigroups we consider in this paper, and indeed are satisfied for any fiber product of simplices over simplices, where the maps in question are surjective on generating sets.

**Proposition 2.6.** *Let  $P_1, P_2, D$  be semigroups with Gorenstein semigroup algebras, and let  $w_1, w_2, w$  be the corresponding generators of the interior sets. Then if  $\pi_i(w_i) = w$ , the semigroup algebra  $P_1 \times_D P_2$  is Gorenstein.*



## 3. THE CLASSICAL PIERI RULE

In this section we describe the algebra  $R(a, b)$  in terms of the representation theory of  $SL_m(\mathbb{C})$ , and an associated flat degeneration to an affine semigroup algebra  $\mathbb{C}[Q(a, b)]$ . In order to define the affine semigroup  $Q(a, b)$  we will recall the Pieri rule for decomposing tensor products of  $SL_m(\mathbb{C})$  representations, and describe how this rule informs the combinatorial commutative algebra of  $R(a, b)$ .

**3.1. The algebra of  $SL_m(\mathbb{C})$  invariant tensors.** First we introduce the algebra  $R_n(SL_m(\mathbb{C}))$  of invariant vectors in  $n$ -fold tensor products of irreducible  $SL_m(\mathbb{C})$  representations. We let  $R_{SL_m(\mathbb{C})}$  denote the Cox ring of the full flag variety of  $SL_m(\mathbb{C})$ , as a vector space, this algebra is a multiplicity-free direct sum of all irreducible representations of  $SL_m(\mathbb{C})$ .

$$(23) \quad R_{SL_m(\mathbb{C})} = \bigoplus_{\lambda \in \Delta} V(\lambda)$$

The multiplication operation in this algebra is given by projecting on the highest weight component of a tensor product.

$$(24) \quad V(\lambda) \otimes V(\eta) \rightarrow V(\lambda + \eta)$$

This algebra is multigraded by the dominant weights  $\lambda \in \Delta$ , and can also be constructed as the algebra of invariants in  $\mathbb{C}[SL_m(\mathbb{C})]$  with respect to the unipotent group of upper triangular matrices. We define  $R_n(SL_m(\mathbb{C}))$  to be the algebra of  $SL_m(\mathbb{C})$  invariants in an  $n$ -fold tensor product of copies of  $R_{SL_m(\mathbb{C})}$ . As a vector space, this is a direct sum of all invariant spaces in  $n$ -fold tensor products of  $SL_m(\mathbb{C})$  representations.

$$(25) \quad R_n(SL_m(\mathbb{C})) = \bigoplus_{\vec{\lambda} \in \Delta^n} [V(\vec{\lambda})]^{SL_m(\mathbb{C})}$$

The coordinate rings of  $\bigwedge^{m-1}(\mathbb{C}^m)$  and  $\mathbb{C}^m$  are graded subalgebras of  $R_{SL_m(\mathbb{C})}$ , they can be identified with the sums of those representations of the form  $V(r\omega_1)$  and  $V(s\omega_{m-1})$ , respectively. It follows that the algebra  $R(a, b)$  is a multigraded subalgebra of  $R_{a+b}(SL_m(\mathbb{C}))$ .

In [M5] and [M4] we describe a flat degeneration of  $R_n(SL_m(\mathbb{C}))$  for each trivalent tree  $\mathcal{T}$  with  $n$  leaves. In [M4], it is shown that the lift of these degenerations to  $R_n(SL_m(\mathbb{C})) \otimes \mathbb{C}[t]$  is compatible with the corresponding degeneration on  $V_{\vec{p}}(a, b)$  under the inclusion map in Equation 17. In particular, for the tree  $\mathcal{T}_0$ ,  $R_n(SL_m(\mathbb{C}))$  degenerates to the algebra of invariants  $[R_3(SL_m(\mathbb{C}))^{\otimes n-2}]^{T^{n-3}}$ , where  $T^{n-3}$  is a product of  $n-3$  copies of the maximal torus  $\mathbb{T} \subset SL_m(\mathbb{C})$ , acting as in Theorem 1.2.

In order to pass this description to  $R(a, b)$ , we define four subalgebras of  $R_3(SL_m(\mathbb{C}))$ . The algebras  $R_3(B, P, B)$  and  $R_3(B, P^*, B)$  are the multigraded subalgebras of  $R_3(SL_m(\mathbb{C}))$  whose components have middle dominant weight equal to a multiple of  $\omega_1$  and  $\omega_{m-1}$ , respectively. The algebras  $R_3(P, P, B)$  and  $R_3(B, P^*, P^*)$  are defined accordingly. We call the algebras  $R_3(B, P, B)$ ,  $R_3(B, P^*, B)$ ,  $R_3(P, P, B)$ , and  $R_3(B, P^*, P^*)$  the Pieri algebras. Theorem 1.11 from [M4] implies that  $R(a, b)$  has a flat degeneration to the following invariant algebra.

