

JAIN-MONRAD CRITERION FOR ROUGH PATHS AND APPLICATIONS

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ABSTRACT. We discuss stochastic calculus for large classes of Gaussian processes, based on rough path analysis. Our key condition is a covariance measure structure combined with a classical criterion due to Jain–Monrad [JM83]. This condition is verified in many examples, even in absence of explicit expressions for the covariance or Volterra kernels. Of special interest are random Fourier series, with covariance given as Fourier series itself, and we formulate conditions directly in terms of the Fourier coefficients. We also establish convergence and rates of convergence in rough path metrics of approximations to such random Fourier series. An application to SPDE is given. Our criterion also leads to an embedding result for Cameron–Martin paths and complementary Young regularity (CYR) of the Cameron–Martin space and Gaussian sample paths. CYR is known to imply Malliavin regularity and also Itô-like probabilistic estimates for stochastic integrals (resp. stochastic differential equations) despite their (rough) pathwise construction. At last, we give an application in the context of non-Markovian Hörmander theory.

INTRODUCTION

There is a lot of interest, from financial mathematics to non-linear SPDE theory, in having a stochastic calculus for non-semimartingales. In the past, much emphasis was laid upon stochastic integration (resp. stochastic differential equations) driven by fractional Brownian motion (fBm), and then general Volterra processes (cf. e.g. [Nua06, Section 5], [Dec05]). More recently, an effort was made to dispense with the Volterra structure (cf. [KRT07, KR10]) leading to a key condition of finite planar (or 2D) variation of the covariance. A completely different approach was started by Lyons [Lyo98] (cf. also [LQ02, LCL07, FV10b, FH13]). In essence, it suffices to have a.s. enough p -variation regularity of sample paths $X.(\omega)$ and *existence of stochastic area(s)*, also subject to some variation type regularity. The problem is then shifted away from developing a general stochastic integration theory to the (arguably) much simpler task of constructing the first few iterated (stochastic) integrals; the rest then follows from deterministic rough path integration theory.

In the case of Gaussian sample paths, a general sufficient condition for the existence of stochastic areas was introduced in [FV10b]. Namely, it was shown that if the covariance of the underlying process is sufficiently regular in terms of finite two-dimensional ρ -variation, the process can be enhanced with stochastic areas in a canonical way. The point is that uniform L^2 -estimates on the stochastic areas (more precisely, smooth approximations thereof) are possible, thanks to two-dimensional Young estimates, as long as $\rho < \rho^* = 2$. It is then fairly straightforward and carried out in detail in [FV10b, Ch. 15] (cf. also [FH13]) to construct a (random) rough path \mathbf{X} associated to X . This setup was rather successful; for instance, it was also used by Hairer [Hai11, Hai13] to construct a spatial rough path associated to the stochastic heat equation in one space dimension

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which laid the foundation to prove well-posedness of certain non-linear SPDE. However, finding bounds for the ρ -variation of the covariance of a stochastic process in concrete examples is not an easy task and checkable conditions have been dearly missing in the literature ¹. Providing such conditions is the first main contribution of the present work.

These conditions immediately apply to known examples such as fractional Brownian motion with Hurst parameter H . In this case, it is known that $\rho = 1/(2H) \vee 1$ and the critical $\rho < 2$ corresponds to $H > 1/4$; sharpness of this condition follows from the well-documented divergence of the Lévy area for $H^* = 1/4$.

Knowing the precise parameter ρ also has other benefits: It was shown (cf. [FV10a]) that finite ρ -variation of the covariance of a Gaussian process implies that the Cameron–Martin space \mathcal{H} can be continuously embedded in the space of paths with finite ρ -variation; in other words,

$$\mathcal{H} \hookrightarrow C^{\rho\text{-var}}$$

holds. In the case $\rho < 3/2$, this embedding assures that the mixed iterated integral between a Gaussian sample path and a Cameron–Martin path can be defined via Young’s integration theory and we thus speak of ”complementary Young regularity” (CYR) here. CYR has many consequences: for instance, it allows for a Malliavin calculus [CF10], [FV10b, Ch. 15, 20] w.r.t. Gaussian rough paths. In fact, SDE solutions - by which we mean solutions to rough differential equations driven by $\mathbf{X}(\omega)$ for a.e. ω - will a.s. be Fréchet-smooth in Cameron–Martin directions as long as CYR holds. This led to the development of a non-Markovian Hörmander theory [CF10, CHLT12]. CYR is important also for other reasons. It is the condition under which one has Stroock-Varadhan type support theorems (see [FV10b, Ch. 19] and the references therein). It is also the key to good probabilistic estimates in (Gaussian) rough path theory. To appreciate this, note that the available pathwise estimates in rough path theory are ill-suited to see the probabilistic cancellations which are the heart of the Itô theory. It was only recently understood that Gaussian isoperimetry (in the form of the Borell-Sudakov-Tsirelson inequality) can bridge this gap (cf. [CLL13] and also [FO10]): In the generic setting of $\rho = 1$, if applied to stochastic integrals (cf. [FR13]) of Lip 1-forms (as it is typical in rough path integration theory), one obtains identical (Gaussian) moment estimates as in the Itô theory. This deteriorates as ρ increases, but exponential integrability - and even better, depending on ρ - remains true². A natural question is whether one can extend CYR to processes which have finite ρ -variation for $\rho \geq 3/2$. In the case of fractional Brownian motion, a direct analysis of its Cameron–Martin paths (using the Volterra structure of fBm) reveals that in this special case the stronger embedding

$$\mathcal{H}^H \hookrightarrow C^{q\text{-var}} \quad \text{for any } q > \frac{1}{H + \frac{1}{2}}$$

holds (cf. [FV06]) which implies CYR for all $H > 1/4$. Another contribution of the present work is to show that this stronger embedding holds in much greater generality and, in particular, even in absence of a Volterra structure of the process under consideration, which readily implies CYR for all $\rho < 2$ and thus closes this gap.

The structure of our article is as follows. In **Part I** we answer in the affirmative the following question: given a multidimensional Gaussian process with covariance of finite ρ -variation, $\rho < 2$, does CYR hold? The caveat here is that the ρ is not related anymore to the ρ -variation of the

¹The situation is easier of course when $\rho = 1$. In this case, the covariance has finite 1-variation iff its mixed distributional derivative is a finite signed measure. In the fBm case this means precisely $H \geq 1/2$.

²Such integrability properties can be crucial in SPDE theory [Hai11] (see the not discussion in [FR13]) and in robust filtering theory [CDFO12, DO13].

covariance but instead to finite *mixed* $(1, \rho)$ -variation; a mild strengthening that we prove not to be restrictive at all in applications. The usefulness of such a result stands and falls with one's ability to verify this condition in concrete cases. The situation is aggravated by the examples from random Fourier series (rFs) where the covariance itself is not known explicitly, but only given as a Fourier series in its own right. A general and checkable condition for finite mixed $(1, \rho)$ -variation is the main result of **Part II**, see Theorem 2.2. Loosely speaking, our condition is a combination of a classical criterion for Gaussian processes to have p -variation sample paths due to Jain–Monrad, with a covariance measure structure condition (the distributional mixed derivative is assumed to be Radon away from the diagonal). We then run through a (long) list of examples (see Examples 2.3 – 2.16) which illustrate the wide applicability of our criterion. (This way, we also recover from general principles previously known results on fBM, such as [FV06].) In **Part III** we apply the results of Part II to study rFs in greater depth. In particular, once we have established finite ρ -variation for the covariance of rFs and therefore the existence of associated (random) rough paths, we ask for convergence (with rates in rough path metrics) of natural approximations given in terms of Fourier multipliers.³ The best rates are obtained by considering the rough paths under consideration as p -rough paths with large p , which also means that one has to go beyond level 2, 3 considerations. Thankfully, we can rely here on general results for Gaussian rough paths established in [FR12]. The main results in Part III are Theorem 3.2 and Theorem 3.15. In **Part IV & V**, we discuss some concrete rFs (resp. random Fourier transforms) arising from (fractional) stochastic heat equations in the study of the stochastic Burgers' [Hai11] and the KPZ [Hai13] equation. Namely, we show how to regard a (fractional, with dissipative term $-(-\partial_{xx})^\alpha$, $\alpha \leq 1$) heat equation with space-time white noise, on bounded intervals subject to various boundary conditions (resp. the entire real line) as an evolution in rough path space. The key here is spatial covariance of finite ρ -variation, where $2\alpha = 1 + 1/\rho$. Note $\rho = 1$ iff $\alpha = 1$ and that $\alpha > 3/4$ is handled by our theory⁴. This type of spatial rough path was first used by Hairer (with $\alpha = 1$, and periodic boundary conditions) to analyze the stochastic Burgers' equation [Hai11]; a similar construction with other boundary conditions (incl. those we handle here) was left as open (technical) problem in [Hai11]. In a recent preprint, Gubinelli et al. [GIP12] consider the fractional stochastic Burgers equation, also with periodic boundary conditions, when $\alpha > 5/6$ based on a direct spatial rough path construction.⁵ Finally, in **Part VI** we illustrate (by the example of a driving rFs) how our results can be used to check the technical conditions put forward in [CHLT12] (cf. also [CF10, HP11]), under which differential equations driven by such Gaussian signals and along Hörmander vector fields possess a smooth density at positive times.

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³Rough path convergence of piecewise linear-, mollifier, Karhunen–Loeve approximation follows from general Gaussian rough path theory [FV10b] and requires no further discussion.

⁴The covariance structure - including local decorrelation as measured by mixed variational regularity - of the fractional SHE in the space variables is similar to fBm with $H = \alpha - 1/2$.

⁵In absence of ρ -variation estimates no conclusions towards CYR and its numerous consequences are drawn in [GIP12], nor do the results allow to use the general body of Gaussian rough path approximation theory [FV10a, FV10b, FR12] based on uniform ρ -variation estimates. That said, the overall aim of [GIP12] was quite different.

which led to Example 2.11. P.K.F. would like to thank S. Tindel for pointing out similar aspects of the present conditions with those from [CHLT12]; this led to Part VI.

Notation. Let $I = [S, T] \subset \mathbb{R}$ be a closed interval. We define the simplex by $\Delta_I := \{(s, t) \mid s \leq t \in I\}$. A dissection D of an interval $I = [S, T]$ is of the form

$$D = (S = t_0 \leq t_1 \leq \dots \leq t_n = T)$$

and we write $\mathcal{D}(I)$ for the family of all such dissections.

We will now very briefly recall the elements of rough paths theory used in this paper. For more details we refer to [FV10b]. Let $T^N(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \dots \oplus (\mathbb{R}^d)^{\otimes N}$ be the truncated step- N tensor algebra. For paths in $T^N(\mathbb{R}^d)$ starting at the fixed point $e := 1 + 0 + \dots + 0$, one may define β -Hölder and p -variation metrics, extending the usual metrics for paths in \mathbb{R}^d starting at zero: The *homogeneous* β -Hölder and p -variation metrics will be denoted by $d_{\beta\text{-Hölder}}$ and $d_{p\text{-var}}$, the *inhomogeneous* ones by $\rho_{\beta\text{-Hölder}}$ and $\rho_{p\text{-var}}$ respectively. Note that both β -Hölder and p -variation metrics induce the same topology on the path spaces. Corresponding norms are defined by $\|\cdot\|_{\beta\text{-Hölder}} = d_{\beta\text{-Hölder}}(\cdot, 0)$ and $\|\cdot\|_{p\text{-var}} = d_{p\text{-var}}(\cdot, 0)$ where 0 denotes the constant e -valued path.

A geometric β -Hölder rough path \mathbf{x} is a path in $T^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)$ which can be approximated by lifts of smooth paths in the $d_{\beta\text{-Hölder}}$ metric; geometric p -rough paths are defined similarly. Given a rough path \mathbf{x} , the projection on the first level is an \mathbb{R}^d -valued path and will be denoted by $\pi_1(\mathbf{x})$. It can be seen that rough paths actually take values in the smaller set $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$, where $G^N(\mathbb{R}^d)$ denotes the free step- N nilpotent Lie group with d generators. The Carnot-Carathéodory metric turns $(G^N(\mathbb{R}^d), d)$ into a metric space. Consequently, we denote by

$$C_0^{0, \beta\text{-Hölder}}(I, G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)) \quad \text{and} \quad C_0^{0, p\text{-var}}(I, G^{\lfloor p \rfloor}(\mathbb{R}^d))$$

the rough paths spaces where $\beta \in (0, 1]$ and $p \in [1, \infty)$. Note that both spaces are Polish spaces.

1. PART I: COMPLEMENTARY YOUNG REGULARITY UNDER MIXED $(1, \rho)$ -VARIATION ASSUMPTION

Let $X : [0, T] \rightarrow \mathbb{R}$ be a real valued, centered, continuous Gaussian process with covariance

$$R_X(s, t) = \mathbb{E}X_s X_t.$$

We will denote the associated Cameron-Martin space by \mathcal{H} . It is well-known that $\mathcal{H} \subset C([0, T], \mathbb{R})$ and each $h \in \mathcal{H}$ is of the form $h_t = \mathbb{E}Z X_t$ with Z being an element of the L^2 -closure of $\text{span}\{X_t \mid t \in [0, T]\}$, a Gaussian random variable. If $h_t = \mathbb{E}Z X_t$, $h'_t = \mathbb{E}Z' X_t$, $\langle h, h' \rangle_{\mathcal{H}} = \mathbb{E}Z Z'$.

For any function $h : [0, T] \rightarrow \mathbb{R}$ we define $h_{s,t} := h_t - h_s$ for all $s, t \in [0, T]$. We recall the definition of mixed right (γ, ρ) -variation given in [Tow02]: For $\gamma, \rho \geq 1$ let

$$(1.1) \quad V_{\gamma, \rho}(R_X; [s, t] \times [u, v]) := \sup_{\substack{(t_i) \in \mathcal{D}([s, t]) \\ (t'_j) \in \mathcal{D}([u, v])}} \left(\sum_{t'_j} \left(\sum_{t_i} \left| R_X \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}},$$

where $\mathcal{D}([s, t])$ denotes the set of all dissections of $[s, t]$ and

$$R_X \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) = \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}}.$$

The notion of the 2D ρ -variation is recovered as $V_\rho = V_{\rho, \rho}$. Recall that V_ρ -regularity plays a key role in Gaussian rough path theory [FV10b, FV10a, FH13] and in particular yields a stochastic integration theory for large classes of multidimensional Gaussian processes. Clearly, $V_{\gamma \vee \rho}(R; A) \leq$

$V_{\gamma,\rho}(R; A) \leq V_{\gamma \wedge \rho}(R; A)$ for all rectangles $A \subseteq [0, T]^2$. As the main result of this section we present the following embedding theorem for the Cameron-Martin space.

Theorem 1. *Assume that the covariance R_X has finite mixed $(1, \rho)$ -variation in 2D sense. Then there is a continuous embedding*

$$\mathcal{H} \hookrightarrow C^{q\text{-var}} \text{ with } q = \frac{1}{\frac{1}{2\rho} + \frac{1}{2}} < 2.$$

More precisely,

$$\|h\|_{q\text{-var}; [s,t]} \leq \|h\|_{\mathcal{H}} \sqrt{V_{1,\rho}(R_X; [s,t]^2)}, \quad \forall [s,t] \subseteq [0,T].$$

The following is then immediate.

Corollary 2. *Assume $\rho \in [1, 2)$. Then complementary Young regularity holds, i.e. we can choose $p > 2\rho$ small enough such that X has a.s. p -variation sample paths, $h \in \mathcal{H}$ has finite q -variation with $1/p + 1/q > 1$.*

We shall see in Part II (as one of many examples) that the assumption of mixed $(1, \rho)$ -variation is met in the case of fBm in the rough regime $H \leq 1/2$ with $\rho = 1/(2H)$. (For instance, example 2.8 applies with $k = 0$ and in fact gives a neat criterion for processes with stationary increments.) It then follows that fractional Cameron-Martin paths enjoy finite $q = \frac{1}{H + \frac{1}{2}}$ -variational regularity, which is consistent (and in fact a mild sharpening) of $q > \frac{1}{H + \frac{1}{2}}$, previously obtained in [FV06] with methods specific to fBm. Let us also note that, for the sole purpose of Theorem 1, it would have been enough to consider identical dissections $(t_i) \equiv (t'_j)$ in the definition of mixed variation $V_{\gamma,\rho}$ in (1.1). The criteria in Theorem 2.2 below would then allow for a mildly simplified proof. On the other hand, this criteria derived in Theorem 2.2 below are also sufficient (and interesting) for finite ρ -variation $V_\rho = V_{\rho,\rho}$ which is the key condition for the construction of Gaussian rough paths needed later on, hence the additional generality of different vertical and horizontal dissections.

Remark 3. Let $X: [0, T] \rightarrow \mathbb{R}^d$ be a multidimensional centered Gaussian process. Then every path h in the associated Cameron-Martin space \mathcal{H} is of the form $h_t = \mathbb{E}Z X_t$ with Z being an element of the L^2 -closure of $\text{span}\{X_t^i \mid t \in I, i = 1, \dots, d\}$ and $\|h\|_{\mathcal{H}} = \|Z\|_{L^2}$. The q -variation of h is finite if and only if the q -variation of every $h^i = \mathbb{E}Z X^i$ is finite and we obtain the bound

$$\|h\|_{q\text{-var}; [s,t]} \leq C \|h\|_{\mathcal{H}} \max_{i=1,\dots,d} \sqrt{V_{1,\rho}(R_{X^i}; [s,t]^2)}$$

where C is a constant depending only on the dimension d .

We now give the proof of Theorem 1. In fact, having identified the importance of mixed variation, the proof is pleasantly short.

