

Exact non-equilibrium DC shot noise in Luttinger liquids and fractional quantum Hall devices

P. Fendley¹, A.W.W. Ludwig^{2†} and H. Saleur^{1*}

¹ *Department of Physics, University of Southern California, Los Angeles CA 90089-0484*

² *Physics Department, University of California, Santa Barbara, CA 93106*

(May 1995, cond-mat/9505031)

A point contact in a Luttinger liquid couples the left- and right-moving channels, producing shot noise. We calculate exactly the DC shot noise at zero temperature in the out-of-equilibrium steady state where current is flowing. Integrability of the interaction ensures the existence of a quasiparticle basis where quasiparticles scatter “one by one” off the point contact. This enables us to apply a direct generalization of the Landauer approach to shot noise to this interacting model. We find a simple relation of the noise to the current and the differential conductance. Our results should be experimentally-testable in a fractional quantum Hall effect device, providing a clear signal of the fractional charge of the Laughlin quasiparticles.

The Luttinger model, describing the low-energy excitations of an interacting one-dimensional fermion gas, is one of the simplest non-Fermi-liquid metals. Experimental observation of this non-Fermi state in 1D quantum wires is difficult since disorder tends to localize these excitations. However, this theory has been proposed to describe the edge states in fractional quantum Hall effect devices [1]. Tunneling through a point contact in a Luttinger liquid is a practically ideal situation for making contact between experiment and theory [2–4]. Very clean measurements of electronic transport properties through point contacts in quantum Hall devices have been performed [5], while there are powerful constraints on the theory because the model is integrable [6]. One can compute the current and conductance through the point contact exactly, even when the system is out of equilibrium [4,7,8]. The experimentally-measured [5] linear-response conductance in such a device agrees well with the exact theoretical prediction [2–4].

The tunneling current and conductance are “spectroscopic” probes of the non-Fermi-liquid state in the leads. These however are not the only transport properties of interest. The current shot noise resulting from the point contact provides another signal of the non-Fermi-liquid state. Here we compute the zero-temperature DC shot noise exactly. This is the first exact computation of noise in a model with interacting electrons.

For weak backscattering, Laughlin quasiparticles hop from one edge to the other at the point contact. For strong backscattering, on the other hand, current transport is caused by electrons. In both limits the tunneling events happen independently, so the shot noise is proportional to the charge of the carriers providing transport. The weak-backscattering limit is thus a direct signal of the fractional charge of the Laughlin quasiparticles in the Hall device [9]. Our non-perturbative results give the noise for any amount of backscattering. Moreover, we find a simple expression of the noise in terms of the

current and conductance.

The shot noise is a function of the (driving) “source-drain” voltage V , and T_B , the scaled point contact interaction strength. Our approach uses the exact bulk and impurity S matrices [10,6] of the Luttinger model. The bulk S matrix describes the scattering of quasiparticles off each other away from the point contact. Knowing it allows us to find the density of these quasiparticles in the Fermi sea at any voltage. The impurity S matrix elements give the probability for tunneling events, which correspond to the scattering of the quasiparticles off the point contact. We find the zero-temperature DC noise using a Landauer-type approach familiar from studies of transport of free electrons [11–13]; the result involves the tunneling probabilities and the quasiparticle density. It is exact in this integrable system because the quasiparticles are transported “one by one”. This means that the scattering of a current of quasiparticles off the point contact can be described by the product of one-particle S matrices, even though the quasiparticles interact.

We first review the bosonized formulation of the Luttinger model, before displaying the appropriate quasiparticle basis. The left- and right-moving channels of the Luttinger model are described by left- and right-moving bosons ϕ_L and ϕ_R defined on a space $-l < x < l$ [14]. The coupling constant of the fermion model is parametrized by ν , which is the filling fraction of the Hall device when $1/\nu$ is an odd integer. In the absence of the point contact, the Hamiltonian is of Tomonaga form:

$$H_0 = \frac{v_F \pi}{\nu} \int_{-l}^l dx [j_L^2 + j_R^2], \quad (1)$$

quadratic in the two individually-conserved $U(1)$ currents, $j_L = -\frac{1}{4\pi}(\partial_t + \partial_x)\phi_L$ and $j_R = \frac{1}{4\pi}(\partial_t - \partial_x)\phi_R$. These currents are the charge densities: for example, $eQ_L = e \int_{-l}^l dx j_L$ is the total charge of the left-moving channel. An applied voltage V imposes a chemical potential difference for the injected left- and right-moving

charge carriers. This results in a term $-\Delta Q eV/2$ in the Hamiltonian, where $\Delta Q \equiv (Q_L - Q_R)$.