$$(26) \quad R_{\mathcal{T}_0}(a, b) = [R_3(P, P, B) \otimes [R_3(B, P, B)]^{\otimes a-2} \otimes [R_3(B, P^*, B)]^{\otimes b-2} \otimes R_3(B, P^*, P^*)]^{T^{a+b-3}}$$

Here, once again,  $T^{a+b-3}$  is a product of  $a+b-3$  copies of the maximal torus of  $SL_m(\mathbb{C})$ , and taking invariants by  $T$  ensures that all graded components of  $R_{\mathcal{T}_0}(a, b)$  are of the following form.

$$(27) \quad \begin{aligned} & V(r_1\omega_1, r_2\omega_1, \lambda_1)^{SL_m(\mathbb{C})} \otimes V(\lambda_1^*, r_3\omega_1, \lambda_2)^{SL_m(\mathbb{C})} \otimes \dots \\ & \dots \otimes V(\lambda_{a-2}^*, r_a\omega_1, \lambda_{a-1})^{SL_m(\mathbb{C})} \otimes V(\lambda_{a-1}^*, s_1\omega_{m-1}, \lambda_a)^{SL_m(\mathbb{C})} \otimes \dots \\ & \dots \otimes V(\lambda_{a+b-4}^*, s_{b-2}\omega_{m-1}, \lambda_{a+b-3})^{SL_m(\mathbb{C})} \otimes V(\lambda_{a+b-3}^*, s_{b-1}\omega_{m-1}, s_b\omega_{m-1})^{SL_m(\mathbb{C})} \end{aligned}$$

The following is the classical Pieri rule, it is a basic component of  $SL_m(\mathbb{C})$  representation theory. We refer the reader to any basic text covering this subject, for example the book of Fulton and Harris, [FH].

**Proposition 3.1.** [The Pieri rule] *The invariant space  $V(\lambda, r\omega_1, \eta)^{SL_m(\mathbb{C})}$  is multiplicity free, and it has dimension 1 precisely when there is a  $GL_m(\mathbb{C})$  weight  $\bar{\lambda}$  such that the following hold.*

- (1)  $\lambda^* = \bar{\lambda} - \bar{\lambda}_m \omega_m$
- (2)  $\sum_j \bar{\lambda}_j - \sum_i \eta_i = r$
- (3)  $\bar{\lambda}_i - \eta_i, \eta_i - \bar{\lambda}_{i+1} \geq 0$

Weights which satisfy these conditions can be arranged in interlacing patterns, as depicted in Figure 2.

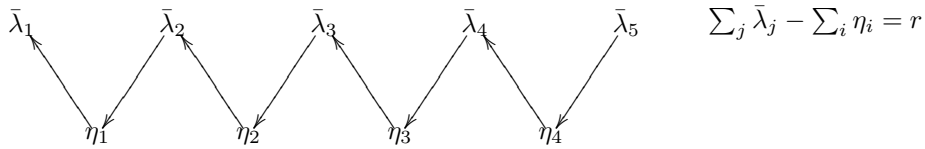


FIGURE 2. Interlacing patterns for  $R_3(B, P, B)$

Interlacing conditions governing the spaces  $V(\lambda, s\omega_{m-1}, \eta)^{SL_m(\mathbb{C})}$  can be derived by duality. For this space to have dimension 1, there must be a  $GL_m(\mathbb{C})$  weight  $\bar{\eta}$  such that the following hold, see Figure 3.

- (1)  $\eta = \bar{\eta} - \bar{\eta}_m \omega_m$
- (2)  $\sum_j \bar{\eta}_j - \sum_i \lambda_i^* = s$
- (3)  $\bar{\eta}_i - \lambda_i^*, \lambda_i^* - \bar{\eta}_{i+1} \geq 0$

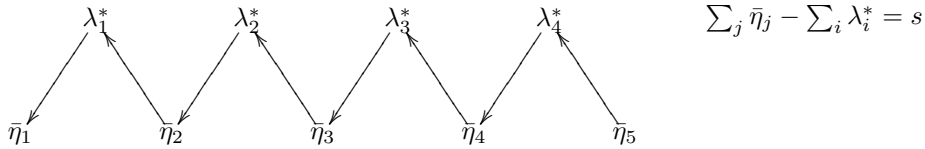


FIGURE 3. Interlacing patterns for  $R_3(B, P^*, B)$

We define boundary maps  $\partial_1, \partial_2$  for these patterns as in the  $K$ -Pieri case. The algebra  $R_3(P, P, B)$  is composed of components of the form  $V(r_1\omega_1, r_2\omega_1, \eta)^{SL_m(\mathbb{C})}$ , following the recipe above, the interlacing inequalities give us patterns as in Figure 4. Dual patterns are depicted in Figure 5 for spaces of the form  $V(\lambda, s_1\omega_{m-1}, s_2\omega_{m-1})^{SL_m(\mathbb{C})}$ .

Notice that both of these diagrams are determined by 3 parameters. Since each multigraded component of the Pieri algebras is dimension 0 or 1, these algebras are all affine semigroup algebras, we refer to their underlying semigroups by  $Q(B, P, B)$ ,  $Q(B, P^*, B)$ ,  $Q(P, P, B)$ , and  $Q(B, P^*, P^*)$ . The following proposition shows that each of these semigroups has a simple structure.