Proof. Let $h = \mathbb{E}Z X \in \mathcal{H}$. Fix a dissection $D = (t_j) \subset [s, t]$, write $h_j \equiv h_{t_j, t_{j+1}}$, $X_j = X_{t_j, t_{j+1}}$ and also $\|h\|_q^q := \sum_j |h_j|^q$. Let q' and ρ' be the conjugate exponents of q and ρ . An easy calculation shows that $\rho' = q'/2$. By duality,

$$\|h\|_q = \sup_{\beta: \|\beta\|_{q'} \leq 1} \sum \beta_j h_j = \sup_{\beta: \|\beta\|_{q'} \leq 1} \mathbb{E} \left(Z \sum_j \beta_j X_j \right)$$

and so by Cauchy–Schwarz

$$\|h\|_q^2 \leq \|h\|_{\mathcal{H}}^2 \sup_{\beta: \|\beta\|_{q'} \leq 1} \sum_{j,k} \beta_j \beta_k \mathbb{E} X_j X_k.$$

Set $R_{j,k} = \mathbb{E} X_j X_k$. Then, using the symmetry of R and Hölder’s inequality,

$$\begin{aligned} \sum_{k,j} \beta_j \beta_k R_{k,j} &\leq \frac{1}{2} \sum_{j,k} \beta_j^2 |R_{j,k}| + \frac{1}{2} \sum_{j,k} \beta_k^2 |R_{j,k}| \\ &= \sum_j \beta_j^2 \sum_k |R_{k,j}| \\ &\leq \|\beta\|_{2\rho'}^2 \left(\sum_j \left(\sum_k |R_{i,k}| \right)^\rho \right)^{\frac{1}{\rho}} \\ &\leq V_{1,\rho}(R; [s, t]^2) \end{aligned}$$

when $\|\beta\|_{2\rho'} = \|\beta\|_{q'} \leq 1$ which shows the claim. \square

2. PART II: JAIN–MONRAD REVISITED

2.1. Preliminaries and motivation from fBm. Let $I \subset \mathbb{R}$ be a compact interval and $R: I \times I \rightarrow \mathbb{R}$ be a symmetric, continuous function. We set $T = |I|$, $D_h := \{(s, t) \in I^2 : |s - t| \leq h\}$ and let $D := D_0$ be the diagonal of I^2 . In this section we will give conditions under which R has finite ρ -variation on $I^2 = I \times I$. For a rectangle $[s, t] \times [u, v] \subseteq I^2$, we define the rectangular increment by

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} = R(s, u) - R(s, v) - R(t, u) + R(t, v)$$

and we set

$$(2.1) \quad \sigma^2(s, t) := R \begin{pmatrix} s, t \\ s, t \end{pmatrix} = R(s, s) + R(t, t) - 2R(s, t),$$

where symmetry of R was used in the last step. Note that

$$\partial_{s,t} \sigma^2 = -2\partial_{s,t} R$$

whenever these mixed derivatives make sense. In many applications R is the covariance function of a zero mean stochastic process X , i.e. $R(s, t) = \mathbb{E} X_s X_t$ and in this case $\sigma^2(s, t) = \text{Var}(X_t - X_s) \geq 0$ is the variance of increments. However, it will be important to conduct the present discussion in a generality that goes beyond covariance functions.

Given a dissection (t_i) of $I = [0, T]$, the square $[0, T]^2$ can be decomposed into little squares $\cup_j [t_i, t_{i+1}]^2$ and off-diagonal rectangles, say $\{Q_j\}$. Then

$$\sum_i \sigma^2(t_i, t_{i+1}) + \sum_j R(Q_j) = R \begin{pmatrix} 0, T \\ 0, T \end{pmatrix} = \sigma^2(0, T) < \infty$$

and the right-hand-side is independent of the dissection. Depending on the behaviour of $\sigma^2(s, t)$, we can or cannot ignore the on-diagonal contributions in the limit mesh $(t_i) \rightarrow 0$. For instance, if $\sigma^2(s, t) = |t - s|^{2H}$ with $H > 1/2$, then

$$\lim_{\text{mesh}(t_i) \rightarrow 0} \sum_i \sigma^2(t_i, t_{i+1}) = 0$$

and with $R(Q_j) \approx \partial_{s,t} R \Delta_j$ for small Q_j , or by direct calculus, we find

$$(2.2) \quad \sigma^2(0, T) = T^{2H} = -\frac{1}{2} \int_0^T \int_0^T \partial_{s,t} |t-s|^{2H} ds dt = H(2H-1) \int_0^T \int_0^T |t-s|^{2H-2} ds dt,$$

noting that $|t-s|^{2H-2} = |t-s|^{-1+2(H-1/2)}$ is integrable at the diagonal (and then everywhere on $[0, T]^2$) iff $H > 1/2$. When $H = 1/2$ this computation fails. Indeed, the prefactor $2H-1 = 0$ combined with the diverging integral effectively leaves us with $0 \cdot \infty$. The reason of course is that $R(Q_j) = 0$ in this case (Brownian increments are uncorrelated) and everything hinges on the (non-vanishing) on-diagonal contribution

$$\sum_i \sigma^2(t_i, t_{i+1}) = \sum_i (t_{i+1} - t_i) = T.$$

As a Schwartz distribution $\partial_{s,t} R = \partial_{s,t} \min(s, t) = \delta_{\{s=t\}}$ is a "Dirac" on the diagonal and indeed with this interpretation as a measure,

$$\sigma^2(0, T) = R \left(\begin{array}{c} 0, T \\ 0, T \end{array} \right) = \int_0^T \int_0^T \delta_{\{s=t\}} ds dt = T.$$

When $H < 1/2$, $\sigma^2(s, t) = |t-s|^{2H}$, the on-diagonal contributions are not only non-vanishing but divergent (as the mesh of (t_i) goes to zero). That is,

$$\sigma^2(0, T) = T^{2H} = \underbrace{\sum_i \sigma^2(t_i, t_{i+1})}_{\rightarrow \infty} + \sum_j R(Q_j)$$

and so, necessarily, $\sum_j R(Q_j) \rightarrow -\infty$. Translated to the calculus setting, this causes (2.2) to fail. Indeed, ignoring the infinite contribution from the diagonal leaves us with

$$T^{2H} \neq H(2H-1) \underbrace{\int_0^T \int_0^T |t-s|^{2H-2} ds dt}_{=+\infty} = -\infty, \quad \text{for } H < 1/2.$$

Let us remark that, with our standing assumption $R \in C([0, T]^2)$ the (distributional) mixed derivative $\partial_{s,t} R$ always exists, i.e.

$$\langle \partial_{s,t} R, \varphi \rangle := \int_0^T \int_0^T R(s, t) \partial_{s,t} \varphi(s, t) ds dt, \quad \forall \varphi \in C_c^\infty((0, T)^2).$$

One can ask if, or when, $\partial_{s,t} R$ is given by a signed and finite (i.e. of finite total variation) Borel measure μ on $[0, T]^2$, say

$$\langle \partial_{s,t} R, \varphi \rangle = \int_{[0, T]^2} \varphi d\mu,$$

with associated Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$. When $H > 1/2$, the answer is affirmative with $\mu = \mu_+ = H(2H-1) |t-s|^{2H-2} ds dt$. For $H = 1/2$, the answer is also affirmative with $\mu = \mu_+ = \delta_{\{s=t\}}$. For $H < 1/2$, the answer is negative.

However, for all values of $H \in (0, 1)$ it is possible to define a (signed) σ -finite measure by

$$\mu(A) := \int_A H(2H-1) |t-s|^{2H-2} ds dt$$

which we shall regard as a signed Radon measure on $(0, T)^2 \setminus D$. Note

$$\mu \equiv \mu_+, \mu \equiv 0, \mu \equiv -\mu_- \text{ for } H > 1/2, H = 0, H < 1/2 \text{ resp.}$$

In general, as seen when $H < 1/2$, μ does not need to be a finite measure on $(0, T)^2 \setminus D$. On the other hand, its restriction to any compact in $(0, T)^2 \setminus D$ is finite so that μ defines a signed Radon measure on $(0, T)^2 \setminus D$. Hence, for all values of $H \in (0, 1)$ the (distributional) mixed derivative $\partial_{s,t} R$ on $(0, T)^2 \setminus D$ is given by the Radon measure μ . (This was certainly observed previously, e.g. in [KR10].)

Care is necessary, for important information has been lost by the restriction to $(0, T)^2 \setminus D$. For instance, nothing was left of Brownian motion ($\mu = 0$). It follows that when $H \leq 1/2$, and in particular in the case $H < 1/2$ where $|\mu| = \mu_-$ has infinite mass on $(0, T)^2 \setminus D$, the on-diagonal information must be captured differently. We shall achieve this by a somewhat classical condition due to Jain–Monrad [JM83, DN98] which imposes “on-diagonal” ρ -variation of σ^2 by

$$v_\rho(\sigma^2; [s, t]) := \sup_{D=(t_i) \in \mathcal{D}([s, t])} \left(\sum_i |\sigma^2(t_i, t_{i+1})|^\rho \right)^{\frac{1}{\rho}} < \infty.$$

Clearly $\rho = 1/2H \geq 1$ in the fBm example with $H \leq 1/2$ but the concept is much more general.

2.2. Main result of the section. Throughout we work on some closed interval $I \subset \mathbb{R}$ with length $T = |I|$.

Condition 2.1 (Jain–Monrad). *Let $\rho \geq 1$ and $\omega: \Delta_I \rightarrow \mathbb{R}_+$ be a super additive function (i.e. $w(s, r) + w(r, t) \leq w(s, t)$ for all $s \leq r \leq t$). We say that $(JM)_{\rho, \omega}$ holds if*

$$|\sigma^2(s, t)| \leq \omega(s, t)^{\frac{1}{\rho}}$$

holds for all $s < t$.

If $v_\rho(\sigma^2; I) < \infty$, we can always set $\omega(s, t) = v_\rho(\sigma^2; [s, t])^\rho$. Conversely, if $(JM)_{\rho, \omega}$ holds, we have $v_\rho(\sigma^2; [s, t]) \leq \omega(s, t)^{\frac{1}{\rho}}$ for all $[s, t] \subseteq I$.

Recall the definition of mixed right (γ, ρ) -variation given in (1.1), noting in particular the triangle inequality: for all rectangles $A \subseteq I^2$

$$(2.3) \quad V_{\gamma, \rho}(R_1 + R_2; A) \leq V_{\gamma, \rho}(R_1; A) + V_{\gamma, \rho}(R_2; A).$$

Recall that a signed Radon measure μ is a locally finite signed Borel measure with decomposition $\mu = \mu_+ - \mu_-$ where μ_\pm are locally finite, non-negative Borel measures one of which has finite mass. For a finite measure μ on $(0, T)^2 \setminus D$ we will consider its extension to $[0, T]^2$ by $\mu(A) := \mu(A \cap (0, T)^2 \setminus D)$ without further notice. We now give the main theorem of this section. For simplicity, we only formulate it for the case $I = [0, T]$.

Theorem 2.2. *Let $R: [0, T]^2 \rightarrow \mathbb{R}$ be a symmetric, continuous function and σ as in (2.1). Assume that the (Schwartz) distributional mixed derivative $\mu := \frac{\partial^2 R}{\partial_t \partial_s} = -\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial_t \partial_s}$ is a Radon measure on $(0, T)^2 \setminus D$ with decomposition $\mu = \mu_+ - \mu_-$.*

Part A: *Assume that*

- (A.i) μ_- has finite mass and a continuous distribution function.
- (A.ii) There exists an $h > 0$ such that $\sigma^2(s, t) \geq 0$ whenever $|t - s| \leq h$.⁶

⁶Automatically true if R is a covariance function.

Then

$$V_1(R; [s, t] \times [u, v]) \leq R \left(\begin{array}{c} s, t \\ u, v \end{array} \right) + 2\mu_-([s, t] \times [u, v]), \quad \forall [s, t] \times [u, v] \subseteq [0, T]^2.$$

Part B: Assume that

(B.i) μ_+ has finite mass and a continuous distribution function.

(B.ii) There exists an $h > 0$ such that⁷

$$2R \left(\begin{array}{c} s, t \\ u, v \end{array} \right) = \sigma^2(s, v) - \sigma^2(s, u) + \sigma^2(u, t) - \sigma^2(v, t) \geq 0, \quad \forall [u, v] \subseteq [s, t] \subseteq I \text{ s.t. } |t - s| \leq h.$$

(B.iii) $(JM)_{\rho, \omega}$ holds.

Then for all $[s, t]^2 \subset D_h$ we have

$$(2.4) \quad V_{1, \rho}(R; [s, t]^2) \leq C \left(\omega^{\frac{1}{\rho}}(s, t) + \mu_+([s, t]^2) \right),$$

for some constant $C = C(\rho)$.

If, in addition, $R: [0, T]^2 \rightarrow \mathbb{R}$ satisfies a Cauchy-Schwarz inequality⁸ then, more generally, there is a constant $C = C(\rho, h, T)$ such that

$$(2.5) \quad V_{1, \rho}(R; [s, t] \times [u, v]) \leq C \left(\omega^{\frac{1}{2\rho}}(s, t) \omega^{\frac{1}{2\rho}}(u, v) + \mu_+([s, t] \times [u, v]) \right),$$

for all rectangles $[s, t] \times [u, v] \subset [0, T]^2$.

The interest in Theorem 2.2 is two-fold. First, it has far-reaching conclusions: mixed $(1, \rho)$ -variation controls ρ -variation which, if applied (componentwise) to the covariance of a Gaussian process (multidimensional, with independent components), is the key quantity for the existence of associated rough paths; here one needs $\rho < 2$ (which corresponds to $H > 1/4$; cf. Example 2.7 below). Moreover, mixed $(1, \rho)$ -variation was seen in Part I to imply *complementary Young regularity*, an extremely important property leading to good probabilistic estimates of rough integrals, as explained in the introduction. It is also required for Stroock–Varadhan type support theorems and is one of the key conditions for the applicability of Malliavin calculus and then non-Markovian Hörmander theory (cf. [CF10, CHLT12]).

Secondly, the theorem is practical because its conditions are easy to check and widely applicable. To illustrate this we now run through a list of examples. Roughly speaking, Part A handles situations similar or nicer than Brownian motion, whereas Part B handles situations similar or worse than Brownian motion. The finite measure $m = \mu_-$ (resp. μ_+) in Part A (resp. B) should be considered as (harmless) perturbation which adds some extra flexibility. Typically m is given by a density, i.e. by the (integrable) negative (resp. positive) part of some locally integrable function. Continuity of the distribution function is then trivial. In fact, $m = 0$ in many interesting examples.

2.3. Examples.

⁷With the exception of bi-fBm, Example 2.11, we typically check (B.ii) by simply showing that $\tau \mapsto \sigma^2(\tau, t + \tau)$ resp. $\sigma^2(t - \tau, t)$ are nondecreasing for all t and $\tau < h$. In particular, in stationary situations where $\sigma^2(s, t) = F(t - s)$ this amounts for F to be non-decreasing on $[0, h]$; conversely it is not hard to see (2.2) implies F non-decreasing on $[0, h/2]$.

⁸I.e. $\left| R \left(\begin{array}{c} s, t \\ u, v \end{array} \right) \right| \leq \left| R \left(\begin{array}{c} s, t \\ s, t \end{array} \right) \right|^{\frac{1}{2}} \left| R \left(\begin{array}{c} u, v \\ u, v \end{array} \right) \right|^{\frac{1}{2}}$, for all $[s, t] \times [u, v] \subseteq I^2$, which is automatically true if R is a covariance function.

2.3.1. Examples handled by Part A.

Example 2.3 (Fractional Brownian motion $H \geq 1/2$). Consider a (standard) fractional Brownian motion B^H , with $\sigma^2(s, t) = |t - s|^{2H}$ in the regime $H > 1/2$. We have, as a measure on $[0, T]^2 \setminus D$,

$$\begin{aligned} \mu &= \mu_+ = H(2H - 1) \int |t - s|^{2H-2} ds dt \geq 0, \quad \text{if } H > 1/2 \\ \mu &= 0, \quad \text{if } H = 1/2, \end{aligned}$$

which clearly yields a Radon measure on $[0, T]^2 \setminus D$ (and even a finite Borel measure on $[0, T]^2$). Note that $\mu_- \equiv 0$ in the decomposition $\mu = \mu_+ - \mu_-$, hence (A.i) holds trivially. Also, since $R(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H})$ is a genuine covariance function, (A.ii) comes for free. It follows that R has finite "Hölder controlled" 1-variation, in the sense that

$$V_1 \left(R; [s, t]^2 \right) \leq R \left(\begin{array}{c} s, t \\ s, t \end{array} \right) = |t - s|^{2H} = O(|t - s|).$$

Example 2.4 (Brownian bridge). Given a standard Brownian motion B , the **Brownian bridge** over $[0, T]$ can be defined as

$$X_t = B_t - \frac{t}{T} B_T \implies R(s, t) = \min(s, t) - st/T.$$

It follows that $\mu = \partial_{s,t} R$, as a measure on $[0, T]^2 \setminus D$, decomposes into $\mu_+ = 0$ and μ_- with (constant) density $1/T$. Part A applies and immediately gives "Hölder controlled" 1-variation, i.e. $V_1 \left(R; [s, t]^2 \right) = O(|t - s|)$.

Example 2.5 (Stationary increments I, Brownian and better regularity). Consider a process with stationary increments in the sense that the variance of its increments is given by

$$\sigma^2(s, t) = F(|t - s|) \geq 0,$$

for some $F \in C^2([0, T])$. A concrete (Gaussian) examples is the **stationary Ornstein-Uhlenbeck process** with $F(x) = 1 - e^{-x}$. In any case, we may expand

$$F(h) = F'(0)h + F''(0)h^2/2 + o(h^2).$$

We compute

$$\partial_{s,t} \sigma^2(s, t) = -F''(|t - s|) + F'(0)2\delta(t - s).$$

so that

$$\frac{\partial^2 R}{\partial s \partial t} = -\frac{1}{2} \frac{\partial^2 \sigma^2}{\partial s \partial t} = \frac{1}{2} F''(|t - s|) \quad \text{on } (0, T)^2 \setminus D.$$

It then follows that (A.i) holds with

$$\mu(A) = \frac{1}{2} \int_A F''(|t - s|) ds dt$$

and we immediately obtain finite (Hölder controlled) 1-variation,

$$V_1 \left(R; [s, t]^2 \right) \leq \sigma^2(s, t) + |F''|_\infty |t - s|^2 = O(|t - s|).$$

For a concrete F , of course, one can compute μ_- and obtain sharper conclusions. This may also be possible if we are in a "better than Brownian" setting, namely $F'(0) = 0$, in which case

$\sigma^2(s, t) = O(|t - s|^2)$. Note that in this case $F''(0) > 0$, unless F is trivial⁹. It follows that, in a neighborhood of the diagonal, $\mu > 0$ and so $\mu_- \equiv 0$. We then have

$$V_1\left(R; [s, t]^2\right) \leq \sigma^2(s, t) = O(|t - s|^2),$$

for $|t - s| \leq \inf\{h > 0 : F''(h) > 0\}$.

