

A RESULT ABOUT PICARD-LEFSCHETZ MONODROMY

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ABSTRACT. Let f and g be reduced homogeneous polynomials in separate sets of variables. We establish a simple formula that relates the eigenspace decomposition of the monodromy operator on the Milnor fiber cohomology of fg to that of f and g separately. We use a relation between local systems and Milnor fiber cohomology that has been established by D. Cohen and A. Suciu.

1. BRIEF INTRODUCTION AND STATEMENT OF RESULTS

Thom-Sebastiani type Theorems have a rich history. This study was initiated by Sebastiani and Thom [12] and improved by others [8, 11]. Perhaps the most successful generalization has been achieved by Némethi [5]. He considers the germs of three holomorphic functions f, g, p at the origin of $\mathbb{C}^n, \mathbb{C}^m$ and \mathbb{C}^2 respectively, and then draws conclusions about the topology of the Milnor fiber [4] of $p(f, g)$. Némethi discovered an expression for the Weil zeta function of $p(f, g)$ in terms of the monodromy representations of f and g as well as the several variable Alexander polynomial of p [6, 7].

In this paper we will investigate the case when f and g are homogeneous polynomials and $p(x, y) = xy$. We will construct a fibration different from those of [11, Theorem 2], and [5, 6, 8]. We will then produce a formula for the eigenspace decomposition of the Picard-Lefschetz monodromy of fg in terms of those of f and g .

The following describes the situation we consider,

Hypothesis 1.1. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ and $g \in \mathbb{C}[y_1, \dots, y_m]$ be homogeneous and reduced of positive degrees r and s respectively.

In this case the restriction $f : \mathbb{C}^n \setminus \{x \mid f(x) = 0\} \rightarrow \mathbb{C}^*$ defines a fibration. The fiber of this fibration is called the Milnor fiber of f , denoted $F_f = f^{-1}(1)$. Lifting the path $t \mapsto \exp(2\pi it) : 0 \leq t \leq 1$ in \mathbb{C}^* induces, through a local trivialization of f , a diffeomorphism of the fiber. This map will be called a *geometric* Picard-Lefschetz(PL) monodromy of f . A geometric PL monodromy of f induces a map on the cohomology algebra $H^*(F_f, \mathbb{C})$ which we will call the *algebraic* PL monodromy of f . In a more general setting we may have any smooth fibration $F \rightarrow M \rightarrow N$. Given a loop in the base space N we may again construct a diffeomorphism of the fiber; in the case when this diffeomorphism is homotopic to the identity for any loop we choose, we say that the fibration has *trivial* geometric monodromy.

For any $M \subseteq \mathbb{C}^t \setminus \{0\}$ we denote by M^* the image of M under the Hopf fibration

$$\rho : \mathbb{C}^t \setminus \{0\} \rightarrow \mathbb{P}^{t-1}.$$

Note that as f is homogeneous, F_f^* is the complement of the projective hypersurface defined by $f = 0$. Let $f = f_1 \cdots f_e$ be a factorization of f into irreducible polynomials. We note that $(\mathbb{C}^n \setminus f^{-1}(0))^*$ has first homology generated by the

meridian circles γ_i around $f_i^{-1}(0) \subset \mathbb{P}^{n-1}$ with orientations determined by the complex orientations. For $\eta^r = 1$ we denote by \mathcal{V}_η^f the rank one local system on $(\mathbb{C} \setminus f^{-1}(0))^* = F_f^*$ induced by the homomorphism

$$(1.1) \quad \Phi_\eta^f : H_1(F_f^*) \rightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$$

that sends $H_1(\rho)(\gamma_i)$ to η . We define \mathcal{W}_η to be the local system on \mathbb{C}^* induced by the representation that sends the standard generator of $\pi_1(\mathbb{C}^*)$ to $\eta \in \mathbb{C}^*$. We will also let $H^*(F_f, \mathbb{C})_\eta$ denote the eigenspace of the algebraic PL monodromy of f with eigenvalue η . Lastly, the symbol $\#$ will be used as a subscript of a continuous function, and denotes the induced homomorphism defined on the fundamental groups.