**Proposition 3.2.** *The algebras  $R_3(B, P, B)$  and  $R_3(B, P^*, B)$  are polynomial algebras on  $2m - 1$  variables. The algebras  $R_3(P, P, B)$  and  $R_3(B, P^*, P^*)$  are polynomial algebras on 3 variables.*

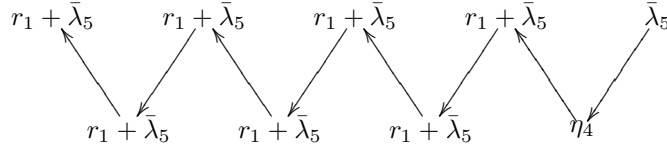


FIGURE 4. Interlacing diagram when two weights are rank 1, here  $\bar{\lambda}_5 = \frac{1}{2}(r_2 + \eta_4 - r_1)$

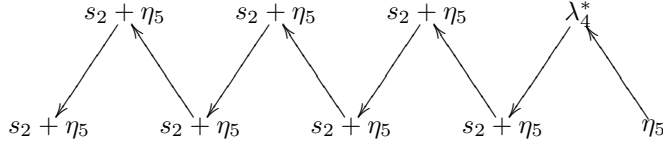


FIGURE 5. Interlacing diagram when two weights are rank 1, here  $\eta_5 = \frac{1}{2}(s_1 + \lambda_4^* - s_2)$

*Proof.* Duality is a linear operation on dominant weights, so it follows that the statement for the algebras  $R_3(B, P^*, B)$  and  $R_3(B, P^*, P^*)$  is a consequence of the statement for  $R_3(B, P, B)$  and  $R_3(P, P, B)$  respectively, so we present proofs for these cases. From Figure 4 we directly compute that the algebra  $R_3(P, P, B)$  is generated by the patterns in Figure 6.

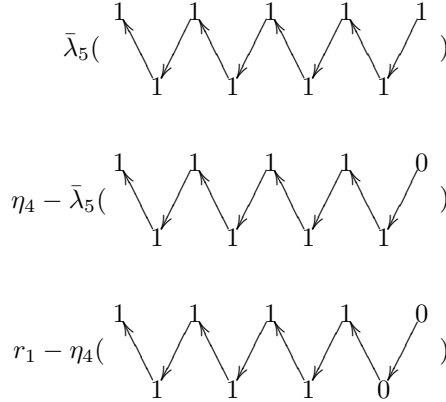


FIGURE 6. Decomposition of an element from  $Q(P, P, B)$

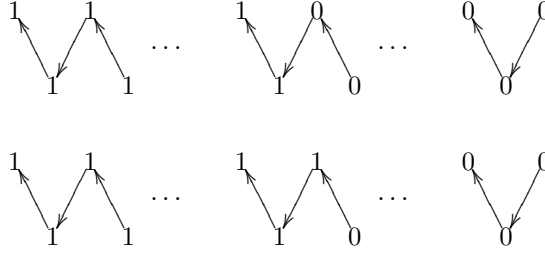
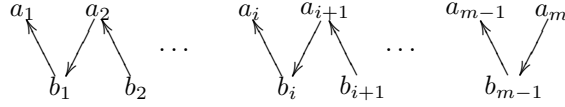
Any element of  $R_3(P, P, B)$  is a unique product of the elements represented by these patterns. We label these elements by recording their boundary fundamental  $GL_m(\mathbb{C})$  weights, from top down they are  $[m, m - 1]$ ,  $[m - 1, m - 1]$ , and  $[m - 1, m - 2]$ . The identity  $1 \in R_3(P, P, B)$  corresponds to the trivial invariant in  $V(0, 0, 0)$ , we label this element with  $[0, 0]$ . In general, we let  $[i, i]$  and  $[i + 1, i]$  denote the patterns in Figure 7.

An interlacing pattern  $\mathbf{b}$  is decomposed uniquely into a sum of these patterns by finding the smallest non-zero entry, and pulling the pattern above with the first 1 occurring at this entry off until this entry is 0.

$$(28) \quad \mathbf{b} = (a_m - 0)[m, m - 1] + (b_{m-1} - a_{m-1})[m - 1, m - 1] + \dots + (a_1 - \sum_{j>1} a_j + b_{j-1})[1, 0]$$

□

The indices  $i, j$  in the generators  $[i, j]$  of these semigroups range between 1 and  $m$ . As  $m$  corresponds to the trivial dominant weight  $(1, \dots, 1)$ , we consider these elements as members of  $\mathbb{Z}/m\mathbb{Z}$ , in particular  $[0, 0] = [m, m]$ . The multigraded components of  $R_{\mathcal{T}_0}(a, b)$  all have dimension 1 or 0, and have dimension

FIGURE 7.  $[i, i]$  (top) and  $[i + 1, i]$  (bottom),  $1 \leq i \leq m - 1$ FIGURE 8. An interlacing pattern  $\mathbf{b}$ .

