EXPANSIVE FLOWS OF SURFACES

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ABSTRACT. We prove that a flow on a compact surface is expansive if and only if the singularities are of saddle type and the union of their separatrices is dense. Moreover we show that such flows are obtained by surgery on the suspension of minimal interval exchange maps.

INTRODUCTION

Expansive homeomorphisms on surfaces are known to be conjugated to pseudo-Anosov maps [7, 12]. In particular the stable leaves form a minimal measured foliation. Here we study flows on compact surfaces that are expansive in the sense of [11]. We show a strong relation between the orbit structure of expansive flows and the stable foliation of a pseudo-Anosov maps. In the lack of saddle connections we prove that expansive flows are suspensions of minimal interval exchange maps. Therefore the orbits of said flows make a minimal measured foliation.

The only published work on the subject of expansive flows of surfaces, known to the author, is [13]. In our context their result means that expansive flows on surfaces must present singular (i.e. equilibrium) points.

The main ideas of the present article arise from relating expansive dynamics, polygonal billiards and singular flows. The expansive properties of polygonal billiards were established in [3]. In this article it is shown that excluding periodic orbits the collision map is expansive and two close non-periodic points are separated after visiting different sides of the polygon. The link between flows with singularities and polygonal billiards is given in [18]. In said work a flow on a compact surface is constructed from any given rational polygonal billiard and fixed direction. The equilibrium points of the flow are singularities of saddle type associated with the corners of the polygon.

Our main result is Theorem 6.1. It states that a flow on a compact surface, in the absence of singularities of saddle type of index 0, is expansive if and only if the singularities are of saddle type and the union of their separatrices is dense in the surface. Roughly speaking the proof is the following. Take two points in different local orbits, consider a separatrix between them and follow them until the end of the separatrix. The separation occurs because the singularity is of

saddle type of negative index. To prove the converse the key step is to show that there are no wandering points. The rest of the argument is quite standard.

Description of the content. Section 1. We introduce the definition of expansive flow and show the equivalence with the definition of k^*-expansive given in [11].

Section 2. We prove that expansive flows on surfaces do not have wandering points.

Section 3. We show that the singular points of expansive flows are of saddle type. Also we show how this kind of singularities helps to the expansive properties of the flow.

Section 4. We show that expansive flows are Cherry flows (in the sense of [4]) and as a consequence we conclude that expansive flows are smoothable. Previously we prove that expansive flows on surfaces do not have periodic orbits and have singular points, moreover the union of their separatrices is dense in the surface.

Section 5. We show that expansiveness is invariant under adding and removing singularities of index 0.

Section 6. We give characterizations of expansive flows on surfaces. In the absence of singularities of index 0 it is shown that expansiveness is equivalent with (1) the singularities are of saddle type and the union of their separatrices is dense and (2) there is a finite and positive number of singularities, $\Omega(\phi) = S$ and there are no periodic orbits. Also we show that the surfaces that admits expansive flows are: the torus with b boundaries, h handles and c cross-cups with $b + h + c > 0$. In particular the torus do not admit expansive flows.

Section 7. We show that the suspension of an interval exchange map f is an expansive flow on a surface S if and only if S is not the torus and f has not periodic points.

Section 8. In the discrete case it is known that the interval and the circle do not admit expansive homeomorphisms (see [9]). Nevertheless it is easy to prove that expansive flows on those spaces are the ones with a finite number of singular points. In particular this shows that expansive flows can posses saddle connections. In fact, Example 8.1 shows that there exist expansive flows with cycles of saddle connections that disconnect the surface. In Theorem 8.3 we give a procedure to construct every expansive flow on a compact surface, the procedure is: suspend some minimal interval exchange maps, add singularities of index 0, make cuts along saddle connections and glue them. This procedure will give and expansive flow if and only if no connected component of the surface obtained is the torus.

Section 9. As an application and a source of examples of expansive flows on surfaces we consider rational polygonal billiards. As explained in [18] such billiard flows can be seen on a compact surface and fixing a

direction $v \in \mathbb{R}^2$ a flow is determined presenting singularities of saddle type in the corners. Theorem 9.1 states that the the associated flow is expansive if and only if there are no periodic orbits with initial direction v in the billiard.

1. Expansive flows

In this section we introduce the definition of expansive flow. Let (X, dist) be a compact metric space and $\phi\colon \mathbb{R} \times X \to X$ be a continuous flow. A point $p \in X$ is said to be a *singularity* of ϕ if $\phi_t(p) = p$ for all $t \in \mathbb{R}$. In other case it is called *regular*. The set of singularities is denoted by Sing.