We will show the following two results:

Lemma 1.2. *Let f and g be as in Hypothesis 1.1. Then there is a fibration $F_{fg}^* \rightarrow F_f^* \times F_g^*$ defined by*

$$[x_1 : \dots : x_n : y_1 : \dots : y_m] \mapsto ([x_1 : \dots : x_n], [y_1 : \dots : y_m])$$

with fiber \mathbb{C}^* and trivial geometric monodromy.

The Leray spectral sequence associated to this fibration will allow us to prove the following formula.

Theorem 1.3. *Let f and g satisfy Hypothesis 1.1. Then,*

$$H^*(F_{fg}, \mathbb{C})_\eta = H^*(F_f, \mathbb{C})_\eta \otimes H^*(F_g, \mathbb{C})_\eta \otimes H^*(\mathbb{C}^*, \mathbb{C}).$$

In the statement above, the tensor symbol is used to mean the tensor product of vectorspaces graded by cohomological degree. Namely, if M_* and N_* are graded vectorspaces then

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j.$$

It clearly follows from Theorem 1.3 that the Weil zeta function of the algebraic PL monodromy of fg is always 1, recovering Némethi's result in our case [5].

2. PROOF OF THEOREM 1.3

We will make heavy use of a Theorem of D.Cohen and A. Suciu [1] that we state here.

Theorem 2.1 (Cohen, Suciu). *Let f be homogeneous and reduced of degree r , and pick $\eta \in \mathbb{C}^*, \eta^r = 1$. Then*

$$H^*(F_f, \mathbb{C})_\eta \cong H^*(F_f^*, \mathcal{V}_\eta^f).$$

□

The equation above simply tells us that an eigenspace of the algebraic PL monodromy of f is isomorphic to the cohomology of a local system defined on the complement of the projective hypersurface $f = 0$. This Theorem leads us to consider the cohomology $H^*(F_{fg}, \mathcal{V}_\eta^{fg})$. We establish Lemma 1.2 to aid in the computation of $H^*(F_{fg}, \mathcal{V}_\eta^{fg})$.

Lemma 2.2. *Let \mathbb{P}^{n+m-1} have coordinates $x_1, \dots, x_n, y_1, \dots, y_m$. Let*

$$M = \mathbb{P}^{n+m-1} \setminus (\mathbb{P}^{n-1} \cup \mathbb{P}^{m-1})$$

be the complement of the projective variety defined by the ideal

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_m).$$

Then the map

$$\phi : M \rightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$$

sending $[x_1 : \dots : x_n : y_1 : \dots : y_m]$ to $([x_1 : \dots : x_n], [y_1 : \dots : y_m])$ makes M a \mathbb{C}^ -bundle over $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ with trivial geometric monodromy.*

Proof. This map is clearly well-defined. When we restrict to the chart defined by $x_j \neq 0$ (resp. $y_j \neq 0$) then ϕ can be interpreted as the Hopf fibration applied to the y 's (resp. x 's). This map is clearly surjective and has trivial geometric monodromy as $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is simply connected. \square

Proof of Lemma 1.2. The restriction of ϕ to F_{fg}^* has image $F_f^* \times F_g^*$ and has trivial geometric monodromy. \square

When we look at the Leray spectral sequence induced by this fibration we obtain the following result.

