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ON MEASURE AND HAUSDORFF DIMENSION OF JULIA SETS FOR HOLOMORPHIC COLLET–ECKMANN MAPS

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Abstract. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map on the Riemann sphere , such that for every f-critical point $c \in J$ which forward trajectory does not contain any other critical point, $|(f^n)'(f(c))|$ grows exponentially fast (Collet–Eckmann condition), there are no parabolic periodic points, and else such that Julia set is not the whole sphere. Then smooth (Riemann) measure of the Julia set is 0.

For f satisfying additionally Masato Tsujii's condition that the average distance of $f^n(c)$ from the set of critical points is not too small, we prove that Hausdorff dimension of Julia set is less than 2. This is the case for $f(z) = z^2 + c$ with c real, $0 \in J$, for a positive line measure set of parameters c.

Introduction

It is well-known that if $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ a rational map of the Riemann sphere is hyperbolic, i.e. expanding on its Julia set $J = J(f)$ namely $|(f^n)'| > 1$ for an integer $n > 0$, then Hausdorff dimension $\mathrm{HD}(J) < 2$.

The same holds for a more general class of subexpanding maps, namely such maps that all critical points in $J(f)$ are non-reccurrent, supposed $J(f) \neq \overline{\mathbb{C}}$, see [U].

On the other hand there is an abundance of rational maps with $J \neq \overline{\mathbb{C}}$ and $\mathrm{HD}(J) = 2$, [Shi].

Recently Chris Bishop and Peter Jones proved that for every finitely generated not geometrically finite Kleinian groups for the Poincaré limit set Λ one has $\mathrm{HD}(\Lambda) = 2$. As geometrically finite exhibits some analogy to subexpanding in the Kleinian Groups – Rational Maps dictionary, the question arised, expressed by Ch. Bishop and M. Lyubich at MSRI Berkeley conference in January 1995, isn't it true for every non-subexpanding rational map with connected Julia set, that $HD(J) = 2$?

Here we give a negative answer. For a large class of "non-uniformly" hyperbolic so called Collet–Eckmann maps, studied in [P1], satisfying an additional Tsujii condition, $\mathrm{HD}(J) < 2$.

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Notation. For a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ denote by $\mathrm{Crit}(f)$ the set of all critical points of f, i.e.points where $f' = 0$. Let $\nu := \sup\{\text{multiplicity of} f^n \text{at} c : c \in \mathrm{Crit}(f) \cap J\}$. Finally denote by $\mathrm{Crit}'(f)$ the set of all critical points of f in $J(f)$ which forward trajectories do not contain other critical points.

We prove in this paper the following results:

Theorem A. Let f be a rational map on the Riemann sphere $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and there exist $\lambda > 1, C > 0$ such that for every f-critical point $c \in \mathrm{Crit}'(f)$

$$|(f^n)'(f(c))| \geq C\lambda^n, \tag{0.1}$$

there are no parabolic periodic points, and $J(f) \neq \overline{\mathbb{C}}$. Then $\mathrm{Vol}(J(f) = 0$, where Vol denotes Riemann measure on $\overline{\mathbb{C}}$.

Theorem B. In the conditions of Theorem A assume additionally that

$$\lim_{t\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \max(0, -\log\big(\mathrm{dist}(f^j(c), \mathrm{Crit}(f))\big) - t) = 0. \tag{0.2}$$

Then $\mathrm{HD}(J(f)) < 2$.

For $f(z) = z^2+c, c \in [-2, 0]$ real, it is proved in [T] that (0.1) and (0.2) are satisfied for a positive measure set of parameters c for which there is no sink in the interval $[c, c^2 + c]$. Tsujii's condition in [T] called there *weak regularity* is in fact apparently stronger than (0.2). The set of subexpanding maps satisfying (0.1) and weak regularity has measure 0, [T]. Thus Theorem B answers Bishop–Lyubich's question.