It is easy to show that if x is a regular point then for all $\delta > 0$ there exist $s > 0$ such that $y = \phi_s(x) \neq x$ and $\mathrm{dist}(\phi_t(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$. Roughly speaking it means that two points in a small orbit segment contradict the expansiveness of the flow. So every definition of *expansive flow* must consider this fact. In [1] it is done in such a way that singular points must be isolated points of the space, therefore no manifold of positive dimension admits an expansive flows (in the sense of [1]) with singular points. In [11] a definition is given in order to study expansive properties of the Lorenz attractor. They used the term k^*-*expansive*. Of course it is weaker than the definition in [1], but just because it allows singular points. Our definition of expansive flow is equivalent with [11] (as we show in Theorem 1.3) but seems to be more insightful.

First we define another distance in X that allow us to say in a formal way that two points are locally in the same orbit. Let

$$\mathrm{dist}_\phi(x, y) = \inf\{\mathrm{diam}(\phi_{[a,b]}(z)) : z \in X, [a, b] \subset \mathbb{R}, x, y \in \phi_{[a,b]}(z)\}$$

if $y \in \phi_\mathbb{R}(x)$ and $\mathrm{dist}_\phi(x, y) = \mathrm{diam}(X)$ if $y \notin \phi_\mathbb{R}(x)$. Consider

$$\beta_0 = \inf\{\mathrm{diam}(\phi_\mathbb{R}(x)) : x \notin \mathrm{Sing}\}.$$

Notice that $\mathrm{dist}_\phi(x, y) < \beta_0$ if and only if x and y are in an orbit segment of diameter less than β_0. Let $\mathcal{H}_0^+(\mathbb{R})$ be the set of increasing homeomorphisms $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0$.

Definition 1.1. We say that ϕ is *expansive* if for all $\beta > 0$ there exists an *expansive constant* $\delta > 0$ such that if $\mathrm{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}_0^+(\mathbb{R})$ then $\mathrm{dist}_\phi(x, y) < \beta$.

If an expansive flow presents a singularity p that is not an isolated point of X, it is easy to see that there exist $t_n \to \infty$ and $x_n \to p$, $x_n \neq p$, such that $\mathrm{dist}_\phi(x_n, \phi_{t_n}(x_n)) \to 0$. That simple remark is the essential difference between our definition of expansive flow and the one given in [1].

In [1] (Theorem 3) they give four equivalent definitions of expansive flows. In fact it is not necessary to assume that the flow do not present

singular points in this Theorem, that is because each item implies that the set of singular points is an isolated finite set (see [16]). Our Definition 1.1 above is the combination of items (ii) and (iii) of this Theorem suggested in [1] after its proof. It may not be equivalent in the presence of equilibrium points. In fact they are equivalent if and only if Sing is an isolated set of the space. In Theorem 1.3 we show that Definition 1.1 coincides with the notion of k^*-*expansive* introduced in [11].

Lemma 1.2. *Suppose that ϕ presents a finite number of singular points and $\beta_0 > 0$. Then for all $\beta \in (0, \beta_0)$ there exist $\delta > 0$ such that if $\mathrm{dist}(\phi_{g(t)}(x), \phi_t(x)) < \delta$ for all $t \in \mathbb{R}$ and a continuous function $g\colon \mathbb{R} \to \mathbb{R}$, $g(0) = 0$, then $\mathrm{dist}_\phi(\phi_{g(t)}(x), \phi_t(x) < \beta$ for all $t \in \mathbb{R}$.*

Proof. By contradiction suppose that there exist $\beta \in (0, \beta_0)$, $\delta_n \to 0$, x_n and continuous functions $g_n\colon \mathbb{R} \to \mathbb{R}$ such that

$$(1) \qquad\qquad \mathrm{dist}(\phi_{g_n(t)}(x_n), \phi_t(x_n)) < \delta_n$$

for all $t \in \mathbb{R}$, for all $n \in \mathbb{N}$, $g_n(0) = 0$ and also suppose that there exists t'_n such that

$$\mathrm{dist}_\phi(\phi_{g_n(t'_n)}(x_n), \phi_{t'_n}(x_n)) > \beta$$

. Since $\phi_{g_n(0)}(x_n) = \phi_0(x_n)$ there exist $t_n \in (0, t'_n)$ such that