Theorem 2.3. *If Hypothesis 1.1 holds then there is a spectral sequence*

$$(2.1) \quad E_2^{i,j} \implies H^{i+j}(F_{fg}, \mathbb{C})_\eta,$$

with $E_2^{i,j} = 0$ for $j \neq 0, 1$, and

$$E_2^{i,0} \cong E_2^{i,1} \cong \bigoplus_{j+k=i} H^j(F_f, \mathbb{C})_\eta \otimes H^k(F_g, \mathbb{C})_\eta.$$

Proof. Let M be the complement of $f = 0$ in \mathbb{C}^n and $f : M \rightarrow \mathbb{C}^*$ be the Milnor fibration. On page 107 of [1] we have the following commutative diagram with exact rows.

$$(2.2) \quad \begin{array}{ccccc} \pi_1(\mathbb{C}^*) & \xrightarrow{\iota} & \pi_1(M) & \longrightarrow & \pi_1(M^*) \\ \downarrow \cong & & \downarrow f_\# & & \downarrow \\ \mathbb{Z} & \xrightarrow{\times r} & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

The top row of this diagram is part of the homotopy sequence associated to the Hopf fibration restricted to M . It also follows from [1] that if $f = f_1 \cdots f_e$ is a factorization of f into distinct irreducible polynomials and if a_i is the homotopy class of a meridian around $f_i = 0$ with orientation determined by the complex orientations, then $f_\#(a_i) = 1$ for all $i = 1, \dots, e$. In this way $H_1(M, \mathbb{Z})$ may be identified with the free \mathbb{Z} module with basis given by the homology classes determined by each of the a_i , and $f_* : H_1(M, \mathbb{Z}) \rightarrow \mathbb{Z}$ may be identified with the matrix $[1, 1, \dots, 1]^T$. Also note by commutativity that if σ is an appropriate choice of a generator of $\pi_1(\mathbb{C}^*)$, then we have $f_\# \circ \iota(\sigma) = r$. Thus in particular $\Phi_\eta^f([\iota(\sigma)]) = \eta^r$, where $[*]$ denotes “the homology class determined by”.

Now recall the fibration from Lemma 1.2:

$$(2.3) \quad \mathbb{C}^* \xrightarrow{\kappa} F_{fg}^* \xrightarrow{\phi} F_f^* \times F_g^* .$$

Recall further that we consider $F_f^* \times F_g^*$ as a subset of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_m$. The open subset $y_1 \neq 0$ of $F_f^* \times F_g^*$ can be thought of as $M^* \times C$ where C is the complement of the hypersurface $g(1, y_2, \dots, y_m) = 0$ in \mathbb{A}^{m-1} . In this way, $\phi^{-1}(M^* \times C)$ is $M \times C$ in \mathbb{A}^{m+n-1} . In fact $\phi|_{\phi^{-1}(F_f^* \times C)}$ may be identified with the Hopf fibration applied to the x 's,

$$(2.4) \quad \mathbb{C}^* \longrightarrow M \times C \longrightarrow M^* \times C .$$

Let $U \subseteq C$ be a contractible subset of C and $\psi = \phi|_{\phi^{-1}(F_f^* \times U)}$, and consider the restriction of (2.4),

$$\mathbb{C}^* \xrightarrow{\lambda} M \times U \xrightarrow{\psi} M^* \times U .$$

Then by the discussion in the first paragraph of this proof we know that

$$(2.5) \quad \Phi_{\eta}^{fg}([\lambda_{\#}(\sigma)]) = \eta^r ,$$

where $[*]$ denotes the image of the homology class of $*$ under the natural map $H^1(F_f^* \times U, \mathbb{Z}) \rightarrow H^1(F_f^* \times F_g^*, \mathbb{Z})$.

Let $\eta^{r+s} = 1$. We will see that the spectral sequence of the Theorem is essentially the Leray spectral sequence

$$(2.6) \quad H^i(F_f^* \times F_g^*, \mathbb{R}^j \phi_*(\mathcal{V}_{\eta}^{fg})) \implies H^{i+j}(F_{fg}^*, \mathcal{V}_{\eta}^{fg}) .$$

To compute $\mathbb{R}^{\ell} \phi_*(\mathcal{V}_{\eta}^{fg})$, we may apply [2, Proposition 6.4.3] to obtain:

- $\mathbb{R}^{\ell} \psi_*(\mathcal{V}_{\eta}^{fg}|_{F_f^* \times U}) = \pi_1^*(\mathcal{V}_{\eta}^f)|_{F_f^* \times U}$ if $\eta^r = 1$ and $\ell = 0, 1$.
- $\mathbb{R}^{\ell} \psi_*(\mathcal{V}_{\eta}^{fg}|_{F_f^* \times U}) = 0$ otherwise.

