Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds

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Abstract

In this paper, we first give an intersection theory for moduli problems for nonlinear elliptic operators with certain precompact space of solutions in differential geometry. Then we apply the theory to constructing Gromov-Witten invariants for general symplectic manifolds.

0 Introduction

This paper is a sequel to [LT1]. In [LT1], by using purely algebraic methods, we developped an intersection theory for moduli problems on smooth algebraic varieties over any algebraically closed fields of characteristic zero. An alternative construction was given in [BF]. The key point in [LT1] is the existence of locally free resolutions of tangent complexes involved. In this paper, we apply the same idea to constructing the intersection theory for moduli problems in the differential category. However, the tool will be analytic this time.

Given a Banach manifold B, a smooth bundle $E \mapsto B$ is Fredholm, if there is a section $s : B \mapsto E$ such that $s^{-1}(0)$ is compact and any linearization of s at any point of $s^{-1}(0)$ is Fredholm of index r. Then one can define

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the determinant line bundle $\det(s)$ over $s^{-1}(0)$. Note that $s : B \mapsto E$ is orientable if $\det(s)$ does. It should be well-known that for any orientable Fredholm bundle $s : B \mapsto E$, one can associate an Euler class $e([s : B \mapsto E])$ in $H_r(B, \mathbb{Z})$, which depends only on the homotopy class $[s : B \mapsto E]$ of $s : B \mapsto E$.

However, its use is very limited. In many problems, such as constructing Donaldson invariants and Gromov-Witten invariants, the zero set $s^{-1}(0)$ is often noncompact, if we insist on smooth Banach manifolds, smooth bundles. For many useful applications, we have to construct Euler classes for spaces, bundles and their sections, which are not necessarily smooth, and prove that they are invariant. In section one, we will give two simple theorems on constructing the Euler classes of so called generalized Fredholm bundles, and more generally, the rational Euler classes of generalized Fredholm orbifold bundles. The main part of this paper is devoted to constructing Gromov-Witten invariants over rational numbers for general symplectic manifolds by establishing the Fredholm properties of the bundle of $(0, 1)$-forms over the space of smooth stable maps (cf. section 2). In fact, we have constructed symplectically invariant Euler classes in the space of stable maps.

The theory of the Gromov-Witten invariants was first established in a systematical and mathematical way by Ruan and the second author in [RT1], [RT2] for semi-positive symplectic manifolds. They actually constructed the invariants over integers. In [LT1] and [BF], the authors constructed the Gromov-Witten invariants for any algebraic manifolds over any closed fields of characteristic zero.

The similar idea was also used by Liu and the second author in proving the Arnold conjecture for nondegenerate Hamiltonian functions on general symplectic manifolds [LiuT].

During the preparation of this paper, we learned that Fukaya and Ono also gave a different construction of the Gromov-Witten invariants and a proof of the Arnold conjecture for nondegenerate Hamiltonian functions for general symplectic manifolds ([FO]). We also learned that one or both of them was also claimed by Hofer and Salamon, and Ruan. Shortly after we finished writting of this paper, we received a preprint [Si] of B. Siebert, in which he gave another different construction of Gromov-Witten invariants for general symplectic manifolds.

We believe that our construction can be also used to constructing the Gauge theory invariants, such as Donaldson invariants. It is also interesting to compare the Gromov-Witten invariants constructed here with algebraic ones in [LT1] (cf. [LT2]). We plan to discuss these in forthcoming papers.

Contents

1 Euler classes for Fredholm bundles

In this section, we collect a few simple theorems, which can be proved easily.

Let B be a topological space. Recall that a topological bundle $\pi : E \mapsto B$ consists of a continuous map π between topological spaces, satisfying: (1) there is a topological subspace Z in E such that $\pi|_Z$ is a homeomorphism from Z onto B; (2) For any $x \in B$, the fiber $E_x = \pi^{-1}(x)$ is a vector space with origin at $(\pi|_Z)^{-1}(x)$. A section of E is a map $s : B \mapsto E$ such that $\pi \cdot s$ is the identity map of B. Clearly, Z defines a section of E, which is usually refered as the 0-section. For any section s, its zero locus in B is $s^{-1}(Z)$, which is also denoted by $s^{-1}(0)$.

A smooth approximation (E_i, U_i) of $s : B \mapsto E$ consists of an open subset U_i in B and a continuous vector subbundle E_i of finite rank over U_i, such that $s^{-1}(E_i) \subset U_i$ is a smooth manifold and $E_i|_{s^{-1}(E_i)}$ is a smooth bundle over $s^{-1}(E_i)$ with $s|_{s^{-1}(E_i)}$ smooth . We say that $s : B \mapsto E$ has a weakly smooth structure $\{(E_i, U_i)\}$ of index r, if (i) each (E_i, U_i) is a smooth approximation of $s : B \mapsto E$; (ii) $\{U_i\}$ is a covering of $s^{-1}(0)$; (iii) $s^{-1}(E_i) \subset U_i$ is of dimension $r + \mathrm{rk}(E_i)$; (iv) For any $x \in s^{-1}(0) \cap U_i \cap U_j$, there is another smooth approximation $E_k \mapsto U_k$ with $x \in U_k$, such that $E_i|_{U_i \cap U_k}$ (resp. $E_j|_{U_j \cap U_k}$) is a subbundle of $E_k|_{U_i \cap U_k}$ (resp. $E_k|_{U_j \cap U_k}$).

Given two smooth structures $\{(E_i, U_i)\}$ and $\{(E'_j, U'_j)\}$ of $s : B \mapsto E$, we say that $\{U_i\}$ is finer than $\{U'_j\}$, if for $x \in s^{-1}(0) \cap U'_j$, there is at least one (E_i, U_i) such that near x, (1) $s^{-1}(E_i) \cap s^{-1}(E'_j)$ is a smooth submanifold in U_i of dimension $\dim s^{-1}(E'_j)$; (2) $E'_j|_{U_i \cap U'_j}$ is a subbundle of $E_i|_{U_i \cap U'_j}$; (3) The restriction $E'_j|_{s^{-1}(E_i) \cap s^{-1}(E'_j)}$ is a smooth subbundle of $E_i|_{s^{-1}(E_i) \cap s^{-1}(E'_j)}$.

We say that E is a generalized Fredholm bundle of index r, if there is a continuous section $s : B \mapsto E$ satisfying the followings:

(1) $s^{-1}(0)$ is compact;
(2) $s : B \mapsto E$ has a weakly smooth structure $\{(E_i, U_i)\}$ of index r;
(3) There is a finitely dimensional vector space F and a bundle homomorphism $\psi_F : B \times F \mapsto E$, such that for any smooth approximation (E_i, U_i), $\psi_F|_{s^{-1}(E_i)\times F}$ is a smooth map from $s^{-1}(E_i) \times F$ into E_i and transverse to s along $s^{-1}(0) \cap U_i$.

Such a section s is called admissible. We call $\{F, (E_i, U_i)\}$ a weakly smooth resolution of $s : B \mapsto E$.

Put $W_F = (s - \psi_F)^{-1}(0) \subset B \times F$. Here by abusing the notation, we also regard s as a section of E over $B \times F$. Then W_F is a smooth manifold of dimension $r + \mathrm{rk}(F)$ near $s^{-1}(0)$, and s lifts to a smooth section $s_F : W_F \mapsto F$, namely, for any $(x, v) \in W_F \subset B \times F$, $s_F(x, v) = v$. Clearly, $s_F^{-1}(0) = s^{-1}(0)$.

Remark 1 *One can also define the weakly C^ℓ-smoothness of $s : B \mapsto E$. We say that $s : B \mapsto E$ is of class C^ℓ ($\ell \geq 1$) if any E_i is a C^ℓ-smooth bundle over a C^ℓ-smooth manifold $s^{-1}(E_i)$, and s, ψ_F are C^ℓ-smooth along $s^{-1}(E_i)$.*

Remark 2 *If $\{F', (E'_j, U'_j)\}$ is another smooth resolution of s, we say that $\{F, (E_i, U_i)\}$ is finer than $\{F', (E'_j, U'_j)\}$, if $F' \subset F$, $\{(E_i, U_i)\}$ is finer than $\{(E'_j, U'_j)\}$ and ψ_F restricts to $\psi_{F'}$ on F'.*

We will identify $\{F', (E'_j, U'_j)\}$ with $\{F, (E_i, U_i)\}$, if there is another smooth structure $\{F'', (E''_k, U''_k)\}$ finer than $\{F', (E'_j, U'_j)\}$ and $\{F, (E_i, U_i)\}$.

Let $s, s' : B \mapsto E$ be two admissible sections. In the following, unless specified, by $s = s'$, we mean that they are the same as continuous sections and their weakly smooth structures are identical.

We say that two generalized Fredholm bundles $s : B \mapsto E$ and $s' : B \mapsto E$ are homotopic to each other, if there is a generalized Fredholm bundle of the form $S : \pi_2^* E \to [0, 1] \times B$ of index $r + 1$, such that $S|_{0\times B} = s$ and $S|_{1\times B} = s'$, where $\pi_2 : [0, 1] \times B \mapsto B$ is the natural projection.

We denote by $[s : B \to E]$ the equivalence class of generalized Fredholm bundles which are homotopic to $s : B \to E$. We also denote by $r(B, E, s)$ the index of the generalized Fredholm bundle $s : B \mapsto E$.

Example 1 *Let B be a Banach manifold (possibly incomplete). Suppose that E is a vector bundle E over B with a section $s : B \to E$ satisfying: 1) $s^{-1}(0)$ is compact; 2) For any $x \in B$, $L_x(s)$ is Fredholm, where $L_x(s)$*

denotes the linearization of s at x with $s(x) = 0$, defined as follows: if $\phi : E|_W \mapsto W \times H$ is any local trivialization near x, then

$$L_x(s)(v) = \phi^{-1}(v(\pi_2 \cdot \phi \cdot s)(x)),$$

where π_2 be the projection from $W \times H$ onto H. Since $s(x) = 0$, $L_x(s)(v)$ is independent of choices of local trivializations of E near x.

The index of $L_x(s)$ is independent of x in B. Therefore, we can denote this index by $r(B, E, s)$.

One can easily show that such a $s : B \mapsto E$ is a generalized Fredholm bundle of index $r(B, E, s)$.

Let $s : B \mapsto E$ be a generalized Fredholm bundle. We can define its determinant bundle $\det(s)$ as follows: let $\{F, (E_i, U_i)\}$ be a smooth resolution of s, then we define $\det(s)$ to be $\det(TW_F) \otimes \det(F)^{-1}|_{s^{-1}(0)}$. For any smooth approximation (E_i, U_i), W_F is a smooth submanifold in $U_i \times F$ and its normal bundle can be canonically identified with $E_i|_{W_F}$ by using the differential $d(s - \psi_F)$. It follows that $det(s)$ can be canonically identified with $\det(TU_i \times F) \otimes \det(E_i)^{-1} \otimes \det(F)^{-1}|_{s^{-1}(0)}$, and consequently, $\det(TU_i) \otimes \det(E_i)^{-1}|_{s^{-1}(0)}$. In particular, $\det(s)$ is independent of choices of $\{F, (E_i, U_i)\}$. Moreover, it implies that $\det(s)(x)$ ($x \in s^{-1}(0)$) can be naturally identified with $\det(L_x(s))$, where $L_x(s)$ denotes the linearization of s from $\bigcup\{T_x U_i | x \in U_i \cap s^{-1}(0)\}$ into E_x defined as in Example 1. The determinant $\det(L_x(s))$ is defined in the standard way by using finitely dimensional approximations.

We say that $s : B \mapsto E$ is orientable if $\det(s)$ is orientable, i.e., it admits a nonvanishing section. Clearly, if $s : B \mapsto E$ is orientable, so is any other bundle in $[s : B \mapsto E]$, so we can simply say that $[s : B \mapsto E]$ is an orientable equivalence class.

Theorem 1.1 *For each oriented equivalence class $[s : B \to E]$ of generalized Fredholm bundles, we can assign an oriented Euler class $e([s : B \to E])$ in $H_r(B, Z)$, where $r = r(B, E, s)$. More precisely, $e([s : B \to E])$ can be represented by an r-dimensional manifold.*

Furthermore, this Euler class satisfies the usual functorial properties for the Euler class of bundles of finite rank.

Proof: First we observe that $s_F^{-1}(0) = s^{-1}(0)$ is compact. Then, by the standard transversality theorem, there is a generic, small section $v : W_F \mapsto$

F, such that $(s_F + v)^{-1}(0)$ is a compact submanifold in $B \times F$ of dimension r. It has a natural orientation induced by $\det(s)$.

We claim that the homology class of $(s_F + v)^{-1}(0)$ is independent of choices of smooth resolutions. Suppose that $\{F', (E'_j, U'_j)\}$ is another smooth resolution which is finer than $\{F, (E_i, U_i)\}$. Then we have another smooth manifold $W_{F'}$ containing W_F as a submanifold.

We may assume that $W_F = W_{F'} \cap B \times F$ and the above v extends to a map from $W_{F'}$ into F.

Let $F' = F \oplus F'/F$ be a splitting. Write $B \times F'$ as $B \times F \times F'/F$, for any $(x, u_1, u_2) \in W_{F'} \backslash W_F$, we have $u_2 \neq 0$. It follows that $(s_{F'} + v)^{-1}(0) = (s_F + v)^{-1}(0)$. Then the claim follows easily.

We define $e([s : B \mapsto E])$ to be the homology class in $H_r(B, \mathbb{Z})$, which is represented by $(s_F + v)^{-1}(0)$ in $B \times F$. Here we have used the fact that B is homotopically equivalent to $B \times F$. We can also regard $e([s : B \mapsto E])$ as the intersection class of W_F with $B \times \{0\}$.

The class $e([s : B \mapsto E])$ is independent of choices of Fredholm sections s in $[s : B \mapsto E]$. In fact, to prove it, we simply repeat the above arguments for any homotopy $S : B \times [0, 1] \mapsto E$ with $S|_0 = s$.

One can show that $e([s : B \mapsto E])$ satisfies all functorial properties of the Euler class. So Theorem 1.1 is proved.

Remark 3 *Theorem 1.1 still holds even if the assumption (3) on $s : B \mapsto E$ is replaced by*
(3)' there are finitely many open subsets $\{V_a\}$ and finitely dimensional vector spaces F_a satisfying: (i) $B = \bigcup_a V_a$; (ii) For each a, there is a bundle map $\psi_a : V_a \times F_a \mapsto E|_{V_a}$ which is transverse to s along $s^{-1}(0)$ for any smooth approximations, or equivalently, (F_a, ψ_a) is a weakly smooth resolution of $s|_{V_a}$; (iii) If $\dim F_a \leq \dim F_b$, we have that $F_a \subset F_b$ and

$$\psi_b|_{(V_a \cap V_b) \times F_a} = \psi_a|_{(V_a \cap V_b) \times F_a}.$$

The proof is not much more difficult than the above one.

We can also regard $\{V_a, F_a\}$ as a resolution of s. However, when the weakly smooth structure of B admits appropriate cut-off functions, this weakened condition is the same as (3).

The assumptions in Theorem 1.1 can be weaken, namely, we do not really need B to be weakly smooth. The following is rather straightforward, if one treats orbifolds like manifolds.

Let B be a topological space. We recall that a topological fibration $\pi : E \mapsto B$ is an orbifold bundle, if there is a covering $\{U_i\}$ of B by open subsets, satisfying: (1) each U_i is of the form $\tilde{U}_i/\Gamma_i$, where Γ_i is a finite group acting on $\tilde{U}_i$; (2) for each i, there is a topological bundle $\tilde{E}_i \mapsto \tilde{U}_i$, such that $E|_{U_i} = \tilde{E}_i/\Gamma_i$; (3) For any i, j, there is a bundle map Φ_{ij} from $\tilde{E}_j|_{\pi_j^{-1}(U_i \cap U_j)}$ to $\tilde{E}_i|_{\pi_i^{-1}(U_i \cap U_j)}$, which descends to the identity map of $E|_{U_i \cap U_j}$, where $\pi_k : \tilde{U}_k \mapsto U_k$ is the natural projection; (4) For each $x \in \pi_j^{-1}(U_i \cap U_j)$, there is a small neighborhood U_x, such that $\Phi_{ij}|_{\pi_j^{-1}(U_x)}$ is an isomorphism from each connected component of $\pi_j^{-1}(U_x)$ onto its image; (5) Each Φ_{ij} is compatible with actions of Γ_i and Γ_j. Any such a $\pi_i : \tilde{U}_i \mapsto U_i$ is called a local uniformization of B. Note that $\Phi_{ij} \cdot \Phi_{ji}$ may not be an identity. It is only a covering map. We will denote by ϕ_{ij} the induced map from $\pi_j^{-1}(U_i \cap U_j)$ to $\pi_i^{-1}(U_i \cap U_j)$.

An orbifold section $s : B \mapsto E$ is a continuous map such that for each i, $s|_{U_i}$ lifts to a section s_i of $\tilde{E}_i$ over $\tilde{U}_i$.

Similarly, we can define orbifold bundle homomorphisms, and the zero set $s^{-1}(0)$ in an obvious way.