1 exactly when each part of the tensor product satisfies the conditions of Proposition 3.1. We use this fact to describe an affine semigroup  $Q(a, b)$  such that  $R_{\mathcal{T}_0}(a, b) = \mathbb{C}[Q(a, b)]$ . We define  $Q(a, b)$  to be the affine semigroup obtained by taking the following fiber product over the boundary maps  $\partial_1, \partial_2$ . The image of these boundary maps is the affine semigroup for the simplicial cone  $\Delta_{m-1}$  with  $m - 1$  generators.

$$(29) \quad \begin{aligned} Q(a, b) &= Q(P, P, B) \times_{\Delta_{m-1}} Q(B, P, B) \times_{\Delta_{m-1}} \cdots \\ &\cdots \times_{\Delta_{m-1}} Q(B, P, B) \times_{\Delta_m} Q(B, P^*, B) \times_{\Delta_{m-1}} \cdots \\ &\cdots \times_{\Delta_{m-1}} Q(B, P^*, B) \times Q(B, P^*, P^*) \end{aligned}$$

The boundary maps  $\partial_1, \partial_2$  were defined to capture the operation of taking invariants by the torus  $T^{a+b-3}$ , so the semigroup algebra of  $Q(a, b)$  can be identified with  $R_{\mathcal{T}_0}(a, b)$  by definition. Proposition 2.2 then implies that elements composed of generators of Pieri semigroups suffice to generate  $Q(a, b)$ , and therefore  $R_{\mathcal{T}_0}(a, b)$ . The semigroups  $Q(P, P, B)$ ,  $Q(B, P, B)$ ,  $Q(B, P^*, B)$ , and  $Q(B, P^*, P^*)$  are generated by elements labelled by pairs  $[i, j]$ , where  $i, j$  are elements of  $\mathbb{Z}/m\mathbb{Z}$ . It is easily checked from the definition of the boundary maps  $\partial_1, \partial_2$  that two of these generators  $[i, j], [k, l]$  can be glued when  $j = k$ . This implies that a generating set of  $Q(a, b)$  is given by  $a + b - 1$  tuples of elements from  $\mathbb{Z}/m\mathbb{Z}$ ,

$$(30) \quad [i_1, i_2, \dots, i_{a+b-2}, i_{a+b-1}]$$

where  $i_1$  and  $i_{a+b-1}$  can be  $m$  or  $m - 1$ , the first  $a - 1$  differences  $i_j - i_{j+1}$  must be 0 or 1 and the last  $b - 1$  differences must be 0 or  $-1$ , all of these conditions holding modulo  $m$ . By Proposition 2.3 the relations on these generators are generated by the swap relations. Since the element  $[0, \dots, 0]$  represents  $1 \in R(a, b)$ , the swap relations tell us that an element of the form  $[i_1, \dots, i_k, 0, i_{k+2}, \dots, i_{a+b-1}]$  may be factored,

$$(31) \quad [i_1, \dots, i_k, 0, i_{k+2}, \dots, i_{a+b-1}][0, \dots, 0, \dots, 0] = [i_1, \dots, i_{k+2}, 0, \dots, 0][0, \dots, 0, i_{k+2}, \dots, i_{a+b-1}]$$

From this it follows that  $Q(a, b)$  is generated by unbroken strings of nonzero entries from  $\mathbb{Z}/m\mathbb{Z}$ .

#### 4. STRUCTURE OF $V_{\mathcal{T}_0}(a, b)$

In this section we study the four  $K$ -Pieri algebras,  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ ,  $V_{0,3}(P, P, B)$ ,  $V_{0,3}(B, P^*, P^*)$ . We will use the results of the previous section to prove that each of these algebras is a polynomial ring, and then use this fact to find a presentation of the algebra  $V_{\mathcal{T}_0}(a, b)$ .

**4.1. Conformal blocks, invariant tensors, and the level valuation.** Conformal blocks  $V_{\mathbb{P}^1, \vec{p}}(\vec{\lambda}, K)$  in the genus 0 case can be recovered as a subspace of the space of  $SL_m(\mathbb{C})$ -invariant vectors in the tensor product  $V(\vec{\lambda}) = V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$ . The precise subspace depends on the data  $\vec{p} \subset \mathbb{P}^1$ . The moduli of 3-marked projective lines is a single point, so in this case there is a unique space of conformal blocks,  $V_{0,3}(\lambda, \eta, \mu, K)$ , which can be explicitly defined as a subspace of  $[V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{SL_m(\mathbb{C})}$  using elements of the classical representation theory of  $SL_m(\mathbb{C})$ .

We consider the copy of  $SL_2(\mathbb{C})$  inside  $SL_m(\mathbb{C})$  which corresponds to the longest root  $\alpha_{1m} = \theta$ . Each representation  $V(\lambda), V(\eta), V(\mu)$  can be decomposed into isotypical components of this subgroup  $V(\lambda) = \bigoplus_{i \geq 0} W_{\lambda, i}$ . In [U], Ueno defines the following subspace  $W_K \subset V(\lambda) \otimes V(\eta) \otimes V(\mu)$ .

$$(32) \quad W_K = \bigoplus_{i+j+k \leq 2K} W_{\lambda, i} \otimes W_{\eta, j} \otimes W_{\mu, k}$$

The following proposition can also be found in [U].