Example 2.6 (Volterra processes I; Brownian and better regularity). Assume $X_t = \int_0^t K(t, r) dB_r$ where $K(t, \cdot)$ is assumed to be square-integrable. For $s < t$ we have

$$\begin{aligned} X_{s,t} &= \int_0^t K(t, r) - K(s, r) 1_{\{r \leq s\}} dB_r, \\ \sigma^2(s, t) &= \mathbb{E}X_{s,t}^2 = \int_0^s (K(t, r) - K(s, r))^2 dr + \int_s^t K(t, r)^2 dr. \end{aligned}$$

We assume a regular situation, by which we shall mean here that K is continuous on the simplex $\{0 \leq s \leq t \leq T\}$, and assuming suitable differentiability properties of K , one computes

$$\partial_{s,t}R = K(s, s) \partial_t K(t, s) + \int_0^s \partial_s K(s, r) \partial_t K(t, r) dr =: f(s, t).$$

If $\mu := f(s, t) ds dt$ defines a Radon measure on $[0, T]^2 \setminus D$, with μ_- having finite mass, Part A is applicable. Rather than imposing technical conditions on K , we verify this in the model case of **Volterra fBm**, $K(t, s) = (t - s)^{H-1/2}$, $H > 1/2$ (As above, there is nothing to do in the Brownian case $H = 1/2$ since then $f \equiv 0$ and so $\mu \equiv 0$). Specializing the above formula for $\partial_{s,t}R$, we have

$$\partial_{s,t}R = (H - 1/2)^2 \int_0^s (t - r)^{H-3/2} (s - r)^{H-3/2} dr =: f(s, t) \geq 0.$$

Since f remains bounded away from the diagonal, it clearly defines a (non-negative!) Radon measure. Trivially, $\mu_- \equiv 0$, and so thanks to Part A

$$V_1\left(R; [s, t]^2\right) \leq \sigma^2(s, t) = O(|t - s|).$$

2.3.2. Examples handled by Part B.

Example 2.7 (Fractional Brownian motion $H \leq 1/2$). Consider a (standard) fractional Brownian motion B^H , with $\sigma^2(s, t) = |t - s|^{2H}$ in the regime $H \leq 1/2$. We compute $\mu = \partial_{s,t}R = (-1/2) \partial_{s,t}\sigma^2$ away from the diagonal and find

$$\mu = -\mu_- = -H(1 - 2H)|t - s|^{2H-2} ds dt \leq 0$$

which clearly yields a Radon measure on $[0, T]^2 \setminus D$. Note that $\mu_+ \equiv 0$ in the decomposition $\mu = \mu_+ - \mu_-$. Conditions (B.ii) and (B.iii) with $\rho = 1/(2H)$, $\omega(s, t) = t - s$ are clear. It follows that the fBm covariance function, $R(s, t) = \frac{1}{2} \left(s^{2H} + t^{2H} - |t - s|^{2H} \right)$, has finite "Hölder controlled" mixed $(1, \rho)$ -variation, in the sense that

$$V_{1,\rho}\left(R; [s, t]^2\right) \leq O(|t - s|^{1/\rho}).$$

⁹Indeed, if $F'(0) = F''(0) = 0$, then $\|X_t - X_s\|_{L^2} = o(t - s)$ which is enough to conclude that X_t is a constant in L^2 , but then $\sigma^2(s, t) = \|X_t - X_s\|_{L^2}^2 = 0$.

Example 2.8 (Stationary increments II, Brownian and worse regularity). *Consider the case*

$$\sigma^2(s, t) = F(|t - s|) \geq 0,$$

with F continuous, non-negative and with $F(0) = 0$. A simple condition on F which generalizes at once the above fBm example and the previous Example 2.5 is **semi-concavity**, i.e.

$$F'' \leq k, \quad \text{in distributional sense on } (0, T) \text{ for some } k \in \mathbb{R},$$

which is tantamount to say that $-F'' + k$ is a (non-negative) Radon measure on $(0, T)$, which in turn induces a signed Radon measure on $[0, T]^2 \setminus D$, given by

$$A \mapsto \int_A (-F''(|t - s|) + k) ds dt - k\lambda(A),$$

where λ is the two-dimensional Lebesgue measure. Then $\mu := \partial_{s,t} R = -\frac{1}{2}\partial_{s,t}\sigma^2$ is also a signed Radon measure, with $\mu_+ \leq \frac{k}{2}\lambda$. Clearly, there will always be some $h > 0$ (depending on F) such that (B.ii) holds. Under the additional assumption $F(t) = O(t^{1/\rho})$ for some $\rho \geq 1$, we then have (B.iii), with $\omega(s, t) = C(t - s)$ and conclude that, with changing constants,

$$V_{1,\rho}(R; [s, t]^2) \leq C \left(|t - s|^{1/\rho} + \frac{k}{2} |t - s|^2 \right) \leq O(|t - s|^{1/\rho}).$$

Example 2.9 (Sums of fBm). *In the previous example, F'' was bounded, as a Schwartz distribution, by an L^∞ -function on $[0, T]^2$, namely by the constant k . But L^1 would be enough. Consider $X = B^{H_1} + B^{H_2}$, a sum of two independent fBm with Hurst parameters $H_1 \geq 1/2 \geq H_2$. A look at our two previous fBm examples reveals that*

$$\mu = \underbrace{H_1(2H_1 - 1)|t - s|^{2H_1 - 2} ds dt}_{=: \mu_+} - \underbrace{H_2(1 - 2H_2)|t - s|^{2H_2 - 2} ds dt}_{=: \mu_-}.$$

We easily check all conditions, in particular (B.iii) holds with $\rho = 1/(2H_2) \geq 1$ and $\omega(s, t) = t - s$. As a consequence,

$$\begin{aligned} V_{1,\rho}(R; [s, t]^2) &\leq C \left(|t - s|^{1/\rho} + H_1(2H_1 - 1) \int_{[s, t]^2} |t' - s'|^{2H_1 - 2} ds' dt' \right) \\ &\leq C \left(|t - s|^{1/\rho} + |t - s|^{2H_1} \right) = O(|t - s|^{1/\rho}). \end{aligned}$$

(Of course, the same conclusion can be obtained from our previous fBm examples, using $R_X = R_{B^{H_1}} + R_{B^{H_2}}$ and then the triangle inequality for the semi-norm $V_{1,\rho}$.)

Example 2.10 (Volterra processes II). *Volterra fBm with $H < 1/2$, i.e. singular kernel $K(t, s) = (t - s)^{H-1/2}$ is also covered by Part B. More generally, it is possible (thanks to the robustness of the conditions of Part B), if tedious, to give technical assumptions on K which guarantee that (B.i-iii) are satisfied. We note that $\rho \geq 1$ of condition (B.iii) is determined from the blow-up behaviour of K near the diagonal.*

Example 2.11 (bifractional Brownian motion). *Consider a **bifractional Brownian motion** (cf. e.g. [HV03, RT06, KRT07]), that is, a centered Gaussian process $B^{H,K}$ on $[0, T]$ with covariance*

function given by¹⁰

$$R(s, t) = \frac{1}{2^K} \left((s^{2H} + t^{2H})^K - |t - s|^{2HK} \right),$$

for some $H \in (0, 1)$ and $K \in (0, 1]$. It is known (cf. [HV03, Proposition 3.1]) that whenever $s < t$

$$(2.6) \quad 2^{-K} |t - s|^{2HK} \leq \sigma^2(s, t) \leq 2^{1-K} |t - s|^{2HK}.$$

We claim that the case $HK \geq \frac{1}{2}$ (resp. $\leq \frac{1}{2}$) is handled by Part A (resp. B) of Theorem 2.2. To this end, first note that

$$\partial_{s,t} R(s, t) = \frac{(2H)^2 K(K-1)}{2^K} \frac{s^{2H-1} t^{2H-1}}{(s^{2H} + t^{2H})^{2-K}} + \frac{2HK(2HK-1)}{2^K} |t - s|^{2HK-2}.$$

The measure

$$-\nu := \frac{(2H)^2 K(K-1)}{2^K} \frac{s^{2H-1} t^{2H-1}}{(s^{2H} + t^{2H})^{2-K}} ds dt$$

has finite mass, as can be easily seen by introducing polar coordinates. This implies that $\partial_{s,t} R(s, t) := \mu$ is a Radon measure on $(0, T)^2 \setminus D$. If $HK \geq \frac{1}{2}$, we have the decomposition $\mu = \mu_+ - \mu_-$ with $\mu_- = \nu$ and we have already seen that (A.i) holds. (A.ii) is trivially satisfied, and with (2.6) we may conclude that

$$V_1(R; [s, t]^2) \leq 2^{1-K} |t - s|^{2HK} + 2\nu([s, t]^2) \quad \text{for all } [s, t] \subseteq [0, T].$$

If $HK \leq \frac{1}{2}$, $\mu_+ \equiv 0$ on $(0, T)^2 \setminus D$, thus (B.i) is satisfied in both cases. (B.ii) is also easy to see. Indeed, since $B^{H,K}$ is a self-similar process with index HK , one can use scaling to see that it is enough to show that for all $t_0 \in \mathbb{R}_+$ and $h_0 \in [0, 1]$, the function

$$h \mapsto R \left(\begin{array}{c} t_0, t_0 + 1 \\ t_0 + h_0, t_0 + h_0 + h \end{array} \right) =: \phi(h)$$

is nonnegative on $[0, 1 - h_0]$. Since $\phi(0) = 0$, it is enough to show that $\phi' \geq 0$ on $(0, 1 - h_0)$ which follows by a simple calculation. Finally, from (2.6) we see that (B.iii) holds with $\rho = \frac{1}{2HK}$ and $\omega(s, t) = |t - s|$, therefore

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{\frac{1}{\rho}}).$$

Example 2.12 (Random Fourier series I: stationary). Consider a stationary random Fourier series¹¹

$$\Psi(t) = \sum_{k=1}^{\infty} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt), \quad t \in [0, 2\pi],$$

with zero-mean, independent Gaussians $\{Y^k \mid k \in \mathbb{Z}\}$ with unit variance. We compute

$$\begin{aligned} R(s, t) &= \sum \alpha_k^2 \sin(ks) \sin(kt) + \alpha_{-k}^2 \cos(ks) \cos(kt) \\ &= \frac{1}{2} \sum (\alpha_k^2 + \alpha_{-k}^2) \cos(k(t-s)) + (\alpha_k^2 - \alpha_{-k}^2) \cos(k(t+s)) \end{aligned}$$

¹⁰As pointed out e.g. in [KRT07] this process does not fit in the Volterra framework.

¹¹We may ignore the (constant, random) zero-mode in the series since we are only interested in properties of the increments of the process.

and note that $\alpha_k^2 \equiv \alpha_{-k}^2$ due to the assumed stationarity of Ψ . This leaves us with

$$\begin{aligned} R(s, t) &= K(|t - s|) \\ \sigma^2(s, t) &= 2(K(0) - K(|t - s|)) =: F(|t - s|), \end{aligned}$$

where

$$K(t) := \sum_{k=1}^{\infty} \alpha_k^2 \cos(kt).$$

In special situations, e.g. when $\alpha_k^2 = 1/k^2$, one can find $K \in C^2([0, 2\pi])$ in closed form, which brings us back to Example 2.5. This is not possible in general, but in view of the above Example 2.8, it would suffice to know that K is convex and $1/\rho$ -Hölder. Conditions on the Fourier-coefficients for this to hold true are known from Fourier analysis (recalled in detail in Section 3 below). For instance, given (eventually) decreasing (α_k^2) , K is $1/\rho$ -Hölder iff $\alpha_k^2 = O\left(k^{-(1+\frac{1}{\rho})}\right)$. In particular, in the model case

$$\alpha_k^2 = \frac{1}{k^{2\alpha}}$$

the desired decay holds true iff

$$2\alpha = 1 + 1/\rho \leftrightarrow \rho = \frac{1}{2\alpha - 1} \geq 1 \text{ (for } \alpha \leq 1).$$

Convexity also holds true here and we conclude that for all $[s, t] \subset [0, 2\pi]$

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}).$$

Example 2.13 (Random Fourier series II: non-stationary, general case). *As seen in the previous example the covariance may be written as*

$$\begin{aligned} R(s, t) &= K(|t - s|) + K(|t + s|) + \tilde{K}(|t - s|) - \tilde{K}(|t + s|) \\ &=: R^-(s, t) + R^+(s, t) + \tilde{R}^-(s, t) - \tilde{R}^+(s, t), \end{aligned}$$

where R^\pm and K are as before and

$$\tilde{K}(t) := \sum_{k=1}^{\infty} \alpha_{-k}^2 \cos(kt).$$

Under the assumption that K, \tilde{K} are convex and $1/\rho$ -Hölder, the cases $R \in \{R^-, \tilde{R}^-\}$ were already handled in the previous example, where we established

$$V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{1/\rho}).$$

We claim that R^+ can be handled with Part A. \tilde{R}^+ may then be treated analogously. Condition (A.i) is simple: Using convexity of K

$$\partial_{s,t} R^+ = K''(t + s) \geq 0, \quad \text{on } [0, T]^2 \setminus D,$$

so that $\mu := \partial_{s,t} R^+ = \mu_+$ is a non-negative (but in general not finite) Radon-measure on $[0, T]^2 \setminus D$. Unlike in previous examples, condition (A.ii) is not trivial, since R^+ is not a covariance function

in general. Nonetheless, we have

$$\begin{aligned} R^+ \left(\begin{array}{c} s, t \\ s, t \end{array} \right) &= K(2t) + K(2s) - (2K(t+s)) \\ &= 2 \left(\frac{K(2t) + K(2s)}{2} - K\left(\frac{2t+2s}{2}\right) \right) \\ &\geq 0, \quad \forall 0 \leq s \leq t \leq \pi, \end{aligned}$$

thanks to convexity of K on $[0, 2\pi]$. This settles condition (A.ii). We conclude that R^+ has finite 1-variation¹²,

$$V_1 \left(R^+; [s, t]^2 \right) \leq R \left(\begin{array}{c} s, t \\ s, t \end{array} \right) = K(2t) + K(2s) - 2K(t+s) = O(|t-s|^{1/\rho}).$$

Since $R = R^- + R^+ + \tilde{R}^- - \tilde{R}^+$ we can now conclude with $V_{1,\rho} \leq V_1$ and the triangle inequality to see that R has (Hölder controlled) mixed $(1, \rho)$ -variation, in the sense that

$$V_{1,\rho} \left(R; [s, t]^2 \right) = O(|t-s|^{1/\rho}),$$

for all $[s, t] \subset [0, \pi]$. (The extension of this estimate to $[0, 2\pi]$ is not difficult¹³.)

Example 2.14 (Fourier fractional Brownian bridge). *Fourier fractional Brownian bridge is the Gaussian process given by the random Fourier series*

$$W_t^\alpha = \sum_{k=1}^{\infty} \frac{Y_k \sin\left(\frac{k}{2}t\right)}{k^\alpha}, \quad \text{for } t \in [0, 2\pi], \quad \alpha \in \left(\frac{1}{2}, 1\right],$$

with Y_k as above. This process arises by replacing the covariance operator of Brownian bridge (the Dirichlet Laplacian $-\Delta$) by its fractional power $(-\Delta)^\alpha$. Clearly, this is a special case of the previous example.

Example 2.15 (Stationary processes: spectral measure). *Let X_t be a stationary, zero-mean process with covariance*

$$R(s, t) = K(|t-s|)$$

for some continuous function K . By a well-known theorem of Bochner,

$$\begin{aligned} K(t) &= \int \cos(t\xi) \mu(d\xi), \\ \sigma^2(t) := \sigma^2(0, t) &= 2(K(0) - K(t)) = 4 \int \sin^2(t\xi/2) \mu(d\xi), \end{aligned}$$

where μ is a finite positive symmetric measure on \mathbb{R} ("spectral measure"). The case of discrete μ corresponds to Example 2.12. Another example is given by the **fractional Ornstein–Uhlenbeck process**,

$$X_t = \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H, \quad t \in \mathbb{R},$$

¹²The situation here is reminiscent of absolutely continuous paths $x = x(t)$ on $[0, T]$ with $\dot{x} \in L^p$ where $1/\rho + 1/p = 1$. Indeed, as may be seen from Hölder's inequality, the L^1 -norm of $\dot{x}|_{[s,t]}$, which equals the 1-variation of x over $[s, t]$, is finite and of order $|t-s|^{1/\rho}$.

¹³Considering the Fourier series with argument shifted by π , gives the same estimate on $[\pi, 2\pi]^2$. In fact, one can also handle the mixed $(1, \rho)$ -variation of R^+ on $[0, \pi] \times [\pi, 2\pi]$ by playing it back to the mixed variation of R^- on $[0, \pi] \times [0, \pi]$, using the fact that K is given by cosine series, hence is even around π .

which should be viewed as the stationary solution to $dX = -\lambda X dt + dB^H$. In this case, it is known that X has a spectral density of the form¹⁴

$$\frac{d\mu}{d\xi} = c_H \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2}.$$

Clearly, the decay of the density is related to the regularity of K . More precisely, writing

$$\begin{aligned} \hat{K}(\xi) &:= \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2} \sim \langle \xi \rangle^{-1-2H} \quad \text{where } \langle \xi \rangle = (1 + \xi^2)^{1/2} \\ \langle \xi \rangle^s \hat{K}(\xi) &\sim \langle \xi \rangle^{s-1-2H} \in L^2 \quad \text{iff } 2(s-1-2H) < -1, \end{aligned}$$

i.e. iff $s < s^* := 1/2 + 2H$. It follows that $K \in H^s$ for any $s < s^*$ and thus by a standard Sobolev embedding K is α -Hölder for $\alpha < s^* - 1/2 = 2H$. Alternatively, and a little sharper, Theorem 7.3.1 in [MR06] tells us that if \hat{K} is regularly varying at ∞ , then

$$\sigma^2(t) \sim C \hat{K}(1/t)/t \text{ as } t \rightarrow 0.$$

Applied to the situation at hand we see that $\sigma^2(t) = O(t^{2H})$, since $\hat{K}(\xi) \sim (1/\xi)^{1+2H}$. With focus on the rough case $H \leq 1/2$, this gives condition (B.iii) with $\rho = 1/(2H)$, $\omega(s, t) = t - s$. Moreover, it can be seen that there is a $T > 0$ such that K is convex on $[0, T]$ (cf. Example 5.3 below), which implies (B.i) and (B.ii) as in Example 2.8. Hence it follows that $V_{1,\rho}(R; [s, t]^2) = O(|t - s|^{2H})$ for all $[s, t] \subseteq [0, T]$.

Example 2.16 (SPDEs). Solutions to the stochastic heat equation on $[0, 2\pi]$ with periodic, Dirichlet or Neumann boundary conditions viewed as a spatial process, i.e. indexed by $x \in [0, 2\pi]$ for fixed time are essentially covered by Example 2.12 (periodic boundary case; the situation of [Hai11, GIP12]) and by Example 2.13 in case of the other boundary conditions. We shall return to this in Section 4 where we also address related questions such as continuity in t of solutions viewed as (random) spatial rough paths.