Let $\pi : E \mapsto B$ be a topological orbifold bundle. We say that E is a generalized Fredholm orbifold bundle of index r, if there is an orbifold section $s : B \mapsto E$ satisfying:

(1) $s^{-1}(0)$ is compact;

(2) For each local uniformization $\pi_i : \tilde{U}_i \mapsto U_i$, s_i has a weakly smooth structure of index r;

(3) For any i, j, Φ_{ij} respects weakly smooth structures of $s_j : \tilde{U}_j \mapsto \tilde{E}_j$ and $s_i : \tilde{U}_i \mapsto \tilde{E}_i$;

(4) For each i, there is a finitely dimensional vector space F_i, on which Γ_i acts, and a Γ_i-equivariant bundle homomorphism $\psi_i : \tilde{U}_i \times F_i \mapsto \tilde{E}_i$, satisfying: (i) together with the weakly smooth structure, F_i provides a weakly smooth resolution of s_i; (ii) For any pair i, j, if $\dim F_j \leq \dim F_i$, then there is an injective bundle homomorphism $\theta_{ij} : \pi_j^{-1}(U_i \cap U_j) \times F_j \mapsto \pi_i^{-1}(U_i \cap U_j) \times F_i$, such that $\tilde{p}_i \cdot \theta_{ij} = \phi_{ij} \cdot \tilde{p}_j$, where $\tilde{p}_i$ denotes the natural projection from $\tilde{U}_i \times F_i$ onto $\tilde{U}_i$, and $\psi_i \cdot \theta_{ij} = \Phi_{ij} \cdot \psi_j$ on $\pi_j^{-1}(U_i \cap U_j) \times F_j$; (iv) If $\dim F_k \leq \dim F_j \leq \dim F_i$, then $\theta_{ik} = \theta_{ij} \cdot \theta_{jk}$ over $\pi_k^{-1}(U_i \cap U_j)$; (v) For any x in $U_i \cap U_j$, θ_{ij} is Γ_x-equivariant near $\pi_j^{-1}(x)$, where Γ_x is the uniformization group of B at x. We will also call $\{F_i, \psi_i\}$ a resolution of s.

Clearly, all $\tilde{U}_i \times F_i/\Gamma_i$ can be glued together to obtain a topological space $V(F)$. There is a natural projection $p_F : V(F) \mapsto B$. In fact, $V(F)$ is a union of finitely many orbifold bundles, so we may call it an orbifold quasi-bundle. Also, all ψ_i can be put together to form a map ψ_F from $V(F)$ into E.

Similarly, one can define notions of homotopy equivalence of generalized Fredholm orbifold bundles. One can also compare weakly smooth structures and resolutions of generalized Fredholm orbifold bundles in the same way as we did before.

For any generalized Fredholm orbifold bundle $s : B \mapsto E$, we can also associate a determinant orbifold bundle, denoted by $\det(s)$, in the same way as we did before. We say that $s : B \mapsto E$ is orientable if $\det(s)$ does, i.e., $\det(s)$ admits a nonvanishing section. Note that the orientability of generalized Fredholm orbifold bundles is a homotopy invariant.

Now we have the following generalization of Theorem 1.1.

Theorem 1.2 *For each equivalence class $[s : B \to E]$ of generalized Fredholm orbifold bundles, we can assign an oriented Euler class $e([s : B \to E])$ in $H_r(B, \mathbb{Q})$, where r is the index.*

Furthermore, this Euler class satisfies the usual functorial properties for the Euler class of bundles of finite rank.

Proof: We will adopt the notations in the above definitions of generalized Fredholm orbifold bundles.

As before, we denote by W_F the zero set $(s - \psi_F)^{-1}(0)$. Here again, we regard s as a section of E over $B \times F$ in an obvious way. For each local uniformization $\pi_i : \tilde{U}_i \mapsto U_i$, $\tilde{W}_i = (s_i - \psi_i)^{-1}(0)$ is smooth in $\tilde{U}_i \times F_i$ near $s_i^{-1}(0)$ and of dimension $r + \mathrm{rk}(F_i)$. Then W_F is obtained by gluing together all $W_i = \tilde{W}_i/\Gamma_i$. More precisely, for any $\dim F_j \leq \dim F_i$, by the above (4), $\tilde{W}_j \cap \left(\pi_j^{-1}(U_i) \times F_j\right)$ is locally embedded into $\tilde{W}_i$ by θ_{ij}, consequently, we can identify $\left(\tilde{W}_j \cap \pi_j^{-1}(U_i) \times F_j\right)/\Gamma_j$ with a smooth suborbifold, simply denoted by $W_i \cap W_j$ if there is no confusion, in W_i. We also denote by π_i the projection from $\tilde{W}_i$ onto W_i.

Furthermore, $V(F)$ pulls back to an orbifold quasi-bundle $V_0(F)$ over W_F. More precisely, $V_0(F) = \bigcup_i V_i$, where each $V_i = \tilde{V}_i/\Gamma_i$ and $\tilde{V}_i = \tilde{W}_i \times F_i$. As above, if $\dim F_j \leq \dim F_i$, θ_{ij} induces an injective bundle map

$$h_{ij} : V_j|_{W_i \cap W_j} \mapsto V_i|_{W_i \cap W_j}.$$

We denote by F_{ij} the orbifold subbundle $h_{ij}(V_j|_{W_i \cap W_j})$. It extends to an orbifold subbundle, still denoted by F_{ij}, of V_i over a small neighborhood of

$W_i \cap W_j$ in W_i. Let $\tilde{F}_{ij}$ be the lifting of F_{ij} in $\tilde{V}_i$ over a small neighborhood of $\pi_i^{-1}(W_i \cap W_j)$ in $\tilde{W}_i$.

We define a continuous section $s_F : W_F \mapsto V_0(F)$ as follows: for each i, $s_F|_{W_i}$ is descended from the section

$$(x, v) \in \tilde{W}_i \subset \tilde{U}_i \times F_i \mapsto v \in F_i.$$

Clearly, $s_F^{-1}(0) = s^{-1}(0)$. Moreover, by the definitions of s_F and W_F, we may assume that for some small neighborhood O of $s_F^{-1}(0)$ in W_F and any $\dim F_j \leq \dim F_i$, F_{ij} is well defined over $O \cap W_i \cap p_F^{-1}(U_j)$ and

$$(s_F|_{W_i})^{-1} \left(F_{ij}|_{O \cap W_i \cap p_F^{-1}(U_j)}\right) = \left(s_F|_{W_j}\right)^{-1} (V_j|_{W_i \cap W_j}).$$

To save the notations, we will simply identify W_F with the 0-section in $V_0(F)$.

Let $p : V_0(F) \mapsto W_F$ be the natural projection induced by p_F. Observe that it induces a homomorphism $\tau : H_*(V_0(F), \mathbb{Q}) \mapsto H_*(B, \mathbb{Q})$. Then we define the Euler class $e([s : B \mapsto E])$ to be $\tau([W_F \cap G(s_F)])$. Here, $[W_F \cap G(s_F)]$ denotes the intersection class of W_F with the graph $G(s_F)$ of s_F in $V_0(F)$. Using the above properties of s_F and standard arguments, one can show that such an intersection exists in $H_*(V_0(F), \mathbb{Q})$. Note that W_F and $V_0(F)$ are unions of finitely many orbifolds and $s_F^{-1}(0)$ is compact.

For the reader's convenience, we will outline construction of this intersection class by constructing its rational cycle representative.

Choose U_i' such that $s_F^{-1}(0) \subset \bigcup_i U_i'$ and its closure $\overline{U}_i'$ is contained in U_i. Put $\tilde{W}_i' = \tilde{W}_i \cap \left(\pi_i^{-1}(U_i') \times F_i\right)$ and $W_i' = \tilde{W}_i'/\Gamma_i$. Then $s_F^{-1}(0) \subset \bigcup_i W_i' = W_F'$. We also put $\tilde{V}_i' = \tilde{W}_i' \times F_i$ and $V_i' = \tilde{V}_i'/\Gamma_i$.

In this proof, by a cocycle Z' of degree m in $p^{-1}(O)$, we mean a union of cycles $Z_i' \subset \overline{V}_i' \cap p^{-1}(O)$ with its boundary in $\partial \overline{V}_i'$ and of dimension $\dim F_i + m$, such that $Z_j' \cap \overline{V}_i'$ is embedded in Z_i' whenever $\dim F_j \leq \dim F_i$, and $Z_i' \cap p^{-1}(U) \subset F_{ij}$, where U is some neighborhood of $O \cap \overline{W}_i' \cap p_F^{-1}(\overline{U}_j')$. For example, W_F and $G(s_F)$ are cocycles of degree r. We say that two cocycles are homologous if they can be deformed to each other through a family of cocycles.

Let m_i be the order of the local uniformization group Γ_i of V_i. Put $m = m_1 \cdots m_\ell$. Then $me([s : B \mapsto E])$ should be the intersection class of $m\, W_F$ with the graph $G(s_F)$ in $V_0(F)$.

We will construct an oriented cocycle Z in $p^{-1}(O)$, which is homologous to mW_F in $p^{-1}(O)$, such that Z is transverse to the graph of s_F in $V_0(F)$.

We will use the induction for this purpose.

Without loss of generality, we may arrange

$$\dim F_1 \le \dim F_2 \le \cdots \le \dim F_k \le \cdots.$$

By perturbing $\tilde{W}_1$ in $\tilde{W}_1 \times F_1$ and averaging over the action of Γ_1, we obtain a cycle $\tilde{Z}_1 \subset \tilde{V}_1 = \tilde{W}_1 \times F_1$, satisfying: (i) $\tilde{Z}_1$ is Γ_1-invariant; (ii) $\tilde{Z}_1 = m_1\tilde{W}_1$ near $\partial\tilde{V}_1$; (iii) $\tilde{Z}_1$ is homologous to $m_1\tilde{W}_1$ in $\pi_1^{-1}(O)$ with fixed boundary; (iv) $\tilde{Z}_1$ is transverse to $G(s_{F,1})$ in an neighborhood of $\overline{\tilde{W}_i' \times F_1}$, where $s_{F,i}$ be the induced section over $\tilde{W}_i$ by s_F.

We extend $\tilde{Z}_1/\Gamma_1$ to a cycle over $\left(\bigcup_{i\ge 2} O \cap W_i \cap p_F^{-1}(U_1)\right) \cup W_1$, such that $\tilde{Z}_1/\Gamma_1$ is contained in F_{i1} over some neighborhood of $O \cap \overline{W_i' \cap p_F^{-1}(U_1')}$ and coincides with $m_1 W_F$ near each $\left(\partial(O \cap W_i \cap p_F^{-1}(U_1))\right) \cap W_i$. Then we can glue $\tilde{Z}_1/\Gamma_1$ and $m_1 W_F$ together to form a cocycle Z_1 in $p^{-1}(O)$.

Now suppose that for $k \ge 1$, we have found cocycles Z_i $(1 \le i \le k)$ in $p^{-1}(O)$ formed by glueing $m_i Z_{i-1} \backslash p^{-1}(W_i)$ and $\tilde{Z}_i/\Gamma_i$, where $\tilde{Z}_i \subset \tilde{V}_i = \tilde{W}_i \times F_i$, satisfying:

(i) $\tilde{Z}_i$ is Γ_i-invariant and coincides with $\pi_i^{-1}(Z_{i-1})$, where $\pi_i : \tilde{V}_i \mapsto V_i$ is the projection, near $\partial\tilde{V}_i$ and $\pi_i^{-1}(\bigcup_{j<i} p^{-1}(W_j))$;

(ii) $\tilde{Z}_i$ is transverse to the graph $G(s_{F,i})$ in an neighborhood of $\overline{\tilde{W}_i' \times F_i}$;

(iii) $\tilde{Z}_i$ is homologous to $\pi_i^{-1}(Z_{i-1})$ in $\tilde{V}_i$ with boundary fixed.

Furthermore, we may assume that for each $l > i$, Z_i is contained in F_{li} over some neighborhood of $O \cap \overline{W_l' \cap U_i'}$.

Now we construct Z_{k+1}. Observe that $p|_{Z_k}$ is a branched covering of Z_k over W_F of order $m_1 \cdots m_k$. It follows that $\pi_{k+1}^{-1}(Z_k) \subset \tilde{V}_{k+1}$, counted with multiplicity, is a branched covering of $W_{k+1} \subset W_F$ of order $m_1 \cdots m_{k+1}$. Then, by the standard transversality theorem and the same arguments as we did for Z_1, we can have a Γ_{k+1}-invariant perturbation $\tilde{Z}_{k+1}$ of $\pi_{k+1}^{-1}(Z_k)$ inside $p^{-1}(O)$, such that all properties for $\tilde{Z}_i$ $(i \le k)$ are satisfied for $\tilde{Z}_{k+1}$. We define Z_{k+1} to be the glueing of $\tilde{Z}_{k+1}/\Gamma_{k+1}$ with $m_{k+1}Z_k$ along the boundary. The method is the same as that in the construction of Z_1, so we omit it.

The orientation of each $\tilde{Z}_i$ (≥ 1) is naturally induced by that of $\det(s)$, as we did before.

Thus by induction, we have constructed an integral cocycle Z in $V_0(F)$ of degree r, homologous to mW_F as we wanted. Moreover, one can show that the intersection of $G(s_F)$ with Z is a well-defined cycle in $V_0(F)$.

We define $e([s : B \mapsto E])$ to be the homology class in $H_r(B, \mathbb{Q})$, which is represented by $\frac{1}{m} Z \cap G(s_F)$. We remind the readers that $G(s_F)$ is the graph of s_F in $V_0(F)$. This class is independent of choice of the admissible orbifold section s in $[s : B \mapsto E]$. In fact, to prove it, we simply repeat the above arguments for any homotopy $S : [0, 1] \times B \mapsto E$ with $S|_{0 \times B} = s$.

One can show that $e([s : B \mapsto E])$ satisfies all functorial properties of the Euler class. So Theorem 1.2 is proved.

Remark 4 *It is very important to know when $e([s : B \mapsto E])$ is an integer class.*

Let us stratify B according to the local unformization group, namely, write B as a disjoint union of B_i, such that the local unformization group is the same at any points of B_i. In fact, each B_i consists of fixed points of local uniformization group of the same type. If $s_i = s|_{B_i} : B_i \mapsto E_i$ is a generalized Fredholm bundle of index $r_i < r$, where E_i is the subbundle of E which consists of fixed points of the local uniformization group, then in the above proof, one can show that $e([s : B \mapsto E])$ is in $H_*(B, \mathbb{Z})$. In the case of Gromov-Witten invariants for rational curves (cf. section 2, 3), if the target manifold is semi-positive, then the above assumptions hold. This explains why the Gromov-Witten invariants in [RT1], [RT2] are integer-valued.

If the above assumptions do not hold, in order to get integral Euler classes, one has to use special properties of certain generalized Fredholm orbifold bundles which arise from concrete applications.

We believe that Theorem 1.1, 1.2 can be generalized to other singular varieties. However, the resulting Euler class may lie in intersection homology groups.

We end this section with a remark. Let us denote formally by ∞_E the rank of E and by ∞_B the dimension of B. Then $r = \infty_B - \infty_E$. The Euler class we defined is Poincare dual to the Chern class $c_{\infty_E}(E)$ (formally) of E. A natural question is whether or not one can construct reasonable cycles, which are Poincare dual to the lower degree Chern classes $c_{\infty_E - i}(E)$ of E, at least under certain assumptions on $[s : B \to E]$. The compactness is the main problem.

2 Gromov-Witten invariants

Let (V, ω, J) be a compact symplectic manifold, where ω is a symplectic form and J is a compatible almost complex structure, i.e.,

$$J^2 = -id \ , \qquad \omega(Ju, Jv) = \omega(u, v) \ , \quad \forall u, v \in TV \ .$$

Then $g(u, v) = \omega(u, Jv)$ defines a Riemannian metric on V. Without loss of generality, we may assume that (V, ω, J) is C^∞-smooth.

As usual, if $2g + k \geq 3$, we denote by $\mathcal{M}_{g,k}$ the moduli space of Riemann surfaces of genus g and with k marked points. Each point of $\mathcal{M}_{g,k}$ can be presented as

$$(\Sigma; x_1, \ldots, x_k)$$

where Σ is a Riemann surface of genus g, $x_1, \ldots, x_k \in \Sigma$ are distinct. We identify $(\Sigma; x_1, \ldots, x_k)$ with $(\Sigma'; x'_1, \ldots, x'_k)$, if there is a biholomorphism $f : \Sigma \to \Sigma'$ carrying x_i to x'_i. Therefore, $\mathcal{M}_{0,3}$ consists of one point.

Let $\overline{\mathcal{M}}_{g,k}$ be the Deligne-Mumford compactification of $\mathcal{M}_{g,k}$. Then $\overline{\mathcal{M}}_{g,k}$ consists of all genus g stable curves with k marked points. It is well-known that $\overline{\mathcal{M}}_{g,k}$ is a Kähler orbifold (cf. [Mu]).

In this section, we will apply Theorem 1.2 to construct the GW-invariant

$$\Psi^V_{(A,g,k)} : H^*(V, \mathbb{Q})^k \times H^*(\overline{\mathcal{M}}_{g,k}, \mathbb{Q}) \to \mathbb{Q} \ .$$

First let us introduce the notion of stable maps.

Definition 2.1 *A stable C^ℓ-map ($\ell \geq 0$) with k marked points is a tuple $(f, \Sigma; x_1, \ldots, x_k)$ satisfying:*

1). $\Sigma = \bigcup_{i=1}^m \sum_i$ is a connected curve with normal crossings and $x_1, \ldots, x_k$ are distinct smooth points in Σ;

2). f is continuous, and each restriction $f|_{\Sigma_i}$ lifts to a C^ℓ-smooth map from the normalization $\tilde{\Sigma}_i$ into V;

3). If the homology class of $f|_{\Sigma_i}$ is zero in $H_2(V, \mathbb{Q})$ and Σ_i is a smooth rational curve, then Σ_i contains at least three of $x_i, \ldots, x_k$ and those points in $\mathrm{Sing}(\Sigma)$*, the latter denotes the singular set of Σ.*

This definition is inspired by the holomorphic stable maps in [KM]. We should also note that $2g + k$ may be less than 3 in the above definition.