A point-contact interaction coupling the right and left channels at the origin $x = 0$ results in backscattering, which in the FQHE edge state model corresponds to tunneling of Laughlin quasiparticles. The resulting Hamiltonian includes the term [2]

$$H_B = \lambda \cos[\phi_L(x=0) - \phi_R(x=0)]. \quad (2)$$

Other allowed terms are irrelevant for $\nu > 1/4$ [2]; our analysis holds for any ν as long as there is only a single relevant operator in H_B . We rewrite the model in terms of two left-moving bosons [17]:

$$\phi^{e,o}(x+t) \equiv \frac{1}{\sqrt{2}}[\phi_L(x,t) \pm \phi_R(-x,t)], \quad (3)$$

where the *even* boson ϕ^e and the *odd* boson ϕ^o have the + and - sign, respectively. The even and odd Hamiltonians are decoupled: H_B involves only the odd boson, while in H_0 j_L, j_R are replaced with j^e, j^o , where $j^{e,o}(x+t) = (1/\sqrt{2})[j_L(x,t) \pm j_R(-x,t)]$. The even and odd charges are thus related to the charges of the original left- and right-moving edges by $\Delta Q = Q_L - Q_R = \sqrt{2}Q^o$ and $Q_L + Q_R = \sqrt{2}Q^e$. Therefore Q^e is the total charge on both edges and is conserved even in the presence of the interaction. The backscattering current thus depends only on the odd boson theory.

Describing the model in terms of quasiparticles allows us to calculate exact transport properties. These quasiparticles span the Hilbert space of the left-moving odd boson; they are the excitations above the ‘‘Fermi sea’’ (at zero voltage). Because the odd boson theory is integrable [6], these quasiparticles have very special properties. The infinite number of conserved quantities which commute with the Hamiltonian in an integrable model define a basis of quasiparticles where the quasiparticles scatter ‘‘one-by-one’’. The conservation laws allow determination of the exact quasiparticle spectrum and the S matrix. This result is already known [10,6,15] for the odd boson Hamiltonian (1) and (2). For any ν , the spectrum contains a kink (+) and an antikink (-). These carry (odd) charges $Q^o = 1/\sqrt{2}$ and $-1/\sqrt{2}$, respectively. At some values of ν , there are chargeless ‘‘breather’’ states, but these do not enter into the zero-temperature analysis below. We parametrize the energy and momentum of these massless left-moving quasiparticles in terms of the rapidity θ defined by $E = -pv_F = Me^\theta/2$, where M is an arbitrary energy which cancels out of physical quantities.

When a positive voltage is turned on, it becomes energetically favorable for positively-charged particles (the kinks) to fill the sea. If the quasiparticles did not interact, all momentum states with $v_F p < eV/2$ would be filled with kinks (no antikinks) at zero temperature. The interaction affects not only the position of the Fermi level

but also the density of quasiparticles. In [4,7] it was shown how to find the shift of the Fermi level due to an applied V exactly, following techniques given for example in [16]. We define the density $\rho(\theta)$ so that $\rho(\theta)d\theta$ is the number of *kinks* per unit length with rapidities between θ and $\theta + d\theta$. The shift of the Fermi level for kinks is given by the quantity A , such that $\rho(\theta) = 0$ for $\theta > A$. Then

$$\tilde{\rho}(s) = \frac{M}{2ihv_F} \frac{K(-s)K(i)}{s-i} e^{(is+1)A}. \quad (4)$$

$$e^A = \frac{eV}{M} \frac{K(0)}{K(i)} \quad (5)$$

where $\tilde{\rho}(s)$ is the Fourier transform of $\rho(\theta)$, and

$$K(s) \equiv \sqrt{\frac{2\pi}{\nu}} \frac{\Gamma(s/[2i(1-\nu)])}{\Gamma(\nu s/[2i(1-\nu)]) \Gamma(1/2 - is/2)} e^{-is\Delta}$$

$$\Delta \equiv \frac{\nu}{2(1-\nu)} \ln \nu - \frac{1}{2} \ln(1-\nu)$$

As required, $\rho(\theta) = 0$ for $\theta > A$, because K is analytic everywhere in the upper half plane (including $s = i\infty$).