2.4. Proof of Theorem 2.2, Part A.

Proof of Theorem 2.2, Part A. From (A.i), the distributional mixed derivative of R on $(0, T)^2 \setminus D$ is given by

$$(2.7) \quad \frac{\partial^2 R}{\partial s \partial t} = \mu_+ - \mu_-$$

where μ_- (trivially extended to $[0, T]^2$ whenever convenient) has finite mass. By assumption, the distribution function of μ_-

$$R^-(s, t) := \mu_-([0, s] \times [0, t]),$$

is continuous. We may then define $R^+ \in C([0, T])$ by imposing the decomposition

$$R = R^+ - R^-.$$

Clearly, the distributional mixed derivatives of R^\pm on $(0, T)^2 \setminus D$ are given by

$$(2.8) \quad \frac{\partial^2 R^\pm}{\partial_t \partial_s} = \mu_\pm.$$

¹⁴This generalizes the well-known fact that the spectral density of the classical OU process is of Cauchy type.

Noting that all rectangular increments of R^- are non-negative, $R^-(A) = \mu_-(A) \geq 0$, we immediately have

$$V_1(R^-; A) = R^-(A) = \mu_-(A)$$

for all $A = [s, t] \times [u, v] \subset [0, T]^2$. On the other hand, any such rectangle A may be split up in finitely many "small squares", say $Q_i = [t_i, t_{i+1}]^2$ with $t_{i+1} - t_i \leq h$ for all i , and a (finite) number of "off-diagonal" rectangles A_j , whose interior does not intersect the diagonal. Since $R(Q_i) = \sigma^2(t_i, t_{i+1}) \geq 0$, by (A.ii), and $R(A_j) \geq -R^-(A_j) = -\mu_-(A_j)$, we have

$$\begin{aligned} R(A) &= \sum_i R(Q_i) + \sum_j R(A_j) \\ &\geq -\sum_j \mu_-(A_j) \geq -\mu_-(A), \end{aligned}$$

for all rectangles A . This implies finite 1-variation over every rectangle $A = [s, t] \times [u, v]$. Indeed, for any dissections (t_i) of $[s, t]$ and (t'_j) of $[u, v]$ we have

$$\begin{aligned} \sum_{t_i, t'_j} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i, t'_j} \left\{ \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} + \mu_-([t_i, t_{i+1}] \times [t'_j, t'_{j+1}]) \right| + \mu_-([t_i, t_{i+1}] \times [t'_j, t'_{j+1}]) \right\} \\ &= R \begin{pmatrix} s, t \\ u, v \end{pmatrix} + 2\mu_-([s, t] \times [u, v]), \end{aligned}$$

and so, for all rectangles A ,

$$V_1(R; A) \leq R(A) + 2\mu_-(A).$$

□

2.5. Proof of Theorem 2.2, Part B. Let us start with a few definitions.

Definition 2.17. For $\gamma, \rho \geq 1$ set

$$V_{\gamma, \rho}^+(R; [s, t] \times [u, v]) := \sup_{(t'_j) \in \mathcal{D}([u, v])} \left(\sum_{(t_i) \in \mathcal{D}([s, t])} \sup_{(t_i) \in \mathcal{D}([s, t])} \left(\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}}$$

and

$$\begin{aligned} V_{\gamma, \rho}^+(R; U_{[s, t]}) &:= \sup_{(t'_j) \in \mathcal{D}([s, t])} \left(\sum_{(t'_j) \in \mathcal{D}([s, t])} \sup_{(t_i) \in \mathcal{D}([s, t'_j])} \left(\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}} \\ V_{\gamma, \rho}^+(R; L_{[s, t]}) &:= \sup_{(t'_j) \in \mathcal{D}([s, t])} \left(\sum_{(t'_j) \in \mathcal{D}([s, t])} \sup_{(t_i) \in \mathcal{D}([t'_{j+1}, t])} \left(\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}} \\ V_{\gamma, \rho}^+(R; D_{[s, t]}) &:= \sup_{(t'_j) \in \mathcal{D}([s, t])} \left(\sum_{(t'_j) \in \mathcal{D}([s, t])} \sup_{(t_i) \in \mathcal{D}([t'_j, t'_{j+1}])} \left(\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}}. \end{aligned}$$

For any rectangle $A \subseteq I^2$ it is easy to see that

$$V_{\gamma, \rho}(R; A) \leq V_{\gamma, \rho}^+(R; A)$$

and also (e.g. as a consequence of [FV11, Theorem 1.i])

$$V_1(R; A) = V_{1,1}^+(R; A).$$

The main reason for introducing V^+ as above is the following lemma:

Lemma 2.18 (Concatenation lemma 1). *Let R be as before. Then*

$$V_{\gamma,\rho}^+(R; [s, t]^2) \leq C (V_{\gamma,\rho}^+(R; U_{[s,t]}) + V_{\gamma,\rho}^+(R; D_{[s,t]}) + V_{\gamma,\rho}^+(R; L_{[s,t]})), \quad \forall [s, t] \subseteq I,$$

for some constant $C = C(\rho, \gamma)$.

Proof. Let (t'_j) be a partition of $[s, t]$. Fix $[t'_j, t'_{j+1}]$ and let (t_i) be a partition of $[s, t]$. By subdividing rectangles which lie on the diagonal into at maximum three parts, we see that

$$\begin{aligned} & 3^{1-\gamma} \sum_{t_i} \left| R \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \\ & \leq \sup_{(t_i) \in \mathcal{D}([s, t'_j])} \sum_{t_i} \left| R \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma + \sup_{(t_i) \in \mathcal{D}([t'_j, t'_{j+1}])} \sum_{t_i} \left| R \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \\ & \quad + \sup_{(t_i) \in \mathcal{D}([t'_{j+1}, t])} \sum_{t_i} \left| R \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma. \end{aligned}$$

Now we take the supremum, sum over t'_j , take the supremum again to see that

$$\begin{aligned} & \sup_{(t'_j) \in \mathcal{D}([s, t])} \left(\sum_{t'_j} \sup_{(t_i) \in \mathcal{D}([s, t])} \left(\sum_{t_i} \left| R \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \right)^{\frac{\rho}{\gamma}} \right)^{\frac{1}{\rho}} \\ & \leq C (V_{\gamma,\rho}^+(R; U_{[s,t]}) + V_{\gamma,\rho}^+(R; D_{[s,t]}) + V_{\gamma,\rho}^+(R; L_{[s,t]})). \end{aligned}$$

□

Lemma 2.19 (Concatenation lemma 2). *Assume that there is an $h > 0$ such that*

$$V_{\gamma,\rho}(R; [s, t] \times [u, v]) \leq \Phi(s, t; u, v)$$

holds for all squares $[s, t] \times [u, v] = [s, t]^2 \subseteq D_h$ and all off-diagonal rectangles $(s, t) \times (u, v) \subseteq I^2 \setminus D$, where $\Phi: \Delta_I \times \Delta_I \rightarrow \mathbb{R}_+$ is a non-decreasing function in $t - s$ and $v - u$. Then there is a constant $C = C(\gamma, \rho, h, T)$ such that

$$V_{\gamma,\rho}(R; [s, t] \times [u, v]) \leq C\Phi(s, t; u, v)$$

holds for all rectangles $[s, t] \times [u, v]$. The constant C can be chosen independently of h and T when considering only rectangles $[s, t] \times [u, v] \subset D_h$. The same is true if one replaces $V_{\gamma,\rho}$ by $V_{\gamma,\rho}^+$.

Proof. Step 1: Consider any square of the form $[s, t]^2 \subseteq I^2$. Then we can subdivide this square in N^2 smaller squares $(A_{i,j})_{i,j=1}^N$ with equal side length \tilde{h} which can be chosen such that $h/2 \leq \tilde{h} \leq h$ and $N \leq M$ where M is a number depending on T and h . Each of these small squares does either lie on the diagonal or its inner part does not intersect with the diagonal. Hence

$$V_{\gamma,\rho}(R; [s, t]^2) \leq c_1(N, \gamma, \rho) \sum_{i,j=1}^N V_{\gamma,\rho}(R; A_{i,j}) \leq c_2(N, \gamma, \rho) \Phi(s, t; u, v).$$

by monotonicity.

Step 2: Let $[s, t] \times [u, v]$ be any rectangle in I^2 . Then we can subdivide it into one square lying on the diagonal and three rectangles for which the inner part does not intersect with the diagonal. We conclude as in step 1. \square

Lemma 2.20. *Let R as before and σ as in (2.1). Then the following two assertions are equivalent:*

(i) $\frac{\partial^2 \sigma^2}{\partial_t \partial_s} = -2 \frac{\partial^2 R}{\partial_t \partial_s} \geq 0$ in the sense of distributions, i.e. for every non-negative $\phi \in C_c^\infty(I^2 \setminus D)$

$$\int_{I^2} \frac{\partial^2 \phi}{\partial_t \partial_s}(s, t) \sigma^2(s, t) ds dt = -2 \int_{I^2} \frac{\partial^2 \phi}{\partial_t \partial_s}(s, t) R(s, t) ds dt \geq 0.$$

(ii) For all off-diagonal rectangles $(s, t) \times (u, v) \subseteq I^2 \setminus D$ we have

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq 0,$$

In addition, if either of the above conditions is satisfied then

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq \sigma^2(u, v), \quad \forall [u, v] \subseteq [s, t] \subseteq I.$$

All assertions remain true if we substitute \leq by \geq in the three inequalities.

Proof. We will only consider the \leq -case. Let $\varphi \in C_c(B_1(0))$ non-negative with $\|\varphi\|_{L^1(\mathbb{R}^2)} = 1$. We then define the standard Dirac sequence $\varphi^\varepsilon((s, t)) := \varepsilon^{-2} \varphi(\frac{1}{\varepsilon}(s, t))$ and observe $\text{supp}(\varphi^\varepsilon) \subseteq B_\varepsilon(0)$. We extend R by 0 to all of \mathbb{R}^2 and set $R^\varepsilon := R * \varphi^\varepsilon$. Then

$$\frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(a, b) = \int_{I^2} R(s, t) \frac{\partial^2}{\partial_t \partial_s}(\varphi^\varepsilon(s - a, t - b)) ds dt.$$

For $(a, b) \in \mathring{\Delta}_I$ we note

$$\text{supp}(\varphi^\varepsilon(s - a, t - b)) \subseteq B_\varepsilon((a, b)) \subseteq \mathring{\Delta}_I$$

for all ε small enough. Hence, $\varphi^\varepsilon(\cdot - a, \cdot - b)$ is an admissible test-function for (i) and thus $\frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(a, b) \leq 0$. Since

$$R^\varepsilon \begin{pmatrix} s, t \\ u, v \end{pmatrix} = \int_{I^2} \mathbb{1}_{[s, t]}(x) \mathbb{1}_{[u, v]}(y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy, \quad \forall s \leq t \leq u \leq v \in I,$$

(ii) follows using continuity of R .

Suppose now that (ii) is satisfied. We may approximate R by $R^\varepsilon \in C^\infty(\Delta_I)$ such that

$$\|R - R^\varepsilon\|_{C(\Delta_I)} \leq \frac{\varepsilon}{4}.$$

By (ii) we have

$$\int_{I^2} \mathbb{1}_{[s, t]}(x) \mathbb{1}_{[u, v]}(y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy = R^\varepsilon \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq \varepsilon,$$

for all $s \leq t \leq u \leq v \in I$. We note that the set of non-negative $f \in L^1(\Delta_I)$ satisfying

$$\int_{\Delta_I} f(x, y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy \leq \varepsilon$$

is a monotone class. By the monotone class theorem we thus have

$$\int_{\Delta_I} f(x, y) \frac{\partial^2 R^\varepsilon}{\partial_t \partial_s}(x, y) dx dy \leq \varepsilon$$

for all non-negative $f \in L^1(\Delta_I)$. Considering non-negative $f \in C_c^\infty(\hat{\Delta}_I)$, a partial integration and letting $\varepsilon \rightarrow 0$ yields (i).

To prove the remaining inequality we note

$$R \begin{pmatrix} s, t \\ u, v \end{pmatrix} = R \begin{pmatrix} s, u \\ u, v \end{pmatrix} + R \begin{pmatrix} u, v \\ u, v \end{pmatrix} + R \begin{pmatrix} v, t \\ u, v \end{pmatrix} \leq R \begin{pmatrix} u, v \\ u, v \end{pmatrix}.$$

□

We are now able to prove part B of our main theorem.

Proof of Theorem 2.2, Part B. We decompose R as in (2.7), (2.8). Note that we now have $m = 2\mu^+$. We start by proving (2.4) by an application of Lemma 2.18: Let (t'_j) be a partition of $[s, t]$. Fix $[t'_j, t'_{j+1}]$ and let (t_i) be a partition of $[s, t'_j]$. Apply Lemma 2.20 with R equal to $-R^-$ and then $-R^+$ to get

$$-R^-(A_{i,j}) \leq 0 \leq R^+(A_{i,j})$$

for all $A_{i,j} = [t'_j, t'_{j+1}] \times [t_i, t_{i+1}]$. Hence, with condition (B.ii) we have

$$\begin{aligned} \sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i} \left| R^- \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| + \left| R^+ \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| \\ &= R^- \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} + R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &= -R \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &= -R \begin{pmatrix} s, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} + R \begin{pmatrix} t'_j, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &\leq \sigma^2(t'_j, t'_{j+1}) + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix} \\ &\leq \omega(t'_j, t'_{j+1})^{\frac{1}{\rho}} + 2R^+ \begin{pmatrix} s, t'_j \\ t'_j, t'_{j+1} \end{pmatrix}. \end{aligned}$$

Taking the supremum over (t_i) , then the ρ th power, summing over (t'_j) and finally taking the supremum over (t'_j) gives

$$\begin{aligned} V_{1,\rho}^+(R; U_{[s,t]}) &\leq C (\omega(s, t) + V_{1,\rho}^+(R^+; U_{[s,t]})^\rho)^{\frac{1}{\rho}} \\ &\leq C \left(\omega^{\frac{1}{\rho}}(s, t) + m([s, t]^2) \right), \end{aligned}$$

for some constant C depending on ρ only. Similarly,

$$V_{1,\rho}^+(R; L_{[s,t]}) \leq C \left(\omega^{\frac{1}{\rho}}(s, t) + m([s, t]^2) \right).$$

Now let (t'_j) be a partition of $[s, t]$, fix $[t'_j, t'_{j+1}]$ and let (t_i) be a partition of $[t'_j, t'_{j+1}]$. By (B.ii), $R(A_{i,j}) \geq 0$ for all $A_{i,j} = [t'_j, t'_{j+1}] \times [t_i, t_{i+1}]$, thus

$$\sum_{t_i} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| = \left| R \begin{pmatrix} t'_j, t'_{j+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| = \sigma^2(t'_j, t'_{j+1})$$

and hence

$$V_{1,\rho}^+(R; D_{[s,t]}) \leq \omega(s,t)^{\frac{1}{\rho}}.$$

By Lemma 2.18 we conclude

$$\begin{aligned} V_{1,\rho}^+(R; [s,t]^2) &\leq C (V_{1,\rho}^+(R; U_{[s,t]}) + V_{1,\rho}^+(R; D_{[s,t]}) + V_{1,\rho}^+(R; L_{[s,t]})) \\ &\leq C \left(\omega^{\frac{1}{\rho}}(s,t) + m([s,t]^2) \right) \end{aligned}$$

and (2.4) has been shown. Note that in fact we may deduce the somewhat stronger conclusion

$$V_{1,\rho}^+(R; [s,t]^2) \leq C \left(\omega^{\frac{1}{\rho}}(s,t) + V_{1,\rho}^+(R^+; [s,t]^2) \right), \quad \forall [s,t]^2 \subseteq D_h.$$

We now establish (2.5). Let $(s,t) \times (u,v) \subseteq I^2 \setminus D$ and let (t_i) be a partition of $[s,t]$ and (t'_j) be a partition of $[u,v]$. By non-negativity of non-overlapping increments,

$$\begin{aligned} \sum_{t_i, t'_j} \left| R \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| &\leq \sum_{t_i, t'_j} \left| R^- \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| + \left| R^+ \begin{pmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{pmatrix} \right| \\ &= R^- \begin{pmatrix} s, t \\ u, v \end{pmatrix} + R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \\ &\leq \left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| + 2R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \end{aligned}$$

Taking the supremum over all partitions, the Cauchy-Schwarz inequality

$$\left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| \leq \left| R \begin{pmatrix} s, t \\ s, t \end{pmatrix} \right|^{\frac{1}{2}} \left| R \begin{pmatrix} u, v \\ u, v \end{pmatrix} \right|^{\frac{1}{2}}$$

gives

$$\begin{aligned} V_1(R; [s,t] \times [u,v]) &\leq \left| R \begin{pmatrix} s, t \\ u, v \end{pmatrix} \right| + 2R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \\ &\leq C \left(\omega(s,t)^{\frac{1}{2\rho}} \omega(u,v)^{\frac{1}{2\rho}} + m([s,t] \times [u,v]) \right) \end{aligned}$$

and Lemma 2.19 finishes the proof. \square

3. PART III: RANDOM FOURIER SERIES

Let us now consider a (formal) random Fourier series

$$(3.1) \quad \Psi(t) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{\infty} \alpha_k Y^k \sin(kt) + \alpha_{-k} Y^{-k} \cos(kt),$$

where Y^k are real-valued, centered random variables with $\mathbb{E}Y^k Y^l = \delta_{k,l}$ for all $k, l \in \mathbb{Z}$ and α_k are real valued coefficients. Since we are interested in properties of the covariance of Ψ , we will formulate our conditions in terms of the squared coefficients $a_k := \alpha_k^2$, $k \in \mathbb{Z}$.

Remark 3.1. *Assume that $\alpha_k^2 = O(|k|^{-(1+\frac{1}{\rho})})$ for some $\rho > 0$. Then (3.1) converges uniformly almost surely and the limit yields a continuous function. Moreover, if the Y_k are Gaussian,*

Ψ has β -Hölder continuous trajectories¹⁵ almost surely for all $\beta < \frac{1}{2\rho}$. This follows from [Kah85, Theorem 7.4.3 and Theorem 5.8.3].