Remark. In [DPU] it is proved that for every rational map $f\overline{\mathbb{C}} \to \overline{\mathbb{C}}, c \in \mathrm{Crit}'$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} -\log \mathrm{dist}(f^j(c), \mathrm{Crit}(f)) \leq C_f$$

where C_f depends only on f. Here in the condition (0.2) it is sufficient, for Theorem B to hold, to have a positive constant instead of 0 on the right hand side, unfortunately apparently much smaller than C_f.

Crucial in proving Theorems A and B is the following intermediate result:

Theorem 0.1 (on the existence of pacim), see [P1]. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ satisfies the assumptions of Theorem A. Let μ be an α-conformal measure on the Julia set $J = J(f)$ for an arbitrary $\alpha > 0$. Assume there exists $0 < \lambda' < \lambda$ such that for every $n \geq 1$ and every $c \in \mathrm{Crit}'(f)$

$$\int \frac{d\mu}{\mathrm{dist}(x, f^n(c))^{(1-1/\nu)\alpha}} < C^{-1}(\lambda')^{\alpha n/\nu}. \tag{0.3}$$

Then there exists an f-invariant probability measure m on J absolutely continuous with respect to μ (pacim).

Recall that a probability measure μ on J is called α-conformal if for every Borel $B \subset J$ on which f is injective $\mu(f(B) = \int_B |f'|^\alpha d\mu$. In particular $|f'|^\alpha$ is Jacobian for f and μ. The number α is called the exponent of the conformal measure.

If $\mathrm{Vol}(J) > 0$ then the restriction of Vol to J, normalized, is 2-conformal and obviously satisfies (0.3). If $\mathrm{HD}(J) = 2$ then by [P1] we know there exists a 2-conformal measure μ on J but we do not know whether it is not too singular, namely whether it satisfies (0.3). Under the additional assumption (0.2) we shall prove that it is so for every α-conformal measure.

Notation. Const will denote various positive constants which may change from one formula to another, even in one string of estimates.

1. More on pacim. Proof of Theorem A.

Proposition 1.1. In the situation of Theorem 0.1 there exists $K > 0$ such that μ-a.e. $\frac{dm}{d\mu} \geq K$.

Proof. In Proof of Theorem 0.1 [P1] one obtains m as a weak* limit of a subsequence of the sequence of measures $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\mu)$.

It is sufficient to prove that there exists $K > 0$ and $n_0 > 0$ such that for μ-a.e. $y \in J(f)$

$$\frac{df_*^n(\mu)}{d\mu}(y) = \mathcal{L}^n(\mathbf{1}) \geq K. \tag{1.1}$$

Here $\mathcal{L}$ denotes the transfer operator, which can be defined for example by $\mathcal{L}(\varphi)(y) = \sum_{f(z)=y} |f'(z)|^{-\alpha}\varphi(z)$. $\mathbf{1}$ is the constant function of value 1. We can assume $y \notin \bigcup_{n>0} f^n(\mathrm{Crit}(f))$ because

$$\mu(\bigcup_{n>0} f^n(\mathrm{Crit}(f))) = 0 \tag{1.2}$$

If a critical value for f^n were an atom then a critical point would have μ measure equal to ∞.

It is sufficient to prove (1.1) for $y \in B(x,\delta) \cap J(f)$ for an *a priori* chosen x and an arbitrarily small δ and next to use the fact that there exists $m \geq 0$ such that $f^m(B(x,\delta)) \supset J(f)$ (called *topological exactness*). Indeed

$$\mathcal{L}^n(\mathbf{1})(w) = \sum_{f^m(y)=w} \mathcal{L}^{n-m}(\mathbf{1})(y)|(f^m)'|^{-\alpha} \geq \ (\sup|(f^m)'|)^{-\alpha}\mathcal{L}^{n-m}(y_0)$$

where $y_0 \in f^{-m}(\{w\}) \cap B(x,\delta)$.