Given $(f, \Sigma; x_1, \ldots, x_k)$ as above, let $\mathrm{Aut}(\Sigma; x_1, \ldots, x_k)$ be the automorphism group of $(\Sigma; x_1, \ldots, x_k)$. Note that if $2g + k \geq 3$, $(\Sigma; x_1, \ldots, x_k) \in \overline{M}_{g,k}$ if and only if $\mathrm{Aut}(\Sigma; x_1, \ldots, x_k)$ is finite. Let $\mathrm{Aut}(f, \Sigma; x_1, \cdots, x_k)$ be the group consisting of all σ in $\mathrm{Aut}(\Sigma; x_1, \ldots, x_k)$ such that $f \cdot \sigma = f$. Clearly, if $(f, \Sigma; x_1, \cdots, x_k)$ is stable, then $\mathrm{Aut}(f, \Sigma; x_1, \cdots, x_k)$ is finite.

We say that two stable maps $(f, \Sigma; x_1, \ldots, x_k)$ and $(f', \Sigma'; x'_1, \ldots, x'_k)$ are equivalent if there is a biholomorphism $\sigma : \Sigma \mapsto \Sigma'$ such that $\sigma(x_i) = x'_i$ $(1 \leq i \leq k)$ and $f' = f \circ \sigma$. We will denote by $[f, \Sigma; x_1, \ldots, x_k]$, usually abbreviated as $[\mathcal{C}]$, the equivalence class of stable maps equivalent to $(f, \Sigma; x_1, \ldots, x_k)$. Note that in case $\Sigma = \Sigma'$, σ is in $\mathrm{Aut}(\Sigma; x_1, \ldots, x_k)$.

The genus of a stable map $(f, \Sigma; x_1, \ldots, x_k)$ is defined to the genus of Σ.

Let $\overline{\mathcal{F}}^{\ell}_A(V, g, k)$ $(\ell \geq 0)$ be the space of equivalence classes $[f, \Sigma; x_1, \ldots, x_k]$ of C^{ℓ} stable maps $(f, \Sigma; x_1, \ldots, x_k)$ of genus g and with total homology class A, which is represented by the image $f(\Sigma)$ in V. Clearly, $\overline{\mathcal{F}}^{\ell}_A(V, g, k)$ is contained in $\overline{\mathcal{F}}^{\ell'}_A(V, g, k)$, if $\ell > \ell'$.

We will also denote $\overline{\mathcal{F}}^{0}_A(V, g, k)$ by $\overline{\mathcal{F}}_A(V, g, k)$.

For any sequence of stable C^{ℓ}-maps $\{(f_i, \Sigma_i; x_{i1}, \cdots, x_{ik})\}$, we say that $(f_i, \Sigma_i; x_{i1}, \cdots, x_{ik})$ converges to $(f_\infty, \Sigma_\infty; x_{\infty 1}, \cdots, x_{\infty k})$ in C^{ℓ}-topology, if there are (1) $(\Sigma_i; \{x_{ij}\})$ converges to $(\Sigma_\infty; \{x_{\infty j}\})$ as marked curves; (2) f_i converges to f_∞ in C^0-topology on Σ_∞; (3) f_i converges to f_∞ in C^{ℓ}-topology on any compact subset outside the singular set of Σ_∞. Let $\mathcal{C}_i$ be any sequence of equivalence classes of C^{ℓ}-stable maps. We say that $[\mathcal{C}_i]$ converges to $[\mathcal{C}_\infty]$, if there are $\mathcal{C}_i = (f_i, \Sigma_i; x_{i1}, \cdots, x_{ik})$ representing $[\mathcal{C}_i]$ and converging to a representative $\mathcal{C}_\infty = (f_\infty, \Sigma_\infty; x_{\infty 1}, \cdots, x_{\infty k})$ of $[\mathcal{C}_\infty]$ in C^{ℓ}-topology.

The topology of $\overline{\mathcal{F}}^{\ell}_A(V, g, k)$ is given by the sequencial convergence in the above sense. One can easly show that the homotopy class of $\overline{\mathcal{F}}^{\ell}_A(V, g, k)$ is independent of ℓ.

We define $\mathcal{F}_A(V, g, k)$ to be the set of all equivalence classes of stable maps with smooth domain. Put $\mathcal{F}^{\ell}_A(V, g, k) = \mathcal{F}_A(V, g, k) \cap \overline{\mathcal{F}}^{\ell}_A(V, g, k)$.

Remark 5 *$\mathcal{F}^{\ell}_A(V, g, k)$ is basically a family of spaces of maps from Riemann surfaces into V. Its topology has been extensively studied in the literature of algebraic topology. Here, one can regard $\overline{\mathcal{F}}_A(V, g, k)$ as a partial compactification of $\mathcal{F}_A(V, g, k)$. This partial compactification seems to have more structures than the original space does. The authors do not know much study on it in the literature. We believe that it deserves more attention.*

If $2g+k \geq 3$, one can define a natural map $\pi_{g,k}$ from $\overline{\mathcal{F}}_A(V,g,k)$ onto $\overline{\mathcal{M}}_{g,k}$ as follows:

$$\pi_{g,k}(f,\Sigma;x_1,\ldots,x_k) = \mathrm{Red}(\Sigma;x_1,\ldots,x_k)$$

where $\mathrm{Red}(\Sigma;x_1,\ldots,x_k)$ is the stable reduction of $(\Sigma;x_1,\ldots,x_k)$, which is obtained by contracting all its non-stable irreducible components. Then, we have $\mathcal{F}_A(V,g,k) = \pi_{g,k}^{-1}(\mathcal{M}_{g,k})$, moreover, we can describe $\mathcal{F}^\ell_A(V,g,k)$ locally as follows: given any $[f,\Sigma;x_1,\cdots,x_k]$ in $\pi_{g,k}^{-1}(\mathcal{M}_{g,k})$. Then the automorphism group $\Gamma = \mathrm{Aut}(\Sigma;x_1,\cdots,x_k)$ is finite. We denote by Γ_0 its subgroup consisting of automorphisms preserving f. Let W_0 be a small neighborhood of $(\Sigma;x_1,\cdots,x_k)$ in $\mathcal{M}_{g,k}$, and $p_{W_0} : \tilde{W}_0 \mapsto W_0$ be the local uniformization. Note that Γ acts on $\tilde{W}_0$ and $W_0 = \tilde{W}_0/\Gamma$. One can show that $[f,\Sigma;x_1,\cdots,x_k]$ has a neighborhood of the form $\tilde{W}_0 \times U/\Gamma_0$, where U is some open neighborhood of 0 in the space $C^\ell(\Sigma,f^*TV)$ of f^*TV-valued C^ℓ-smooth functions. Note that Γ_0 acts on $C^\ell(\Sigma,f^*TV)$ naturally. Therefore, $\mathcal{F}^\ell_A(V,g,k)$ is a Banach orbifold.

Without much more difficulty, one can also show that $\mathcal{F}^\ell_A(V,g,k)$ is a Banach orbifold, even if $2g+k<3$. However, it seems to be much harder to prove that $\overline{\mathcal{F}}^\ell_A(V,g,k)$ is smooth. Fortunately, we can avoid it by exploring its weakly smooth structure.

Next we define a generalized bundle E over $\overline{\mathcal{F}}^1_A(V,g,k)$.

In the following, we will often denote by $\mathcal{C}$ a stable map $(f,\Sigma;x_1,\ldots,x_k)$, $f_\mathcal{C}$ the map f and $\Sigma_\mathcal{C}$ the connected curve Σ.

We define $\wedge^{0,1}_\mathcal{C}$ as follows: if $\Sigma_\mathcal{C}$ is smooth, then $\wedge^{0,1}_\mathcal{C}$ consists of all continuous sections ν in $\mathrm{Hom}(T\Sigma_\mathcal{C}, f^*_\mathcal{C}TV)$ with $\nu\cdot j_\mathcal{C} = -J\cdot\nu$, where $j_\mathcal{C}$ denotes the complex structure on $\Sigma_\mathcal{C}$. In other words, $\wedge^{0,1}_\mathcal{C}$ consists of all $f^*_\mathcal{C}TV$-valued (0,1)-forms ν over $\Sigma_\mathcal{C}$. In general, $\wedge^{0,1}_\mathcal{C}$ consists of all $f^*_\mathcal{C}TV$-valued (0,1)-form ν over the normalization of $\Sigma_\mathcal{C}$, more precisely, if $\Sigma_\mathcal{C}$ has nodes $q_1,\cdots,q_s$, then $\wedge^{0,1}_\mathcal{C}$ consists of all $f^*_\mathcal{C}TV$-valued (0,1)-form ν over $\mathrm{Reg}(\Sigma_\mathcal{C})$ of $\Sigma_\mathcal{C}$ satisfying: for each i, if D_1 and D_2 are the two local components of $\Sigma_\mathcal{C}$ near q_i, then $\nu|_{D_1}$, $\nu|_{D_2}$ can be extended continuously across q_i.

Let $\mathcal{C} = (f,\Sigma;\{x_i\})$ and $\mathcal{C}' = (f',\Sigma';\{x'_i\})$ be two equivalent stable maps, and σ be the biholomorphism from Σ' to Σ such that $\sigma(x'_i) = x_i$ and $f' = f\cdot\sigma$. For convenience, we sometimes denote $\mathcal{C}'$ by $\sigma^*(\mathcal{C})$. One can show

$$\wedge^{0,1}_{\mathcal{C}'} = \sigma^*\left(\wedge^{0,1}_\mathcal{C}\right).$$

It follows that $\wedge^{0,1}_\mathcal{C}$ descends to a space $E_{[\mathcal{C}]}$ over the equivalence class of $\mathcal{C}$.

We put $E = \bigcup_{[\mathcal{C}]} E_{[\mathcal{C}]}$ and equip it with the continuous topology. Then E is a topological fibration over $\overline{\mathcal{F}}^1_A(V, g, k)$.

For simplicity, we will also use E to denote the restriction of E to $\overline{\mathcal{F}}^\ell_A(V, g, k)$ for any $\ell > 1$.

There is a natural section $\Phi([\mathcal{C}]) : \overline{\mathcal{F}}^1_A(V, g, k) \mapsto E$, i.e., the Cauchy-Riemann equation, defined as follows: for any C^1-smooth equivalence class $[\mathcal{C}] \in \overline{\mathcal{F}}^1_A(V, g, k)$, we define $\Phi([\mathcal{C}])$ to be represented by

$$df_{\mathcal{C}} + J \cdot df_{\mathcal{C}} \cdot j_{\mathcal{C}} \in E_{\mathcal{C}},$$

where $j_{\mathcal{C}}$ denotes the conformal structure of $\Sigma_{\mathcal{C}}$. Sometimes, by abusing the notations, we simply write

$$\Phi(\mathcal{C}) = df_{\mathcal{C}} + J \cdot df_{\mathcal{C}} \cdot j_{\mathcal{C}}.$$

Then we have

Proposition 2.2 *For any $\ell \geq 2$, the section $\Phi : \overline{\mathcal{F}}^\ell_A(V, g, k) \mapsto E$ gives rise to a generalized Fredholm orbifold bundle with the natural orientation and of index $2c_1(V)(A) + 2k + (2n - 6)(1 - g)$.*

We will postpone the proof of proposition 2.2 to section 3.

Let ω' be another symplectic form on V and J' be one of its compatible almost complex structure, Recall that (ω', J') is deformation equivalent to (ω, J), if there is a smooth family of symplectic forms ω_s and compatible almost complex structures J_s $(0 \leq s \leq 1)$ such that $(\omega_0, J_0) = (\omega, J)$ and $(\omega_1, J_1) = (\omega', J')$.

Proposition 2.3 *Let $\Phi' : \overline{\mathcal{F}}^\ell_A(V, g, k) \mapsto E$ be the admissible section induced by the Cauchy-Riemann equation of J', where (ω', J') is given as above. Assume that (ω', J') is deformation equivalent to (ω, J). Then Φ' is homotopic to Φ as generalized Fredholm orbifold bundles.*

The proof of this proposition is identical to that of Proposition 2.2.

Using the last two propositions, we can construct symplectic invariants, particularly, the GW-invariants.

In the following, if $2g + k < 3$, for convenience, we denote by $\overline{\mathcal{M}}_{g,k}$ the topological space of one point.

Notice that for any $\ell > 0$, $\overline{\mathcal{F}}^\ell_A(V, g, k)$ is homotopically equivalent to $\overline{\mathcal{F}}_A(V, g, k)$. Then we can deduce from Theorem 1.2

Theorem 2.4 *Let (V,ω,J) be a compact symplectic manifold with compatible almost complex structure. Then for each g,k and $A\in H_2(V,\mathbb{Z})$, there is a symplectically invariant homomorphism*

$$\rho^V_{A,g,k}: H^*(\overline{\mathcal{M}}_{g,k},\mathbb{Q}) \mapsto H_*(\overline{\mathcal{F}}_A(V,g,k),\mathbb{Q}),$$

satisfying: for any α, β in $H^*(\overline{\mathcal{M}}_{g,k},\mathbb{Q})$,

$$\rho^V_{A,g,k}(\alpha\cup\beta) = \rho^V_{A,g,k}(\alpha)/\pi^*_{g,k}\beta,$$

where $\pi_{g,k}: \overline{\mathcal{F}}_A(V,g,k)\mapsto\overline{\mathcal{M}}_{g,k}$ is defined as above. We usually write $\rho^V_{A,g,k}(1)$ as $e_A(V,g,k)$, which is a symplectically invariant class in

$$H_{2c_1(V)(A)+2k+(2n-6)(1-g)}(\overline{\mathcal{F}}_A(V,g,k),\mathbb{Q}).$$

Furthermore, if $A=0$, then for any β in $\overline{\mathcal{M}}_{g,k}$, we have that $\rho^V_{A,g,k}(\beta)$ takes values in $\tau_*(H_*(\overline{\mathcal{M}}_{g,k}\times V,\mathbb{Q}))$, where $\tau:\overline{\mathcal{M}}_{g,k}\times V\mapsto\overline{F}_A(V,g,k)$ is the natural embedding of constant maps.

Proof: By Proposition 2.2, $\Phi:\overline{\mathcal{F}}_A(V,g,k)\mapsto E$ is a genaralized Fredholm orbifold bundle of index r, where $r=2c_1(V)(A)+2k+(2n-6)(1-g)$. By Proposition 2.3, its homotopy class is independent of choices of (ω,J). It follows from Theorem 1.2 that there is an Euler class $e([\Phi:\overline{\mathcal{F}}_A(V,g,k)\mapsto E])$ in $H_r(\overline{\mathcal{F}}_A(V,g,k),\mathbb{Q})$. Then $\rho^V_{A,g,k}$ is obtained by taking slant product of this Euler class by cohomological classes in $H^*(\overline{\mathcal{M}}_{g,k},\mathbb{Q})$. All the properties can be easily checked.

Remark 6 *We conjecture that the invariant $\rho^V_{g,k}$ is integer-valued, i.e., for any α in $H^r(\overline{\mathcal{M}}_{g,k},\mathbb{Z})$, $\rho^V_{g,k}(\alpha)$ is in $H_{2c_1(V)(A)+2k+2n(1-g)-r}(\overline{\mathcal{F}}_A(V,g,k),\mathbb{Z})$.*

In order to define the GW-invariants, we observe that there is an evaluation map

$$\begin{aligned} ev:\overline{\mathcal{F}}_A(V,g,k) &\mapsto V^k,\\ ev(f,\Sigma;x_1,\ldots,x_k) &= (f(x_1),\ldots,f(x_k)), \end{aligned}$$

then we can define the GW-invariants

$$\Psi^V_{(A,g,k)}: H^*(V,\mathbb{Q})^k\times H^*(\overline{M}_{g,k},\mathbb{Q})\to\mathbb{Q}\ ,$$

namely, for any $\alpha_1,\ldots,\alpha_k\in H^*(V,\mathbb{Q})$, $\beta\in H^*(\overline{M}_{g,k},\mathbb{Q})$,

$$\Psi^V_{(A,g,k)}(\beta;\alpha_1,\ldots,\alpha_k) = \mathrm{ev}^*\left(\pi_1^*\alpha_1\wedge\cdots\wedge\pi_k^*\alpha_k\right)\left(\rho^V_{A,g,k}(\beta)\right)$$

Theorem 2.5 *The GW-invariants* $\Psi^V_{(A,g,k)}$ *are symplectic invariants satisfying the composition law and the basic properties* (cf. [KM], [RT2], [T], [Wi1]).

Remark 7 *It is believed that* $\Psi^V_{(A,g,k)}$ *is also integer-valued. In fact, it is true for semi-positive symplectic manifolds (cf. [RT1], [RT2]).*

Example 2 *Let* (V,ω,J) *be a symplectic manifold as above and* ω *be an integer class. Then for any holomorphic curve* $C \subset V$,

$$\int_C \omega \geq 1$$

We say that a pseudo-holomorphic map $f : S^2 \to V$ *is a line if* $\int_{S^2} f^*\omega = 1$. *Let* A *be the homology class of lines, then the moduli space of lines is compact modulo automorphisms of* S^2.