Our main Theorem on random Fourier series is

Theorem 3.2. *Consider the random Fourier series (3.1) with (a_k) satisfying $\Delta^2(k^2 a_k) \leq 0$ for all $k \in \mathbb{Z}$,*

$$\lim_{k \rightarrow \pm\infty} k^3 |\Delta^2 a_k| + k^2 |\Delta a_k| = 0,$$

$a_k = O\left(|k|^{-(1+\frac{1}{\rho})}\right)$ for some $\rho \geq 1$ for $k \rightarrow \pm\infty$ and a_k, a_{-k} non-increasing¹⁶ for $k \geq 1$. Then the covariance R_Ψ of Ψ has finite Hölder controlled $(1, \rho)$ -variation and there is a constant $C > 0$ such that

$$(3.2) \quad V_{1,\rho}(R_\Psi; [s, t] \times [u, v]) \leq C |t - s|^{\frac{1}{2\rho}} |v - u|^{\frac{1}{2\rho}}, \quad \forall [s, t] \times [u, v] \subseteq [0, 2\pi]^2.$$

The constant C depends only on ρ and C_1 , where $C_1 \geq \sup_{k \in \mathbb{Z}} a_k |k|^{1+\frac{1}{\rho}}$.

Note that the model case $(a_k) = (|k|^{2\alpha})$ for $\alpha \in (\frac{1}{2}, 1]$ is contained as a special case in Theorem 3.2.

Proof of Theorem 3.2: Note first that, as already seen in Remark 3.1, Ψ exists as an uniformly almost sure limit. Since $(a_k) \in l^1(\mathbb{Z})$ we have $(\alpha_k) \in l^2(\mathbb{Z})$, thus for fixed $t \in [0, 2\pi]$, Ψ exists also as a convergent sum in $L^2(\Omega)$. Set $Q_1 = [0, \pi]^2$, $Q_2 = [0, \pi] \times [\pi, 2\pi]$, $Q_3 = [\pi, 2\pi]^2$ and $Q_4 = [\pi, 2\pi] \times [0, \pi]$. We first show that (3.2) holds provided $[s, t] \times [u, v] \subseteq Q_i$ for some $i = 1, \dots, 4$. We recall from Example 2.13

$$(3.3) \quad R_\Psi(s, t) = R^-(s, t) + R^+(s, t) + \tilde{R}^-(s, t) - \tilde{R}^+(s, t)$$

and using the triangle inequality it is enough to show the estimate (3.2) for R^\pm, \tilde{R}^\pm separately. From Lemma 3.3 below we know that K, \tilde{K} are convex on $[0, 2\pi]$ and non-increasing on $[0, \pi]$. By Lemma 3.4 below, K is $\frac{1}{\rho}$ -Hölder continuous. As we have seen in Example 2.13 this implies the claim for $[s, t]^2 \subseteq Q_1$. It is easy to see that all R^\pm, \tilde{R}^\pm satisfy the Cauchy-Schwarz inequality, thus we can use Theorem 2.2, part B to conclude (3.2) for R^-, \tilde{R}^- if $[s, t] \times [u, v] \subseteq Q_1$. Furthermore,

$$R^+ \begin{pmatrix} s, t \\ u, v \end{pmatrix} \leq \sqrt{R^+ \begin{pmatrix} s, t \\ s, t \end{pmatrix}} \sqrt{R^+ \begin{pmatrix} u, v \\ u, v \end{pmatrix}} \leq \|K\|_{1/\rho\text{-Hölder}} |t - s|^{\frac{1}{2\rho}} |u - v|^{\frac{1}{2\rho}}$$

as seen in Example 2.13. The same is true for \tilde{R}^+ which shows (3.2) for R^+, \tilde{R}^+ and $[s, t] \times [u, v] \subseteq Q_1$. The process $t \mapsto \Psi_{t+\pi}$ has the same covariance as Ψ , thus the estimate (3.2) also holds for $[s, t] \times [u, v] \subseteq Q_3$. By symmetry considerations, if E is any rectangle in Q_2 or Q_4 , there is a rectangle \bar{E} in Q_1 (or in Q_3) with the same side length such that $R^+(E) = R^-(\bar{E})$, $\tilde{R}^+(E) = \tilde{R}^-(\bar{E})$ and vice versa for R^-, \tilde{R}^- . Thus (3.2) also holds for $[s, t] \times [u, v] \subseteq Q_i$ for $i = 2, 4$. The general case just follows by subdividing a given rectangle $[s, t] \times [u, v]$ in at maximum four rectangles lying in Q_1, \dots, Q_4 and using the estimates above (which only leads to a larger constant). This proves the theorem. \square

¹⁵If $\beta = n + \tilde{\beta}$ for some $\tilde{\beta} \in (0, 1]$, this means that the trajectories are n -times differentiable and the n -th derivative is $\tilde{\beta}$ -Hölder continuous.

¹⁶The monotonicity of a_k, a_{-k} is required for the sole purpose of using Lemma 3.4 below. In fact, it can be dropped when we use Sobolev embeddings instead (cf. Remark 3.5 below). However, we may only conclude finite $(1, \rho')$ -variation for any $\rho' > \rho$ in this case.

3.1. Convexity, monotonicity and Hölder regularity of cosine series. We start by deriving conditions for convexity and monotonicity of cosine series

$$(3.4) \quad K(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt).$$

In the following let Δ , Δ^2 be the first and second forward-difference operators, i.e. for a sequence $\{a_k\}_{k \in \mathbb{N}}$

$$\Delta a_k := a_{k+1} - a_k$$

and $\Delta^2 := \Delta \circ \Delta$. Moreover, let

$$D_n(t) := 1 + 2 \sum_{k=1}^n \cos(kt), \quad t \in \mathbb{R}$$

be the Dirichlet kernel and

$$F_n(t) := \sum_{k=0}^n D_k(t), \quad t \in \mathbb{R},$$

be the unnormalized Fejér kernel.

Lemma 3.3. *Let $\{a_k\}_{k \in \mathbb{N}}$ be such that $\Delta^2(k^2 a_k) \leq 0$, $\forall k \in \mathbb{N}$ and*

$$(3.5) \quad \lim_{k \rightarrow \infty} k^3 |\Delta^2 a_k| + k^2 |\Delta a_k| + k |a_k| = 0.$$

Then the cosine series (3.4) exists locally uniformly in $(0, 2\pi)$, is convex on $[0, 2\pi]$ and decreasing on $[0, \pi]$.

Proof. The proof follows ideas from [Kra11], we include it for the readers convenience. We first note that since

$$\Delta(k^2 a_k) = k^2 \Delta a_k + (2k+1)a_{k+1}$$

and

$$\Delta^2(k^2 a_k) = k^2 \Delta^2 a_k + 2(2k+1)\Delta a_{k+1} + 2a_{k+2}$$

assumption (3.5) is equivalent to

$$(3.6) \quad \lim_{k \rightarrow \infty} |k \Delta^2(k^2 a_k)| + |\Delta(k^2 a_k)| + k |a_k| = 0.$$

Using the Abel transformation we observe

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos(kt) = \frac{1}{2} \sum_{k=0}^n \Delta a_{k+1} D_k(t) + \frac{1}{2} a_{n+1} D_n(t).$$

By the assumptions and (3.6) we have $\sum_{k=1}^{\infty} |\Delta a_k| < \infty$. Since $\sup_{n \in \mathbb{N}} D_n(t)$ is bounded locally uniformly on $(0, 2\pi)$ and $a_n \rightarrow 0$ we observe that

$$K(t) := \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) = \frac{1}{2} \sum_{k=0}^{\infty} \Delta a_k D_k(t)$$

exists locally uniformly and is continuous in $(0, 2\pi)$.

The Cesàro means of the sequence $S_n(t)$ are given by

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) a_k \cos(kt).$$

By Fejér's Theorem [Zyg59, Theorem III.3.4] and continuity of K , $\sigma_n \rightarrow K$ locally uniformly in $(0, 2\pi)$. Hence, $\sigma_n'' \rightarrow K''$ in the space of distributions on $(0, 2\pi)$. Clearly,

$$\sigma_n''(t) = - \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) k^2 a_k \cos(kt).$$

Let $\beta_k := \left(1 - \frac{k}{n+1}\right) k^2 a_k$. Using summation by parts twice we obtain

$$\begin{aligned} 2\sigma_n''(t) &= \sum_{k=0}^n \Delta \beta_k D_k(t) \\ &= \Delta \beta_n F_n(t) - \sum_{k=0}^{n-1} \Delta^2 \beta_k F_k(t) \\ &= - \sum_{k=0}^{n-1} \Delta^2 (k^2 a_k) F_k(t) - \sum_{k=0}^{n-1} \left(\frac{k \Delta^2 (k^2 a_k)}{n+1} - \frac{2\Delta((k+1)^2 a_{k+1})}{n+1} \right) F_k(t) + \frac{n^2}{n+1} a_n F_n(t). \end{aligned}$$

By (3.6) and $0 \leq F_n(t) \leq \frac{C}{t^2} + \frac{C}{(2\pi-t)^2}$ it follows

$$\liminf_{n \rightarrow \infty} \inf_{t \in [\varepsilon, 2\pi - \varepsilon]} \sigma_n''(t) \geq 0, \quad \forall \varepsilon > 0.$$

For any non-negative test-function $\varphi \in C_c^\infty(0, 2\pi)$ Fatou's Lemma implies

$$K''(\varphi) = \lim_{n \rightarrow \infty} \sigma_n''(\varphi) \geq \int_0^{2\pi} \liminf_{n \rightarrow \infty} \sigma_n''(t) \varphi(t) dt \geq 0,$$

i.e. K'' is a non-negative distribution on $(0, 2\pi)$. Thus, K is convex on $[0, 2\pi]$.

Assume now that K is not decreasing on $[0, \pi]$, i.e. there are $s < t \in [0, \pi]$ such that $K(s) < K(t)$. Since K is given as a cosine series, we have $K(s) = K(s')$ and $K(t) = K(t')$ for $s' = 2\pi - s$ and $t' = 2\pi - t$. Choose $\lambda \in (0, 1)$ such that $\lambda s + (1 - \lambda)s' = t$. Then

$$K(\lambda s + (1 - \lambda)s') = K(t) > K(s) = \lambda K(s) + (1 - \lambda)K(s')$$

which is a contradiction to the convexity of K . □

As concerning Hölder regularity of cosine series we recall

Lemma 3.4 ([Lor48], Satz 8). *A cosine series (3.4) with non-increasing coefficients $a_k \downarrow 0$ for $k \rightarrow \infty$ is $\frac{1}{\rho}$ -Hölder continuous iff $a_k = O\left(k^{-(1+\frac{1}{\rho})}\right)$ for $k \rightarrow \infty$.*

Remark 3.5. *The above Lemma gives a sharper result than what is obtained by usual Sobolev embeddings. Indeed: Recall that an L^2 function on the torus with Fourier coefficients (a_k) is in the Sobolev space H^s iff $((1 + |k|^s)a_k) \in l^2$. By a standard Sobolev embedding (here in dimension 1), such functions are $(s - 1/2)$ -Hölder, provided $s > 1/2$. Hence, a cosine series (3.4) with coefficients $a_k = O\left(k^{-(1+\frac{1}{\rho})}\right)$ for $k \rightarrow \infty$ is α -Hölder for all $\alpha < 1/\rho$.*

3.2. Stability under approximation. We now aim to prove stability of the estimates provided in Theorem 3.2 under approximations of Ψ . These stability properties will be used in Section 3.4 to prove the convergence (in rough path topology) of Galerkin and hyper-viscosity approximations of random Fourier series. Let us consider

$$(3.7) \quad \tilde{\Psi}(t) = \frac{\alpha_0 \beta_0 Y^0}{2} + \sum_{k=1}^{\infty} \alpha_k \beta_k Y^k \sin(kt) + \alpha_{-k} \beta_{-k} Y^{-k} \cos(kt),$$

with Y^k as above and $(\alpha_k), (\beta_k)$ real-valued sequences. In the applications, the multiplication of the coefficients by β_k will correspond to a smoothing of Ψ . We thus aim to prove that the estimates given in Theorem 3.2 remain true uniformly for $(b_k) = (\beta_k^2)$ in an appropriate class of sequences. This will naturally lead to the following

Definition 3.6. (1) A sequence $(b_k)_{k \in \mathbb{Z}}$ is negligible if there are finite, signed, real Borel measures μ_1, μ_2 on $\mathcal{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ such that

$$b_k = \int_0^{2\pi} \cos(kr) \mu_1(dr), \quad b_{-k} = \int_0^{2\pi} \cos(kr) \mu_2(dr), \quad \forall k \in \mathbb{N}.$$

- (2) A family of sequences (b_k^r) is uniformly negligible if each (b_k^r) is negligible with associated measures μ_1^r, μ_2^r being uniformly bounded in total variation norm.
- (3) For two bounded sequences $(a_k), (c_k)$ we write $(a_k) \preceq (c_k)$ if there is a negligible sequence (b_k) such that $a_k = c_k b_k$ for every $k \in \mathbb{Z}$.

As will be seen below, our results are uniform relative to “negligible” perturbations as in (3.7). Let us give some (simple) examples of negligible sequences:

- (1) $(b_k) \equiv C$, with $\mu_1 = \mu_2 = C\delta_0$.
- (2) $(b_k) \in l^1(\mathbb{Z})$, with $\mu_1 = \sum_{k=1}^{\infty} b_k \cos(kt) dt$ and $\mu_2 = \sum_{k=1}^{\infty} b_{-k} \cos(kt) dt$.

Moreover, $(b_k) = (\frac{1}{k})$ and $(b_k) = (e^{k^2\tau})$, $\tau > 0$ will be shown to be (uniformly) negligible below. In Section 3.3 below, we give sufficient criteria for (uniform) negligibility of sequences.

Proposition 3.7. Consider the random Fourier series (3.7) with (a_k) satisfying the assumptions of Theorem 3.2. Let (b_k) be negligible. Then

$$V_{1,\rho}(R_{\tilde{\Psi}}; [s, t]^2) \leq C |t - s|^{\frac{1}{\rho}}, \quad \forall [s, t]^2 \subseteq [0, 2\pi]^2.$$

The constant C depends only on ρ , the constant $C_1 = \sup_{k \in \mathbb{Z}} a_k |k|^{1+\frac{1}{\rho}}$ and a constant C_2 which bounds $\|\mu_1\|_{TV}$ and $\|\mu_2\|_{TV}$ with μ_1, μ_2 corresponding to (b_k) (cf. Definition 3.6).

This proposition is a special case of Proposition 3.8 below. Consider another random Fourier series

$$\Phi(t) = \frac{\gamma_0 Z^0}{2} + \sum_{k=1}^{\infty} \gamma_k Z^k \sin(kt) + \gamma_{-k} Z^{-k} \cos(kt),$$

and assume that the Z^k fulfill the same conditions as the Y^k . Furthermore, assume that $\{Y^k, Z^k\}_{k \in \mathbb{Z}}$ are uncorrelated random variables and set $c_k := \gamma_k^2$, $\varrho_k := \mathbb{E}Y^k Z^k$ and

$$R_{\Psi, \Phi}(s, t) := \mathbb{E}\Psi(s)\Phi(t).$$

Then the following holds:

Proposition 3.8. *Assume that there is a sequence (d_k) satisfying the assumptions of Theorem 3.2 such that*

$$(b_k) := \left(\frac{\alpha_k \gamma_k \varrho_k}{d_k} \right)$$

is negligible with associated measures μ_1, μ_2 . Then

$$V_\rho(R_{\Psi, \Phi}; [s, t]^2) \leq V_{1, \rho}(R_{\Psi, \Phi}; [s, t]^2) \leq C|t - s|^{\frac{1}{\rho}}, \quad \forall [s, t]^2 \subseteq [0, 2\pi]^2.$$

The constant C depends only on ρ , the constant $C_1 = \sup_{k \in \mathbb{Z}} d_k |k|^{1 + \frac{1}{\rho}}$ and a constant C_2 which bounds $\|\mu_1\|_{TV}$ and $\|\mu_2\|_{TV}$.

Proof. Arguing as for Theorem 3.2 we observe

$$\begin{aligned} V_{1, \rho}(R_{\Psi, \Phi}; [s, t] \times [u, v]) &\lesssim V_{1, \rho}(R^-; [s, t] \times [u, v]) + V_{1, \rho}(R^+; [s, t] \times [u, v]) \\ &\quad + V_{1, \rho}(\tilde{R}^-; [s, t] \times [u, v]) + V_{1, \rho}(\tilde{R}^+; [s, t] \times [u, v]), \end{aligned}$$

with $R^-(s, t) = K(t - s)$, $R^+(s, t) = K(t + s)$, $\tilde{R}^-(s, t) = \tilde{K}(t - s)$ and $\tilde{R}^+(s, t) = \tilde{K}(t + s)$

$$K(t) := \frac{1}{2} \sum_{k=1}^{\infty} d_{-k} b_{-k} \cos(kt), \quad \tilde{K}(t) := \frac{1}{2} \sum_{k=1}^{\infty} d_k b_k \cos(kt).$$

We thus need to estimate the mixed $(1, \rho)$ -variation of cosine series under multiplication with negligible sequences. This is the content of Proposition 3.11 below, which finishes the proof. \square

In the following let $\mathcal{M}(\mathcal{S}^1)$ be the space of signed, real Borel-measures on the circle \mathcal{S}^1 with finite total variation $\|\cdot\|_{TV}$. Define $\mathcal{M}^w(\mathcal{S}^1)$ to be $\mathcal{M}(\mathcal{S}^1)$ endowed with the topology of weak convergence. For $B \in L^1(\mathcal{S}^1)$ we set $\mu_B := B dt \in \mathcal{M}(\mathcal{S}^1)$ to be the associated measure with density B .

Lemma 3.9. *Let $\mu \in \mathcal{M}(\mathcal{S}^1)$, $R: \mathcal{S}^1 \times I \rightarrow \mathbb{R}$ and set $R_\mu(s, t) := (R(\cdot, t) * \mu)(s)$. Then*

$$V_{\gamma, \rho}(R_\mu; [s, t] \times [u, v]) \leq \|\mu\|_{TV} \sup_{x \in \mathcal{S}^1} V_{\gamma, \rho}(R; [s - x, t - x] \times [u, v])$$

for all $[s, t] \times [u, v] \subseteq \mathcal{S}^1 \times I$ and $1 \leq \gamma \leq \rho$.