Recall the estimate from [P1]. For an arbitrary $\gamma > 1$ there exists $C > 0$ such that for every $x \in J(f)$

$$\mathcal{L}^n(\mathbf{1})(x) \leq C + C \sum_{c\in\mathrm{Crit}(f)\cap J} \sum_{j=0}^{\infty} \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^j(f(c)))^{(1-1/\nu)\alpha}}. \tag{1.3}$$

By the assumptions (0.1) and (0.3) the above function is μ-integrable if γ is small enough.

Pay attention to the assumption (0.3). It concerns only $c \in \mathrm{Crit}'$. Fortunately there is only a finite number of summands in (1.3) for which $f^{j_0}(c) \in \mathrm{Crit}, j_0 \geq j$. Each summand is integrable because up to a constant it is bounded by $\mathcal{L}^j(\mathbf{1})$.

So

$$\sum_{c\in \mathrm{Crit}(f)\cap J} \sum_{j=s}^{\infty} \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^j(f(c)))^{(1-1/\nu)\alpha}} \to 0 \quad \mu - \text{a.e. as} s \to \infty. \tag{1.4}$$

Fix from now on an arbitrary $x \in J(f)$ for which (1.4) holds, $(dm/d\mu)(x) \geq 1$ and $x \notin \bigcup_{n>0} \varphi^n(\mathrm{Crit}(f))$ (possible by (1.2) and by $\int (dm/d\mu)d\mu = 1$).

We need now to repeat from [P1] a part of Proof of Theorem 0.1:

For every $y \in B(x, \delta)$ and $n > 0$

$$\begin{aligned} \mathcal{L}^n(\mathbf{1})(y) = & \sum_{y'\in f^{-n}(y),\mathrm{regular}} |(f^n)'(y')|^{-\alpha} + \sum_{(y',s)\mathrm{singular}} \mathcal{L}^{n-s}(\mathbf{1})(y')|(f^s)'(y')|^{-\alpha} \\ = & \sum_{\mathrm{reg},y} + \sum_{\mathrm{sing},y} . \end{aligned} \tag{1.5}$$

We shall recall the definitions of *regular* and *singular*: Take an arbitrary subexponentially decreasing sequence of positive numbers $b_j, j = 1, 2 \dots$ with $\sum b_j = 1/100$. Denote by $B_{[k}$ the disc $B(x, (\prod_{j=1}^k (1 - b_j))2\delta)$. We call s the *essentially critical time* for a sequence of compatible components $W_j = \mathrm{Comp} f^{-j}(B_{[j})$, compatible means $f(W_j) \subset W_{j-1}$, if there exists a critical point $c \in W_s$ such that $f^s(c) \in B_{[s}$.

We call y' *regular* in (1.5) if for the sequence of compatible components $W_s, s = 0, 1, \dots, n, W_n \ni y'$ no $s < n$ is essentially critical.

We call a pair (y', s) *singular* if $f^s(y') = y$ and for the sequence of compatible components W_j, for $j = 0, 1, \dots, s$, with $W_s \ni y'$, the integer s is the first (i.e., the only) essentially critical time.

If δ is small enough then all s in $\sum_{\mathrm{sing},x}$ are sufficiently large that $\sum_{\mathrm{sing},x} \leq 1/2$. This follows from the estimates in [P1]; here is the idea of the proof: Transforming $\sum_{\mathrm{sing},x}$ in (1.5) using the induction hypothesis (1.3) we obtain the summands

$$C \frac{\gamma^j \lambda^{-j\alpha/\nu}}{\mathrm{dist}(x, f^{s+j-1}(f(c)))^{(1-1/\nu)\alpha}}, \ j = 0, \dots, n - s$$

multiplied by

$$\mathrm{Const}|(f^{s-1})'(x')|^{-\alpha/\nu} a_s < \gamma^{s-1} \lambda^{-(s-1)\alpha/\nu}.$$

The numbers a_s are constants arising from distortion estimates, related to b_s. The numbers γ^s swallow them and other constants.