Here $\pi_1 : F_f^* \times F_g^* \rightarrow F_f^*$, and $\pi_2 : F_f^* \times F_g^* \rightarrow F_g^*$ are the natural projections. Note that the equation (2.5) calculates what A. Dimca calls the total monodromy operator on [2, p. 210]. A symmetric argument holds with f replaced by g (note that $\eta^r = \eta^{-s}$ as $\eta^{r+s} = 1$) and we may conclude:

- $\mathbb{R}^{\ell} \phi_*(\mathcal{V}_{\eta}^{fg}) = \pi_1^*(\mathcal{V}_{\eta}^f) \otimes \pi_2^*(\mathcal{V}_{\eta}^g) := \mathcal{V}_{\eta}^f \boxtimes \mathcal{V}_{\eta}^g$ if $\eta^r = 1$ and $\ell = 0, 1$.
- $\mathbb{R}^{\ell} \phi_*(\mathcal{V}_{\eta}^{fg}) = 0$ otherwise.

Therefore, if $\eta^r \neq 1, \eta^{r+s} = 1$ the spectral sequence (2.6) is zero. Hence $H^*(F_{fg}, \mathbb{C})_{\eta} = 0$. In this case $H^*(F_f, \mathbb{C})_{\eta}$ is also zero and the Theorem is proved. When $\eta^r = 1$ we may now apply the Künneth formula [2, Theorem 4.3.14] to obtain that for $j = 0, 1$ one has

$$H^{\ell}(F_f^* \times F_g^*, \mathbb{R}^j \phi_*(\mathcal{V}_{\eta}^{fg})) = \bigoplus_{i+k=\ell} H^i(F_f^*, \mathcal{V}_{\eta}^f) \otimes H^k(F_g^*, \mathcal{V}_{\eta}^g),$$

while the left hand side is zero for other j . These are exactly the $E_2^{\ell, j}$ terms of the spectral sequence of our Theorem, and we know that it converges to $H^*(F_{fg}^*, \mathcal{V}_{\eta}^{fg}) \cong H^*(F_{fg}, \mathbb{C})_{\eta}$. This establishes the Theorem for $\eta^{r+s} = 1$.

It may be noted that when $\eta^{r+s} \neq 1$ then either η^r or η^s is not equal to one. If $\eta^r \neq 1$ then $H^*(F_f, \mathbb{C})_{\eta}$ is zero as well as $H^*(F_{fg}, \mathbb{C})_{\eta}$. If $\eta^s \neq 1$ then $H^*(F_g, \mathbb{C})_{\eta}$ is zero as well as $H^*(F_{fg}, \mathbb{C})_{\eta}$ and the Theorem follows in these cases. \square

We will now concern ourselves with computing $H^i(F_{fg}, \mathbb{C})$. We first consider a variant of [8, Theorem 4] and [11, Theorem 2] in our homogeneous case. We tacitly use the embedding $F_{fg} \subset \mathbb{C}^n \times \mathbb{C}^m$ in the following statement.

Lemma 2.4. *The map $\widehat{f} : F_{fg} \rightarrow \mathbb{C}^*$ defined by $(x, y) \mapsto f(x)$ is a fibration with fiber $F_f \times F_g$. A geometric PL monodromy of this fibration is,*

$$(x, y) \mapsto \left(\exp\left(\frac{2\pi i}{r}\right)x, \exp\left(-\frac{2\pi i}{s}\right)y \right).$$

Proof. Since f and g are homogeneous we have $f(\exp(\frac{t}{r})x) = \exp(t)f(x)$ and $g(\exp(\frac{t}{s})y) = \exp(t)g(y)$. We also note that if $f(x)g(y) = 1$ then $f(x) = g(y)^{-1}$. These two properties prove the Lemma. \square

We have a direct consequence, included here for completeness.