$$\mathrm{dist}_\phi(\phi_{g_n(t_n)}(x_n), \phi_{t_n}(x_n)) = \beta$$

Let $a_n = \phi_{g_n(t_n)}(x_n)$ and $b_n = \phi_{t_n}(x_n)$. By equation (1) we have that $\mathrm{dist}(a_n, b_n) < \delta_n \to 0$. Then we can suppose that $a_n, b_n \to c$. Also assume that $\phi_{s_n}(a_n) = b_n$ with $s_n > 0$ and $\mathrm{diam}(\phi_{[0,s_n]}(a_n)) = \beta$. If $\{s_n\}_{n\in\mathbb{N}}$ is bounded it is easy to prove that c is a periodic point and $\mathrm{diam}\,\phi_\mathbb{R}(c) = \beta$, contradicting $\beta < \beta_0$. If $\{s_n\}_{n\in\mathbb{N}}$ is not bounded, eventually taking a subsequence, we have that $s_n \to +\infty$. Also we can suppose that $\phi_{[0,s_n]}(a_n)$ converges to a compact set $K \subset X$ considering the Hausdorff distance[1] between compact subsets of X. Since $a_n, \phi_{s_n}(a_n) \to c$, it is easy to see that K is an invariant set. Also we have that $\mathrm{diam}(K) = \beta$ and K is connected, therefore K is an infinite set. By hypothesis, the flow presents a finite number of singularities and then there is a regular point in K whose orbit has diameter less or equal than β, again the contradiction is that $\beta < \beta_0$. $\qquad\square$

Theorem 1.3. *The following statements are equivalent:*
  (1) *ϕ is expansive,*
  (2) *ϕ is k^*-expansive, i.e. for all $\varepsilon > 0$ there exist an expansive constant $\delta > 0$ such that if $\mathrm{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}_0^+(\mathbb{R})$ then there exist $s, t_0 \in \mathbb{R}$ such that $|s| < \varepsilon$ and $\phi_{h(t_0)}(x) = \phi_{t_0+s}(y)$.*

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[1]If $A, B \subset X$ are compact sets the Hausdorff distance between A and B is $d_H(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} \mathrm{dist}(a,b), \sup_{b\in B} \inf_{a\in A} \mathrm{dist}(a,b)\}$.

Proof. Notice that in both cases the number of singular points is finite and $\beta_0 > 0$ if X is not a finite set. If X is a finite set the proof is trivial.

$(1 \Rightarrow 2)$ Expansiveness easily implies that if $\gamma > 0$ is smaller than an expansive constant then for all $x \notin \text{Sing}$ there exists $t \in \mathbb{R}$ such that $\phi_t(x) \notin B_\gamma(\text{Sing})$.

Fix $\varepsilon > 0$. Now we will show that there exist $\beta > 0$ such that if $\text{dist}(x, \text{Sing}) \geq \gamma$ and $\text{diam}(\phi_{[0,s]}(x)) < \beta$ then $|s| < \varepsilon$. By contradiction, suppose that there exist $x_n \notin B_\gamma(\text{Sing})$ and $t_n > \varepsilon$ such that $\text{diam}(\phi_{[0,t_n]}(x_n)) \to 0$. Eventually taking a subsequence we can suppose that $x_n \to z$. Notice that z is a regular point and then there exists $\varepsilon' \in (0, \varepsilon)$ such that $\phi_{\varepsilon'}(z) \neq z$. By the continuity of ϕ we have that $\phi_{\varepsilon'}(x_n)$ converges to $\phi_{\varepsilon'}(z)$. This is a contradiction because $\text{diam}(\phi_{[0,t_n]}(x_n)) \to 0$.

For a value β as in the previous paragraph there exist an expansive constant δ, by hypothesis. We will show that δ is an expansive constant for the value ε fixed before. Suppose that $\text{dist}(\phi_{h(t)}(y), \phi_t(x)) < \delta$ for all $t \in \mathbb{R}$, some $x, y \in X$ and $h \in \mathcal{H}_0^+(\mathbb{R})$. Without loss of generality we can suppose that $x \notin \text{Sing}$. Let $t_0 \in \mathbb{R}$ be such that $\phi_{t_0}(x) \notin B_\gamma(\text{Sing})$. If we consider $h' \in \mathcal{H}_0^+(\mathbb{R})$ given by $h'(t) = h(t + t_0) - h(t_0)$ we have that