Proof. Let $(t_i), (t'_j)$ be partitions of $[s, t]$ resp. $[u, v]$. From Jensen's inequality,

$$\begin{aligned} \left| R_\mu \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma &\leq \left(\int_{\mathcal{S}^1} \left| R \left(\begin{array}{c} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{array} \right) \right| d|\mu|(x) \right)^\gamma \\ &\leq \|\mu\|_{TV}^\gamma \int_{\mathcal{S}^1} \left| R \left(\begin{array}{c} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma d \frac{|\mu|(x)}{\|\mu\|_{TV}}. \end{aligned}$$

Summing over t_i and using again Jensen's inequality for $\frac{\rho}{\gamma}$ yields

$$\begin{aligned} \sum_{t'_j} \left(\sum_{t_i} \left| R_\mu \left(\begin{array}{c} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \right)^{\frac{\rho}{\gamma}} &\leq \|\mu\|_{TV}^\rho \int_{\mathcal{S}^1} \sum_{t'_j} \left(\sum_{t_i} \left| R \left(\begin{array}{c} t_i - x, t_{i+1} - x \\ t'_j, t'_{j+1} \end{array} \right) \right|^\gamma \right)^{\frac{\rho}{\gamma}} d \frac{|\mu|(x)}{\|\mu\|_{TV}} \\ &\leq \|\mu\|_{TV}^\rho \int_{\mathcal{S}^1} V_{\gamma, \rho}^\rho(R; [s - x, t - x] \times [u, v]) d \frac{|\mu|(x)}{\|\mu\|_{TV}} \\ &\leq \|\mu\|_{TV}^\rho \sup_{x \in \mathcal{S}^1} V_{\gamma, \rho}^\rho(R; [s - x, t - x] \times [u, v]). \end{aligned}$$

Taking the supremum over all partitions yields the inequality. \square

Remark 3.10. In many cases, $x \mapsto V_{\gamma,\rho}(R; [s-x, t-x] \times [s, t])$ attains its maximum at $x = 0$. In this case our inequality above reads

$$V_{\gamma,\rho}(R * \mu; [s, t]^2) \leq \|\mu\|_{TV} V_{\gamma,\rho}(R; [s, t]^2)$$

for all squares $[s, t]^2 \subseteq [0, 2\pi]^2$. Lemma 3.9 can thus be interpreted as a Young-inequality for the mixed (γ, ρ) -variation of a function with two arguments. If $\mu = \delta_0$, we have $b_k = 1$ for every k and the estimate is thus sharp.

Proposition 3.11. Let $R_\mu^+, R_\mu^- : [0, 2\pi]^2 \rightarrow \mathbb{R}$ be continuous functions of the form

$$R_\mu^\pm(s, t) = \frac{a_0 b_0}{2} + \sum_{k=1}^{\infty} a_k b_k \cos(k(s \pm t))$$

with a_k, b_k being real-valued coefficients such that $\sum_{k=1}^{\infty} |a_k| < \infty$ and assume that there is a measure $\mu \in \mathcal{M}(\mathcal{S}^1)$ such that

$$b_k = \int_0^{2\pi} \cos(kr) \mu(dr).$$

Set

$$R^\pm(t, s) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k(t \pm s)).$$

Then for every $1 \leq \gamma \leq \rho$,

$$V_{\gamma,\rho}(R_\mu^\pm; [s, t] \times [u, v]) \leq \|\mu\|_{TV} \sup_{0 \leq z \leq 2\pi} V_{\gamma,\rho}(R^\pm; [s-z, t-z] \times [u, v])$$

for every $[s, t] \times [u, v] \subseteq [0, 2\pi]^2$.

Proof. Let $a_{-k} := a_k, b_{-k} := b_k$ for $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} |a_k| < \infty$ we observe

$$R_\mu^\pm(s, t) = \frac{1}{2} \sum_{k \in \mathbb{Z}} a_k b_k e^{ik(t \pm s)} = (R^\pm(\cdot, t) * \mu)(s)$$

and the estimate is thus a direct consequence from Lemma 3.9. \square

3.3. (Uniform) negligibility. In order to use Proposition 3.11 to control the (γ, ρ) -variation of $R(s, t)$, we need to control $\|\mu\|_{TV}$. We recall

Lemma 3.12. Let $\{b_k\}_{k \in \mathbb{N}}$ be a sequence satisfying $b_k \rightarrow b \in \mathbb{R}$ for $k \rightarrow \infty$ and let $S_n(t) := \frac{b_0}{2} + \sum_{k=1}^n b_k \cos(kt)$. Assume one of the following conditions

- (1) $\sum_{k=1}^{\infty} |b_k - b| < \infty$.
- (2) there exists a non-increasing sequence A_k such that $\sum_{k=0}^{\infty} A_k < \infty$ and $|\Delta b_k| \leq A_k$ for all $k \geq 0$.
- (3) b_k is quasi-convex, i.e.

$$\sum_{k=0}^{\infty} (k+1) |\Delta^2 b_k| < \infty.$$

Then, $B(t) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos(kt)$ exists locally uniformly on $(0, 2\pi)$ and the right hand side is the Fourier series of B . Moreover,

$$\mu_{S_n} \rightharpoonup \mu_B + b\delta_0 =: \mu, \quad \text{weakly in } \mathcal{M}(\mathcal{S}^1)$$

and $b_k = \int_0^{2\pi} \cos(kr) \mu(dr)$. Moreover, there is a numerical constant $C > 0$ such that

$$(3.8) \quad \|\mu\|_{TV} \leq |b| + C \begin{cases} \sum_{k=0}^{\infty} |b_k - b|, & \text{in case (1)} \\ \sum_{k=0}^{\infty} A_k, & \text{in case (2)} \\ \sum_{k=0}^{\infty} (k+1) |\Delta^2 b_k|, & \text{in case (3)}. \end{cases}$$

Proof. The case $b = 0$ is classical (cf. [Tel73] for (2), [Kol23] for (3)). The case $b \neq 0$ may be reduced to $b = 0$ by noting that $bD_n(t) \rightarrow 2\pi b\delta_0$ in $\mathcal{M}^w(\mathcal{S}^1)$, where D_n is the Dirichlet kernel. \square

Lemma 3.12 in combination with Proposition 3.11 allows to derive bounds on the ρ -variation of covariance functions of here discussed type depending on μ only via its total variation norm. Since we will use this to prove uniform estimates, we will need the following uniform estimates on the L^1 -norm of cosine series.

Lemma 3.13. *Let $b \in C^1(0, \infty)$ with $b(r) \rightarrow 0$ for $r \rightarrow \infty$ and $b_k^\tau := b(\tau^m k)$ for some $\tau, m > 0$. If*

- (1) *b is convex, non-increasing, then b_k^τ satisfies the assumptions of Lemma 3.12, (1), $B^\tau(t) = \frac{b_0^\tau}{2} + \sum_{k=1}^{\infty} b_k^\tau \cos(kt)$ exists locally uniformly in $(0, 2\pi)$ and*

$$\|B^\tau\|_{L^1([0, 2\pi])} \leq C b_0,$$

for some $C > 0$.

- (2) *$b \in C^2(0, \infty)$ with $r \mapsto r|b''(r)|$ being integrable, then b_k^τ satisfies the assumptions of Lemma 3.12, (2) and*

$$\|B^\tau\|_{L^1([0, 2\pi])} \leq C \int_0^\infty r|b''(r)|dr,$$

for some $C > 0$ with B^τ as in (1).

Proof. (1): Since $b, |b'|$ are non-increasing $\Delta b_k^\tau \leq 0$ and $-\Delta b_k$ is non-increasing. We set $A_k := -\Delta b_k$. Clearly, $\sum_{k=0}^{\infty} A_k = 2b_0$ and the claim follows from Lemma 3.12.

- (2): Let $b^\tau(r) := b(\tau^m r)$ and observe

$$\Delta^2 b_k^\tau = \int_{k+1}^{k+2} \int_{s-1}^s (b^\tau)''(r) dr ds.$$

Since $(b^\tau)'' dr = \tau^m b''(\tau^m r) d(\tau^m r)$, elementary calculations show

$$\sum_{k=0}^{\infty} (k+1) |\Delta^2 b_k^\tau| \leq 2 \int_0^\infty r|b''(r)|dr$$

and Lemma 3.12 finishes the proof. \square

3.4. Random Fourier series as rough paths. We now return to the initial problem of showing the existence of a lift to vector-valued versions of (3.1) to a process with values in a rough paths space.

We will use the following notation: If $(A_k) = (a_k^{i,j})$ is a sequence of matrices and (b_k) is a sequence of real numbers, $(A_k) \preceq (b_k)$ means that $(a_k^{i,j}) \preceq (b_k)$ for every i, j . If $(A_k) = (A_k^1, \dots, A_k^m)$ is a sequence of vectors whose entries are matrices or real numbers, we will write $(A_k) \preceq (b_k)$ if $(A_k^i) \preceq (b_k)$ for all $i = 1, \dots, m$.

Let $\Psi = (\Psi^1, \dots, \Psi^d)$ where the Ψ^i are given as random Fourier series

$$(3.9) \quad \Psi^i(t) = \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^{\infty} \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt),$$

with $(Y^{k,i})_{k \in \mathbb{Z}, i=1, \dots, d}$ being independent, $\mathcal{N}(0, 1)$ distributed random variables. As before, set $a_k^i := (\alpha_k^i)^2$ and $(a_k) := (a_k^1, \dots, a_k^d)$. Our main existence result is the following:

Theorem 3.14. *Assume $(a_k) \preceq (|k|^{-(1+1/\rho)})$ for some $\rho \in [1, 2)$ with associated measures μ_1^i, μ_2^i , $i = 1, \dots, d$, as in Definition 3.6 and let $K \geq \max_{i=1, \dots, d} \{\|\mu_1^i\|_{TV}, \|\mu_2^i\|_{TV}\}$. Then for every $\beta < \frac{1}{2\rho}$, there exists a continuous $G^{[1/\beta]}(\mathbb{R}^d)$ -valued process Ψ such that*

- (1) Ψ has geometric β -Hölder rough sample paths, i.e. $\Psi \in C_0^{0, \beta - \text{Hö}}([0, 2\pi], G^{[1/\beta]}(\mathbb{R}^d))$ almost surely,
- (2) Ψ lifts Ψ in the sense that $\pi_1(\Psi_t) = \Psi_t - \Psi_0$,
- (3) there is a $C = C(\rho, K)$ such that for all $s < t$ in $[0, 2\pi]$ and $q \in [1, \infty)$,

$$|d(\Psi_s, \Psi_t)|_{L^q} \leq C \sqrt{q} |t - s|^{\frac{1}{2\rho}},$$

- (4) there exists $\eta = \eta(\rho, K, \beta) > 0$, such that

$$\mathbb{E} e^{\eta \|\Psi\|_{\beta - \text{Hö}; [0, 2\pi]^2}} < \infty.$$

Proof. By assumption,

$$\Psi^i(t) = \frac{\beta_0^i Y^{0,i}}{2} + \sum_{k=1}^{\infty} \beta_k^i |k|^{-(\frac{1}{2} + \frac{1}{2\rho})} Y^{k,i} \sin(kt) + \beta_{-k}^i |k|^{-(\frac{1}{2} + \frac{1}{2\rho})} Y^{-k,i} \cos(kt)$$

for every $i = 1, \dots, d$ where $(\beta_k^i) = ((\beta_k^i)^2)$ is a negligible sequence. Hence, we may apply Proposition 3.7 to see that the covariance of Ψ^i has finite Hölder dominated ρ -variation for every i , thus [FV10a, Theorem 35] applies. \square

We will now compare the lifts of two random Fourier series $\Psi = (\Psi^1, \dots, \Psi^d)$ and $\tilde{\Psi} = (\tilde{\Psi}^1, \dots, \tilde{\Psi}^d)$ with

$$\begin{aligned} \Psi^i(t) &= \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^{\infty} \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt), \\ \tilde{\Psi}^i(t) &= \frac{\tilde{\alpha}_0^i \tilde{Y}^{0,i}}{2} + \sum_{k=1}^{\infty} \tilde{\alpha}_k^i \tilde{Y}^{k,i} \sin(kt) + \tilde{\alpha}_{-k}^i \tilde{Y}^{-k,i} \cos(kt). \end{aligned}$$

We make the following assumption:

$$\{(Y^{k,i}, \tilde{Y}^{k,i}) : k \in \mathbb{Z}, i = 1, \dots, d\}$$

are independent, normally distributed random vectors with $Y^{k,i}, \tilde{Y}^{k,i} \sim \mathcal{N}(0, 1)$ for all $k \in \mathbb{Z}$ and $i = 1, \dots, d$. It follows that $\mathbb{E}Y^{k,i}\tilde{Y}^{l,j} = 0$ for $k \neq l$ or $i \neq j$ and we set $\varrho_k^i := \mathbb{E}Y^{k,i}\tilde{Y}^{k,i}$. As before, let $a_k^i := (\alpha_k^i)^2$ and $\tilde{a}_k^i := (\tilde{\alpha}_k^i)^2$. Define the matrix

$$A_k^i := \begin{pmatrix} a_k^i & \alpha_k^i \tilde{\alpha}_k^i \varrho_k^i \\ \alpha_k^i \tilde{\alpha}_k^i \varrho_k^i & \tilde{a}_k^i \end{pmatrix}$$

and set $A_k := (A_k^1, \dots, A_k^d)$.

Theorem 3.15. *Assume that $(A_k) \preceq (|k|^{-(1+1/\rho)})$ for some $\rho \in [1, 2)$ and that the total variation of all associated measures is bounded by a constant K . Then we can lift Ψ and $\tilde{\Psi}$ to processes with values in a rough paths space as in Theorem 3.14 and for all $\gamma < 1 - \frac{\rho}{2}$ and $\beta < \frac{1}{\rho}(\frac{1}{2} - \gamma)$ there is a constant $C = C(\rho, K, \beta, \gamma)$ such that*

$$|\rho_{\beta-H\ddot{o}l}(\Psi, \tilde{\Psi})|_{L^q} \leq Cq^{\frac{1}{2}[1/\beta]} \left(\sup_{t \in [0, 2\pi]} \mathbb{E}|\Psi(t) - \tilde{\Psi}(t)|^2 \right)^\gamma$$

for all $q \in [1, \infty)$.

Proof. The existence of the lifted processes Ψ and $\tilde{\Psi}$ follows from Theorem 3.14. We aim to apply [FR12, Theorem 5]¹⁷. To do this, we need to show that the joint covariance function of the process $(\Psi, \tilde{\Psi})$ has finite, Hölder dominated ρ -variation, bounded by a constant depending only on the parameters above. From independence of the components, it suffices to estimate the ρ -variation of $R_{\Psi^i, \tilde{\Psi}^i}(s, t) = \mathbb{E}\Psi^i(s)\tilde{\Psi}^i(t)$ for every $i = 1, \dots, d$. This can be done using Proposition 3.8. \square

As an application, we consider the *truncated random Fourier series*, i.e. we define $\Psi^N = (\Psi^{1,N}, \dots, \Psi^{d,N})$ by

$$(3.10) \quad \Psi^{i,N}(t) = \frac{\alpha_0^i Y^{0,i}}{2} + \sum_{k=1}^N \alpha_k^i Y^{k,i} \sin(kt) + \alpha_{-k}^i Y^{-k,i} \cos(kt), \quad \text{for } i = 1, \dots, d.$$

It is then easy to show that convergence also holds for the corresponding rough paths lifts and we can even give an upper bound for the order of convergence.

Corollary 3.16. *Under the assumptions of Theorem 3.14, choose some $\eta < \frac{1}{\rho} - \frac{1}{2}$ and $\beta < \frac{1}{2\rho} - \eta$. Then there is a constant $C = C(\rho, K, \beta, \eta)$ such that*

$$|\rho_{\beta-H\ddot{o}l}(\Psi, \Psi^N)|_{L^q} \leq Cq^{\frac{1}{2}[1/\beta]} \left(\frac{1}{N} \right)^\eta$$

for every $N \in \mathbb{N}$, $q \in [1, \infty)$. In particular, $\rho_{\beta-H\ddot{o}l}(\Psi, \Psi^N) \rightarrow 0$ for $N \rightarrow \infty$ almost surely and in L^q for any $q \in [1, \infty)$ with rate η .

Remark 3.17. *We emphasize that Ψ, Ψ^N are lifted to level $[1/\beta]$ above. In particular, a “good” rate η forces β to be small so that, in general, it is not enough to work with 3 levels, as is the usual setting in Gaussian rough paths theory.*

¹⁷Strictly speaking, [FR12, Theorem 5] assumes that $\tilde{\Psi}$ is a certain approximation of Ψ . However, it is shown in [RX12] that this is not necessary and [FR12, Theorem 5] can be used more generally to give an upper bound for the distance between Ψ and $\tilde{\Psi}$ as we need it here.

Proof of Corollary 3.16. We aim to apply Theorem 3.15 with $\tilde{\alpha}_k^i = \mathbb{1}_{|k| \leq N} \alpha_k^i$ and $\varrho_k^i \equiv 1$. We will first show that $(a_k^i \mathbb{1}_{|k| \leq N}) \preceq (|k|^{-(1+1/\rho')})$ for every $\rho' > \rho$, uniformly over i and N . Indeed, we have

$$a_k^i \mathbb{1}_{|k| \leq N} = (a_k^i |k|^{1+1/\rho}) (|k|^{1/\rho' - 1/\rho} \mathbb{1}_{|k| \leq N}) |k|^{-(1+1/\rho')}$$

and since $(a_k^i) \preceq (|k|^{-(1+1/\rho)})$ for all $i = 1, \dots, d$ it suffices to show that $(|k|^{-\varepsilon} \mathbb{1}_{|k| \leq N})$ is uniformly negligible for every $\varepsilon > 0$. Therefore, we need to show that the cosine series

$$B^N(x) = \sum_{k=1}^{\infty} |k|^{-\varepsilon} \mathbb{1}_{|k| \leq N} \cos(kx) = \sum_{k=1}^N |k|^{-\varepsilon} \cos(kx)$$

is uniformly bounded in $L^1([0, 2\pi])$. Since $\Delta k^{-\varepsilon} = O(k^{-\varepsilon-1})$ and $\lim_{k \rightarrow \infty} \log(k) k^{-\varepsilon} = 0$ we can apply the Sidon-Telyakovskii Theorem (cf. [Tel73, Theorem 4]) to obtain $B^N \rightarrow B$ for $N \rightarrow \infty$ in $L^1([0, 2\pi])$ which proves the uniform negligibility and we may apply Theorem 3.15 for every $\rho' > \rho$. Furthermore,

$$\begin{aligned} \mathbb{E}|\Psi(t) - \Psi^N(t)|^2 &= \sum_{k=N+1}^{\infty} a_k \sin^2(kt) + a_{-k} \cos^2(kt) \\ &\leq 2 \sum_{|k| \geq N+1} a_k \lesssim 4 \sum_{k=N+1}^{\infty} k^{-(1+1/\rho)} \lesssim \left(\frac{1}{N}\right)^{\frac{1}{\rho}}. \end{aligned}$$

For given η , we choose ρ' such that $\eta < \frac{1}{\rho'} - \frac{1}{2} < \frac{1}{\rho} - \frac{1}{2}$ and apply Theorem 3.15 to conclude the L^q convergence. The almost sure convergence follows by a standard Borel-Cantelli argument (cf. [FR12, Theorem 6, p. 41]). \square

4. PART IV. APPLICATIONS TO SPDE

In this section we will apply our results on random Fourier series to construct spatial rough path lifts of stationary Ornstein-Uhlenbeck processes corresponding to the \mathbb{R}^d -valued (generalized) fractional stochastic heat equation with Dirichlet, Neumann or periodic boundary conditions

$$(4.1) \quad d\Psi_t = -(-\Delta)^\alpha \Psi_t dt + dW_t, \quad \text{on } [0, T] \times [0, 2\pi]$$

where the fractional Laplacian $(-\Delta)^\alpha$ acts on each component of Ψ_t and $\alpha \in (0, 1]$. We will start by first considering the fractional stochastic heat equation with Dirichlet boundary conditions, proving the existence of (continuous) spatial rough paths lifts and stability under approximations. Then we comment on Neumann boundary conditions and on more general equations for periodic boundary conditions.