Without the backscattering interaction, the odd charge is conserved ($\partial_t \Delta Q = 0$). Because there are no antikinks at zero temperature, all the current arises from the kinks moving to the left at the fermi velocity:

$$I_0(V) = ev_F \int_{-\infty}^A d\theta \rho(\theta) = ev_F \tilde{\rho}(0) = \nu \frac{e^2}{h} V \quad (6)$$

The backscattering current $I_B(V)$ is the rate at which the charge of the left-moving edge is depleted due to backscattering off the impurity. This decreases the total current $I = I_0 + I_B$. By symmetry, $\partial_t Q_L = -\partial_t Q_R$, so $I_B = \partial_t (\frac{e}{2} \Delta Q) = \partial_t (\frac{e}{\sqrt{2}} Q^o)$. In the even/odd basis, tunneling corresponds to the violation of odd charge conservation at the contact. In the quasiparticle basis, this corresponds (at $T = 0$) to a kink scattering off the contact into an antikink.

At $T = 0$, both the backscattering current and the DC shot noise can be written entirely in terms of the tunneling probability and the density ρ of *kinks* only. The impurity S matrix element $S_{jk}(p/T_B)$ describes a single quasiparticle of type j and momentum p scattering elastically off the point contact into a quasiparticle of type k . Here $T_B \propto \lambda^{1/(1-\nu)}$ is the crossover scale introduced by the interaction. We define $T_B \equiv Me^{\theta_B}/2$ so that the S matrix elements are functions of $\theta - \theta_B$. These were derived exactly in [6]; the tunneling probability is given by

$$|S_{+-}(\theta - \theta_B)|^2 = \frac{1}{1 + \exp[2(1-\nu)(\theta - \theta_B)/\nu]} \quad (7)$$

and $|S_{++}|^2 = 1 - |S_{+-}|^2$. A simple kinetic equation then gives [4,7], when specialized to $T = 0$

$$I_B(V, T_B) = -ev_F \int_{-\infty}^A d\theta \rho(\theta) |S_{+-}(\theta - \theta_B)|^2 \quad (8)$$

The differential conductance is $G = \partial_V I$. Using the explicit expressions for ρ and $|S_{+-}|^2$, one finds power series expressions for $I(V, T_B) = I_0(V) + I_B(V, T_B)$; they are

$$I(V, T_B) = \frac{e^2 V}{h} \left[\nu - \nu^2 \sum_{n=1}^{\infty} a_n(\nu) \left(\frac{eV}{T'_B} \right)^{2n(\nu-1)} \right] \quad (9)$$

$$I_B(V, T_B) = \frac{e^2 V}{h} \sum_{n=1}^{\infty} a_n(1/\nu) \left(\frac{eV}{T'_B} \right)^{2n(\frac{1}{\nu}-1)} \quad (10)$$

where

$$a_n(\nu) = (-1)^{n+1} \frac{\sqrt{\pi} \Gamma(\frac{n}{\nu})}{2\Gamma(n)\Gamma(\frac{3}{2} + n(\frac{1}{\nu} - 1))}$$

$$T'_B \equiv T_B \frac{2\sqrt{\pi}\Gamma(1/[2(1-\nu)])}{\Gamma(\nu/[2(1-\nu)])}$$

The expansion (9) is appropriate for $T_B/V < 1$, while (10) is appropriate for $T_B/V > 1$; notice the strong barrier/weak barrier duality [7].

Finding the shot noise requires a more detailed analysis. We follow arguments given in [13] for the free-electron case, and show that they can be directly generalized to our interacting quasiparticles, due to the constraints of integrability. To make the calculation of the noise precise, we first examine the quantum-mechanical current operator $j(t)$, which includes the current with its fluctuations. The system is not in equilibrium because current is flowing, but it is in a steady state, so the current $I = \langle j(t) \rangle$ does not depend on time. The current fluctuations in frequency space are characterized by the correlator

$$C(\omega) = \frac{1}{2} \int dt e^{i\omega t} \langle \{j(t), j(0)\} \rangle. \quad (11)$$

We will focus on the noise at zero frequency (the DC limit), which we denote by $\langle I^2 \rangle \equiv C(0)$.