$$\text{dist}(\phi_{h'(t)}(\phi_{h(t_0)}(y)), \phi_t(\phi_{t_0}(x))) =$$
$$\text{dist}(\phi_{h(t+t_0)-h(t_0)}(\phi_{h(t_0)}(y)), \phi_t(\phi_{t_0}(x))) =$$
$$\text{dist}(\phi_{h(t+t_0)}(y), \phi_{t+t_0}(x)) < \delta$$

for all $t \in \mathbb{R}$. We conclude by hypothesis that $\text{dist}_\phi(\phi_{h(t_0)}(y), \phi_{t_0}(x)) < \beta$. Since $\phi_{t_0}(x) \notin B_\gamma(\text{Sing})$, there exists $s \in (-\varepsilon, \varepsilon)$ such that $\phi_{h(t_0)}(y) = \phi_{t_0+s}(x)$ and the proof ends.

$(2 \Rightarrow 1)$ Given any $\beta > 0$, without loss of generality we suppose $\beta \in (0, \beta_0)$. We apply Lemma 1.2 to some value $\beta' \in (0, \beta)$ and we have the associated value $\delta' > 0$. Let $\delta'' \in (0, \delta')$. By the continuity of the flow there exists $\varepsilon > 0$ such that

$$(2) \qquad \text{if } \text{diam}(\phi_{[a,b]}(x)) < \beta' \text{ then } \text{diam}(\phi_{[a-\varepsilon,b+\varepsilon]}(x)) < \beta$$

Assuming that ε and δ'' are sufficiently small we can suppose that

$$(3) \qquad \text{if } \text{dist}(x, y) < \delta'' \text{ then } \text{dist}(\phi_s(x), y) < \delta' \text{ for all } s \in (-\varepsilon, \varepsilon)$$

By hypothesis there exists an expansive constant δ''' associated with ε. We will show that any positive value $\delta < \min\{\delta', \delta'', \delta'''\}$ is an expansive constant for the value β fixed before. Suppose that

$$(4) \qquad \text{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta, \text{ for all } t \in \mathbb{R} \text{ and some } h \in \mathcal{H}_0^+(\mathbb{R})$$

Then by hypothesis there exist $s, t_0 \in \mathbb{R}$ such that $\phi_{h(t_0)}(x) = \phi_{t_0+s}(y)$ and $|s| < \varepsilon$. Define $z = \phi_{h(t_0)}(x)$ and $g(t) = h(t_0 + t) - h(t_0)$. By (4)

we have that $\text{dist}(\phi_{g(t)}(z), \phi_{t-s}(z)) < \delta$ for all $t \in \mathbb{R}$, that is because

$$\phi_{g(t)}(z) = \phi_{h(t_0+t)-h(t_0)}(\phi_{h(t_0)}(x)) = \phi_{h(t_0+t)}(x) \text{ and}$$
$$\phi_{t-s}(z) = \phi_{t-s}(\phi_{h(t_0)}(x)) = \phi_{t+t_0}(\phi_{h(t_0)-s-t_0}(x)) = \phi_{t+t_0}(y)$$

Since $|s| < \varepsilon$ if we consider (3) we have that $\text{dist}(\phi_{g(t)}(z), \phi_t(z)) < \delta'$ for all $t \in \mathbb{R}$, because $\delta < \delta''$. Then applying Lemma 1.2 we have that

$$\text{dist}_\phi(\phi_{g(-t_0)}(z), \phi_{-t_0}(z)) < \beta'$$

This points are x and $\phi_s(y)$ respectively. Finally, by condition (2) we have that $\text{dist}_\phi(x, y) < \beta$. $\qquad\square$

Remark 1.4. Notice that in the previous proof (2⇒1) we have shown the following result: if ϕ has a finite number of singularities and $\beta_0 > 0$ then for all $\beta > 0$ there exist $\varepsilon, \delta > 0$ such that if $\text{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$ with $x, y \in X$, $h \in \mathcal{H}_0^+(\mathbb{R})$ and $\phi_{h(t_0)}(x) = \phi_{t_0+s}(y)$ for some $t_0 \in \mathbb{R}$ and $s \in (-\varepsilon, \varepsilon)$ then $\text{dist}_\phi(x, y) < \beta$.

The equivalence of the definitions of expansive flow considered in [1] and k^*−expansive given in [11], in the lack of singular points, was shown in [16].