**Proposition 4.1.** *The space of conformal blocks  $V_{0,3}(\lambda, \eta, \mu, K)$  can be identified with the space  $W_K \cap [V(\lambda) \otimes V(\eta) \otimes V(\mu)]^{SL_m(\mathbb{C})}$ .*

Proposition 4.1 can be used to define a function  $v_\theta : R_3(SL_m(\mathbb{C})) \rightarrow \mathbb{R}$  as follows. On pure graded components of  $R_3(SL_m(\mathbb{C}))$  we define  $v_\theta$  on  $T \in V(\lambda, \eta, \mu)^{SL_m(\mathbb{C})}$ , using the subspace  $W_K$ ,

$$(33) \quad v_\theta(T) = \min\{K | T \in W_K\},$$

this is extended to mixed grade components using the *max* function,

$$(34) \quad T = \sum T_i, T_i \in V(\lambda_i, \eta_i, \mu_i), v_\theta(T) = \max\{\dots, v_\theta(T_i), \dots\}.$$

By convention we set  $v_\theta(0) = -\infty$ . This function was studied in [M10], where it was shown to be a valuation on  $R_3(SL_m(\mathbb{C}))$ , see also page 87 in [G]. This means that  $v_\theta$  has the following properties.

- (1)  $v_\theta(ab) = v_\theta(a) + v_\theta(b)$
- (2)  $v_\theta(a + b) \leq \max\{v_\theta(a), v_\theta(b)\}$
- (3)  $v_\theta(C) = 0, C \in \mathbb{C}^*$

In terms of this function, the graded components of  $V_{0,3}(SL_m(\mathbb{C}))$  are the following subspaces of the graded components of  $R_3(SL_m(\mathbb{C}))$ .

$$(35) \quad V_{0,3}(\lambda, \eta, \mu, K) = \{T | v_\theta(T) \leq K\} \subset [V(\lambda^*) \otimes V(\eta^*) \otimes V(\mu^*)]^{SL_m(\mathbb{C})}$$

There are corresponding inclusions of the  $K$ -Pieri algebras into polynomial rings over the Pieri algebras.

$$(36) \quad V_{0,3}(B, P, B) \subset R_3(B, P, B) \otimes \mathbb{C}[t]$$

$$(37) \quad V_{0,3}(B, P^*B) \subset R_3(B, P^*, B) \otimes \mathbb{C}[t]$$

$$(38) \quad V_{0,3}(P, P, B) \subset R_3(P, P, B) \otimes \mathbb{C}[t]$$

$$(39) \quad V_{0,3}(B, P^*, P^*) \subset R_3(B, P^*, P^*) \otimes \mathbb{C}[t]$$

**4.2. Generators of the  $K$ -Pieri algebras.** Now our task is to establish explicitly the  $v_\theta$  value for a non-trivial vector in each graded component of the Pieri algebras, this will allow us to compute a presentation for the  $K$ -Pieri algebras. In order to do this, we find  $v_\theta(T)$  for each generator  $T$  of  $R_3(B, P, B)$ ,  $R_3(B, P^*, B)$ ,  $R_3(P, P, B)$ , and  $R_3(B, P^*, P^*)$ , and use this calculation to find the  $v_\theta$  value for each monomial in the generators.

The generators of the Pieri algebras are precisely the unique invariants in tensor products of the form  $\bigwedge^k(\mathbb{C}^m) \otimes \mathbb{C}^m \otimes \bigwedge^j(\mathbb{C}^m)$  and  $\bigwedge^k(\mathbb{C}^m) \otimes \bigwedge^{m-1}(\mathbb{C}^m) \otimes \bigwedge^j(\mathbb{C}^m)$ , where  $k + j + 1 \in m\mathbb{Z}$ , and  $\bigwedge^k(\mathbb{C}^m) \otimes \bigwedge^j(\mathbb{C}^m)$ , where  $k + j \in m\mathbb{Z}$ . Note that the sum of these indices must be equal to  $2m$  or  $m$ . Both invariant types can be viewed as a representation of the determinant, and have the following form as alternating tensors.

$$(40) \quad \begin{aligned} T_{i,1,j} &= \sum_{|I|=i, |J|=j} (-1)^{\sigma(I,k,J)} z_I \otimes z_k \otimes z_J \\ T_{i,m-1,j} &= \sum_{|I|=i, |K|=m-1, |J|=j} (-1)^{\sigma(I,K,J)} z_I \otimes z_K \otimes z_J \\ P_{i,j} &= \sum_{|I|=i, |J|=j} (-1)^{\sigma(I,J)} z_I \otimes z_J \end{aligned}$$

Here  $z_I$  is a wedge product of basis vectors over the indicated index set, and  $(-1)^{\sigma(I,K,J)}$  is a sign function. The following is well known in the type  $A$  conformal blocks literature.