On the other hand, $G = \mathrm{Aut}(S^2)$ *acts naturally on* $\mathrm{Map}_A(S^2,V)$ *and the bundle* $\wedge^{0,1}(TV)$ *over* $\mathrm{Map}_A(S^2,V)$. *Therefore, we have a Fredholm bundle* E *over* $B = \mathrm{Map}_A(S^2,V)_0/G$, *where* $\mathrm{Map}_A(S^2,V)_0$ *denotes the space of maps which are generically immersive. The Cauchy-Riemann equation descends to a section of* $E \to B$. *One can show that*

$$\Psi^V_{(A,0,3)}(\alpha_1,\alpha_2,\alpha_3) = \Big(\mathrm{ev}(\pi^{-1}(e(B,E))) \cap (\alpha_1^* \times \alpha_2^* \times \alpha_3^*)\Big) \textit{ in } V^3$$

where $\pi : \mathrm{Map}_A(S^2,V) \mapsto B$ *is the natural projection, and* $\alpha_1^* \times \alpha_2^* \times \alpha_3^*$ *are Poincare duals of* $\alpha_1,\alpha_2,\alpha_3$.

Note that $e(B,E) \in H_r(B,\mathbb{Z})$ *for* $r = 2(c_1(V)\cdot A + n - 3)$.

We end up this section with two basic decomposition properties of the symplectic invariant $\rho^V_{A,g,k}$.

Let $\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \mapsto \overline{\mathcal{M}}_{g,k}$, where $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_1 + k_1 \geq 2$, $2g_2 + k_2 \geq 2$, be the map by glueing the k_1+1-th marked point of the first factor to the first marked point of the second factor. We denote by $PD(\sigma)$ the Poincare dual of $\mathrm{Im}(\sigma)$. The composition law expresses $\rho^V_{A,g,k}(PD(\sigma))$ in terms of $\rho^V_{A_1,g_1,k_1+1}$ and $\rho^V_{A_2,g_2,k_2+1}$ with $A = A_1 + A_2$.

Given any decomposition $A = A_1 + A_2$, there is an natural map

$$p : \overline{\mathcal{F}}_{A_1}(V,g_1,k_1+1) \times \overline{\mathcal{F}}_{A_2}(V,g_2,k_2+1) \mapsto V \times V,$$
$$p([h_1,\Sigma_1;x_1,\cdots,x_{k_1+1}],[h_2,\Sigma_2;y_1,\cdots,y_{k+2+1}]) = (h_1(x_{k_1+1}),h_2(y_1)).$$

Let Δ be the diagonal in $V \times V$. Then there is an obvious map π from $p^{-1}(\Delta)$ onto $\pi_{g,k}^{-1}(\mathrm{Im}(\sigma))$ by identifying x_{k_1+1} with y_1.

Clearly, $\rho_{A,g,k}^V(PD(\sigma))$ can be regarded as a class in $H_*(\pi_{g,k}^{-1}(\mathrm{Im}(\sigma)), \mathbb{Q})$.

On the other hand, if $\{u_i\}$ is any basis of $H^*(V, \mathbb{Z})$ and $\{u_i^*\}$ is its dual basis, then we have a homology class

$$\sum_i \rho_{A_1,g_1,k_1+1}^V / ev^* \pi_{k_1+1}^* u_i \otimes \rho_{g_2,k_2+1}^V / ev^* \pi_1^* u_i^*$$

in $H_*(p^{-1}(\Delta), \mathbb{Q})$.

The first composition law for $\rho_{A,g,k}^V$ is given by the equation:

$$\rho_{A,g,k}^V(PD(\sigma)) = \pi_* \left(\sum_{A=A_1+A_2} \sum_i \rho_{A_1,g_1,k_1+1}^V / ev^* \pi_{k_1+1}^* u_i \otimes \rho_{A_2,g_2,k_2+1}^V / ev^* \pi_1^* u_i^* \right).$$

The second composition law for $\rho_{A,g,k}^V$ arises from the map $\theta : \overline{\mathcal{M}}_{g-1,k+2} \mapsto \overline{\mathcal{M}}_{g,k}$, which is obtained by glueing the last two marked points, in a similar way.

As above, we define

$$\begin{aligned} &p : \overline{\mathcal{F}}_A(V, g-1, k+2) \mapsto V \times V, \\ &p([h, \Sigma; x_1, \cdots, x_{k+2}]) = (h(x_{k+1}), h(x_{k+2})). \end{aligned}$$

We also have the resolution $\pi : p^{-1}(\Delta) \mapsto \pi_{g,k}^{-1}(\mathrm{Im}(\theta))$. Then we have

$$\rho_{A,g,k}^V(PD(\theta)) = \pi_* \left(\sum_i \rho_{A,g-1,k+2}^V / \pi_{k+1}^* u_i \wedge \pi_{k+2}^* u_i^* \right).$$

3 The proof of Proposition 2.2 and 2.3

In the section, we prove Proposition 2.2 in details. The same arguments can be applied to proving Proposition 2.3. We will omit its proof except a few comments at the end of this section.

Fix any $\ell \geq 2$. Let $(f, \Sigma; x_1, \ldots, x_k)$ be a stable C^ℓ-map representing a point in $\overline{\mathcal{F}}_A^\ell(V, g, k)$. Since the structure of $\mathcal{F}_A^\ell(V, g, k)$ is clear (cf. section 2), we may assume that $[f, \Sigma; x_1, \ldots, x_k]$ is in $\overline{\mathcal{F}}_A^\ell(V, g, k) \backslash \overline{\mathcal{F}}_A^\ell(V, g, k)$.

The components of Σ can be grouped into two parts: the principal part and the bubbling part. The principal part consists of those components of genus bigger than zero and those rational components, which contain at least three of $x_1, \cdots, x_k$ and the points in $\mathrm{Sing}(\Sigma)$. Other non-stable rational components consist in the bubbling part.

By adding one or two marked points to each bubbling component, we obtain a stable curve $(\Sigma; x_1, \cdots, x_k, z_1, \ldots, z_l)$ in $\overline{\mathcal{M}}_{g,k+l}$, where $z_1, \ldots, z_l$ are added points.

Let W be a small neighborhood of $(\Sigma; \{x_i\}, \{z_j\})$ in $\overline{\mathcal{M}}_{g,k+l}$, and $\tilde{W}$ be the uniformization of W. Then $W = \tilde{W}/\Gamma$, where $\Gamma = \mathrm{Aut}(\Sigma; \{x_i\}, \{z_j\})$.

If $2g + k \geq 3$, we can express $\tilde{W}$ as follows: by contracting the bubbling part, we obtain the stable reduction $(\Sigma'; y_1, \ldots, y_k)$ of $(\Sigma; x_1, \ldots, x_k)$. Let W_0 be a small neighborhood of $(\Sigma'; y_1, \ldots, y_k)$ in $\overline{\mathcal{M}}_{g,k}$. Let $\tilde{W}_0$ be the uniformization of W_0. Then $W_0 = \tilde{W}_0/\Gamma$ and $\tilde{W} = W \times_{W_0} \tilde{W}_0$. In particular, $\tilde{W} = W$ is smooth whenever W_0 is smooth.

Let $\tilde{\mathcal{U}}$ be the universal family of curves over $\tilde{W}$. Clearly, $\tilde{\mathcal{U}}$ is smooth. We fix a metric g on $\tilde{\mathcal{U}}$. For any two maps h_1, h_2 from fibers of $\mathcal{U}$ over $\tilde{W}$, we define the distance

$$\begin{aligned} d_{\tilde{W}}(h_1, h_2) = &\ \sup_{x \in \mathrm{Dom(h_1)}} \sup_{d_g(y,x) = d_g(x, \mathrm{Dom(h_2)})} d_V(h_1(x), h_2(y)) \\ + &\ \sup_{y \in \mathrm{Dom(h_2)}} \sup_{d_g(x,y) = d_g(y, \mathrm{Dom(h_1)})} d_V(h_1(x), h_2(y)), \end{aligned}$$

where $d_g(\cdot, \cdot)$, $d_V(\cdot, \cdot)$ are distance functions of g and V.

Let Σ_j be any non-stable component of $(\Sigma; x_1, \ldots, x_k)$, then by the definition, the homology class of $f(\Sigma_j)$ is nontrivial. It follows that there is at least one regular value for $f|_{\Sigma_j}$. Therefore, we may choose $z_1, \ldots, z_l$, such that for each i, $f^{-1}(f(z_i))$ consists of finitely many immersive points.

Choose local hypersurfaces $H_1, \ldots, H_l$, such that each H_i intersects $\mathrm{Im}(f)$ transversally at $f(z_i)$.

Fix a small $\delta > 0$, we define

$$\begin{aligned} \mathrm{Map}_\delta(W) = &\ \{(\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \mid (\tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\}) \in \tilde{W}, d_{\tilde{W}}(\tilde{f}, f) < \delta, \\ &\ \tilde{f} \text{ is } C^0 \text{ on } \tilde{\Sigma} \text{ and } C^\ell \end{aligned}$$

We will equip it with the topology: any sequence $(h_a, \Sigma_a; \{x_{ia}\}, \{z_{ja}\})$ converges to $(h_\infty, \Sigma_\infty; \{x_{i\infty}\}, \{z_{j\infty}\})$, if $(\Sigma_a; \{x_{ia}\}, \{z_{ja}\})$ converges to the stable curve $(\Sigma_\infty; \{x_{i\infty}\}, \{z_{j\infty}\})$ in $\tilde{W}$, and h_a converges to h_∞ in C^0-topology everywhere and C^ℓ-topology outside $\mathrm{Sing}(\Sigma_\infty)$.

We denote by $\mathrm{Sing}(\tilde{\mathcal{U}})$ the union of singularities of the fibers of $\tilde{\mathcal{U}}$ over $\tilde{W}$. Let K be any compact subset in $\tilde{\mathcal{U}} \setminus \mathrm{Sing}(\tilde{\mathcal{U}})$ of the form: there is a diffeomorphism $\psi_K : (K \cap \Sigma) \times \tilde{W} \mapsto K$ such that $\psi_K((K \cap \Sigma) \times \{t\})$ lies in the fiber of $\tilde{\mathcal{U}}$ over t. Then we define

$$\begin{aligned}&\mathrm{Map}_\delta(W,K)\\=\ &\{(\tilde f,\tilde\Sigma;\{\tilde x_i\},\{\tilde z_j\})\in \mathrm{Map}_\delta(W)\mid ||(\tilde f\cdot\psi_K-f)|_{K\cap\Sigma\times\{0\}}||_{C^\ell}<\delta\}.\end{aligned}$$

Clearly, each $Map_\delta(W,K)$ is open in $Map_\delta(W)$.

By forgetting added marked points, each point in $\mathrm{Map}_\delta(W)$ gives rise to a stable map $\mathcal{C}$ and consequently, an equivalence class $[\mathcal{C}]$ in $\overline{\mathcal{F}}^\ell_A(V,g,k)$. Let p_W be such a projection map into $\overline{\mathcal{F}}^\ell_A(V,g,k)$. We denote by $\mathrm{Map}_\delta(W_0,K)$ the image of $\mathrm{Map}_\delta(W,K)$ in $\overline{\mathcal{F}}^\ell_A(V,g,k)$ under the projection p_W.

Let $\mathrm{Aut}(\mathcal{C})$ be the automorphism group of the stable map $\mathcal{C}$. It is a subgroup of Γ, so it is finite and acts on $\tilde{\mathcal{U}}$. Let us denote by $m(\mathcal{C})$ its order.

¿From now on, K always denotes a compact set in $\tilde{\mathcal{U}}\setminus\mathrm{Sing}(\tilde{\mathcal{U}})$ containing an open neighborhood of $\bigcup_j f^{-1}(f(z_j))$. Moreover, we may assume that K is invariant under the action of $\mathrm{Aut}(\mathcal{C})$.

Lemma 3.1 *If $\delta>0$ is sufficiently small, then the map $p_W|_{\mathrm{Map}_\delta(W,K)}$ is finite-to-one of the order $m(\mathcal{C})$, and $\mathrm{Map}_\delta(W_0,K)$ is an open neighborhood of $\mathcal{C}$ in $\overline{\mathcal{F}}^\ell_A(V,g,k)$.*

Furthermore, there is a canonical action of $\mathrm{Aut}(\mathcal{C})$ on $\mathrm{Map}_\delta(W,K)$ with the quotient $\mathrm{Map}_\delta(W_0,K)$. In particuar, if $\mathcal{C}$ has trivial automorphism group, then $p_W|_{\mathrm{Map}_\delta(W,K)}$ is actually one-to-one.

Proof: Suppose that $(h',\Sigma';\{x'_i\},\{z'_j\})$ and $(h'',\Sigma'';\{x''_i\},\{z''_j\})$, which are close to $\mathcal{C}$, have the same image under the projection p_W. Then there is a biholomorphism $\sigma:\Sigma'\mapsto\Sigma''$, such that $h'=h''\cdot\sigma$ and $\sigma(x'_i)=x''_i$. Since h',h'' are close to f, σ has to be close to an automorphism of $\mathcal{C}$. Since $h'(z'_j),h''(z''_j)\in H_j$ for $1\le j\le l$, we have that $\sigma(z'_j)$ and z''_j are close to $f^{-1}(f(z_j))$. Since f is transverse to H_j for each j, p_W is finite-to-one of order no more than $m(\mathcal{C})$.

Let us construct the action of $\mathrm{Aut}(\mathcal{C})$ on $\mathrm{Map}_\delta(W,K)$ with $\mathrm{Map}_\delta(W_0,K)$ as its quotient. In fact, let $\tau\in\mathrm{Aut}(\mathcal{C})$ and $\mathcal{C}'=(h',\Sigma';\{x'_i\},\{z'_j\})$ in $\mathrm{Map}_\delta(W,K)$. If $\mathcal{C}'$ is very close to $\mathcal{C}$, then there is a unique sequence $\{z_{\tau j}\}$ in $\tau(\Sigma')$ such that $h'(\tau^{-1}(z_{\tau j}))\in H_j$ and $z_{\tau j}$ is very close to z_j. We put

$$\tau_*(\mathcal{C}')=(h'\cdot\tau^{-1},\tau(\Sigma');\{\tau(x'_i)\},\{z_{\tau j}\}),$$

then $\tau_*(\mathcal{C})\in\mathrm{Map}_\delta(W,K)$. Clearly, if τ' is another one in $\mathrm{Aut}(\mathcal{C})$, we have that

$$(\tau\cdot\tau')_*(\mathcal{C}')=\tau_*(\tau'_*(\mathcal{C}')).$$

It follows that there is an natural action of $\mathrm{Aut}(\mathcal{C})$ on $\mathrm{Map}_\delta(W,K)$. Clearly, the quotient is $\mathrm{Map}_\delta(W_0,K)$. It also follows that p_W is of order $m(\mathcal{C})$.

If $(h,\Sigma';x_1,\ldots,x_k)$ is a stable map very close to $\mathcal{C}$, then h is immersive near z_j and there are unique z'_j in Σ' near z_j, such that $h(z'_j)\in H_j$. It follows that $(h,\Sigma';\{x'_i\},\{z'_j\})$ is in $\mathrm{Map}_\delta(W,K)$, so $\mathrm{Map}_\delta(W_0,K)$ is a neighborhood of $[\mathcal{C}]$ in $\mathcal{F}^\ell_A(V,g,k)$. The lemma is proved.

Recall that a TV-valued, $(0,1)$-form over the universal family $\tilde{\mathcal{U}}$ of curves is a continuous section ν in $Hom(\pi_1^*T\tilde{\mathcal{U}},\pi_2^*TV)$ satisfying: $\nu\cdot j_{\tilde{\mathcal{U}}}=-J\cdot\nu$, where $j_{\tilde{\mathcal{U}}}$ denotes the complex structure on $\tilde{\mathcal{U}}$. We denote by $\Gamma^{0,1}_\ell(\tilde{\mathcal{U}},TV)$ the space of such $(0,1)$-forms, which are C^ℓ smooth and vanish near $\mathrm{Sing}(\tilde{\mathcal{U}})$.

Note that $E|_{\mathrm{Map}_\delta(W_0,K)}\mapsto \mathrm{Map}_\delta(W_0,K)$ lifts to a topological bundle, denoted by $E|_{\mathrm{Map}_\delta(W,K)}$, or simply E if no possible confusions, over $\mathrm{Map}_\delta(W,K)$. In order to prove Proposition 2.2, we need to show that each $E|_{\mathrm{Map}_\delta(W,K)}$ is a generalized Fredholm bundle over $\mathrm{Map}_\delta(W,K)$.

Let Φ be defined by the Cauchy-Riemann equation in section 2. It lifts to a section, still denoted by Φ, of E over $\mathrm{Map}_\delta(W)$, explicitely,

$$\Phi(\tilde{f},\tilde{\Sigma};\{\tilde{x}_i\},\{\tilde{z}_j\})=d\tilde{f}+J\cdot d\tilde{f}\cdot j_{\tilde{\Sigma}}.$$

Let $L_{\tilde{f}}$ be the linearization of Φ at $\tilde{f}$. Then, for any vector field u over $\tilde{f}(\tilde{\Sigma})$,

$$L_{\tilde{f}}(u)=du+J(\tilde{f})\cdot du\cdot j_{\tilde{\sigma}}+\nabla_uJ\cdot d\tilde{f}\cdot j_{\tilde{\Sigma}}.$$

We denote by r the distance function to the singular set $\mathrm{Sing}(\tilde{\mathcal{U}})$ with respect to g.