Theorem 2.5. *Let f and g satisfy Hypothesis 1.1, then*

$$H^\ell(F_{fg}, \mathbb{C}) \cong \bigoplus_{\lambda=\ell-1}^{\ell} \left(\bigoplus_{\substack{\eta^{r+s}=1 \\ i+j=\lambda}} H^i(F_f, \mathbb{C})_\eta \otimes H^j(F_g, \mathbb{C})_{\eta^{-1}} \right)$$

where the inner sum runs over all possible η .

Proof. The algebraic PL monodromy operator of \widehat{f} is

$$T_f \otimes T_g^{-1} : H^*(F_f, \mathbb{C}) \otimes H^*(F_g, \mathbb{C}) \rightarrow H^*(F_f, \mathbb{C}) \otimes H^*(F_g, \mathbb{C})$$

where T_f, T_g are the algebraic PL monodromy operators of the respective fibers. Since T_f, T_g are of finite order, they are diagonalizable. We let $\{a_i\}$ (resp. $\{b_j\}$) be a homogeneous basis of $H^*(F_f, \mathbb{C})$ (resp. $H^*(F_g, \mathbb{C})$) that are eigenvectors of T_f (resp. T_g), with eigenvalue α_i (resp. β_j). In such a case $\{a_i \otimes b_j\}$ are a basis of eigenvectors for $T_f \otimes T_g^{-1}$ with eigenvalue $\alpha_i \beta_j^{-1}$. This shows that

$$\mathbb{R}^\ell \widehat{f}_*(\mathbb{C}_{F_{fg}}) = \bigoplus_{\substack{(\alpha_i, \beta_j) \\ \deg(a_i) + \deg(b_j) = \ell}} \mathcal{W}_{\alpha_i \beta_j^{-1}}.$$

Ergo, since non-constant rank one local systems on \mathbb{C}^* have no cohomology

$$H^p(\mathbb{C}^*, \mathbb{R}^\ell \psi_*(\mathbb{C}_{F_{fg}})) = \bigoplus_{\substack{\alpha_i \beta_j^{-1} = 1 \\ \deg(a_i) + \deg(b_j) = \ell}} \mathbb{C}$$

for $p = 0, 1$, and the left hand side is zero for $p \neq 0, 1$.

Now we may consider the Leray spectral sequence associated with the fibration of Lemma 2.4. Since the base of this fibration is \mathbb{C}^* , the spectral sequence has only two columns, and thus converges on the second page. This yields the Theorem. \square

Proof of Theorem 1.3. Theorem 1.3 now follows from Theorems 2.3 and 2.5. To see this we denote by Γ_η^ℓ the vector space $\bigoplus_{i+j=\ell} H^i(F_f, \mathbb{C})_\eta \otimes H^j(F_g, \mathbb{C})_\eta$. Theorem 2.3 shows the existence of the following exact sequence,

$$\Gamma_\eta^{\ell-2} \xrightarrow{d_2} \Gamma_\eta^\ell \longrightarrow H^\ell(F_{fg}, \mathbb{C})_\eta \longrightarrow \Gamma_\eta^{\ell-1} \xrightarrow{d_2} \Gamma_\eta^{\ell+1}.$$

By [1, Proposition 1.1], $H^\ell(F_f, \mathbb{C})_\eta \cong H^\ell(F_f, \mathbb{C})_{\eta^{-1}}$. Hence

$$H^\ell(F_{fg}, \mathbb{C}) \cong \bigoplus_{\substack{j=\ell-1 \\ \eta^{r+s}=1}}^{\ell} \Gamma_\eta^j,$$

by Theorem 2.5, and so the second differential, d_2 is zero. \square

3. A FEW EXAMPLES

The reader may wish to consult [10] for definitions of terms that involve hyperplane arrangements.