2. WANDERING POINTS

In this section we shall prove that expansive flows on surfaces do not present wandering points. We consider a continuous flow ϕ acting on a compact surface S. First we will recall some basic tools.

Consider an embedded segment $l \subset S$. We say that l is a *local cross section* of time $\tau > 0$ for the flow if ϕ maps $[-\tau, \tau] \times l$ homeomorphically onto $\phi_{[-\tau,\tau]}(l)$. If $a, b \colon l \to (0, \tau)$ are continuous then the set $U = \{\phi_t(x) : x \in l \text{ and } t \in (-a(x), b(x))\}$ is called a *flow box*. In [17] it is shown that every regular point belongs to a local cross section.

Lemma 2.1. *If the flow ϕ presents a finite number of singularities then $S \setminus Sing = \cup_{i=1}^{+\infty} U_i$ where:*

- *each U_i is a flow box,*
- *each compact subset of $S \setminus Sing$ is contained in finitely many flow boxes U_i,*
- *if $i \neq j$ then $U_i \cap U_j \subset \partial U_i \cap \partial U_j$*

Proof. See Proposition 4.3 of [5]. $\qquad\square$

As usual we define the ω-limit set of a point x as the $\omega(x) = \{a \in S : \exists t_n \to +\infty / \phi_{t_n}(x) \to a\}$.

Lemma 2.2. *Let $l = [a, b]$ and l' be two compact local cross sections and $\tau \colon [a, b] \to \mathbb{R}$ be a continuous function such that $\phi_{\tau(x)}(x) \in l'$ for all $x \in [a, b]$ and $\lim_{x \to b} \tau(x) = +\infty$. Then $\omega(b) \subset Sing$.*

Proof. By contradiction suppose that there exist a regular point $y \in \omega(b)$. We can assume that $y \notin l \cup l'$. Consider a compact local cross section j such that $y \in j$ and since l and l' are compact we can suppose that $j \cap l = j \cap l' = \emptyset$. For all $x \in [a, b)$ consider the set $T_x = \{t \in [0, \tau(x)] : \phi_t(x) \in j\}$ and define $N(x) \in \mathbb{Z}$ as the number of points in T_x. The continuity of ϕ implies that N is continuous at x if the points in $\phi_{T_x}(x)$ are not in the boundary of the interval j. Therefore N has at most two points of discontinuity and then N is bounded. On the other hand there is an infinite number of values of $t > 0$ such that $\phi_t(b) \in j$ because $y \in \omega(b)$. We also have that $\tau(x) \to +\infty$ as $x \to b$, thus $\lim_{x \to b} N(x) = \infty$, which is an absurd. $\qquad\square$

If p is a singularity and x is a regular point such that $\phi_t(x) \to p$ as $t \to +\infty$ (resp. $t \to -\infty$) then the orbit of x is said to be a *stable* (resp. *unstable*) *separatrix* of p. We say that a point $x \in S$ is *stable* if for all $\varepsilon > 0$ there exist $\delta > 0$ such that if $y \in B_\delta(x)$ then $\mathrm{dist}(\phi_t(y), \phi_t(x)) < \varepsilon$ for all $t > 0$. A point $x \in S$ is *asymptotically stable* if it is stable and there exists $r > 0$ such that if $y \in B_r(x)$ then $\mathrm{dist}(\phi_t(y), \phi_t(x)) \to 0$ as $t \to +\infty$.

Lemma 2.3. *If a singularity of ϕ presents an infinite number of separatrices then at least one of them is asymptotically stable.*

Proof. It follows by the arguments in [6] (pag. 161). $\qquad\square$

We say that $x \in S$ is a *wandering* point if there exist a neighborhood U of x and $\tau > 0$ such that $\phi_t(U) \cap U = \emptyset$ for all $t > \tau$. We denote by $\Omega(\phi)$ the set of non-wandering points.

Proposition 2.4. *If ϕ is expansive then $\Omega(\phi) = S$.*

Proof. By contradiction suppose that there are wandering points. Then it is easy to see that there exists a local cross section l such that $\phi_t(l) \cap l = \emptyset$ for all $t \neq 0$. We will show that there exists a subsegment $l' \subset l$ that contradicts the expansiveness for positive values of t and then arguing the same for the opposite flow, we arrive to a contradiction. Fix an expansive constant $\delta > 0$. We study two possible cases.