For any smooth section $u\in\Gamma^0(\tilde{\Sigma},\tilde{f}^*TV)$, we define the norm

$$||u||_{1,p}=\left(\int_{\tilde{\Sigma}}(|u|^p+|\nabla u|^p)d\mu\right)^{\frac{1}{p}}+\left(\int_{\tilde{\Sigma}}r^{-\frac{2(p-2)}{p}}|\nabla u|^2d\mu\right)^{\frac{1}{2}},$$

where $p\geq 2$ and $\Gamma^0(\tilde{\Sigma},\tilde{f}^*TV)$ is the space of continuous sections of $\tilde{f}^*TV$ over $\tilde{\Sigma}$, and all norms, covariant derivatives are taken with respect to the metric $g|_{\tilde{\Sigma}}$. If $\tilde{\Sigma}$ has more than one components, then u consists of continuous sections of components which have the same value at each node.

Then we define

$$L^{1,p}(\tilde{\Sigma},\tilde{f}^*TV)=\{u\in\Gamma^0(\tilde{\Sigma},\tilde{f}^*TV)\mid ||u||_{1,p}<\infty\}.$$

Lemma 3.2 *For any $p \geq 2$, there is a uniform constant $c(p)$ such that for any fiber $\tilde{\Sigma}$ of $\tilde{\mathcal{U}}$ over $\tilde{W}$, and any u in $L^{1,p}(\tilde{\Sigma}, \tilde{f}^*TV)$, we have*

$$||u||_{C^0} \leq c(p)||u||_{1,p}.$$

Proof: We observe that any small geodesic ball of $\tilde{\Sigma}$ is uniformly equivalent to an euclidean ball or the union of two euclidean annuli of the same size. Then the lemma follows from the standard Sobolev Embedding Theorem.

It follows that $L^{1,p}(\tilde{\Sigma}, \tilde{f}^*TV)$ is complete for $p > 2$.

On the other hand, for any $v \in \mathrm{Hom}(T\tilde{\Sigma}, \tilde{f}^*TV)$, we define

$$||v||_p = \left(\int_{\tilde{\Sigma}} |v|^p d\mu\right)^{\frac{1}{p}} + \left(\int_{\tilde{\Sigma}} r^{-\frac{2(p-2)}{p}} |v|^2 d\mu\right)^{\frac{1}{2}},$$

where all norms and derivatives are taken with respect to $g|_{\tilde{\Sigma}}$, too. Then we put

$$L^p(\wedge^{0,1}(\tilde{f}^*TV)) = \{v \in \mathrm{Hom}(T\tilde{\Sigma}, \tilde{f}^*TV) \mid J \cdot v = -v \cdot j_{\tilde{\Sigma}},\ ||v||_p < \infty\}.$$

For any $(\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\})$ in $\mathrm{Map}_\delta(W)$, $L_{\tilde{f}}$ maps the space $L^{1,p}(\tilde{\Sigma}, \tilde{f}^*TV)$ into $L^p(\wedge^{0,1}(\tilde{f}^*TV))$. Let $L^*_{\tilde{f}}$ be its adjoint operator with respect to the L^2-inner product on $L^2(\wedge^{0,1}(\tilde{f}^*TV))$, more explicitly, for any $\tilde{f}^*TV$-valued (0,1)-form v,

$$L^*_{\tilde{f}} v = -e_1(v_1) - e_2(v_2) + B_{\tilde{f}}(v),$$

where $\{e_1, e_2\}$ is any orthonormal basis of $\tilde{\Sigma}$ with $j_{\tilde{\Sigma}}(e_1) = e_2$, $v_i = v(e_i)$ $(i = 1, 2)$ and $B_{\tilde{f}}(v)$ is an operator of order 0, defined by

$$2g_V(u, B_{\tilde{f}}(v)) = g_V((\nabla_u J)e_2(\tilde{f}), v_1) - g_V((\nabla_u J)e_1(\tilde{f}), v_2)$$

for any $u \in L^{1,2}(\tilde{\Sigma}, \tilde{f}^*TV)$. We denote by $\mathrm{Coker}(L_{\tilde{f}})$ the space of all v in $L^2(\wedge^{0,1}(\tilde{f}^*TV))$ such that $L^*_{\tilde{f}}(v) = 0$. Then by the standard elliptic theory, it is a finite dimensional subspace in $L^p(\wedge^{0,1}(\tilde{f}^*TV))$ for any p.

Lemma 3.3 *For any $v \in L^p(\wedge^{0,1}(\tilde{f}^*TV))$ $(p \geq 2)$, there are $v_0 \in \mathrm{Coker}(L_{\tilde{f}})$ and $u \in L^{1,p}(\tilde{\Sigma}, \tilde{f}^*TV)$, such that $L_{\tilde{f}} u = v - v_0$.*

Proof: By the definition, one can find $u \in L^{1,2}(\tilde{\Sigma}, \tilde{f}^*TV)$ and $v_0 \in \mathrm{Coker}(L_{\tilde{f}})$, such that $L_{\tilde{f}}u = v - v_0$. Then the lemma follows from the standard elliptic theory.

Let $\mathcal{C}$ be the fixed holomorphic, stable C^ℓ-map, in particular, $\Phi(\mathcal{C}) = 0$.

For any $v \in \Gamma^{0,1}_{\ell-1}(\mathcal{U}, TV)$, we define its restriction $v|_{\tilde{\mathcal{C}}}$ to a stable map $\tilde{\mathcal{C}}$ as follows: let $\tilde{\mathcal{C}} = (\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\})$, then for any $x \in \tilde{\Sigma}$, we define

$$v|_{\tilde{\mathcal{C}}}(x) = v(x, \tilde{f}(x)).$$

Let S be any finitely dimensional subspace in $\Gamma^{0,1}_{\ell-1}(\tilde{\mathcal{U}}, TV)$ ($\ell \geq 2$). We define

$$S|_{\mathcal{C}} = \{v|_{\mathcal{C}} \mid v \in S\}.$$

Then we can define E_S over $\mathrm{Map}_\delta(W, K)$ as follows: for any $\tilde{\mathcal{C}}$ in $\mathrm{Map}_\delta(W, K)$, $E_S|_{\tilde{\mathcal{C}}} = S|_{\tilde{\mathcal{C}}}$.

Assume that $\dim S = \dim S|_{\mathcal{C}}$. Then if K in $\tilde{\mathcal{U}} \backslash \mathrm{Sing}(\tilde{\mathcal{U}})$ is sufficiently large and δ is sufficiently small, E_S is a bundle of rank $\dim S$ over $\mathrm{Map}_\delta(W, K)$.

The following is the main technical result of this section.

Proposition 3.4 *Let S be as above. Suppose that its restriction $S|_{\mathcal{C}}$ to $\mathcal{C}$ is transverse to $L_{f_{\mathcal{C}}}$, i.e., if $v_1, \cdots, v_s$ span S, then $v_1|_{\mathcal{C}}, \cdots, v_s|_{\mathcal{C}}$ and $\mathrm{Im}(L_{f_{\mathcal{C}}})$ generate $L^{1,p}(\wedge^{0,1} f^*TV)$. Then by shrinking W if necessary, if δ is sufficiently small and K is sufficiently large, $\Phi^{-1}(E_S)$ is a smooth submanifold, which contains $\mathcal{C}$, in $\mathrm{Map}_\delta(W, K)$ and of dimension $2c_1(V)(A) + 2(n-3)(1-g) + 2k + \dim S$. Moreover, $E_S \mapsto \Phi^{-1}(E_S)$ is a smooth bundle.*

Remark 8 *Suppose that W and S are invariant under the natural action of $\mathrm{Aut}(\mathcal{C})$. Clearly, there is an induced action of $\mathrm{Aut}(\mathcal{C})$ on both $\Phi^{-1}(E_S)$ and the total space of the bundle E_S over $\Phi^{-1}(E_S)$.*

Now let us prove Proposition 3.4. The tool is the Implicit Function Theorem.

Let $\mathcal{C}$ be the stable map $(f, \Sigma; x_1, \cdots, x_k)$ in $\mathrm{Map}(W)$ as given in Proposition 3.4. We denote by $q_1, \cdots, q_s$ those nodes in Σ. Recall that $z_1, \cdots, z_l$ be the added points such that $f(z_i) \in H_i$.

Fix an ν in S such that its restriction to $\Sigma_{\mathcal{C}}$ is 0. In fact, for proving Proposition 3.4, we suffice to take $\nu = 0$.

First we want to construct a family of approximated (J,ν_t)-maps $\tilde{f}_t$ parametrized by $t \in \tilde{W}$, where $\nu_t = \nu|_{\Sigma_t}$. Note that a (J,ν_t)-map is a smooth $\tilde{f} : \Sigma_t \mapsto V$ satisfying the inhomogeneous Cauchy-Riemann equation:

$$\Phi(\tilde{f})(y) = \nu(y, \tilde{f}(y)), \qquad y \in \Sigma_t.$$

For any q_i $(1 \leq i \leq s)$, by shrinking $\tilde{W}$ if necessary, we may choose coordinates w_{i1}, w_{i2}, as well as t in $\tilde{W}$, near $\mathcal{C}$, such that the fiber

$$(\Sigma_t; x_1(t), \cdots, x_k(t), z_1(t), \cdots, z_l(t))$$

of $\tilde{\mathcal{U}}$ over t is locally given by the equation

$$w_{i1} w_{i2} = \epsilon_i(t), \quad |w_{i1}| < 1, \ |w_{i2}| < 1,$$

where ϵ_i is a C^∞-smooth function of t. For any y in Σ_t, if $|w_{i1}(y)| > L\sqrt{|\epsilon(t)|}$ or $|w_{i2}(y)| > L\sqrt{|\epsilon(t)|}$ for all i, where L is a large number, then there is a unique $\pi_t(y)$ in $\Sigma = \Sigma_0$ such that $d_g(y, \pi_t(y)) = d_g(y, \Sigma)$. Note that if y is not in the coordinate chart given by w_{i1}, w_{i2}, then we simply set $w_{i1}(y) = w_{i2}(y) = \infty$.

The following lemma follows from straightforward computations.

Lemma 3.5 *For any $k > 0$, there is a uniform constant a_k such that for any y in Σ_t with either $|w_{i1}(y)| > L\sqrt{|\epsilon(t)|}$ or $|w_{i2}(y)| > L\sqrt{|\epsilon(t)|}$ for all i,*

$$|\nabla^k(\pi_t - id|_{\Sigma_t})|(y) \leq a_k \min_i\{|t|, \frac{|\epsilon_i(t)|}{d_g(y,q_i)^{k+1}}\},$$

where ∇ denotes the covariant derivative with respect to g, and both π_t, id are regarded as maps from Σ_t into $\tilde{\mathcal{U}}$.

Let us introduce a complex structure $\tilde{J}$ on $\tilde{\mathcal{U}} \times V$ as follows: for any $u_1 \in T\tilde{\mathcal{U}} \subset T(\tilde{\mathcal{U}} \times V)$,

$$\tilde{J}(u_1) = j_{\mathcal{U}}(u_1) + \nu(j_{\mathcal{U}}(u_1));$$

For any u_2 in $TV \subset T(\tilde{\mathcal{U}} \times V)$, we put $\tilde{J}(u_2) = J(u_2)$.

Define $F : \Sigma \mapsto \tilde{\mathcal{U}} \times V$ by assigning y in Σ to $(y, f(y))$. We call F the graph map of f. One can show that F is $\tilde{J}$-holomorphic. In fact, for any given $\tilde{f} : \Sigma_t \mapsto V$, it is a (J,ν_t)-map if and only if its graph map is $\tilde{J}$-holomorphic (cf. [Gr]).

Put $p_i = F(q_i)$. Without loss of generality, we may assume that

$$F(\{w_{i1}w_{i2} = 0 | |w_{i1}| < 1, |w_{i2}| < 1\})$$

is contained in a coordinate chart $(u_1, \cdots, u_{2N})$ of $\tilde{\mathcal{U}} \times V$ near p_i. We may further assume that

$$\begin{aligned} \tilde{J}(\frac{\partial}{\partial u_i}) &= \frac{\partial}{\partial u_{N+i}} + \mathcal{O}(|u|), \\ \tilde{J}(\frac{\partial}{\partial u_{N+i}}) &= -\frac{\partial}{\partial u_i} + \mathcal{O}(|u|), \end{aligned}$$

where $i = 1, 2, \ldots, N$ and $|u| = \sqrt{\sum_{i=1}^{2n} |u_i|^2}$. The curve $F(\Sigma)$ has two components near p_i, which intersect transversally there. Then by changing $u_1, \cdots, u_{2N}$ appropriately, we may assume that in complex coordinates $u_1 + \sqrt{-1}u_{N+1}$, ..., $u_N + \sqrt{-1}u_{2N}$,

$$F(w_{i1}, w_{i2}) = (w_{i1}, w_{i2}, 0, 0, \cdots, 0) + \mathcal{O}(|w_{1i}|^2 + |w_{i2}|^2).$$

Using this same formula, one can easily extend F to a neighborhood of q_i in $\mathcal{U}$.

Lemma 3.6 *Let $\pi_2 : \tilde{\mathcal{U}} \times V \mapsto V$ be the natural projection. Then there is a uniform constant a such that for any y in Σ_t with $\frac{1}{2} \le |w_{i1}(y)| \le 1$ or $\frac{1}{2} \le |w_{i2}(y)| \le 1$,*

$$|\pi_2(F(y)) - f(\pi_t(y))|_{C^2} \le a|\epsilon_i(t)|.$$

This lemma can be easily proved by straightforward computations.

Let $\eta : \mathbb{R}^1 \mapsto \mathbb{R}^1$ be a cut-off function satisfying: $\eta(x) = 0$ for $|x| \le 1$, $\eta(x) = 1$ for $|x| > 2$, and $|\eta^{(k)}(x)| \le 2^k$.

We define $\tilde{f}_t(y)$, where $y \in \Sigma_t$, as follows: if either $|w_{i1}(y)| > 1$ or $|w_{i2}(y)| > 1$ for all i, put $\tilde{f}_t(y) = f(\pi_t(y))$; If for some i, $|w_{i1}(y)| < \frac{1}{2}$ and $|w_{i2}(y)| < \frac{1}{2}$, then we define $\tilde{f}_t(y) = \pi_2(F(y))$; If $\frac{1}{2} \le |w_{i1}(y)| \le 1$ or $\frac{1}{2} \le |w_{i2}(y)| \le 1$, we define $\tilde{f}_t(y)$ to be

$$\exp_{f(q_i)}\left(\eta(2d_g(y, q_i))\exp^{-1}_{f(q_i)} f(\pi_t(y)) + (1 - \eta(2d_g(y, q_i)))\exp^{-1}_{f(q_i)}\pi_2(F(y))\right).$$

Since f is continuous at each q_i, $\tilde{f}_t$ is continuous.

Lemma 3.7 *There is a uniform constant a_f such that for any $0 \le k \le \ell$ and $y \in \Sigma_t$,*

$$\begin{aligned} |\nabla^k \tilde{f}_t|(y) &\le a_f \min\{|t|, \tfrac{|\epsilon(t)|}{d_g(y,q_i)^{k+1}}\}, \\ |\nabla^{k-1}(\Phi(\tilde{f}_t) - \nu(\cdot, \tilde{f}_t(\cdot)))|(y) &\le a_f \min_i\{|t|, \tfrac{|\epsilon_i(t)|}{d_g(y,q_i)^k}\}. \end{aligned}$$

In particular, $|\nabla \tilde{f}_t|$ is uniformly bounded.

Proof: By Lemma 3.5 and 3.6, we suffice to prove those estimates near a given node, say q_i. Assume that $|w_{i1}|(y), |w_{i2}|(y) < \frac{1}{2}$. Then $\tilde{f}_t(y) = \pi_2(F(y))$. Let us prove the second estimate. The proof for the first is identical. We omit it.

We may assume that $|w_{i1}(y)| \geq |w_{i2}(y)|$. Let J_0 be the standard complex structure in the coordinate chart $\{u_1, \cdots u_{2N}\}$, i.e.,

$$J_0(\frac{\partial}{\partial u_i}) = \frac{\partial}{\partial u_{N+i}}, J_0(\frac{\partial}{\partial u_{N+i}}) = -\frac{\partial}{\partial u_i},$$

where $i = 1, \cdots, N$. Note that F is holomorphic with respect to J_0. Then we can deduce

$$\begin{aligned} & \Phi(\tilde{f}_t)(y) - \nu(y, \tilde{f}_t(y)) \\ = \ & d\pi_2 \cdot \left(dF + \tilde{J} \cdot dF \cdot j_{\Sigma_t}\right)(w_{i1}(y), w_{i2}(y), 0, \cdots, 0) \\ = \ & d\pi_2 \cdot (\tilde{J} - J_0) \cdot dF \cdot j_{\Sigma_t}(w_{i1}(y), w_{i2}(y), 0, \cdots, 0) \\ \leq \ & c|\nabla F||\tilde{J} - J_0|(w_{i1}(y), w_{i2}(y), 0, \cdots, 0), \end{aligned}$$

where c is some unfiorm constant. It follows that

$$|\Phi(\tilde{f}_t)(y) - \nu(y, \tilde{f}_t(y)| \leq \frac{c|\epsilon(t)|}{d_g(y, q_i)}.$$

Similarly, one can deduce other cases of the second estimate from the above identity.

For any t small, we denote by g_t the induced metric on Σ_t by g. Note that r is the distance function from $\mathrm{Sing}(\tilde{\mathcal{U}})$ with respect to g.