If a (spatial) rough path lift of (4.1) has been constructed one can view (4.1) as an evolution in a rough path space, a point of view which has proven extremely fruitful in solving new classes of until now ill-posed stochastic PDE [Hai11, HW11, Hai13], arising for example in path sampling problems for \mathbb{R}^d -valued SDE [HSVW05, HSV07, Hai11].

For a variant of (4.1) with $\alpha = 1$, Hairer proved in [Hai11] finite 1-variation of the covariance of the stationary solution to (4.1), that is of $(x, y) \mapsto \mathbb{E}\Psi(t, x) \Psi(t, y)$, general theory then gives the existence of a “canonical, level 2” rough path Ψ lifting Ψ (cf. Theorem 3.14). See also [GIP12]. It is clear that in the case $\alpha = 1$ the Brownian-like regularity of $x \mapsto \Psi(t, x; \omega)$ is due to the competition between the smoothing effects of the Laplacian and the roughness of space-time white noise. Truncation of the higher noise modes (or suitable “coloring”) leads to better spatial regularity; on

the other hand, replacing Δ by a fractional Laplacian, i.e. considering (4.1) for some $\alpha \in (0, 1)$ dampens the smoothing effect and $x \mapsto \Psi(t, x; \omega)$ will have “rougher” regularity properties than a standard Brownian motion. One thus expects ρ -variation regularity for the spatial covariance of $x \mapsto \Psi(t, x; \omega)$ for (4.1) only for some $\rho > 1$ and subsequently only the existence of a “rougher” rough path, i.e. necessarily with higher p than before.

As we shall see below, (4.1) is handled, as spatial rough path with a number of precise estimates, provided

$$\alpha > \alpha^* = \frac{3}{4}.$$

More precisely, the resulting (geometric rough) path enjoys $\frac{1}{p}$ -Hölder regularity for any $p > 2\rho = \frac{2}{2\alpha-1}$. When $\alpha > \frac{5}{6}$ we have $\rho = \frac{1}{2\alpha-1} < \frac{3}{2}$ and can pick $p < 3$. The resulting rough path can then be realized as a “level 2” rough path. In the general case (similar to $H \in (\frac{1}{4}, \frac{1}{3}]$ in the fBm setting) one must go beyond the stochastic area and control the third level iterated integrals. Our approach, which crucially passes through ρ -variation, combined with existing theory, has many advantages: the notoriously difficult third-level computation need not be repeated in the present context; leave alone the higher level computations needed for rates. A satisfactory approximation theory is also available, based on *uniform* ρ -variation estimates (cf. Section 4.2 below).

4.1. Fractional stochastic heat equation with Dirichlet boundary conditions. We consider

$$(4.2) \quad d\Psi_t = -(-\Delta)^\alpha \Psi_t dt + dW_t, \quad \text{on } [0, T] \times [0, 2\pi]$$

on $[0, 2\pi]$ endowed with Dirichlet boundary conditions. Neumann and periodic boundary conditions may be treated analogously (cf. Section 4.3 below). We have the following orthogonal basis of eigenvectors with corresponding eigenvalues of $-\Delta$ on $L^2([0, 2\pi])$:

$$e_k(x) = \sin\left(\frac{k}{2}x\right), \quad \tau_k = \left(\frac{k}{2}\right)^2, \quad k \in \mathbb{N}.$$

and take $W_t = \sum_{k \in \mathbb{N}} \beta_t^k e_k(x)$. The fractional Laplacian has eigenvalues $\lambda_k := \tau_k^\alpha$ for $k \in \mathbb{N}$ and (informal) Fourier expansion of the stationary solution Ψ to (4.1) leads to the random Fourier series

$$(4.3) \quad \Psi(t, x) = \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin\left(k\frac{x}{2}\right),$$

with $\alpha_k = \frac{1}{\sqrt{2\lambda_k}}$ and Y_t^k being a decoupled, infinite system of d -dimensional, stationary, normalized Ornstein-Uhlenbeck processes satisfying

$$(4.4) \quad dY_t^k = -\lambda_k Y_t^k dt + \sqrt{2\lambda_k} d\beta_t^k.$$

Clearly (4.3) gives a well-defined and continuous random field and solves (4.2) in the sense of standard SPDE theory (cf. e.g. [Wal86, DPZ92]). Note $\mathbb{E}Y_t^k \otimes Y_s^l = e^{-\lambda_k|t-s|} \delta_{k,l} \text{Id}$ and set

$$a_k = \alpha_k^2 = \frac{1}{2\lambda_k} = 2^{2\alpha-1} \frac{1}{k^{2\alpha}}, \quad k \in \mathbb{N}.$$

As an immediate consequence of our results on random Fourier series we get

Proposition 4.1. *Suppose $\alpha \in (\frac{1}{2}, 1]$. Then*

- (1) For every $t \geq 0$, the spatial process $x \mapsto \Psi(t, x)$ is a centered Gaussian process which admits a continuous modification (which we denote by the same symbol) with covariance R_Ψ of finite mixed $(1, \rho)$ -variation for all $\rho \geq \frac{1}{2\alpha-1}$ and all conclusions of Theorem 3.2 hold.
- (2) If $\alpha > \frac{3}{4}$, the process $x \mapsto \Psi(t, x)$ lifts to a process with geometric β -Hölder rough paths

$$\Psi(t) \in C_0^{0, \beta\text{-Hölder}}([0, 2\pi], G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d))$$

almost surely for every $\beta < \alpha - \frac{1}{2}$.

- (3) Choose γ and β such that

$$\gamma < 1 - \frac{3}{4\alpha}, \quad \beta < \alpha - \frac{1}{2} - \frac{2\alpha\gamma}{2\alpha - 1}.$$

Then there is a γ -Hölder continuous modification of the map

$$(4.5) \quad \begin{aligned} \Psi : [0, T] &\rightarrow C_0^{0, \beta\text{-Hölder}}([0, 2\pi], G^{\lfloor 1/\beta \rfloor}(\mathbb{R}^d)) \\ t &\mapsto \Psi(t). \end{aligned}$$

Remark 4.2. In (3), we observe a “trade-off” between the parameters β and γ : If we want a “good” time regularity (i.e. large γ), we have to take β small which is tantamount to working in a rough paths space with many “levels” of formal iterated integrals. For instance, when $\alpha = 1$, we can get arbitrarily close to $\frac{1}{4}$ in time, at the price of working with arbitrary many levels. On the other hand, if we insist to work with the first 3 levels only (or 2 levels in case $\alpha > 5/6$), which is the standard setting in Gaussian rough path theory, we only get poor time regularity of the evolution in rough path space.

Proof of Proposition 4.1. Since Ψ is a rescaling of

$$\tilde{\Psi}(t, x) = \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin(kx) = \Psi(t, 2x),$$

it is enough to consider $\tilde{\Psi}$.

- (1): Clearly $x \mapsto \tilde{\Psi}(t, x)$ is centered and Gaussian. Due to (3.3) and Lemma 3.4, we have

$$\sigma_t^2(x, y) = \mathbb{E} \left| \tilde{\Psi}(t, x) - \tilde{\Psi}(t, y) \right|^2 \lesssim |x - y|^{2\alpha-1},$$

which implies that there is a continuous modification of $\tilde{\Psi}$. Theorem 3.2 implies the claim.

- (2): Follows from Theorem 3.14.

(3): We will derive the existence of a continuous modification by application of Kolmogorov’s continuity theorem. Therefore, we need an estimate on a q -th moment of the distance in the $\rho_{\beta\text{-Hölder}}$ metric of the rough paths $\tilde{\Psi}(t)$, $\tilde{\Psi}(s)$ at different times $0 \leq s < t \leq T$. Such an estimate can be obtained by applying Theorem 3.15. Let $0 \leq s \leq t \leq T$, $\tau := |t - s|$ and set $A_k = (A_k^1, \dots, A_k^d)$ where

$$A_k^i := \begin{pmatrix} a_k & a_k e^{-\lambda_k \tau} \\ a_k e^{-\lambda_k \tau} & a_k \end{pmatrix}$$

for $i = 1, \dots, d$. We claim that $(A_k) \preceq (|k|^{-2\alpha})$ uniformly in τ . Defining $b(r) := e^{-(\frac{r}{2})^{2\alpha}}$ we note $b_k^\tau = e^{-\lambda_k \tau} = b(k\tau^{\frac{1}{2\alpha}})$ and b is convex, non-increasing. Lemma 3.13 then implies that $(e^{-\lambda_k \tau})$ is

uniformly negligible which shows the claim. Hence, we can apply Theorem 3.15 and obtain

$$|\rho_{\beta-\text{H\"{o}l}}(\tilde{\Psi}(t), \tilde{\Psi}(s))|_{L^q} \leq Cq^{\frac{1}{2} \lfloor \frac{1}{\beta} \rfloor} \left(\sup_{x \in [0, 2\pi]} \mathbb{E} |\tilde{\Psi}(t, x) - \tilde{\Psi}(s, x)|^2 \right)^\theta$$

for all $\theta < \frac{4\alpha-3}{4\alpha-2}$, $\beta < \alpha - \frac{1}{2} - \theta$ and all $q \in [1, \infty)$. In order to estimate the right hand side we note

$$\begin{aligned} \mathbb{E} |\tilde{\Psi}^1(t, x) - \tilde{\Psi}^1(s, x)|^2 &= \mathbb{E} |\tilde{\Psi}^1(t, x)|^2 + \mathbb{E} |\tilde{\Psi}^1(s, x)|^2 - 2\mathbb{E} \tilde{\Psi}^1(t, x) \tilde{\Psi}^1(s, x) \\ &\leq 2 \sum_{k=1}^{\infty} a_k (1 - e^{-\lambda_k \tau}) \sin^2(kx) \\ &\leq 2 \sum_{k=1}^{\infty} a_k |1 - e^{-\lambda_k \tau}| \\ &\leq C \sum_{k \leq N} |t - s| + CN^{1-2\alpha'} \sum_{k > N} a_k k^{2\alpha'-1} \\ &\leq C(N|t - s| + N^{1-2\alpha'}) \end{aligned}$$

for all $\alpha' < \alpha$. We then choose $N \sim |t - s|^{-\frac{1}{2\alpha'}}$ to obtain

$$\mathbb{E} |\tilde{\Psi}(t, x) - \tilde{\Psi}(s, x)|^2 \leq C|t - s|^{1 - \frac{1}{2\alpha'}},$$

thus we can choose $\gamma < 1 - \frac{3}{4\alpha}$ and $\beta < \alpha - \frac{1}{2} - \frac{2\alpha\gamma}{2\alpha-1}$ to obtain

$$|\rho_{\beta-\text{H\"{o}l}}(\tilde{\Psi}(t), \tilde{\Psi}(s))|_{L^q} \leq Cq^{\frac{1}{2} \lfloor \frac{1}{\beta} \rfloor} |t - s|^\gamma,$$

for all $q \in [1, \infty)$. Kolmogorov's continuity Theorem gives the result. \square

4.2. Stability and approximations. Due to the "contraction principle" in the form of Proposition 3.11, the estimates on the ρ -variation of the covariance of random Fourier series derived in Section 3 are robust with respect to approximations. In order to emphasize this point, in this section we consider Galerkin and hyper-viscosity approximations to Ψ with Ψ as in Section 4.1 and prove strong convergence of the corresponding rough paths lifts. Recall that by the general theory of rough paths this immediately implies the strong convergence of the corresponding stochastic integrals as well as of solutions to rough differential equations (cf. e.g. [BFRS13, Hai11]).

4.2.1. Galerkin approximations. The Galerkin approximation Ψ_t^N of Ψ_t is defined to be the projection of Ψ onto the N -dimensional subspace spanned by $\{e_k\}_{k=1, \dots, N}$. This process solves the SPDE

$$(4.6) \quad d\Psi_t^N = -(P_N(-\Delta)^\alpha) \Psi_t^N dt + dP_N W_t,$$

where $P_N(-\Delta)^\alpha$ has the eigenvalues $\left(\frac{k}{2}\right)^{2\alpha} \mathbb{1}_{k \leq N}$ and $P_N W_t$ has the covariance operator Q^N given by $Q^N e_k = \mathbb{1}_{k \leq N} e_k$. The process Ψ^N can be written as the truncated Fourier series

$$\Psi^N(t, x) = \sum_{k=1}^N \alpha_k Y_t^k \sin\left(k \frac{x}{2}\right),$$

with $\alpha_k = 2^{\alpha-1/2} k^{-\alpha}$ and Y^k Ornstein-Uhlenbeck processes as in Section 4.1.

One easily checks that we can lift the spatial sample paths of Ψ_t^N to Gaussian rough paths and find continuous modifications of $t \mapsto \Psi_t^N$. Moreover, we can prove the following strong convergence result:

Proposition 4.3. *Assume $\alpha > \frac{3}{4}$ and choose $\eta < 2\alpha - \frac{3}{2}$ and $\beta < \alpha - \frac{1}{2} - \eta$. Then there is a constant $C = C(\alpha, \beta, \eta)$ such that*

$$|\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^N(t))|_{L^q} \leq Cq^{\frac{1}{2}[\frac{1}{\beta}]} \left(\frac{1}{N}\right)^\eta$$

for all $t \in [0, T]$, $N \in \mathbb{N}$, $q \in [1, \infty)$. In particular, for every $t \in [0, T]$, $\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^N(t)) \rightarrow 0$ for $N \rightarrow \infty$ almost surely and in L^q for any $q \in [1, \infty)$ with rate η .

Proof. Follows from Corollary 3.16. \square

4.2.2. *Hyper-viscosity approximations.* The hyper-viscosity approximation $\Psi^\varepsilon = (\Psi^{\varepsilon,1}, \dots, \Psi^{\varepsilon,d})$ is the solution to

$$(4.7) \quad d\Psi_t^\varepsilon = -((-\Delta)^\alpha + \varepsilon(-\Delta)^\theta)\Psi_t^\varepsilon dt + dW_t,$$

for some (large) $\theta \geq 1$ and $\varepsilon > 0$. Again, it is easy to see that we can lift the spatial sample paths of Ψ_t^ε to Gaussian rough paths and find continuous modifications of $t \mapsto \Psi_t^\varepsilon$.

Proposition 4.4. *Assume $\alpha > \frac{3}{4}$ and $\theta > \alpha$. Choose $\beta < \alpha - \frac{1}{2}$. Then there is a function $r_{\alpha,\beta,\theta}: \mathbb{R} \rightarrow \mathbb{R}_+$ such that $r(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$ and a constant $C = C(\alpha, \beta, \theta)$ such that*

$$|\rho_{\beta\text{-H\"{o}l}}(\Psi(t), \Psi^\varepsilon(t))|_{L^q} \leq Cq^{\frac{1}{2}[\frac{1}{\beta}]}r(\varepsilon)$$

for every $t \in [0, T]$, $\varepsilon > 0$ and $q \in [1, \infty)$.

Proof. As before, Ψ_t^ε has the form of a random Fourier series where the k -th Fourier coefficients are given by $\alpha_k^\varepsilon Y_t^{\varepsilon,k}$ with $\alpha_k^\varepsilon = \frac{1}{\sqrt{2\lambda_k^\varepsilon}}$,

$$\lambda_k^\varepsilon = \left(\frac{k}{2}\right)^{2\alpha} + \varepsilon \left(\frac{k}{2}\right)^{2\theta}$$

and $t \mapsto Y_t^{\varepsilon,k}$ are d -dimensional, stationary Ornstein-Uhlenbeck processes with independent components, each component being centered with variance 1 and correlation

$$\mathbb{E}Y_t^{\varepsilon,k} \otimes Y_t^{\varepsilon,l} = 2\frac{\sqrt{\lambda_k \lambda_k^\varepsilon}}{\lambda_k + \lambda_k^\varepsilon} \delta_{k,l} \text{Id}.$$

From Theorem 3.15, we know that it is sufficient to show that $(A_k^\varepsilon) \preceq (|k|^{-2\alpha})$ uniformly over $\varepsilon > 0$ where

$$A_k^\varepsilon := \begin{pmatrix} \alpha_k^2 & \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon \\ \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon & (\alpha_k^\varepsilon)^2 \end{pmatrix}, \quad \varrho_k^\varepsilon := 2\frac{\sqrt{\lambda_k \lambda_k^\varepsilon}}{\lambda_k + \lambda_k^\varepsilon}$$

and that

$$(4.8) \quad \sup_{t \in [0, T]} \sup_{x \in [0, 2\pi]} \mathbb{E}|\Psi(t, x) - \Psi^\varepsilon(t, x)|^2 \rightarrow 0, \quad \text{for } \varepsilon \rightarrow 0.$$

For the first claim, we have to show that the series

$$\sum_{k=1}^{\infty} (\alpha_k^\varepsilon)^2 k^{2\alpha} \cos(kx), \quad \sum_{k=1}^{\infty} \alpha_k \alpha_k^\varepsilon \varrho_k^\varepsilon k^{2\alpha} \cos(kx)$$

are uniformly bounded in L^1 which can be done using Lemma 3.13 (2). Showing (4.8) follows by writing down the left hand side as a Fourier series and bounding it uniformly in x and t by an infinite series. Then we send $\varepsilon \rightarrow 0$, using the dominated convergence theorem. \square

4.3. Various generalizations.

4.3.1. *Generalized fractional stochastic heat equation on periodic domains.* Based on the stability results for the mixed $(1, \rho)$ -variation of the covariance of random Fourier series developed in Section 3 one may consider more general fractional stochastic heat type equations and different types of boundary conditions. As an example let us consider generalized fractional stochastic heat equations on $[0, 2\pi]$ with periodic boundary conditions. An orthogonal basis of eigenvectors and corresponding eigenvalues of $-\Delta$ on $L^2([0, 2\pi])$ is given by

$$\tau_k = k^2, \quad e_k(x) := \begin{cases} \sin(kx), & k > 0 \\ \frac{1}{2}, & k = 0 \\ \cos(kx), & k < 0. \end{cases}$$

Via the spectral theorem we may define $A = f(-\Delta)$ for each Borel measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, still having e_k as a basis of eigenvectors and eigenvalues $f(\tau_k)$.