Case 1. Suppose that there is an infinite number of points in stable separatrices in l. Since there is a finite number of singularities, there exists $p \in \mathrm{Sing}$ with infinitely many separatrices meeting l. By Lemma 2.3 one of them is asymptotically stable. Take $x \in l$ in a asymptotically stable separatrix. Therefore there exist $\mu > 0$ such that if $\mathrm{dist}(x, y) < \mu$ then $\mathrm{dist}(\phi_t(x), \phi_t(y) < \delta/2$ for all $t \geq 0$. Finally take $l' \subset l \cap B_\mu(x)$. For all $y, z \in l'$ it holds that $\mathrm{dist}(\phi_t(y), \phi_t(z)) < \delta$ for all $t \geq 0$.

Case 2. Suppose that there is a finite number of stable separatrices meeting l. Then there exist $l' \subset l$ such that for all $x \in l'$ it holds $\omega(x)$ is not a singular point. In this case we say that l' satisfies condition (1). For each singularity p consider a disc D_p around p of diameter less than

$\delta/2$. Take the covering $\{U_i : i \in \mathbb{N}\}$ of $S \setminus \operatorname{Sing}$ given by Lemma 2.1. Making a subdivision of each U_i we can suppose that $\operatorname{diam}(U_i) < \delta/2$ for all $i \in \mathbb{N}$. Reordering the flow boxes we can suppose that the sets D_p and $U_1, \ldots, U_N$ make a finite covering of the surface. Considering l' smaller we can also suppose that orbits of the orbit segments in ∂U_i do not meet l', call it condition (2). Let a_i be the local cross section of ∂U_i where the flow enters to the flow box. We can also suppose that l' do not meet any a_i.

Fix $x, y \in l'$ and define $A = \cup_{i=1}^{N} a_i$. Hence there exist two increasing and divergent sequences $t_n, s_n \in \mathbb{R}^+$ such that $\{t_n : n \in \mathbb{N}\} = \{t \in \mathbb{R}^+ : \phi_t(x) \in A\}$ and $\{s_n : n \in \mathbb{N}\} = \{t \in \mathbb{R}^+ : \phi_t(y) \in A\}$. Let $I, J \colon \mathbb{N} \to \{1, \ldots, N\}$ the functions given by: $\phi_{t_n}(x) \in a_{I(n)}$ and $\phi_{s_n}(y) \in a_{J(n)}$.

By induction we will show that $I = J$. *Base case.* Let $l'' = [x, y] \subset l'$ be the segment determined by the points x and y and

$$X = \{z \in l'' : \exists t > 0 / \phi_{[0,t]}(z) \cap a_{J(1)} = \emptyset \text{ and } \phi_t(z) \in a_{I(1)}\}$$

We will prove that $y \in X$. We have that $x \in X$. Also X is an open set in l'' by condition (2). Let Y be the connected component of X that contains x. Then Y is an interval. Let u be the extreme of Y, $u \neq x$. We will show that $u \in Y$ and therefore $u = y$ and $y \in Y$. By the definition of X we have that there exist a continuous function $T \colon [x, u) \to \mathbb{R}^+$ such that $\phi_{T(z)}(z) \in a_{I(1)}$ and $\phi_{[0,T(z)]}(z) \cap a_{J(1)} = \emptyset$. By condition (1) we have that $T(z)$ can not converge to ∞ as $z \to u$, because of Lemma 2.2. Hence there exist $z_n \in [x, u)$ such that $z_n \to u$ and $T(z_n) \to T_u$. Then $\phi_{T(z_n)}(z_n) \to \phi_{T_u}(u)$. On the other side $\phi_{[0,T_u]}(u)$ do not meet $a_{J(1)}$, because condition (2) implies that if n is sufficiently big, $\phi_{[0,T(z_n)]}(z_n) \cap a_{J(1)} \neq \emptyset$. Then we have that $u \in Y$ and $I(1) = J(1)$. *Inductive step.* In order to repeat the previous argument note that if $I(k) = J(k)$ for all $k = 1, \ldots, K$, if we define $l_K = [\phi_{t_K}(x), \phi_{s_K(y)}] \subset a_{I(K)}$, then l_K also verifies conditions (1) and (2).