**Proposition 4.2.** *Each generating invariant of the Pieri algebras has  $v_\theta$  value equal to 1.*

*Proof.* To evaluate  $v_\theta$  on these elements, we reduce to the case where the sum of the indices is  $m$ , as the  $2m$  case is related to this case by duality. In this case, each of the terms in the expansion above is of the form  $z_I \otimes z_K \otimes z_J$  with  $I \amalg J \amalg K = [m]$ .

When this term is considered with respect to the  $\theta$  subgroup isomorphic to  $SL_2(\mathbb{C})$ , its properties are determined by whether or not the indices  $1, n$  show up in the sets  $I, J, K$ . There are two possibilities. If  $1$  and  $n$  are in separate index sets, the term corresponds to an invariant in the  $SL_2(\mathbb{C})$  representation  $V(1) \otimes V(1)$ , and therefore has  $v_\theta$  value  $1 = \frac{1}{2}(1+1)$ . If both  $1$  and  $n$  appear in the same index set, the term is the trivial invariant in  $V(0) = \bigwedge^2(V(1))$ , and has  $v_\theta$  value  $0$ . The value on the sums of these terms is therefore  $1$ .  $\square$

We focus briefly on a particular  $K$ -Pieri algebra,  $V_{0,3}(B, P, B)$ . We associate each generator  $[i, i], [i+1, i]$  to the elements  $[i, i]t, [i+1, i]t \in V_{0,3}(B, P, B) \subset R_3(B, P, B) \otimes \mathbb{C}[t]$ . Now consider a non-trivial element  $\mathbf{b}t^K \in V_{0,3}(\eta, r\omega_1, \lambda, K) \subset (V(\eta, r\omega_1, \lambda)^{SL_m(\mathbb{C})})t^K$ . The invariant represented by  $\mathbf{b}$  has a monomial decomposition into the generators above.

$$(41) \quad \mathbf{b} = \prod [i+1, i]^{n_i} [j, j]^{m_j}$$

Multiplicative properties of valuations then give us  $v_\theta(\mathbf{b}) = \sum n_i + \sum m_j$ . This implies that in the  $K$ -Pieri algebra we have

$$(42) \quad \mathbf{b}t^K = \prod ([i+1, i]t)^{n_i} ([j, j]t)^{m_j} t^{K - \sum n_i - \sum m_j}.$$

The same argument applies for the other  $K$ -Pieri algebras. As a consequence we obtain the following theorem.

**Theorem 4.3.** *The algebras  $V_{0,3}(B, P, B)$  and  $V_{0,3}(B, P^*, B)$  are polynomial algebras on  $2m$  generators. The algebras  $V_{0,3}(P, P, B)$  and  $V_{0,3}(B, P^*, P^*)$  are polynomial algebras on  $4$  generators.*

The generators of these algebras are the generators of the corresponding Pieri algebras, along with the identity  $[0, 0]$ , multiplied by the auxiliary variable  $t$ . This also gives the following characterization of the non-zero spaces  $V_{0,3}(\lambda, r\omega_1, \mu, L)$ .

**Corollary 4.4.** *The space of conformal blocks  $V_{0,3}(\lambda, r\omega_1, \mu, K)$  is multiplicity free, and dimension 1 if and only if the conditions in Proposition 1.4 hold.*

*Proof.* This follows from equation 42 and the fact that the  $v_\theta$  of the patterns  $[i, i]$ ,  $[i + 1, i]$  are given by the first entry on the top row.

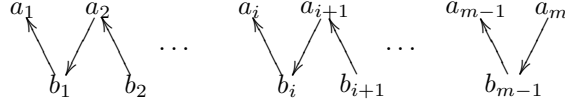


FIGURE 9.

$$(43) \quad \mathbf{b} = (a_m - 0)[m, m - 1] + (b_{m-1} - a_{m-1})[m - 1, m - 1] + \dots + (a_1 - \sum_{j>1} a_j + b_{j-1})[1, 0]$$

□

**4.3. Fiber product structure of  $P(a, b)$ .** We are now in a position to construct the affine semigroup  $P(a, b)$ . As polynomial algebras, the  $K$ -Pieri algebras  $V_{0,3}(B, P, B)$ ,  $V_{0,3}(B, P^*, B)$ ,  $V_{0,3}(P, P, B)$ , and  $V_{0,3}(B, P^*, P^*)$ , are affine semigroup algebras on simplicial cones, with  $2m, 2m, 4, 4$  generators respectively. We construct  $P(a, b)$  as a toric fiber product of the corresponding semigroups following a procedure similar to the construction of  $Q(a, b)$ . The only extra detail is that, in addition to the boundary maps  $\partial_1, \partial_2$  on the interlacing patterns for these algebras, we must also include the level data. This implies that two interlacing patterns can be glued only if they share the same boundary weight data and level. The semigroup  $P(a, b)$  is obtained as fiber product over the affine semigroup of the simplicial cone  $\Delta_m$  with  $m$  generators.