In order to be able to consider stationary Ornstein-Uhlenbeck processes we need to shift the spectrum of A to be strictly negative. Hence, we consider \mathbb{R}^d -valued SPDE of the form

$$(4.9) \quad d\Psi_t = (-A - \lambda)\Psi_t dt + dW_t, \quad \text{with } \lambda > 0,$$

where W_t is a (possibly) colored Wiener process with covariance operator having e_k as basis of eigenvectors and σ_k as eigenvalues. An (informal) Fourier expansion of the stationary solution Ψ to (4.9) leads to the random Fourier series

$$(4.10) \quad \Psi(t, x) = \frac{\alpha_0 Y_0}{2} + \sum_{k=1}^{\infty} \alpha_k Y_t^k \sin(kx) + \alpha_{-k} Y_t^{-k} \cos(kx),$$

with $\alpha_k = \alpha_{-k} = \sqrt{\frac{\sigma_k}{2(\lambda + f(\tau_k))}}$ and Y_t^k as in (4.4). Supposing (a_k) to be eventually non-increasing and $(a_k) \preceq (k^{-2\alpha})$ for some $\alpha \in (\frac{1}{2}, 1]$. Then analogous results to Proposition 4.1 may be established under various assumptions on σ_k and $f(\tau_k)$, by means of the stability results given in Section 3.

Example 4.5. *We consider the stochastic fractional heat equation with (possibly) colored noise on the 1-dimensional torus, i.e.*

$$(4.11) \quad d\Psi_t^i = -((-\Delta)^\alpha \Psi_t^i + \lambda) dt + d(-\Delta)^{-\frac{\gamma}{2}} W_t^i, \quad i = 1, \dots, d,$$

where $\alpha \in (0, 1]$, $\gamma \geq 0$, $\lambda > 0$ and W_t is a cylindrical Wiener process. Hence, $f(\tau_k) = |k|^{2\alpha}$ and $\sigma_k = |k|^{-2\gamma}$. By elementary calculations we see $\left(\frac{\sigma_k}{\lambda + f(\tau_k)}\right) \preceq (k^{-(2\gamma + 2\alpha)})$ and thus the conclusions of Proposition 4.1 hold if $2\gamma + 2\alpha > \frac{3}{2}$.

4.3.2. *Neumann boundary conditions.* In the case of homogeneous Neumann boundary conditions an orthogonal basis of eigenvectors of $-\Delta$ on $L^2([0, 2\pi])$ is given by

$$e_k(x) = \cos\left(\frac{k}{2}x\right), \quad \tau_k = \left(\frac{k}{2}\right)^2, \quad k \in \mathbb{N} \cup \{0\}.$$

In order to be able to consider stationary Ornstein-Uhlenbeck processes we need to shift the spectrum, i.e. we consider

$$d\Psi_t = -((-\Delta)^\alpha + 1)\Psi_t dt + dW_t.$$

We may then proceed as for Dirichlet boundary conditions, resolving additional difficulties due to the shift of the spectrum as in the proof of Proposition 4.1.

5. PART V: THE CONTINUOUS CASE

In some cases, the covariance function of a Gaussian process X is given as the cosine transform of some function f . For example, this is the case if the spectral measure of a stationary process has a density f with respect to the Lebesgue measure (cf. Example 2.15 and [MR06, Chapter 5.6]). In this case, we may obtain similar results as for random Fourier series. The key is a continuous version of Lemma 3.3 which we are now going to present. For a (symmetric) function $f \in L^1(\mathbb{R})$ let \hat{f} denote its (real) Fourier transform. Then the following holds:

Lemma 5.1. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be symmetric and in $L^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$. Assume $\hat{f} \in L^1(\mathbb{R})$ and*

$$\lim_{\xi \rightarrow \infty} |\xi^3 f''(\xi)| + |\xi^2 f'(\xi)| + |\xi f(\xi)| = 0$$

and that there is an $x_0 \in (0, \infty]$ such that

$$\limsup_{R \rightarrow \infty} \int_0^R \frac{\partial^2}{\partial \xi^2} (f(\xi) \xi^2) F_\xi(x) d\xi \leq 0,$$

for all $x \in (0, x_0)$ where $F_\xi(x) = \frac{1 - \cos(\xi x)}{x^2}$ denotes the Féjer kernel. Then \hat{f} is a convex function on $[0, x_0)$.

Proof. Since the proof is very similar to Lemma 3.3 we just sketch it briefly. By Féjer's Theorem for Fourier transforms (cf. [Kör89, Theorem 49.3]),

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R \left(1 - \frac{|\xi|}{R}\right) \hat{g}(\xi) e^{ix\xi} d\xi = g(x),$$

for all x provided $g \in C(\mathbb{R}) \cap L^1(\mathbb{R})$. Setting $g = \hat{f}$, we obtain from Fourier inversion

$$\sigma_R(x) := \int_{-R}^R \left(1 - \frac{|\xi|}{R}\right) f(\xi) e^{ix\xi} d\xi \rightarrow \hat{f}(x), \quad \text{for } R \rightarrow \infty.$$

Applying integration by parts twice, our assumptions imply that

$$\liminf_{R \rightarrow \infty} \sigma_R''(x) \geq 0$$

for all $x \in (0, x_0)$. This implies the claim. \square

Remark 5.2. *Note that for a given $f \in L^1(\mathbb{R})$, it does not follow that also $\hat{f} \in L^1(\mathbb{R})$. However, Bernstein's Theorem states that the Fourier transform of functions f in the Sobolev space $H^s(\mathbb{R})$ is contained in $L^1(\mathbb{R})$ for all $s > \frac{1}{2}$ (cf. [Hör83, Corollary 7.9.4]).*

Example 5.3. *Consider the covariance R of a fractional Ornstein-Uhlenbeck process with Hurst parameter $H \in (0, 1/2]$ (cf. Example 2.15). Then $R(s, t) = K(t - s)$ with*

$$K(x) = \int f(\xi) \cos(\xi x) d\xi, \quad f(\xi) = c_H \frac{|\xi|^{1-2H}}{\lambda^2 + \xi^2}, \quad \lambda > 0.$$

We prove that there is an $x_0 > 0$ such that K is convex on $[0, x_0)$. Since $f(\xi) = O(\xi^{-1-2H})$, $f \in H^s$ for any $s < 2H + 1/2$ and Bernstein's Theorem implies that $\hat{f} \in L^1$ for any $H > 0$. An easy calculation shows that $g := \partial_{\xi, \xi}^2(f(\xi)\xi^2)$ is nonpositive on $[\xi_0, \infty)$ for some $\xi_0 > 0$ and that $g(\xi) = O(-\xi^{-1-2H})$. It follows that

$$\limsup_{R \rightarrow \infty} \int_0^R \frac{\partial^2}{\partial \xi^2}(f(\xi)\xi^2)F_\xi(x) d\xi = \int_0^\infty \frac{\partial^2}{\partial \xi^2}(f(\xi)\xi^2)F_\xi(x) d\xi.$$

Note first that $\int_0^{\xi_0} g(\xi)F_\xi(x) d\xi$ is uniformly bounded for $x \searrow 0$. Furthermore $\lim_{x \rightarrow 0} F_\xi(x) = \xi^2/2$, and Fatou's Lemma gives

$$\liminf_{x \rightarrow 0} \int_{\xi_0}^\infty -g(\xi)F_\xi(x) d\xi \geq -\frac{1}{2} \int_{\xi_0}^\infty \xi^2 g(\xi) d\xi = +\infty.$$

Hence

$$\lim_{x \rightarrow 0} \int_0^\infty g(\xi)F_\xi(x) d\xi = -\infty,$$

thus there is an x_0 such that $\int_0^\infty g(\xi)F_\xi(x) d\xi \leq 0$ for all $x \in (0, x_0)$ and we can apply Lemma 5.1 to conclude that K is convex on $[0, x_0)$.

Example 5.4. Consider the SPDE

$$d\Psi_t = -((-\Delta)^\alpha + \lambda)\Psi_t dt + dW_t, \quad \text{on } \mathbb{R},$$

for some $\alpha \in (0, 1]$, $\lambda > 0$. The stationary solution can be written down explicitly (cf. [Wal86]), namely

$$\Psi_t(x) = \int_{-\infty}^t \int_{\mathbb{R}} K_{t-s}(x, y) W(ds, dy),$$

where K is the fractional heat kernel operator associated to $-((-\Delta)^\alpha + \lambda)$ with Fourier transform given by

$$\hat{K}_t(\xi) = e^{-t(|\xi|^{2\alpha} - \lambda)}.$$

After some calculations, one sees that the covariance R of the spatial process $x \mapsto \Psi_t(x)$ for every time point t is given by $R(x, y) = K(x - y)$ where

$$K(x) = \int f(\xi) \cos(\xi x) d\xi, \quad f(\xi) = \frac{1}{2|\xi|^{2\alpha} + 2\lambda}.$$

With a similar calculation as in Example 5.3, one can see that K is convex in a neighborhood around 0. It is easy to see that $\sigma^2(x) = 2(K(x) - K(0)) = O(|x|^{2\alpha-1})$ (using, for instance, [MR06, Theorem 7.3.1]), hence we are in the situation of Example 2.8 and we may conclude that

$$V_{1,\rho}(R_X; [x, y]) = O(|y - x|^{2\alpha-1})$$

for $|y - x|$ small enough. Applying [FV10b, Theorem 35], we see that Ψ_t can be lifted, for every fixed time point t , to a process Ψ_t with sample paths in $C_0^{0,\beta-H\text{öl}}([x, y], G^{[1/\beta]}(\mathbb{R}^d))$, every $\beta < \alpha - 1/2$, provided $\alpha > 3/4$ and $|y - x|$ is small enough. By concatenation one has the existence of spatial rough paths lifts on all compact intervals in \mathbb{R} .

6. PART VI: APPLICATION TO NON-MARKOVIAN HÖRMANDER THEORY

Consider a (rough) differential equation

$$(6.1) \quad dY = V(Y) d\mathbf{X}$$

driven by a (Gaussian) rough path \mathbf{X} along a vector field $V = (V_1, \dots, V_d)$, started at $Y_0 = y_0 \in \mathbb{R}^e$. Assume V to be bounded with bounded derivatives of all orders such that Hörmander's condition $\text{Lie}(V_1, \dots, V_d)|_{y_0} = \mathbb{R}^e$ holds¹⁸. If X is sufficiently non-degenerate (e.g. fBm) one can hope for a density of Y_t at positive times $t > 0$. This has been achieved in a series of papers starting with Baudoin–Hairer [BH07] (with \mathbf{X} fBm for $H > 1/2$), followed by [CF10, HP11, CHLT12] which dealt, respectively, with general Gaussian signals \mathbf{X} ($\rho < 2$ subject to CYR¹⁹), fBm for $H > 1/4$ and then again general Gaussian signals ($\rho < 2$ subject to CYR), now with a smoothness result. The general case ([CF10, CHLT12]) requires a number of assumptions on \mathbf{X} that are not always easy to check. To wit, even if X is fBm-like, in the sense that

$$\sigma^2(s, t) = F(t - s) \geq 0$$

with F being concave and $F(t) = O(t^{2H})$, already the CYR is unclear in the afore-mentioned references [CF10, CHLT12]. Indeed, CYR for fBm (in case $H > 1/4$) relies on the variation embedding theorem [FV06] which is not applicable in this more general situation. Our results provide a convenient way to check the assumptions of [CHLT12]. Let us illustrate how to proceed by the concrete example of an RDE driven by a (Gaussian) process (with i.i.d. components) with stationary increments.

Theorem 6.1. *Assume $F(t) = O(t^{1/\rho})$ with $\rho \in [1, 2)$ as $t \downarrow 0$, F concave and non-zero. Then*

$$F'_-(T) > 0, \quad \text{for some } T > 0$$

and Y_t in (6.1) has a smooth density for all $t \in (0, T]$.

This applies in particular when X is given as a random Fourier series as in Example 2.12 (with $\rho < 2$) or as a fractional Ornstein-Uhlenbeck process with Hurst parameter $H \in (1/4, 1/2]$ as in Example 2.15.

Proof. By assumption, F is not identically equal to zero. In order to see that $F'_-(T) > 0$ for some T small enough assume the opposite, i.e. $F'_-(t) \leq 0$ for all $t > 0$. Then

$$\frac{F(t+h) - F(t)}{h} \leq F'_-(t) \leq 0, \quad \forall h, t > 0$$

and thus F is nonincreasing. Since $F(0) = 0$ and $F \geq 0$ this implies that F is trivial and gives the desired contradiction. We now proceed by checking the conditions from [CHLT12]. Condition 1 (CYR; [CF10, CHLT12]) follows from Theorem 2.2, applied as in Example 2.8 which yields mixed $(1, \rho)$ -variation and thus, cf. Part 1, complementary Young regularity. For Condition 2 from

¹⁸We may also include a drift vector V_0 in which case we mean the weak Hörmander condition.

¹⁹Complementary Young regularity for Cameron–Martin paths h : That is, h has finite q -variation and $X(\omega)$ has finite p -variation a.s. with $1/p + 1/q > 1$.

[CHLT12]²⁰ we first note that (cf. Lemma 6.2 below)

$$\begin{aligned}
(6.2) \quad 2 \operatorname{Var}(X_{s,t} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) &\geq 2R \begin{pmatrix} s, t \\ 0, T \end{pmatrix} \\
&= \sigma^2(0, t) - \sigma^2(0, s) + \sigma^2(s, T) - \sigma^2(t, T) \\
&= F(t) - F(s) + F(T - s) + F(T - t) \\
&\geq 2F'_-(T)(t - s)
\end{aligned}$$

where we used (thanks to concavity) that the left side derivative of F at T exists and

$$\inf_{0 \leq s < t \leq T} \frac{F(t) - F(s)}{t - s} = F'_-(T).$$

By assumption, $F'_-(T) > 0$ and so Condition 2 holds with $\alpha = 1$. Also note that $F'_-(T)$ is non-increasing in T , thus Condition 2 remains valid upon decreasing T .

Next, we prove that [CHLT12, Condition 4, p.10] is satisfied. Due to concavity of F and Lemma 2.20, X has non-positively correlated increments and it is enough to show that (cf. [CHLT13, p.11])

$$\mathbb{E}X_{0,S}X_{s,t} = R \begin{pmatrix} 0, S \\ s, t \end{pmatrix} \geq 0, \quad \forall [s, t] \subseteq [0, S] \subseteq [0, T],$$

which is clear from our condition (B.ii) of Theorem 2.2, which was seen to be verified in the present situation in Example 2.8. We also note that [CHLT12, Condition 3, p.10] is implied by Condition 4 (cf. [CHLT12, Corollary 6.8]). In conclusion, [CHLT12, Theorem 3.5] implies the claim. \square

Lemma 6.2. *The estimate (6.2) holds.*

Proof. Let $\mathcal{G} := \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}$. Since X is Gaussian, $\operatorname{Var}(X_{s,t} | \mathcal{G})$ is deterministic and we have

$$\begin{aligned}
\operatorname{Var}(X_{s,t} | \mathcal{G}) &= \|\operatorname{Var}(X_{s,t} | \mathcal{G})\|_2 \\
&= \inf_{Y \in L^2(\Omega, \mathcal{G}, \mathbb{P})} \|X_{s,t} - Y\|_2^2.
\end{aligned}$$

It is enough to consider

$$Y_n = \sum_{i=1}^{k_n} a_i^n X_{t_i^n, t_{i+1}^n}$$

²⁰Recall: $\exists c, \alpha > 0 : \forall 0 \leq s < t \leq T : \operatorname{Var}(X_{s,t} | \mathcal{F}_{0,s} \vee \mathcal{F}_{t,T}) \geq c(t - s)^\alpha$

and to estimate $\|X_{s,t} - Y_n\|_2^2$ (cf. [CHLT12, proof of Lemma 4.1]). For this we note, using symmetry of R

$$\begin{aligned}
\|X_{s,t} - Y_n\|_2^2 &= \mathbb{E} \left| X_{s,t} - \sum_i a_i^n X_{t_i^n, t_{i+1}^n} \right|^2 \\
&= \mathbb{E} \left| \sum_i \tilde{a}_i^n X_{\tilde{t}_i^n, \tilde{t}_{i+1}^n} \right|^2 \\
&= \sum_{i,j} \tilde{a}_i^n \tilde{a}_j^n \mathbb{E} X_{\tilde{t}_i^n, \tilde{t}_{i+1}^n} X_{\tilde{t}_j^n, \tilde{t}_{j+1}^n} \\
&= \sum_{i,j} \tilde{a}_i^n \tilde{a}_j^n R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{array} \right) \\
&\geq - \sum_i \sum_{j \neq i} |\tilde{a}_i^n| |\tilde{a}_j^n| \left| R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{array} \right) \right| + \sum_i (\tilde{a}_i^n)^2 R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_i^n, \tilde{t}_{i+1}^n \end{array} \right) \\
&\geq - \sum_i \sum_{j \neq i} (\tilde{a}_i^n)^2 \left| R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{array} \right) \right| + \sum_i (\tilde{a}_i^n)^2 R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_i^n, \tilde{t}_{i+1}^n \end{array} \right).
\end{aligned}$$

Hence, due to non-positively correlated increments we get

$$\begin{aligned}
\|X_{s,t} - Y_n\|_2^2 &= \sum_i (\tilde{a}_i^n)^2 \sum_j R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ \tilde{t}_j^n, \tilde{t}_{j+1}^n \end{array} \right) \\
&= \sum_i (\tilde{a}_i^n)^2 R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ 0, T \end{array} \right) \\
&= R \left(\begin{array}{c} s, t \\ 0, T \end{array} \right) + \sum_i (\tilde{a}_i^n)^2 R \left(\begin{array}{c} \tilde{t}_i^n, \tilde{t}_{i+1}^n \\ 0, T \end{array} \right) \\
&\geq R \left(\begin{array}{c} s, t \\ 0, T \end{array} \right).
\end{aligned}$$

□

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