Let $h \colon [0, +\infty) \to [0, +\infty)$ be such that $h(0) = 0$, $h(t_n) = s_n$ for all $n \in \mathbb{N}$ and extended linearly. In this way h in a homeomorphism. We will show that $\operatorname{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta$ for all $t \geq 0$. Fix $n \in \mathbb{N}$ and let

$$t^* = \sup\{t \geq t_n : \phi_{[t_n, t]}(x) \subset U_i \text{ para algún } i = 1, \ldots N\}.$$

Let i_0 be such that $\phi_{t^*}(x) \in b_{i_0}$. If $t^* \geq t_{n+1}$ then both segments $\phi_{[t_n, t_{n+1}]}(x)$ and $\phi_{[s_n, s_{n+1}]}(y)$ are contained in U_{i_0}. Therefore

$$\operatorname{dist}(\phi_t(x), \phi_{h(t)}(y) < \delta$$

for all $t \in [t_n, t_{n+1}]$, because $\operatorname{diam}(U_{i_0}) < \delta/2$. Suppose that $t^* < t_{n+1}$. Then $x^* = \phi_{t^*}(x) \in D_p$ for some $p \in \operatorname{Sing}$. Hence there exist $s^* \geq s_n$ such that $y^* = \phi_{s^*}(y) \in b_{i_0}$ and $\phi_{[s_n, s^*]}(y)$ are contained in U_{i_0}. Then $[x^*, y^*] \subset b_i$ is contained in D_p. Therefore $\phi_{[t^*, t_{n+1}]}(x)$ and $\phi_{[s^*, s_{n+1}]}(y)$ are contained in D_p. It only rest to note that $\operatorname{diam}(D_p \cup U_{i_0}) < \delta$. $\square$

3. Singularities of saddle type

An isolated singularity is said to be of *saddle type* if it has a positive and finite number of separatrices.

Proposition 3.1. *If ϕ is expansive then its singularities are of saddle type.*

Proof. Expansiveness easily implies that the singularities are isolated. It is easy to see that if $p \in \text{Sing}$ is stable then p is asymptotically stable. But there are no asymptotically stable singularity because by Proposition 2.4 there are no wandering points. So p is not stable and we can apply the arguments in the proof of Lemma 8.1 in [6] to conclude that there exists at least one separatrix associated to p. Again by Proposition 2.4 there are no wandering points, thus there can not be an asymptotically stable separatrix. Then by Lemma 2.3 there is a finite number of separatrices. $\qquad\square$

Now we will show the local behavior of the flow near a singularity of saddle type $p \in \text{Sing}$. Consider an embedded disc $D \subset S$ such that $\text{Sing}\cap\text{clos}(D) = \{p\}$. If D is small enough we can suppose that there is no separatrix of p contained in $\text{clos}(D)$. Using the Poincaré-Bendixon Theorem we have that if $x \in D$ and $\phi_{\mathbb{R}^+}(x) \subset D$ or $\phi_{\mathbb{R}^-}(x) \subset D$ then x belongs to a separatrix of p. Moreover if $\phi_{\mathbb{R}}(x) \subset D$ then $x = p$. Now consider the set

$$U = \{x \in D : \phi_{\mathbb{R}^+}(x) \nsubseteq \text{clos}(D) \text{ and } \phi_{\mathbb{R}^-}(x) \nsubseteq \text{clos}(D)\}$$

It is easy to see that U is an open set. The connected components of U will be called *hyperbolic sectors*. Since p has a finite number of separatrices it is easy to see that there is a finite number of hyperbolic sectors in D. The *index* of a singularity of saddle type is $1\text{-}n_h/2$, being n_h the number of hyperbolic sectors in D_p. If $p \in \partial S$ similar considerations can be made and we define the index of p as $1 - n_h$. In [6] Theorem 9.1 it is shown that this definition coincides with the usual notion of index for singular points not lying in the boundary of the surface.

As usual we define the *stable* and the *unstable set* of a singular point p as

$$W^s(p) = \{x \in S : \phi_t(x) \to p \text{ as } t \to +\infty\}$$
$$W^u(p) = \{x \in S : \phi_t(x) \to p \text{ as } t \to -\infty\}$$

respectively.