For any smooth section $u \in \Gamma^0(\Sigma_t, \tilde{f}_t^* TV)$, we recall

$$||u||_{1,p} = \left(\int_{\Sigma_t} (|u|^p + |\nabla u|^p) d\mu_t\right)^{\frac{1}{p}} + \left(\int_{\Sigma_t} r^{-\frac{2(p-2)}{p}} |\nabla u|^2 d\mu_t\right)^{\frac{1}{2}},$$

and

$$L^{1,p}(\Sigma_t, \tilde{f}_t^* TV) = \{u \in \Gamma^0(\Sigma_t, \tilde{f}_t^* TV) \mid ||u||_{1,p} < \infty\},$$

where $p \geq 2$ and $\Gamma^0(\Sigma_t, \tilde{f}_t^* TV)$ is the space of continuous sections of $\tilde{f}_t^* TV$ over Σ_t. If Σ_t has more than one components, then u consists of sections which are continuous over each of its components and have the same value at each node.

We put

$$L^{1,p} = \{(u, t) \mid u \in L^{1,p}(\Sigma_t, \tilde{f}_t^* TV)\}.$$

It is a topological bundle over $\mathcal{U}$.

On the other hand, for any $v \in \mathrm{Hom}(T\Sigma_t, \tilde{f}_t^* TV)$, we have

$$||v||_p = \left(\int_{\Sigma_t} |v|^p d\mu_t\right)^{\frac{1}{p}} + \left(\int_{\Sigma_t} r^{-\frac{2(p-2)}{p}} |v|^2 d\mu_t\right)^{\frac{1}{2}},$$

and

$$L^p(\wedge^{0,1}(\tilde{f}_t^* TV)) = \{v \in \mathrm{Hom}(T\Sigma_t, \tilde{f}^* TV) \mid J \cdot v = -v \cdot j_{\Sigma_t},\ ||v||_p < \infty\}.$$

As above, we put $L^p(\wedge^{0,1}(TV))$ to be the union of all $L^p(\wedge^{0,1}(\tilde{f}_t^* TV))$ with $t \in \tilde{W}$. It is another topological bundle over $\mathcal{U}$.

Furthermore, if $C_0^\ell(\tilde{\mathcal{U}}, TV)$ denotes the space of all C^ℓ-smooth sections, which vanish near $\mathrm{Sing}(\tilde{\mathcal{U}})$, of $\pi_2^* TV$ over $\mathcal{U} \times V$, then there is an embedding of $C_0^\ell(\tilde{\mathcal{U}}, TV)$ into $L^{1,p}$, where $\pi_2 : \tilde{\mathcal{U}} \times V \mapsto V$ is the natural projection.

Similarly, there is an embedding of $\Gamma_{\ell-1}^{0,1}(\tilde{\mathcal{U}}, TV)$ into $L^p(\wedge^{0,1} TV)$.

Note that both $C_0^\ell(\tilde{\mathcal{U}}, TV)$ and $\Gamma_{\ell-1}^{0,1}(\tilde{\mathcal{U}}, TV)$ are bundles over $\tilde{\mathcal{U}}$.

By straightforward computations, we can deduce from Lemma 3.7.

Lemma 3.8 *For any $p > 2$, we have*

$$||\Phi(\tilde{f}_t) - \nu(\cdot, \tilde{f}_t(\cdot))||_p \leq c|t|^{\frac{1}{2}},$$

where c is a uniform constant.

Next we define a map from $L^{1,p}$ into $L^p(\wedge^{0,1}(TV))$ as follows: for any (u, t) in $L^{1,p}$,

$$\Psi(u, t) = \Phi(exp_{\tilde{f}_t} u),$$

where $exp_{\tilde{f}_t} u$ denotes the function which takes value $exp_{\tilde{f}_t(x)} u(x)$ at x.

Clearly, this map Ψ is well-defined, and maps $C_0^\ell(\tilde{\mathcal{U}}, TV)$ into $\Gamma_{\ell-1}^{0,1}(\tilde{\mathcal{U}}, TV)$.

Remark 9 *Here we have used the fact that $(x, t) \mapsto (x, t, \tilde{f}_t(x))$ defines a smooth map from $\tilde{\mathcal{U}}$ into $\tilde{\mathcal{U}} \times V$.*

Now let us study the linearization $L_t = D_u \Psi$ of Ψ at $(0, t)$: for any u, we have

$$L_t(u) = du + J(\tilde{f}_t) \cdot du \cdot j_{\Sigma_t} + \nabla_u J(\tilde{f}_t) \cdot d\tilde{f}_t \cdot j_{\Sigma_t}.$$

First we want to establish uniform elliptic estimates for L_t.

Lemma 3.9 *There is a uniform constant c such that for any (u,t) in $L^{1,p}$, we have*

$$||u||_{1,p} \le c(||L_t(u)||_p + ||u||_{1,2}).$$

Proof: We may assume that $p > 2$, otherwise, the lemma is trivially true. Without loss of generality, we may further assume that $r \le \frac{1}{2}$ if both w_{i1} and w_{i2} are less than $\frac{1}{2}$ for some i.

Let η be a cut-off function satisfying: $\eta(x) = 0$ for $|x| \le \frac{1}{4}$, $\eta(x) = 1$ for $|x| > \frac{1}{2}$, and $|\eta'| \le 2$.

Put $\tilde{u} = \eta(r)u$. It vanishes whenever $|w_{i1}|, |w_{i2}| \le \frac{1}{2}$. Moreover, we have

$$L_t(\tilde{u}) = \eta(r)L_t(u) + \eta'(r)\left(u\,dr + (J(\tilde{f}_t)u)\,dr \cdot j_{\Sigma_t}\right).$$

Since Σ_t has uniformly bounded geometry in the region where $r \ge \frac{1}{4}$, we can apply the standard L^p-estimate for 1^{st}-order elliptic operators and obtain

$$||\tilde{u}||_{1,p} \le c(||L_t(\tilde{u})||_p + ||\tilde{u}||_{1,2}).$$

Together with the previous identity, we deduce

$$||\tilde{u}||_{1,p} \le c(||L_t(u)||_p + ||u||_p + ||u||_{1,2}).$$

Note that c always denotes a uniform constant, which may depend on p. By the Sobolev inequality in dimension two ($\dim \Sigma_t = 2$), we have $||u||_p \le c||u||_{1,2}$, hence,

$$||\tilde{u}||_{1,p} \le c(||L_t(u)||_p + ||u||_{1,2}).$$

Therefore, we suffice to show that for each i,

$$\begin{aligned} \left(\int_{|w_{i1}|,|w_{i2}|\le\frac{1}{2}} |\nabla u|^p d\mu_t\right)^{\frac{1}{p}} &\le c(||L_t(u)||_p + ||u||_{1,2}), \\ \left(\int_{|w_{i1}|,|w_{i2}|\le\frac{1}{2}} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} |\nabla u|^2 d\mu_t\right)^{\frac{1}{2}} &\le c(||L_t(u)||_p + ||u||_{1,2}). \end{aligned}$$

Let us first prove the second inequality. Without loss of generality, we assume that $\epsilon_i(t) \ne 0$. Write $w_{i1} = \rho e^{\sqrt{-1}\theta}$, then $w_{i2} = \frac{|\epsilon_i(t)|}{\rho} e^{\sqrt{-1}(\theta+\theta_0)}$, where $\epsilon_i(t) = |\epsilon_i(t)| e^{\sqrt{-1}\theta_0}$. Hence, $|w_{i1}|^2 + |w_{i2}|^2 = \rho^2 + \frac{|\epsilon_i(t)|^2}{\rho^2}$. Moreover, $|w_{i1}|, |w_{i2}| \le 1$ whenever $|\epsilon_i(t)| \le \rho \le 1$.

Put u_i to be zero if either $\rho > 1$ or $\rho < |\epsilon_i(t)|$, and $(1 - \eta(\frac{r}{2}))u$ otherwise. In terms of ρ and θ, we have the following expression:

$$L_t(u_i)(\frac{\partial}{\partial \rho}) = \frac{\partial u_i}{\partial \rho} + \frac{1}{\rho} J(\tilde{f}_t)\left(\frac{\partial u_i}{\partial \theta}\right) + \frac{1}{\rho}(\nabla_{u_i} J)\frac{\partial \tilde{f}_t}{\partial \theta}.$$

It follows that

$$\begin{array}{rl} & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|\frac{\partial u_i}{\partial\rho}+\frac{1}{\rho}J(\tilde{f}_t)\left(\frac{\partial u_i}{\partial\theta}\right)|^2 d\mu_t \\ \leq & c\left(||L_t(u_i)||_p^2+\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|u_i|^2|\nabla\tilde{f}_t|^2 d\mu_t\right) \\ \leq & c\left(||L_t(u)||_p^2+||u||_{1,2}^2+\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|u|^2 d\mu_t\right) \end{array}$$

Notice that the integral

$$\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p-1}}d\mu_t$$

is bounded by a constant depending only on p. However, by the Sobolev Embedding Theorem, we have

$$\left(\int_{|w_{i1}|,|w_{i2}|\leq 1}|u_i|^{2p}d\mu_t\right)^{\frac{1}{p}}\leq c(p)||u||_{1,2}^2.$$

It follows

$$\begin{array}{rl} & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|\frac{\partial u_i}{\partial\rho}+\frac{1}{\rho}J(\tilde{f}_t)\left(\frac{\partial u_i}{\partial\theta}\right)|^2 d\mu_t \\ \leq & c(||L_t(u)||_p^2+||u||_{1,2}^2). \end{array}$$

We have

$$\begin{array}{rl} & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\left(|\frac{\partial u_i}{\partial\rho}|^2+\frac{1}{\rho^2}|\frac{\partial u_i}{\partial\theta}|^2\right)d\mu_t \\ = & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\left(|\frac{\partial u_i}{\partial\rho}+\frac{1}{\rho}J(\tilde{f}_t)\left(\frac{\partial u_i}{\partial\theta}\right)|^2\right. \\ & \left.-2\langle\frac{\partial u_i}{\partial\rho},\frac{1}{\rho}J(\tilde{f}_t)\left(\frac{\partial u_i}{\partial\theta}\right)\rangle\right)d\mu_t \end{array}$$

Using integration by parts, we derive

$$\begin{array}{rl} & \left|\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle\frac{\partial u_i}{\partial\rho},\frac{1}{\rho}J_0\left(\frac{\partial u_i}{\partial\theta}\right)\rangle d\mu_t\right| \\ \leq & \frac{p-2}{p}\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|\langle u_i-a(\rho),\frac{1}{\rho^2}J_0\left(\frac{\partial u_i}{\partial\theta}\right)\rangle|d\mu_t, \end{array}$$

where $J_0=J(q_i)$ and $a(\rho)$ is any function on ρ. Using the Poincare inequality on the unit circle and choosing $a(\rho)$ appropriately, we can show that the last integral is no bigger than

$$\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\frac{1}{\rho^2}|\frac{\partial u_i}{\partial\theta}|^2 d\mu_t,$$

which is the same as the integral

$$
\begin{aligned}
& \tfrac{1}{2}\int_{|w_{i1}|,|w_{i2}|\le 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}} \\
& \quad \left(|\tfrac{\partial u_i}{\partial \rho} - L_t(u_i) - \tfrac{1}{\rho}(\nabla_{u_i}J)\tfrac{\partial \tilde f}{\partial \theta}|^2 + \tfrac{1}{\rho^2}|\tfrac{\partial u_i}{\partial \theta}|^2\right) d\mu_t \\
\le \; & c\,(||u_i|^2_{1,2} + ||L_t u_i||_p^2) \\
& \quad + \left(\tfrac{1}{2}+\tfrac{1}{4p}\right)\int_{|w_{i1}|,|w_{i2}|\le 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\left(|\tfrac{\partial u_i}{\partial \rho})|^2 + \tfrac{1}{\rho^2}|\tfrac{\partial u_i}{\partial \theta}|^2\right) d\mu_t.
\end{aligned}
$$

On the other hand, by the above arguments, one can also show that

$$
\begin{aligned}
& \left|\int_{|w_{i1}|,|w_{i2}|\le 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle \tfrac{\partial u_i}{\partial \rho}, \tfrac{1}{\rho}(J-J_0)\left(\tfrac{\partial u_i}{\partial \theta}\right)\rangle d\mu_t\right| \\
\le \; & c||u_i||_{1,2} + \tfrac{1}{2p}\int_{|w_{i1}|,|w_{i2}|\le 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\left(|\tfrac{\partial u_i}{\partial \rho}|^2 + \tfrac{1}{\rho^2}|\tfrac{\partial u_i}{\partial \theta}|^2\right) d\mu_t.
\end{aligned}
$$

Combining all the above inequalities, we can deduce the second inequality we wanted.

To obtain the first from the second, we decompose the region $\{|\epsilon_i(t)| \le \rho \le 1$ into subannuli $\{\delta_j \le \rho \le \delta_{j-1}\}$, where $j = 1, \cdots, m$, $\delta_0 = 1$, $\delta_m = |\epsilon_i(t)|$ and $1 \le \frac{\delta_{j-1}}{\delta_j} < 2$.

On each subannulus $\{\delta_j \le \rho \le \delta_{j-1}\}$, the scaled metric $\delta_j^{-2} g_t$ has bounded geometry, we can apply the standard L^p-estimate and obtain

$$
\begin{aligned}
& \int_{\delta_j\le\rho\le\delta_{j-1}} |\nabla u_i|^p d\mu_t \\
\le c \; & \left(\int_{\delta_j\le\rho\le\delta_{j-1}} |L_t u_i|^p d\mu_t + \left(\delta_j^{-\frac{2p-4}{p}}\int_{\delta_j\le\rho\le\delta_{j-1}} |\nabla u_i|^2 d\mu_t\right)^{\frac{p}{2}}\right).
\end{aligned}
$$

Clearly, the first inequality we wanted follows by suming up these over j. The lemma is proved.

Lemma 3.10 *Let S be as in Proposition 3.4 and t sufficiently small. Then for any $p > 2$ and v in $L^p(\wedge^{0,1}\tilde f_t^* TV)$, there are u in $L^{1,p}(\tilde f_t^* TV)$ and v_0 in S, satisfying:*

$$
\begin{aligned}
& L_t u = v - v_0, \\
& \max\{||u||_{1,p}, ||v_0||_p\} \le c||v||_p,
\end{aligned}
$$

where c is a uniform constant.

Proof: First we prove that there are u, v_0 such that $L_t u = v - v_0$ for sufficiently small t. If not, we can find a sequence $\{t_j\}$ with $\lim t_j = 0$ and

v_j in $\mathrm{Coker}(L_{t_j})$, such that each v_j is perpendicular to S with respect to the L^2-metric on $L^2(\wedge^{0,1}\tilde f_{t_j}^* TV)$.

Note that $v_j \in L^p(\wedge^{0,1}\tilde f_{t_j}^* TV)$ for any p. We normalize $||v_j||_p = 1$.

By using standard elliptic estimates (cf. [GT]), one can easily show that v_j converges to some v_∞ in $L^p(\wedge^{0,1}\tilde f_{\mathcal{C}}^* TV)$ outside the singular set of $\Sigma_{\mathcal{C}}$. Clearly, v_∞ is perpendicular to S and $L_0^* v_\infty = 0$, so by our assumptions, $v_\infty = 0$. It follows that for any compact subset $K' \subset \mathcal{U}$ with $K' \cap \mathrm{Sing}(\Sigma_{\mathcal{C}}) = \emptyset$, we have

$$\int_{K'\cap\Sigma_{t_j}} |v_j|^p d\mu_{t_j} + \int_{K'\cap\Sigma_{t_j}} r^{-\frac{2(p-2)}{p}} |v_j|^2 d\mu_{t_j} \mapsto 0, \quad \text{as } j \mapsto \infty.$$

Put $t = t_j$ for any fixed j. Let $\epsilon_i(t), w_{i1}, w_{i2}$ be as above, near some node q_i of $\Sigma_{\mathcal{C}}$. As before, without loss of generality, we assume that $\epsilon_i(t) \neq 0$. Write $w_{i1} = \rho e^{\sqrt{-1}\theta}$, then $w_{i2} = \frac{|\epsilon_i(t)|}{\rho} e^{\sqrt{-1}(\theta+\theta_0)}$, where $\epsilon_i(t) = |\epsilon_i(t)| e^{\sqrt{-1}\theta_0}$. Hence, $|w_{i1}|^2 + |w_{i2}|^2 = \rho^2 + \frac{|\epsilon_i(t)|^2}{\rho^2}$. Moreover, $|w_{i1}|, |w_{i2}| \leq 1$ whenever $|\epsilon_i(t)| \leq \rho \leq 1$.