In our quasiparticle approach, the current is thought of as a series of individual quasiparticles. Since the model is interacting, the quasiparticles are correlated, but at zero temperature every kink state with rapidity (parametrizing momentum) less than A is filled, and the remaining kink states as well as all antikink states are empty. Thus without the point contact there is no noise in this steady state. (Even with the impurity there is no noise in the even current, only in the odd current.) When the backscattering is included, shot noise occurs because there are two possible outcomes when a given quasiparticle hits the impurity. We can describe the DC shot noise from a quasiparticle approach, because the scattering off the point contact is elastic and one-by-one. As we discussed above, when a left mover backscatters into a right mover, in the even/odd basis this corresponds to an odd-boson kink scattering into an antikink. Thus as we turn

on the interaction (and the voltage in order to generate a population of kinks) the impurity will scatter some of these into antikinks. The bath of kinks (the battery) is large so it is not depleted by the scattering.

Consider a single kink of momentum p , and define $f = 1$ if this kink turns into an antikink when it scatters off the impurity, and $f = 0$ if it scatters into a kink. By definition of the impurity S matrix elements, the average over many events is $\langle f \rangle = |S_{+-}(p/T_B)|^2$. In the quasiparticle approach, the noise is then proportional to the the fluctuation of f

$$\langle I^2 \rangle = e^2 \sum \langle (f - \bar{f})^2 \rangle = e^2 \sum (\langle f^2 \rangle - \langle f \rangle^2)$$

where the sum is over all kink states which hit the point contact per unit time. The crucial point to notice is that because f is either 0 or 1, $\langle f^2 \rangle = \langle f \rangle$ [13]. We replace the sum with an integral over momenta, obtaining

$$\langle I^2 \rangle = e^2 v_F \int_{-\infty}^A d\theta \rho(\theta) |S_{+-}(\theta - \theta_B)|^2 \times (1 - |S_{+-}(\theta - \theta_B)|^2) \quad (12)$$

This form of the DC shot noise is formally the same as for non-interacting electrons [11–13]; but here the density of states $\rho(\theta)$ given by (5) is non-trivial.

Since we have explicit expressions for $|S_{+-}|^2$ and ρ , power series expressions for the noise analogous to (9) and (10) can be found directly from (12). This however is not necessary because this expression can be related directly to the current. The specific form (7) of the transmission amplitude means that we can write

$$|S_{+-}|^2(1 - |S_{+-}|^2) = \frac{\nu}{2(1-\nu)} \frac{\partial |S_{+-}|^2}{\partial \theta_B}. \quad (13)$$

Since neither ρ nor A depends on θ_B , we can pull the ∂_{θ_B} out of the integral. Using the expressions (6) and (8) for $I(V, T_B)$ yields

$$\langle I^2 \rangle = -\frac{\nu e}{2(1-\nu)} T_B \partial_{T_B} I(V, T_B) \quad (14)$$

This expression is a very simple analog of the fluctuation-dissipation theorem for a Luttinger liquid at zero temperature. Since $I(V, T_B)/V$ is a function of only V/T_B , we find another form of this relation:

$$\langle I^2 \rangle = \frac{\nu e}{2(1-\nu)} V^2 \partial_V \frac{I}{V} = \frac{\nu e}{2(1-\nu)} (VG - I). \quad (15)$$

It is certainly conceivable that (14) and (15) hold for other models, because they are a consequence of the simple identity (13). The noise is plotted in figure 1. Remarkably, due to the simple form of (15), the extrema of $\langle I^2 \rangle/T_B$ and G occur at the same value of T_B/V .

We can compare these exact equations with known results in the limits of weak ($|S_{+-}|^2$ small) and strong

($|S_{+-}|^2$ near 1) backscattering. In the weak backscattering limit, the events happen seldomly and thus should be uncorrelated. Thus the shot noise should obey the formula for non-interacting particles (see e.g. [11–13])

$$\langle I^2 \rangle \approx \nu e(I_0 - I) \quad T_B \text{ small};$$

the ν appears because the original Luttinger fermions (the Laughlin quasiparticles in the FQHE edge realization) being scattered have charge νe . Inserting the small- T_B expansion from (9) into (14), one easily verifies this. Measuring the noise therefore gives a direct measurement of the fractional charge [9]. In the strong backscattering limit, one can check that the leading irrelevant operator contributing to the current corresponds to the tunneling of quasiparticles of charge e . In the FQHE this is the tunneling of physical electrons between two separate systems (in the strong backscattering limit the point contact splits the system in two). Thus

$$\langle I^2 \rangle \approx eI \quad T_B \text{ large},$$

which is easily verified using the large- T_B expansion (10).