$$(44) \quad \begin{aligned} P(a, b) = & P_3(P, P, B) \times_{\Delta^m} \times P_3(B, P, B) \times_{\Delta^m} \times \dots \\ & \dots \times_{\Delta^m} P_3(B, P, B) \times_{\Delta^m} P_3(B, P^*) \times_{\Delta^m} \dots \\ & \dots \times_{\Delta^m} P_3(B, P^*, B) \times_{\Delta^m} P_3(B, P^*, P^*) \end{aligned}$$

By Proposition 2.2, a generating set of  $P(a, b)$  is obtained by taking all tuples of level 1 generating elements  $([i_1, j_1], \dots, [i_{a+b-2}, j_{a+b-2}])$ , where  $j_k = i_{k+1}$ . Note that, as all generators are tuples of level 1 elements, and any product of level 1 elements is a level 2 element, this is already a minimal generating set of  $P(a, b)$ . By the descriptions of the Pieri algebra generators above, the generators of  $P(a, b)$  are exactly those  $a + b - 1$  tuples which satisfy the following properties, all elements are members of  $\mathbb{Z}/m\mathbb{Z}$ .

- (1) The difference  $i_k - i_{k+1}$  is  $\{1, 0\}$  for  $1 \leq k \leq a - 1$ .
- (2) The difference  $i_k - i_{k+1}$  is  $\{-1, 0\}$  for  $a \leq k \leq a + b - 2$ .
- (3) The first and last entries are members of  $\{m - 1, m\}$ .

By Proposition 2.3, the relations for  $P(a, b)$  are generated by the swap relations among these generators. This gives the presentation of  $V_{\mathcal{T}_0}(a, b)$ , and completes the proof of Theorem 1.9. Note that Propositions 2.6 and 2.4 also imply that both  $V_{\mathcal{T}_0}(a, b)$  and  $R_{\mathcal{T}_0}(a, b)$  are Gorenstein algebras with quadratic, square-free Gröbner bases. This in turn implies the following.

**Theorem 4.5.** *The algebras  $R(a, b)$  and  $V_{\vec{p}}(a, b)$  are Gorenstein, Koszul algebras when  $\vec{p}$  is generic.*

Furthermore, since  $V_{\vec{p}}(a, b)$  is graded by the level data, and the only elements of level 0 are multiples of the identity  $1 \in V_{\vec{p}}(a, b)$ , the level 1 elements constitute a minimal generating set of this algebra.

5. INTERLACING PATTERNS

In this section we describe elements of the fiber product semigroups  $P(a, b)$  and  $Q(a, b)$  as labellings of interlacing diagrams. We have defined the interlacing patterns for the  $K$ -Pieri and Pieri semigroups and their boundary maps  $\partial_1, \partial_2$  specifically with the gluing operation in mind. We slightly modify these boundary maps, after taking  $\partial_1$ , or  $\partial_2$ , we always then pass from a positive  $GL_m(\mathbb{C})$  weight to the underlying  $SL_m(\mathbb{C})$  weight.

We consider a tensor product,  $V(\lambda, r_1\omega_1, \eta)^{SL_m(\mathbb{C})} \otimes V(\eta^*, r_2\omega_1, \mu)^{SL_m(\mathbb{C})}$ . By the Pieri rule, this space is non-zero if and only if there are two interlacing patterns  $\mathbf{b}_1, \mathbf{b}_2$  such that  $\partial_1(\mathbf{b}_1) = \eta, \partial_2(\mathbf{b}_1) = \lambda, \partial_1(\mathbf{b}_2) = \mu$ , and  $\partial_2(\mathbf{b}_2) = \eta^*$ . These conditions can be represented with a 2-step interlacing diagram with entries as depicted in Figure 5.

$$\frac{1}{5}(r_1 + r_2 + \sum_j \mu_j + \sum_i \lambda_i - 5\lambda_1) = \bar{\lambda}_5$$

$$\frac{1}{5}(r_2 + \sum_j \mu_j - \sum_i \eta_i) = \bar{\eta}_5$$

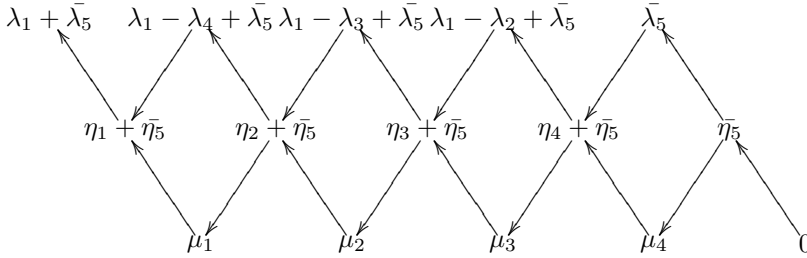


FIGURE 10. gluing interlacing patterns

Notice that the top two rows of this diagram have been modified from what they would have been had we considered the space  $V(\lambda, r_1\omega_1, \eta)^{SL_m(\mathbb{C})}$  by itself. In particular, the the following diagram has been added to the top two rows.