Using $L_{t_j}^* v_j = 0$, we have

$$\begin{array}{rl} & \int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} |\frac{\partial v_{j\rho}}{\partial\rho} + \frac{1}{\rho}\frac{\partial v_{j\theta}}{\partial\theta}|^2 d\mu_t \\ \leq & c\int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} |v_j|^2 d\mu_t, \end{array}$$

where $v_{j\rho} = v_j(\frac{\partial}{\partial\rho})$ and $v_{j\theta} = v_j(\frac{\partial}{\partial\theta})$. Note that

$$v_{j\rho} = J(\tilde f_t) v_{j\theta}, \quad v_{j\theta} = -J(\tilde f_t) v_{j\rho}.$$

It follows that

$$\begin{array}{rl} & \int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} |\frac{\partial v_{j\rho}}{\partial\rho} - \frac{1}{\rho} J(\tilde f_t)\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)|^2 d\mu_t \\ \leq & c\int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} |v_j|^2 d\mu_t. \end{array}$$

We have

$$\begin{array}{l} \displaystyle\int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} \left(|\frac{\partial v_{j\rho}}{\partial\rho}|^2 + \frac{1}{\rho^2}|\frac{\partial v_{j\rho}}{\partial\theta}|^2 \right) d\mu_t \\ \displaystyle = \int_{|w_{i1}|,|w_{i2}|\leq 1} (|w_{i1}|^2 + |w_{i2}|^2)^{-\frac{p-2}{p}} \left(|\frac{\partial v_{j\rho}}{\partial\rho} - \frac{1}{\rho} J(\tilde f_t)\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)|^2 \right. \\ \displaystyle \qquad\qquad \left. +2\langle \frac{\partial v_{j\rho}}{\partial\rho}, \frac{1}{\rho} J(\tilde f_t)\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)\rangle \right) d\mu_t \end{array}$$

Using integration by parts, we derive

$$\begin{aligned} & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle\frac{\partial v_{j\rho}}{\partial\rho},\frac{1}{\rho}J_0\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)\rangle d\mu_t \\ = & \int_{|w_{i1}|=1 \text{ or } |w_{i2}|=1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle v_{j\rho},\frac{1}{\rho}J_0\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)\rangle d\mu_t \\ + & \frac{p-2}{p}\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle v_{j\rho}-a(\rho),\frac{1}{\rho^2}J_0\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)\rangle d\mu_t, \end{aligned}$$

where $J_0 = J(q_i)$ and $a(\rho)$ is any function on ρ. Using the Poincare inequality on the unit circle and choosing $a(\rho)$ appropriately, we can show that the last integral is no bigger than

$$\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\frac{1}{\rho^2}|\frac{\partial v_{j\rho}}{\partial\theta}|^2 d\mu_t.$$

On the other hand, one may assume that for j sufficiently large,

$$\begin{aligned} & \int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\langle\frac{\partial v_{j\rho}}{\partial\rho},\frac{1}{\rho}(J-J_0)\left(\frac{\partial v_{j\rho}}{\partial\theta}\right)\rangle d\mu_t \\ \leq & \frac{1}{2p}\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}\left(|\frac{\partial v_{j\rho}}{\partial\rho}|^2+\frac{1}{\rho^2}|\frac{\partial v_{j\rho}}{\partial\theta}|^2\right)d\mu_t. \end{aligned}$$

Combining all above estimates, we have

$$\begin{aligned} & \lim_{j\to\infty}\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|\nabla v_j|^2 d\mu_t \\ = & \lim_{j\to\infty}\int_{|w_{i1}|,|w_{i2}|\leq 1}(|w_{i1}|^2+|w_{i2}|^2)^{-\frac{p-2}{p}}|v_j|^2 d\mu_t \\ = & 0. \end{aligned}$$

Then one can deduce from this that $\lim_{j\to\infty}||v_j||_p = 0$. A contradiction! Therefore, we have proved the first part.

Let us prove the estimate by contradiction. Suppose that it is not true, then there are u_i in $L^{1,p}(\tilde{f}_{t_i}^*TV)$ and v_{0i} in S satisfying:
(1) $\max\{||u_i||_{1,p}, ||v_{0i}||_p\} = 1$;
(2) u_i are perpedicular to $\mathrm{Ker}(\pi_S\cdot L_{t_i})$, where π_S is the projection onto the orthogonal complement of S in $L^{1,2}(\tilde{f}_{t_i}^*TV)$;
(3) $\lim_{i\to\infty}||L_{t_i}u_i + v_{0i}||_p = 0$.

We may choose t_i such that $\lim_i t_i = t_\infty$ exists.

By (1) and the Sobolev Embedding Theorem, we may assume that u_i converges to u_∞ in the $L^{1,2}$-norm. We may further assume that v_{0i} converges to some $v_{0\infty}$. Note that $L_f u_\infty = v_{0\infty}$.

If $v_{0\infty}\neq 0$, then $u_\infty\neq 0$. Then $u_\infty\in\mathrm{Ker}(\pi_S\cdot L_f)$, which is impossible. Therefore, we have $v_{0\infty} = 0$. This implies that $\lim||u_i||_{1,p} = 1$. It follows from Lemma 3.9 that $||u_i||_{1,2}$ is uniformly bounded away from zero. Then one can show that u_∞ is in $\mathrm{Ker}(\pi_S \cdot L_f)$, a contradiction! The lemma is proved.

Let P be a finitely dimensional subspace in $C_0^\ell(\tilde{\mathcal{U}}, TV)$. Then for any map $\tilde{f} : \tilde{\Sigma} \mapsto V$, where $\tilde{\Sigma}$ is a fiber of $\tilde{\mathcal{U}}$ over $\tilde{W}$, we define $u|_{\tilde{f}}$ by

$$u|_{\tilde{f}}(x) = u(x, \tilde{f}(x)), \quad \text{for any } x \in \tilde{\Sigma},$$

and

$$P_{\tilde{f}} \;=\; \{u|_{\tilde{f}} \mid u \in P\}.$$

We assume that $\dim P = \dim P_f$ and $q_S(\mathrm{Ker}(\pi_S \cdot L_0)) = P_f$, where π_S is defined in the proof of Lemma 3.10 and $q_S : L^{1,2}(\tilde{\Sigma}, \tilde{f}^*TV) \mapsto P_{\tilde{f}}$ is the projection with respect to the L^2-inner product.

One can easily deduce from the above lemma the following.

Lemma 3.11 *Let P and S be as above and t be sufficiently small. Then for any $p > 2$, $u_0 \in P_{\tilde{f}_t}$ and v in $L^p(\wedge^{0,1}\tilde{f}_t^*TV)$, there are unique u in $L^{1,p}(\tilde{f}_t^*TV)$ and v_0 in S, satisfying:*

$$\begin{aligned} &q_S(u) = u_0, \quad L_t(u) = v - v_0, \\ &\max\{||u||_{1,p}, ||v_0||_p\} \le c \max\{||u_0||_{1,p}, ||v||_p\}, \end{aligned}$$

where c is a uniform constant.

Proof of Proposition 3.4: We have the following expansion:

$$\Psi(u,t) = \Psi(0,t) + L_t u + H_t(u),$$

where $H_t(u)$ is the term of higher order satisfying: $||H_t(u)||_p \le c||u||_{C^0}||u||_{1,p}$ for some uniform constant c, which depends only on the derivatives of J. By the Sobolev Embedding Theorem, it follows

$$||H_t(u)||_p \le c||u||_{1,p}^2.$$

Also note that $\Psi(0,t) = \Phi(\tilde{f}_t)$.

Consider the map $\Xi : L^{1,p} \times E_S \mapsto L^p(\wedge^{0,1}TV) \times E_P$, defined by

$$\Xi(u,t,v_0) = (\Psi(u,t) + v_0, q_S(u)).$$

Note that E_P is the bundle induced by P over $\tilde{W}$ with fibers $P_{\tilde{f}_t}$. The linearization of Ξ at $(0, t, 0)$ is the map

$$\begin{array}{rcl} D\Xi : L^{1,p}(\Sigma_t, \tilde{f}_t^* TV) \times S_{\tilde{f}_t} & \mapsto & L^p(\wedge^{0,1} \tilde{f}_t^* TV) \times P_{\tilde{f}_t}, \\ (u, v_0) & \mapsto & (L_t(u) + v_0, q_S(u)). \end{array}$$

By Lemma 3.11, it is an isomorphism with uniformly bounded inverse. Therefore, by Lemma 3.8 and the Implicit Function Theorem, there is an $\epsilon_0 > 0$ such that for any $(0, u_0) \in L^p(\wedge^{0,1} \tilde{f}_t^* TV) \times P_{\tilde{f}_t}$ with $||u_0||_{1,p} < \epsilon_0$ and $d_{\tilde{W}}(t, 0) < \epsilon_0$, there is a unique (u, t, v_0) satisfying:

$$\begin{array}{l} \Xi(u, t, v_0) = (0, u_0), \\ \max\{||u||_{1,p}, ||v_0||_p\} \le c||u_0||_{1,p}, \end{array}$$

where c is some uniform constant.

It follows that if W is sufficiently small, the subset

$$\{(u, t) \in L^{1,p} | \pi_S \cdot \Psi(u, t) = 0, ||u||_{1,p} < \epsilon_0\}$$

is parametrized by u_0 in P and $t \in \tilde{W}$. In particular, it is a smooth manifold of dimension $\dim S + 2c_1(V)(A) + 2n(1 - g) + 2k + 2l$. Note that by our choice of P, we have

$$\dim P = \dim S + 2c_1(V)(A) + 2n(1 - g).$$

We define $Y_{\epsilon_0}(S, W)$ to be

$$\{(u, t) \in L^{1,p} | \pi_S \cdot \Psi(u, t) = 0, ||u||_{1,p} < \epsilon_0, \exp_{\tilde{f}_t(z_j)} u(z_j) \in H_j\},$$

where z_j $(1 \le j \le l)$ are added points given at the beginning of this section. Then $Y_{\epsilon_0}(S, W)$ is a smooth manifold of dimension

$$\dim S + 2c_1(V)(A) + 2n(1 - g) + 2k.$$

We claim that for δ sufficiently small and K is sufficiently large, $\Phi^{-1}(E_S)$ is an open set in $Y_{\epsilon_0}(S, W)$.

Let $(\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\})$ be in $\Phi^{-1}(E_S)$. We denote by t the corresponding point $(\tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\})$ in $\tilde{W}$.

Using the fact that $d(\tilde{f}, f) \le \delta$, we can write $\tilde{f}(x) = \exp_{\tilde{f}_t(x)} u(x)$ for some $\tilde{f}_t^* TV$-valued function u. We suffice to show that $u \in L^{1,p}(\Sigma_t, \tilde{f}_t^* TV)$ and $||u||_{1,p} < \epsilon_0$. It follows from the following lemma.

Lemma 3.12 *For any $p > 2$, there is a uniform constant c such that*

$$\int_{\tilde{\Sigma}} r^{\frac{2(p-2)}{p}} |u|^p d\mu \leq c||u||_{C^0(K)},$$

where r is the distance function from the set of nodes as we used before.

Write $v_0 = \Phi(\tilde{f}) \in S$. Then $||v_0||_{C^0(K)} \leq c\delta$ for some uniform constant c. By our choice of S, it follows that if δ is sufficiently small, $||v_0||_{C^1} << \epsilon_0$. Then Lemma 3.12 can be proved by asymptotic analyses near nodes of Σ or the arguments in the proof of Lemma 3.9.

Finally, by differentiating $\pi_S \cdot \Phi(\tilde{f}) = 0$ on t and using Lemma 3.11, one can show that $E_S \mapsto \Phi^{-1}(E_S)$ is a smooth bundle and $\Phi|_{\Phi^{-1}(E_S)}$ is a smooth section. This is essentially the smooth dependence of solutions, which are produced by the Implicit Function Theorem, on parameters.

Proposition 3.4 is proved.

Proof of Proposition 2.2: We first need to construct a covering of $\Phi^{-1}(0)$ by open subsets, which will be parametrized by $[\mathcal{C}] = [f, \Sigma; \{x_i\}] \in \Phi^{-1}(0)$, a small number $\delta > 0$, an neighborhood W_0 of the stable reduction $\mathrm{Red}(\Sigma; \{x_i\})$ of $(\Sigma; \{x_i\})$ in $\overline{\mathcal{M}}_{g,k}$, a compact subset K in the universal family $\tilde{\mathcal{U}}$ of curves over $\tilde{W}$. Here W, $\tilde{W}$ are given as before. We define

$$U_\delta([\mathcal{C}], W_0, K) = \mathrm{Map}_\delta(W_0, K),$$

where $\mathrm{Map}_\delta(W_0, K)$ is given in Lemma 3.1.

Each $U_\delta([\mathcal{C}], W_0, K)$ is of the form $\mathrm{Map}_\delta(W, K)/\Gamma$, where $\Gamma = \mathrm{Aut}(\mathcal{C})$ and $\mathrm{Map}_\delta(W, K)$ were given as in Lemma 3.1. We put

$$\tilde{U}_\delta([\mathcal{C}], W_0, K) = \mathrm{Map}_\delta(W, K).$$

It is the uniformization of $U_\delta([\mathcal{C}], W_0, K)$. Therefore, we have shown that $\overline{\mathcal{F}}^\ell_A(V, g, k)$ is a topological orbifold.

Let E be the space of TV-valued $(0,1)$-forms defined in section 2. For each $U_\delta([\mathcal{C}], W_0, K)$, as we have already seen, E can be lifted to a topological bundle $E|_{\tilde{U}_\delta([\mathcal{C}], W_0, K)}$ over $\tilde{U}_\delta([\mathcal{C}], W_0, K)$. For the reader's convenience, we recall briefly the definition of this lifted bundle: for any $\tilde{\mathcal{C}} = (\tilde{f}, \tilde{\Sigma}; \{\tilde{x}_i\}, \{\tilde{z}_j\})$ in $\tilde{U}_\delta([\mathcal{C}], W_0, K)$, the fiber of $E|_{\tilde{U}_\delta([\mathcal{C}], W_0, K)}$ at $\tilde{\mathcal{C}}$ consists of all $C^{\ell-1}$-smooth, $\tilde{f}^*TV$-valued $(0,1)$-forms on $\tilde{\Sigma}$. When one passes from $\tilde{U}_\delta([\mathcal{C}], W_0, K)$ to another local uniformization $\tilde{U}_{\delta'}([\mathcal{C}'], W_0', K')$, there is an obvious bundle transition map, which lifts the identity map on E, from $E|_{\tilde{U}_\delta([\mathcal{C}], W_0, K)}$ into

$E|_{\tilde{U}_{\delta'}([\mathcal{C}'],W_0',K')}$. Moreover, those transition maps satisfy all properties listed in section 1. Therefore, we have a topological orbifold bundle E over $\overline{\mathcal{F}}^{\ell}_A(V,g,k)$, which is locally described by those $E|_{\tilde{U}_\delta([\mathcal{C}],W_0,K)}$.

The Cauchy-Riemann operator Φ can be canonically lifted to each local uniformization $\tilde{U}_\delta([\mathcal{C}],W_0,K)$.

Now let us check that $\Phi : \overline{\mathcal{F}}^{\ell}_A(V,g,k) \mapsto E$ satisfy all properties (1) - (4) in the definition of generalized Fredholm orbifold bundles.

All those $U_\delta([\mathcal{C}],W_0,K)$ cover the moduli space $\Phi^{-1}(0)$ in $\overline{\mathcal{F}}^{\ell}_A(V,g,k)$. By the Gromov Compactness Theorem (cf. [Gr], [PW], [Ye], and also [RT1], Proposition 3.1), $\Phi^{-1}(0)$ is compact in $\overline{\mathcal{F}}^{\ell}_A(V,g,k)$ $(\ell \geq 2)$.

For any $S \subset \Gamma^{0,1}(\mathcal{U},TV)$ with properties stated in Proposition 3.4, we can define a bundle E_S of finite rank as before, where W_0, δ are small and K is big. Moreover, we assume that S is invariant under the action of $\mathrm{Aut}(\mathcal{C})$. By Proposition 3.4, $(E_S, \Phi^{-1}(E_S))$ is a smooth approximation of $\tilde{U}_\delta([\mathcal{C}],W_0,K)$. Furthermore, $(E_S, \Phi^{-1}(E_S))$ is invariant under the action of $\mathrm{Aut}(\mathcal{C})$. We denote such a smooth approximation by

$$(\tilde{E}_{\delta,S}([\mathcal{C}],W_0,K), \tilde{X}_{\delta,S}([\mathcal{C}],W_0,K)).$$

One can easily show that all the smooth approximations of the form

$$(\tilde{E}_{\delta,S}([\mathcal{C}],W_0,K), \tilde{X}_{\delta,S}([\mathcal{C}],W_0,K))$$

are compatible with above transition maps between local uniformizations $\{\tilde{U}_\delta([\mathcal{C}],W,K)\}$. Therefore, $\Phi : \overline{\mathcal{F}}^{\ell}_A(V,g,k) \mapsto E$ is weakly smooth. Its index can be computed by the Atiyah-Singer Index Theorem and is equal to $2c_1(V)(A) + (2n-3)(1-g) + 2k$.

We put

$$E_{\delta,S}([\mathcal{C}],W_0,K) = \tilde{E}_{\delta,S}([\mathcal{C}],W_0,K)/\Gamma,$$
$$X_{\delta,S}([\mathcal{C}],W_0,K) = \tilde{X}_{\delta,S}([\mathcal{C}],W_0,K)/\Gamma,$$

where $\Gamma = \mathrm{Aut}(\mathcal{C})$.

We claim that $\Phi^{-1}(0)$ can be covered by finitely many smooth approximations of the form $X_{\delta,S}([\mathcal{C}],W_0,K)$. This is the same as saying that for each $[\mathcal{C}]$ in $\Phi^{-1}(0)$, there is a small neighborhood U such that for some smooth approximation $X_{\delta,S}([\mathcal{C}],W_0,K)$,

$$[\mathcal{C}] \in U \cap \Phi^{-1}(0) \subset X_{\delta,S}([\mathcal{C}],W_0,K).$$

This follows from our construction of $X_{\delta,S}([\mathcal{C}],W_0,K)$ and the following lemma.

Lemma 3.13 *Let $\{f_i\}$ be a sequence of J-holomorphic maps with fixed homology class A, then by taking a subsequence if necessary, we may have that f_i converges to some holomorphic map f_∞, which may be reducible, such that $||f_i||_{1,p}$ is uniformly bounded for any $p > 2$.*

Proof: This lemma was in fact essentially proved in [RT1], section 6. It is also true for any sequence of harmonic maps (cf. [CT]).

By the Gromov Compactness Theorem, we may assume that f_i converges to f_∞ in the topology of $\overline{\mathcal{F}}^\ell_A(V, g, k)$. Then we suffice to show that $||f_i||_{1,p}$ is uniformly bounded.