(There is a minor inaccuracy here: (s, x') is a singular pair where the summand appears, provided x' is not in the forward trajectory of another critical point, otherwise one moves back to it, see [P1] for details.)

Now $\sum_{\mathrm{sing},x} \leq 1/2$ follows from (1.4).

The result is that $\sum_{\mathrm{reg},x} \geq 1/2$. So by the uniformly bounded distortion along regular branches of f^{-n} on $B(x,\delta)$ we obtain

$$\mathcal{L}^n(\mathbf{1})(y) \geq \sum_{\mathrm{reg},y} \geq \mathrm{Const} \sum_{\mathrm{reg},x} \geq \mathrm{Const} > 0$$

The name *regular* concerned formally $y' \in f^{-n}(y)$ but in fact it concerns the branch of f^{-n} mapping y to y' not depending on $y \in B(x,\delta)$.

By distortion of any branch g of f^{-n} on a set U we mean

$$\sup_{z\in B} |g'(z)| / \inf_{z \in B} |g'(z)|.$$

This proves Proposition 1.1. □

Corollary 1.2 In the situation of Theorem 0.1 for measure-theoretic entropy $\mathrm{h}_m(f) > 0$.

Proof. Denote $dm/d\mu$ by u.

Consider an open set $U \subset \overline{\mathbb{C}}$ intersecting $J(f)$ such that there exist two branches g_1 and g_2 of f^{-1} on it. Then by the f-invariance of m we have $\mathrm{Jac}_m(g_1) + \mathrm{Jac}_m(g_2) \leq 1$ ($= 1$ if we considered all branches of f^{-1}). $\mathrm{Jac}_m(g_i)$ means Jacobian with respect to m for g_i.

We have $m(U) > 0$ because μ does not vanish on open sets in J (by the topological exactness of f on J) and by Proposition 1.2. At m-a.e. $x \in U$

$$\mathrm{Jac}_m(g_i)(x) = u(g_i(x)) |g_i'(x)| u(x)^{-1} > 0,$$

(we used here also (1.4)).

Hence $\mathrm{Jac}_m(g_i) < 1$, so $\mathrm{Jac}_m(f) > 1$ on the set $g_i(U), i = 1,2$ of positive measure m. Now we use Rokchlin's formula and obtain

$$\mathrm{h}_m(f) = \int \log \mathrm{Jac}_m(f) dm > 0.$$ □

Let $\chi_m = \int \log |f'| dm$ denote characteristic Lyapunov exponent.

Corollary 1.3 In the situation of Theorem 0.1, $\chi_m > 0$.

Proof. This Corollary follows from Ruelle's inequality $\mathrm{h}_m(f) < 2\chi_m$, see [R].

Proof of Theorem A. Suppose $\mathrm{Vol}(J(f)) > 0$. After normalization we obtain a 2-conformal measure μ on $J(f)$ and by Theorem 0.1 and Corollary 1.3 a pacim m with $\chi_m > 0$. By Pesin's Theory [Pesin] in the iteration in dimension 1 case [Le] ([Le] is on the real case, but the complex one is similar), for m-a.e. x, there exists a sequence of integers $n_j \to \infty$ and $r > 0$ such that for every j there exists a univalent branch g_j of f^{-n_j} on $B_j := B(f^{n_j}(x), r)$ mapping $f^{n_j}(x)$ to x and g_j has distortion bounded by a constant. By $\chi_m > 0$ $\operatorname{diam} g_j(B(f^{n_j}(x), r) \to 0$. (This follows also automatically from the previous assertions by the definition of Julia set [GPS].) Now we can forget about the invariant measure m and go back to Vol. Because $J(f)$ is nowhere dense in $\overline{\mathbb{C}}$, there exists $\varepsilon > 0$ such that for every $z \in J(f)$

$$\frac{\operatorname{Vol}(B(z,r) \setminus J(f))}{\operatorname{Vol}(B(z,r))} > \varepsilon.$$