The interactions between the quasiparticles prevent any naive application of this formalism outside zero temperature. However, observe that if either $V = 0$ or $T_B = 0$, the noise is given by the familiar Johnson-Nyquist formula $\langle I^2 \rangle = 2GT$. Since this vanishes at $T = 0$ and our zero-temperature result vanishes when either $V = 0$ or $T_B = 0$, a good approximation to the noise for arbitrary values of V, T_B and T should be given by adding the two types of contributions:

$$\langle I^2 \rangle(T) \approx \frac{\nu e}{2(1-\nu)}(VG - I) + 2GT. \quad (16)$$

It is conceivable that such a formula, with added terms of the form $TG^m(G - I/V)^n$ ($m, n \neq 0$), might be exact, generalizing the above analog of the fluctuation dissipation theorem.

We have seen that integrability permits the exact calculation of non-equilibrium transport properties through (interacting) point contacts in a Luttinger liquid, generalizing notions of ballistic transport used previously only for non-interacting electrons. It would be very interesting to compare our exact findings with future experiments on shot noise in Quantum Hall devices, identifying the fractional charge of the quasiparticles experimentally. It would also be most interesting to extend these results to non-zero frequency ω in order to check the recent perturbative result that the noise has a singularity in ω [18]. This, however, might require a more complicated formalism based on form factors.

† A. P. Sloan Fellow. * Packard Fellow.

We thank M.P.A. Fisher for useful discussions. This work was supported by the Packard Foundation, the A.P. Sloan Foundation, the National Young Investigator

program (NSF-PHY-9357207) and the DOE (DE-FG03-84ER40168).

-
- [1] X.G. Wen, Phys. Rev. B41 (1990) 12838; Phys. Rev. B43 (1991) 11025; Phys. Rev. B44 (1991) 5708.
 - [2] C.L. Kane, M.P.A. Fisher, Phys. Rev. B46 (1992) 15233.
 - [3] K. Moon, H. Yi, C.L. Kane, S.M. Girvin and M.P.A. Fisher, Phys. Rev. Lett. 71 (1993) 4381.
 - [4] P. Fendley, A.W.W. Ludwig and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005, cond-mat/9408068.
 - [5] F.P. Milliken, C.P. Umbach and R.A. Webb, “Indications of a Luttinger Liquid in the Fractional Quantum Hall Regime”, to appear in Solid State Communications.
 - [6] S. Ghoshal and A.B. Zamolodchikov, Int. J. Mod. Phys. A9 (1994) 3841, hep-th/9306002.
 - [7] P. Fendley, A.W.W. Ludwig and H. Saleur, “Exact non-equilibrium transport through point contacts in quantum wires and fractional quantum Hall devices”, cond-mat/9503172.
 - [8] P. Fendley, F. Lesage and H. Saleur, hep-th/9409176, to appear in J. Stat. Phys.
 - [9] C.L. Kane and M.P.A. Fisher, Phys. Rev. Lett. 72 (1994) 724.
 - [10] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
 - [11] G.B. Lesovik, JETP Lett. 49 (1989) 594.
 - [12] M. Buttiker, Phys. Rev. Lett. 65 (1990) 8184.
 - [13] R. Landauer, Physica D38 (1989) 226; R. Landauer and Th. Martin, Physica B175 (1991) 167.
 - [14] A. Luther and I. Peschel, Phys. Rev. B12 (1975) 3908; J. Sólyom, Adv. Phys. 28 (1979) 201; F.D.M. Haldane, J. Phys. C14 (1981) 2585.
 - [15] P. Fendley, H. Saleur and N.P. Warner, Nucl. Phys. B430 (1994) 577, hep-th/9406125.
 - [16] C.N. Yang and C.P. Yang, Phys. Rev. 150 (1967) 321,327; G. Japaridze, A. Nersisyan and P. Wiegmann, Nucl. Phys. B230 (1984) 511.
 - [17] E. Wong and I. Affleck, Nucl. Phys. B417 (1994) 403, cond-mat/9311040; I. Affleck and A.W.W. Ludwig, J. Phys. A27 (1994) 5375.
 - [18] C.de C. Chamon, D.E. Freed and X.G. Wen, Phys. Rev. B51 (1995) 2363, cond-mat/9408064.

Figure 1. The exact DC noise $\langle I^2 \rangle/V$ as a function of backscattering and driving voltage T_B/V .

DC shot noise (units of e^3/h)