Let Σ_i be the domain of f_i and q be an node of Σ_∞, which is the domain of f_∞. Near q, Σ_i can be locally described by coordinates w_1, w_2 with $w_1 w_2 = \epsilon_i$ and $|w_1|, |w_2| \leq 1$. Note that $\lim_{i\to\infty} \epsilon_i = 0$.

We may assume that $\epsilon_i > 0$ and $f_i(w_1, w_2)$ is very close to q. Write $w_1 = se^{\sqrt{-1}\theta}$, where $\epsilon_i \leq s \leq 1$. Then the Cauchy-Riemann equation becomes

$$\frac{\partial f_i}{\partial s} + \frac{1}{s} J(f_i) \frac{\partial f_i}{\partial \theta} = 0.$$

By the same arguments as in the proof of Lemma 3.9, we can deduce that for any $p > 2$,

$$\int_{\epsilon_i \leq s \leq 1} (|w_1|^2 + |w_2|^2)^{\frac{p-2}{p}} |\nabla f_i|^2 s ds \wedge d\theta \leq c_p,$$

where c_p is a constant depending only on p. It follows that $||f_i||_{1,p}$ is uniformly bounded. The lemma is proved.

Now let us construct a resolution $\{F_i, \psi_i\}$ of $\Phi^{-1}(0)$. We cover $\Phi^{-1}(0)$ by smooth approximations $\{(E_{\delta_i,S_i}([\mathcal{C}_i], W_{0i}, K_i), X_{\delta_i,S_i}([\mathcal{C}_i], W_{0i}, K_i))\}$, where $1 \leq i \leq m$. For each i, we have

$$X_{\delta_i,S_i}([\mathcal{C}_i], W_{0i}, K_i) = \tilde{X}_{\delta_i,S_i}([\mathcal{C}_i], W_{0i}, K_i)/\Gamma_i,$$

where Γ_i is the automorphism group of $\mathcal{C}_i$ and

$$\tilde{X}_{\delta_i,S_i}([\mathcal{C}_i], W_{0i}, K_i) = \Phi^{-1}(E_{S_i}) \subset \tilde{U}_{\delta_i}([\mathcal{C}_i], W_{0i}, K_i) = \mathrm{Map}_{\delta_i}(W_i, K_i).$$

For each i, choose $W'_{0i} \subset W_{0i}$ such that if W'_i denotes the corresponding subset in W_i, then all the $X_{\delta_i,S_i}([\mathcal{C}_i], W'_{0i}, K_i)$ still cover $\Phi^{-1}(0)$.

As we have seen before, any vector $v \in S_i$ induces a section, denoted by v_s, of E_{S_i} over $\mathrm{Map}_\infty(W_i)$.

Let us construct F_i, ψ_i inductively. For $i = 1$, we simply define $F_1 = S_1$ and

$$\begin{aligned} \psi_1 : \quad & \mathrm{Map}_\infty(W_1) \times F_1 \mapsto E_1, \\ & \psi_1(\tilde{\mathcal{C}}, v) = \eta_1(\Sigma_{\tilde{\mathcal{C}}}) v_s(\tilde{\mathcal{C}}). \end{aligned}$$

Here η_i is a smooth function on $\tilde{W}_i$ satisfying: $\eta_i \equiv 1$ on $\tilde{W}_i'$ and $\eta_i = 0$ near $\partial \tilde{W}_i$.

Suppose that we have defined F_i, ψ_i for $1 \leq i \leq l-1$. Let us define F_l, ψ_l. For each $i < l$ and $v \in F_i$, $\psi_i(v)$ induces a section, say $v_{s,l,i}$, of $\tilde{E}_l$ over $\tilde{U}_{\delta_l}(\mathcal{C}_l, W_{0l}, K_l)$. Let F_l be the vector space spanned by S_l and $\sigma^*(v_{s,l,i})$ ($\sigma \in \Gamma_l$, $i < l$). We define $\psi_l(\tilde{\mathcal{C}}, v)$ to be $\eta_l v_s(\tilde{\mathcal{C}})$ if $v \in S_l$ and $v_{s,l,i}(\sigma(\tilde{\mathcal{C}}))$ if $v = \sigma^*(v_{s,l,i})$. Clearly, ψ_l is Γ_l-equivariant. Thus we can construct $\{F_i, \psi_i\}_{1 \leq i \leq m}$.

The smooth structure of $\Phi : \overline{\mathcal{F}}^\ell_A(V, g, k) \mapsto E$ is given by all those smooth approximations $(\tilde{E}_{\delta,S}([\mathcal{C}], W_0, K), \tilde{X}_{\delta,S}([\mathcal{C}], W_0, K))$ satisfying: if $\mathcal{U}$ (resp. $\mathcal{U}_i$) is the universal family of curves over $\tilde{W}$ (resp. $\tilde{W}_i$) corresponding to W_0 (resp. W_{0i}), then $S|_{\mathcal{U} \cap \mathcal{U}_i}$ contains the image of ψ_i for each i. One can show that with all these $(\tilde{E}_{\delta,S}([\mathcal{C}], W_0, K), \tilde{X}_{\delta,S}([\mathcal{C}], W_0, K))$, $\{F_i, \psi_i\}$ satisfies all properties required for a smooth resolution. Therefore, $\Phi : \overline{\mathcal{F}}^\ell_A(V, g, k) \mapsto E$ is a generalized Fredholm orbifold bundle.

Finally, let us give the natural orientation of $\det(\Phi)$ (cf. [R], [RT1]). We first notice that for any $\mathcal{C}$ representing a point in $\Phi^{-1}(0)$, $\det(\Phi)|_{\mathcal{C}}$ can be naturally identified with the determinant of the linear Fredholm operator $L_{\mathcal{C}}\Phi$, where $L_{\mathcal{C}}\Phi$ denotes the linearization of Φ at $\mathcal{C}$. It can be written as $\overline{\partial}_{\mathcal{C}} + B_{\mathcal{C}}$, where $\overline{\partial}_{\mathcal{C}}$ is a J-linear operator and $B_{\mathcal{C}}$ is an operator of zero order. It follows that $L_{\mathcal{C}}\Phi$ can connected to $\overline{\partial}_{\mathcal{C}}$ through the canonical path $\{\overline{\partial}_{\mathcal{C}} + tB_{\mathcal{C}}\}_{0 \leq t \leq 1}$, so $\det(\Phi)$ is naturally isomorphic to the determinant $\det(\overline{\partial})$ of the family of operators $\{\overline{\partial}_{\mathcal{C}}\}_{\mathcal{C}}$. However, since each $\overline{\partial}_{\mathcal{C}}$ is J-invariant, there is a natural orientation on $\det(\overline{\partial})$. It follows that $\det(\Phi)$ can be naturally oriented.

Proposition 2.2 is proved.

Remark 10 *In fact, in this concrete case, we can construct the Euler class* $e([\Phi : \overline{\mathcal{F}}^\ell_A(V, g, k) \mapsto E])$ *without using Theorem 1.2. We can use the arguments in the proof of Theorem 1.2 and smooth approximations*

$$(\tilde{E}_{\delta,S}([\mathcal{C}], W_0, K), \tilde{X}_{\delta,S}([\mathcal{C}], W_0, K))$$

to construct a $\mathbb{Q}$*-cycle. This* $\mathbb{Q}$*-cycle will lie in a finite covering of* $\Phi^{-1}(0)$ *by finitely dimensional smooth approximations. Here we do need smooth* *properties of* $(\tilde{E}_{\delta,S}([\mathcal{C}], W_0, K), \tilde{X}_{\delta,S}([\mathcal{C}], W_0, K))$ *during changes of local uniformizations.*

Proof of Proposition 2.3: The proof is identical to that of Proposition 2.2. So we omit the details.

Let $\Phi : \overline{F}^{\ell}_{A}(V, g, k) \mapsto E$ be the generalized Fredholm orbifold bundle as above, and $\Phi' : \overline{F}^{\ell}_{A}(V, g, k) \mapsto E$ is another one induced by the almost complex structure J'.

Let $\{J_t\}$ be the family of almost complex structures joining J to J'. Consider

$$\begin{aligned} \Psi : [0,1] \times \overline{F}^{\ell}_{A}(V, g, k) &\mapsto E, \\ (t, (f, \Sigma; \{x_i\})) &\mapsto df + J_t(f) \cdot df \cdot j. \end{aligned}$$

Then $\Psi|_{\{0\}\times \overline{F}^{\ell}_{A}(V,g,k)} = \Phi$ and $\Psi|_{\{1\}\times \overline{F}^{\ell}_{A}(V,g,k)} = \Phi'$.

Using the same arguments as above, one can prove that $\Psi : [0,1] \times \overline{F}^{\ell}_{A}(V, g, k) \mapsto E$ is a generalized Fredholm orbifold bundle. Moreover, one can equip this bundle a weakly smooth structure which restricts to the given smooth structures of Φ and Φ' along the boundary $\{0,1\} \times \overline{F}^{\ell}_{A}(V, g, k)$.

This follows that Φ is homotopic to Φ'. That is just what Proposition 2.3 claims.

4 One more example

In this section, we consider a simpler example: the Seiberg-Witten invariants of 4-manifolds. The Seiberg-Witten invariants have found many striking applications in the study of 4-dimensional topology (cf. [Wi], [Ta], [KM], [KST]). Here we just give a different approach to defining the Seiberg-Witten invariants, which seems to be of independent interest.

We first fix the notation we will use. Let X be a compact oriented smooth 4-manifold and let c be a spin^c structure on X with the associated spin^c bundles W^+ and W^-. Let

$$\rho : \Lambda^+ \otimes \mathbb{C} \longrightarrow \mathfrak{sl}(W^+)$$

be the isomorphism induced by the Clifford multiplication, where $\mathfrak{sl}(W^+)$ is the associated $PSL(2, \mathbb{C})$ bundle of W^+. There is also a pairing

$$W^+ \times \bar{W}^+ \longrightarrow \mathfrak{sl}(W^+)$$

that is modeled on the map $\mathbb{C}\times\bar{\mathbb{C}}^2 \to \mathfrak{sl}(\mathbb{C}^2)$ sending (v,w) to $i(v\bar{w}^t)_0$, where the subscript means the traceless part.

Now the Seiberg-Witten invariants is defined as follows. We first fix a Riemannian metric g. Let L be the determinant line bundle of W^+ and let $\mathcal{A}(\mathcal{L})$ be the space of unitary connections on L. Then $A \in \mathcal{A}(\mathcal{L})$ induces a Dirac operator $\Gamma(W^+) \to \Gamma(W^-)$. Now let $\tilde{\mathcal{B}}$ be the Banach manifold $\mathcal{A}(L) \times \Gamma(W^+)$ and let $\tilde{\mathcal{E}}$ be the constant vector bundle over $\tilde{\mathcal{B}}$ with fiber $\Gamma(W^+) \times \Gamma(\mathfrak{sl}(W^+))$. We define a section $\tilde{f} : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$ via

$$\tilde{f}(\varphi, A) = (D_A\varphi, \rho(F_A^+) - i\sigma(\varphi,\varphi)).$$

Now let $p_0 \in X$ be fixed and let $\mathcal{G}_0 = \mathrm{Map}_{p_0}(X, S^1)$ be the pointed gauge group of L. Note that $\mathcal{G}_0$ also acts on $\Gamma(W^+)$ via scalar automorphism of W^+. Hence $\mathcal{G}_0$ acts freely on $\tilde{B}$ and it lifts to an action on $\tilde{\mathcal{E}}$. We let $\mathcal{B} = \tilde{\mathcal{B}}/\mathcal{G}_0$ and $\mathcal{E} = \tilde{\mathcal{E}}/\mathcal{G}_0$. Since $\tilde{f}$ is equivariant under $\mathcal{G}_0$, $\tilde{f}$ descends to a section

$$f : \mathcal{E} \longrightarrow \mathcal{B}\,.$$

Note that $f^{-1}(0)$ is compact. Now let $\mathcal{G}$ be the full gauge group. Then $\mathcal{G}/\mathcal{G}_0 \cong S^1$ acts on $\mathcal{E} \to \mathcal{B}$ and the section f is S^1-equivariant as well. The Seiberg-Witten invariants of X is the S^1-equivariant version of Euler class of $[f : \mathcal{B} \to \mathcal{E}]$ defined in section 1. More precisely, The universal line bundle on $\mathcal{A}(L)$ descends to a complex line bundle $\mathcal{L}$ on $\mathcal{B}$ and the Seiberg-Witten invariants of X is

$$SW : H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$$

defined by

$$SW(L) =< \mathrm{Euler}(\mathcal{E}, f)^{S^1}, c_1(\mathcal{L})^k >\,,$$

where

$$k = \frac{1}{4}(c_1(L)^2 - (2\chi + 3\sigma)),$$

is the Fredholm index of $f/S^1 : \mathcal{B}/S^1 \to \mathcal{E}/S^1$.

In case the zero locus $f^{-1}(0)$ is disjoint from the fixed point set of S^1, which is

$$\mathcal{B}^{S^1} = \mathcal{A}(L)/\mathcal{G}_0 \times \{0\} \subset \mathcal{A}(L) \times \Gamma(W^+)/\mathcal{G}_0,$$

then we can work with $[f/S^1 : \mathcal{B}/S^1 \to \mathcal{E}/S^1]$. Let $\mathcal{B}' = (\mathcal{B} - \mathcal{B}^{S^1})/S^1$, let $\mathcal{E}' = (\mathcal{E}|_{\mathcal{B}'})/S^1$ and let $f' = (f|_{\mathcal{B}'})/S^1$. Then $[f' : \mathcal{B}' \to \mathcal{E}']$ is a Fredholm operator as defined in section 1. The Seiberg-Witten invariant then is

$$SW(L) =< e[f' : \mathcal{B}' \to \mathcal{E}'], c_1(\mathcal{L}')^k >$$

where $\mathcal{L}'$ is the descend of $\mathcal{L}|_{\mathcal{B}-\mathcal{B}^{S^1}}$ to $\mathcal{B}'$.

We now look at the general case. Let

$$\mathcal{E}|_{\mathcal{B}^{S^1}} = \oplus_{i=\infty}^{\infty} \mathcal{F}_i$$

be the spectral decomposition of the restriction of $\mathcal{E}$ to $\mathcal{B}^{S^1}$. Namely, $\mathcal{F}_i \subset \mathcal{E}|_{\mathcal{B}^{S^1}}$ is S^1-invariant and the S^1 action on $\mathcal{F}_i$ has weight i. Then $f|_{\mathcal{B}^{S^1}}$ factor through $\mathcal{F}_0 \subset \mathcal{E}|_{\mathcal{B}^{S^1}}$. We denote this section by f_0. Let $k+1$ be the Fredholm index of f and let l be the Fredholm index of $f_0 : T_z\mathcal{B}^{S^1} \to \mathcal{F}_{0,z}$.

Lemma 4.1 *Assume $l < 0$, then any S^1-equivariant Fredholm section $f : \mathcal{B} \to \mathcal{E}$ is homotopic to an S^1-equivariant Fredholm section $g : \mathcal{B} \to \mathcal{E}$ so that $g^{-1}(0) \cap \mathcal{B}^{S^1} = \emptyset$.*

Proof: The proof is straightforward. We first look at the the restriction of f to $\mathcal{B}^{S^1}$. As we mentioned, it factor through $\mathcal{F}_0$. Let $h : \mathcal{B}^{S^1} \to \mathcal{F}_0$ be this map. Then since dh has negative Fredholm index, by Theorem 1.1, h is homotopic to $\tilde{h} : \mathcal{B}^{S^1} \to \mathcal{F}_0$ so that its vanishing locus is empty. Clearly, for some S^1 invariant neighborhood U of $\mathcal{B}^{S^1} \subset \mathcal{B}$, we can extend this homotopy, and thus $\tilde{h}$, within the category of Fredholm operators, to an S^1-equivariant $g : \mathcal{B} \to \mathcal{E}$ so that the restriction of g to $\mathcal{B}^{S^1}$ is $\tilde{h}$ and $g|_{\mathcal{B}-U} = f|_{\mathcal{B}-U}$. This proves the Lemma.

After having g given by the Lemma, we reduce the situation to when $g^{-1}(0) \cap \mathcal{B}^{S^1} = \emptyset$. Thus as before, we can define the equivariant Euler class $e[f : \mathcal{B}^{S^1} \to \mathcal{E}]^{S^1}$ represented by a smooth k dimensional submanifold in $(\mathcal{B} - \mathcal{B}^{S^1})/S^1$ to be the class $e[g/S^1 : \mathcal{B}' \to \mathcal{E}']$. When $l < -1$, then the above argument shows that any two such representatives of $e[f : \mathcal{B} \to \mathcal{E}]^{S^1}$ in $\mathcal{B}'$ are coborbant to each other in $\mathcal{B}'$. Therefore, they represent a well-defined corbordism class.

We now apply the above construction to the Seiberg-Witten invariant, the fixed point set $\mathcal{B}^{S^1}$ in $\mathcal{B}$ is $\mathcal{A}(L) \times \{0\}$. The $\mathcal{F}_0 \subset \mathcal{E}|_{\mathcal{B}^{S^1}}$ in this case is the subbundle $\Gamma(\mathfrak{sl}(W^+))$ and the restriction section is $\rho(F_A^+)$, whose Fredholm index is $-b_2^+$. Therefore, when $b_2^+ > 1$, the Seigerg-Witten invariant is well defined and can be represented by a smooth submanifold in $(\mathcal{B} - \mathcal{B}^{S^1})/S^1$.

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