Bounded distortion for g_j on $B(z,r)$, $z = f^{n_j}(x)$ allows to deduce that the same part of each small disc$\approx g_j(B_j)$ around x is outside $J(f)$, up to multiplication by a constant. This is so because we can write for every $X \subset B(z,r), y \in B(z,r)$

$$\operatorname{Vol} g_j(X) \approx |g_j'(y)|^2 \operatorname{Vol}(X) \tag{1.6}$$

where $\approx$ means up to the multiplication by a uniformly bounded factor. So x is not a density point of $J(f)$. On the other hand a.e. point is a density point. So $\operatorname{Vol} J(f) = 0$ and we arrived at a contradiction. □

2. Proof of Theorem B.

As mentioned in the Introduction, the following result is crucial:

Lemma 2.1. Under the conditions of Theorem B, i.e. in the situation of Theorem A and assuming (0.2), the condition (0.3) holds for every α-conformal measure, $\alpha > 0$.

Proof. Step 1. Denote the expression from (0.2)

$$\max\big(0, -\log \inf_{c \in \operatorname{Crit}'(f)} \operatorname{dist}(f^n(c), \operatorname{Crit}(f)) - t\big)$$

by $\varphi_t(n)$. Consider the following union of open-closed intervals

$$A_t' := \bigcup_n (n, n + \varphi_t(n) \cdot K_f] \text{ and write } A_t := \mathbb{Z}_+ \setminus A_t',$$

for an arbitrary constant $K_f > \nu / \log \lambda$.

By (0.2) for every $a > 0$ there exist $t > 0$ and $n(a,t)$ such that for every $n \geq n(a,t)$

$$A_t \cap [n, n(1+a)] \neq \emptyset \tag{2.1}$$

Moreover, fixed an arbitrary integer $M > 0$, we can guarantee for every $n' \geq n(1+a), n \geq n(a,t)$

$$\sharp(A_t \cap \{j \in [n, n'] : j \text{divisable by} M\}) \geq \frac{1}{2M}(n' - n). \tag{2.2}$$

Observe that for every n_0, n, (2.1) transforms into

$$[n_0 + n, n_0 + n + a(n_0 + n)] = [n_0 + n, n_0 + n + a(\frac{n_0}{n} + 1)n].$$

The result is that if $n \geq bn_0$ for an arbitrary $b > 0$ then

$$A_t \cap [n_0 + n, n_0 + n + a(b^{-1} + 1)n] \neq \emptyset. \tag{2.3}$$

Denote in the sequel $a(b^{-1} + 1)$ by a'.

Step 2. Observe now that if $n \in A_t$ then for every $c \in \mathrm{Crit}'(f)$ there exist branches $g_s, s = 1, 2, \dots, n-1$ of f^{-s} on $B_n := B(f^n(c), \delta)$ with uniformly bounded distortions, where $\delta = \varepsilon \exp -t\nu$ for a constant ε small enough. Sometimes to exhibit the dependence on n we shall write $g_{s,n}$.

Indeed, define g_s on $B_{[s} = B(f^n(c), \prod_{j=1}^s (1-b_j) 2\delta)$ for $s = 1, 2, \dots, n-1$ according to the procedure described in Proof of Theorem A. If there is an obstruction, namely s an essential critical time, then for every $z \in B_{[s}$

$$|g'_{s-1}(z)| \leq \lambda^{-s} \vartheta^s \leq \exp(-s\nu/K_f) \tag{2.4}$$

for $\vartheta > 1$ arbitrarily close to 1 (in particular such that $K_f > \frac{\nu}{\log\lambda - \log\vartheta}$) and for s large enough. The constant ϑ takes care of distortion. (2.4) holds for $z = f^s(q)$, where q is the critical point making s critical time, without ϑ by (0.1) (with the constant C instead). The small number ε takes care of s small, which cannot be then essential critical.

The inequality (2.4) and rooting ($1/\nu$ to pass from $s-1$ to s) imply $\varphi_t(f^{n-s}(c)) \geq s/K_f$, so $n \notin A_t$, a contradiction.