Example 3.1. Let $r = 4, s = 5, f = x_1x_2(x_1 + x_2)(x_1 + 2x_2), g = y_1y_2(y_1 + y_2)(y_1 + 2y_2)(y_1 + 3y_2)$. Note that f and g define generic central line arrangements. The Weil zeta function of any generic hyperplane arrangement singularity is presented in [9]. To compute the Weil zeta function of any hyperplane arrangement singularity one may use [4, Theorem 9.6], the formula $\chi(F_h) = \text{deg}(h)\chi(F_h^*)$, and the algorithm [10, Theorem 5.87(c)]. This method is practical for low dimensions and is simple for line arrangements. Also the Weil zeta function of any generic hyperplane arrangement singularity is presented in [9]. Since the Milnor fiber of f and g is connected we may easily write down tables expressing the eigenspace decomposition of the algebraic (PL) monodromy as follows. Note that we express $\exp(2\pi i/5)$ as ω .

$\eta \setminus j$	0	1
1	1	3
i	0	2
-1	0	2
$-i$	0	2

$\eta \setminus j$	0	1
1	1	4
ω	0	3
ω^2	0	3
ω^3	0	3
ω^4	0	3

Now our Theorem 1.3 immediately yields the following table for $\dim(H^j(F_{fg}, \mathbb{C})_\eta)$:

$\eta \setminus j$	0	1	2	3
1	1	8	19	12

where there is a zero for every other η and j . In this example even though the algebraic PL monodromy of f and g have non-unity eigenvalues the algebraic PL monodromy of fg has one as the only eigenvalue.

This behavior is not uncommon. Theorem 1.3 guarantees that $H^*(F_f, \mathbb{C})_\eta$ will not contribute to $H^*(F_{fg}, \mathbb{C})$ if $H^*(F_g, \mathbb{C})_\eta$ is zero. We state this observation as the following Corollary of Theorem 1.3.

Corollary 3.2. *We assume the conditions of Hypothesis 1.1. $H^*(F_{fg}, \mathbb{C})_\eta \neq 0$ if and only if $H^*(F_f, \mathbb{C})_\eta \neq 0$ and $H^*(F_g, \mathbb{C})_\eta \neq 0$. In particular if $H^*(F_{fg}, \mathbb{C})_\eta \neq 0$ then $\eta^{\text{gcd}(r,s)} = 1$. \square*

Here we give an if and only if condition for the vanishing of $H^j(F_{fg}, \mathbb{C})_\eta$. The second paragraph of remark 3.2 of [3] only provides the second sentence of this Corollary.

Example 3.3. Let $r = 3, s = 6, f = x_1^3 + x_2^3 + x_3^3$ and $g = y_1y_2(y_1 + y_2)(y_1 + 2y_2)(y_1 + 3y_2)(y_1 + 4y_2)$. The eigenspace decomposition of f is discussed in [4, section 9] and g is a line arrangement so we obtain

$\eta \setminus j$	0	1	2
1	1	0	2
ω^2	0	0	3
ω^4	0	0	3

$\eta \setminus j$	0	1
1	1	5
ω	0	4
ω^2	0	4
ω^3	0	4
ω^4	0	4
ω^5	0	4

where $\omega = \exp(2\pi i/6)$. Applying our Theorem 1.3 yields the following table for $\dim(H^j(F_{fg}, \mathbb{C})_\eta)$.

$\eta \setminus j$	0	1	2	3	4
1	1	6	7	12	10
ω^2	0	0	0	12	12
ω^4	0	0	0	12	12

ACKNOWLEDGMENTS

The author would like to thank his advisor Uli Walther, and is grateful for helpful conversations with D. Arapura.

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