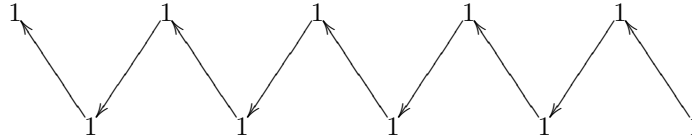


FIGURE 11. The null diagram

This does not change the boundary values of the pattern as it represents the trivial tensor product invariant in  $\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$ . This operation is performed to make the gluing procedure easier, namely one pattern is modified until its boundary weights match, which is always possible when  $\partial_1(\mathbf{b}_1) = \partial_2(\mathbf{b}_2)^*$ . Then it is simply stacked on top of the lower pattern.

Patterns as in Figure 5 label the graded components of the algebra  $[R(B, P, B) \otimes R(B, P, B)]^T$ . To represent more complicated tensor products, we simply stack the appropriate patterns, taking care to modify pairs of rows by the null pattern to correctly identify the weights when necessary. We define  $L_m(a, b)$  to be the "wedge" interlacing pattern in Figure 12, with rows of length  $m$ ,  $a$  steps before the switch in direction, and  $b$  steps after the switch in direction.

The next two propositions then follow from our combinatorial descriptions of the algebras  $R_{\mathcal{T}_0}(a, b)$  and  $V_{\mathcal{T}_0}(a, b)$ . The level condition in Proposition 1.4 has been altered to account for the possible addition of the null diagram to a pair of rows.

**Proposition 5.1.** *There is a basis of the space of invariants  $V(\vec{r}\omega_1, \vec{s}\omega_{m-1})^{SL_m(\mathbb{C})}$  in bijection with the labellings of  $L_m(a, b)$  which satisfy the following properties.*



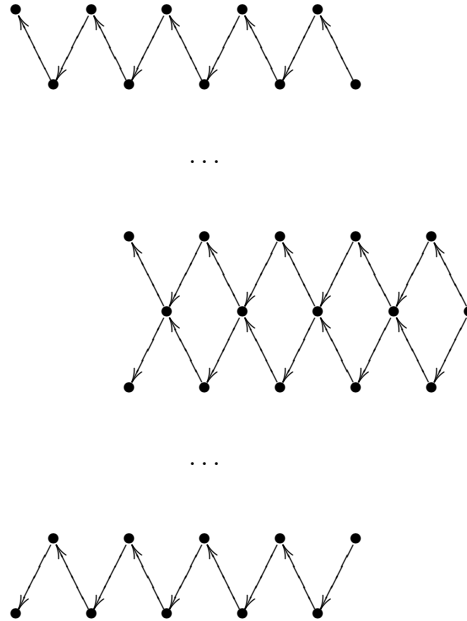


FIGURE 12. The diagram  $L_m(a, b)$

- (1) The boundary value of the top row is  $r_1\omega_1$ .
- (2) The boundary value of the bottom row is  $s_b\omega_{m-1}$ .
- (3) The difference between the sums of the  $i$  and  $i + 1$ st rows is  $r_{i+1}$  if  $1 \leq i \leq a$ , and  $s_{i+1}$  if  $1 \leq i \leq a + b$

**Proposition 5.2.** *There is a basis of the space  $V_{\vec{p}}(\vec{r}\omega_1, \vec{s}\omega_{m-1}, K)$  in bijection with the labellings of  $L_m(a, b)$  which satisfy the following properties.*

- (1) The boundary value of the top row is  $r_1\omega_1$ .
- (2) The boundary value of the bottom row is  $s_b\omega_{m-1}$ .
- (3) The difference between the sums of the  $i$  and  $i + 1$ st rows is  $r_{i+1}$  if  $1 \leq i \leq a$ , and  $s_{i+1}$  if  $1 \leq i \leq a + b$
- (4) For rows  $1 \leq i \leq a$ , the difference between the first entry of row  $i$  and the last entry of row  $i + 1$  is  $\leq K$ .
- (5) For rows  $a + 1 \leq i \leq a + b$ , the difference between the last entry of row  $i$  and the first entry of row  $i + 1$  is  $\leq K$ .

The labellings in Proposition 5.2 are the lattice points in a polytope  $P(\vec{r}, \vec{s}, K)$ . The graded affine semigroup algebra determined by this polytope,  $\mathbb{C}[P(\vec{r}, \vec{s}, L)]$  is the degeneration of the projective coordinate  $R_{\vec{p}}(\vec{r}, \vec{s}, L)$  induced from the degeneration of  $V_{\vec{p}}(a, b)$  to  $V_{\mathcal{T}_0}(a, b)$ . This algebra is considerably more complicated than  $V_{\mathcal{T}_0}(a, b)$ , it would be interesting to find a generating set of lattice points for this semigroup, or a classification of which  $P(\vec{r}, \vec{s}, L)$  yield a Gorenstein algebra.

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Christopher Manon:  
 Department of Mathematics,  
 George Mason University  
 Fairfax, VA 22030 USA