Step 3. By uniformly bounded distortion for the maps $g_{j,n}$, $n \in A_t$ we obtain (compare (1.6)) for every $n_0 > 0$ large enough, $c \in \mathrm{Crit}'$

$$\mu B(f^{n_0}(c), r_j) \approx r_j^\alpha \tag{2.5}$$

for a sequence $r_j, j = 1, 2, \dots$ such that

$$r_1 > \exp -Lbn_0 \tag{2.6}$$

$$r_{j+1} > r_j^{1+\sigma} \tag{2.7}$$

$$\text{and } r_{j+1} < r_j/2. \tag{2.8}$$

Here $L := 2\sup|f'|$ and b, σ are arbitrarily close to 0.

Indeed, we can find r_j satisfying the conditions above by taking

$$r_j := \mathrm{diam} g_{n_j, n_0+n_j}(B(f^{n_0+n_j}(c), \delta))$$

where $n_j \in A_t$ are taken consecutively so that

$$n_{j+1} \in [n_j + (1+\vartheta)n_j, n_j + (1+\vartheta)(1+a')n_j \text{ for } j \geq 2 \text{ and}$$

$$n_1 \in [n_0 + bn_0, n_0 + bn_0 + a'bn_0],$$

where $\vartheta > 0$ is an arbitrary constant close enough to 0.

This gives

$$r_{j+1}/r_j \geq \exp(-2(\log L)a'n_j) \tag{2.9}$$

To conclude we need to know that r_j shrink exponentially fast with $n_j \to \infty$. For that we need the following fact (see for example [GPS], find the analogous fact in Proof of Theorem A):

(*) For every $r > 0$ small enough and $\xi, C > 0$ there exists m_0 such that for every $m \geq m_0, x \in J(f)$ and branch g of f^{-m} on $B(x,r)$ having distortion less than C we have $\text{diam} g(B(x,r)) < \xi r$.

Apply now (2.2) to $n = n_0, n' = n_j + n_0$. We obtain a "telescope": For all consecutive $\tau_1, \tau_2, \dots, \tau_{k(j)} \in A_t \cap [n_0, n_j + n_0]$ divisible by M

$$g_{\tau_{i+1}-\tau_i, \tau_{i+1}}(B(f^{\tau_{i+1}}(c), \delta)) \subset B(f^{\tau_i}(c), \delta/2)$$

for $M \geq m_0$ from (*).

Hence using (2.2)

$$r_j \leq 2^{-n_j/2M}. \tag{2.10}$$

Denote $2a' \log L$ by γ and $(\log 2)/2M$ by γ'. (2.9) and (2.10) give

$$r_{j+1} \geq r_j \exp -\gamma n_j \geq r_j (\exp -\gamma' n_j)^{\gamma/\gamma'} \geq r_j^{1+\gamma/\gamma'}.$$

As γ' is a constant and γ can be done arbitrarily small if a is small enough, we obtain (2.7).

The condition (2.8) follows from the fact that for n_0 large enough all $n_{j+1} - n_j$ are large enough to apply (*).

Conclusion. We obtain

$$\int \frac{d\mu}{\text{dist}(x, f^{n_0}(c))^{(1-1/\nu)\alpha}} \leq$$

$$\mu(\overline{\mathbb{C}} \setminus B(f^{n_0}(c), r_1)) \frac{1}{r_1^{(1-1/\nu)\alpha}} + \sum_{j\geq 2} \mu(B(f^{n_0}(c), r_{j-1}) \setminus B(f^{n_0}(c), r_j)) \frac{1}{r_j^{(1-1/\nu)\alpha}} \leq$$

$$\exp(Lbn_0(1-1/\nu)\alpha) + \text{Const} \sum_{j\geq 2} \frac{r_{j-1}^{\alpha}}{r_j^{(1-1/\nu)\alpha}} \leq$$

$$(\exp(Lb(1-1/\nu)\alpha))^{n_0} + \text{Const} \sum_{j\geq 2} r_{j-1}^{\alpha} r_{j-1}^{-(1-1/\nu)\alpha(1+\sigma)}.$$

The latter series has summands decreasing exponentially fast for σ small enough so it sums up to a constant, hence the first summand dominates. We obtain the bound by $(\lambda')^{n_0}$ with $\lambda' > 1$ arbitrarily close to 1. Thus (0.3) has been proved. □

Remark 2.2. The only result in our disposal on the abundance of non-subexpanding maps satisfying (0.1) and (0.2) is Tsujii's one concerning $z^2 + c, c$ real (see Introduction). For this class however the exponential convergence of

$$\text{diamComp} f^{-n_j}(B(f^{n_j+n_0}(0), \delta)$$

to 0 follows from [N] (the component containing $f^{n_0}(c)$). So restricting our interests to this class we could skip (2.2) and considerations leading to (2.10) above.

By [N] $\operatorname{diam}\big(\mathrm{Comp}(f^{-n}(B(x,\delta)))\cap\mathbb{R}\big) < C\tilde{\lambda}^{-n}$ for some constants $C>0, \tilde{\lambda}>1, \delta$ small enough and every component Comp. Just the uniform convergence of the diameters to 0 as $n\to\infty$ follows from [P1], but I do not know how fast is it.

Proof of Theorem B. Suppose that $\mathrm{HD}(J)=2$. Then there exist a 2-conformal measure μ on J. This follows from the existence of an α-conformal measure for $\alpha=\mathrm{HD}_{\mathrm{ess}}(J)$, where $\mathrm{HD}_{\mathrm{ess}}$ is the essential Hausdorff dimension which can be defined for example as supremum of Hausdorff dimensions of expanding Cantor sets in J, see [DU] [P2] and [PUbook], and from $\mathrm{HD}_{\mathrm{ess}}(J)=\mathrm{HD}(J)$, see [P1]. The former holds for every rational map, the latter was proved in [P1] only for Collet–Eckmann maps.

By Lemma 2.1 the condition (0.3) holds, hence there exists a pacim $m \ll \mu$. Moreover $\chi_m>0$ by Corollary 1.3. As in Proof of Theorem A by Pesin Theory there exists $X\subset J$, $m(X)=\mu(X)=1$, such that for every $x\in X$ there exists a sequence of integers $n_j(x)\to\infty$, $r>0$ and univalent branches g_j of f^{-n_j} on $B(f^{n_j}(x),r)$ mapping f^{n_j} to x with uniformly bounded distortion. Write $B_{x,j}:=g_j(B(f^{n_j}(x),r))$.

We obtain for every $x\in X$ applying (1.6) to Vol and μ (similarly as in Proof of Theorem A)

$$\mu(B_{x,j}) \le \mathrm{Const}\mathrm{Vol}(B_{x,j}) \le \mathrm{Const}(\mathrm{Vol}B(x,\mathrm{diam}B_{x,j})).$$

If $\mathrm{Vol}X=0$ then there exists a covering of X by discs $B(x_t,\mathrm{diam}B_{x_t,j_t}), t=1,2,\dots$ which union has $\mathrm{Vol}<\varepsilon$ for ε arbitrarily close to 0, of multiplicity less than a universal constant (Besicovitch's theorem). Hence

$$\varepsilon \ge \mathrm{Const}\sum_t \mathrm{Vol}B(x_t,\mathrm{diam}B_{x_t,j_t}) \ge \mathrm{Const}\mu\sum_t B_{x_t,j_t} \ge 1,$$

a contradiction. Hence $\mathrm{Vol}J \ge \mathrm{Vol}X > 0$.

This contradicts Theorem A that $\mathrm{Vol}J=0$ and the proof is over.

Remark that we could end the proof directly: As in Proof of Theorem A we show that no point of X is a point of density of the Vol measure. Hence $\mathrm{Vol}X=0$. (I owe this remark to M. Urbański.) □

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