Definition 3.2. If $p \in \text{Sing}$ is of saddle type we say that an embedded disc (or half disc if $p \in \partial S$) D_p is an *adapted* neighborhood of p if $\overline{D}_p \cap \text{Sing} = \{p\}$ and $\partial D_p = \cup_{i=1}^{i=n}(\alpha_i \cup \beta_i \cup \gamma_i^+ \cup \gamma_i^-)$. Where α_i and β_i are orbit segments and $\gamma_i^\pm$ are local cross sections, such that there exist $x_i \in \gamma_i^+$ and $y_i \in \gamma_i^-$ such that $W^u(p) \setminus \{p\} = \bigcup_{i=0}^{n} \phi_{\mathbb{R}}(x_i)$, $W^s(p) \setminus \{p\} = \cup_{i=0}^{n}\phi_{\mathbb{R}}(y_i)$. See figure 1.
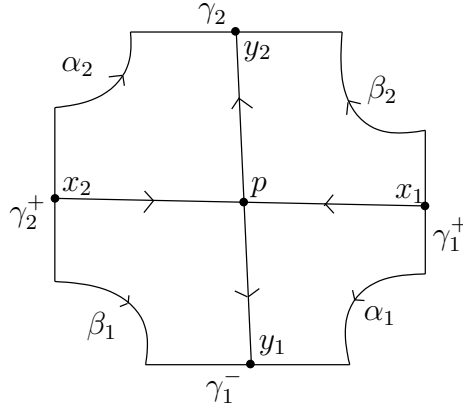
FIGURE 1. An adapted neighborhood of a singularity of index -1.

Considering the analysis of hyperbolic sectors made in [6] (pag. 167) we have that every singularity of saddle type has an adapted neighborhood. The following result shows how a singularity of negative index gives a kind of local expansiveness.

Lemma 3.3. *Suppose that p is a singularity of saddle type of negative index and take an adapted neighborhood D_p of p. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in D_p$ and $\operatorname{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ and for some $h \in \mathcal{H}_0^+(\mathbb{R})$, then x and y belongs to the same hyperbolic sector or there exist $t_0, s \in \mathbb{R}$ such that $\phi_{h(t_0)}(y) = \phi_{t_0+s}(x)$ and $|s| < \varepsilon$.*

Proof. We give the proof for the case $p \notin \partial S$, the other case is similar. In the notation of the Definition 3.2 define $W = \{p\} \cup_{i=1}^{n_h/2} \phi_{\mathbb{R}^+}(x_i) \cup_{i=1}^{n_h/2} \phi_{\mathbb{R}^-}(y_i)$. It is the complement of the hyperbolic sectors in D_p. Consider $\delta > 0$ such that the following conditions hold:

(1) if $z \in W$ and $z \in B_\delta(x_i)$ or $z \in B_\delta(y_i)$ then there exist $s \in (-\varepsilon, \varepsilon)$ such that $\phi_s(z) = x_i$ ($\phi_s(z) = y_i$) and
(2) for all hyperbolic sector U_j and $\gamma_i^\pm$ if $\operatorname{dist}(\gamma_i^\pm, U_j) < \delta$ then $\operatorname{dist}(\gamma_i^\pm, U_j) = 0$.

Now we study three possible cases.

Case 1: $x \in W$, $y \notin W$. Since $y \notin W$ there exist $t' > 0$ such that $\phi_{h(t')}(y) \in \gamma_i^+$ for some i. Without loss of generality suppose that $\lim_{t \to +\infty} \phi_t(x) = p$. Since the index of p is negative, there exist a hyperbolic sector U_k such that $\phi_t(x) \in \partial U_k$ for all $t \geq 0$ and $\operatorname{dist}(U_k, \gamma_i^+) \geq \delta$. Therefore $\operatorname{dist}(\phi_{h(t')}(y), \phi_{t'}(x)) \geq \delta_1$.

Case 2: $x, y \in W$. If $x = y = p$ the case is trivial. If $x, y \in \phi_{\mathbb{R}^+}(x_i)$ (or $\phi_{\mathbb{R}^-}(y_i)$, the argument is the same) we conclude by condition (1). If x and y are in different connected components of $W \setminus \{p\}$ we conclude again by condition (2).

Case 3: $x, y \notin W$ but not in the same hyperbolic sector. We conclude by condition (2). $\qquad\square$

4. Smooth models for expansive flows

In this section we will show that continuous expansive flows on C^∞ compact surfaces are topologically equivalent to flows of C^∞ class. Two flows $\phi\colon \mathbb{R} \times S \to S$ and $\phi'\colon \mathbb{R} \times S' \to S'$ are *topological equivalent* if there exists a homeomorphisms from S to S' preserving the orbits.

A regular point $x \in S$ is said to be *periodic* if there exists $t > 0$ such that $x = \phi_t(x)$. We do not consider singularities as periodic points.

Proposition 4.1. *Expansive flows on compact surfaces do not have periodic points.*

Proof. By contradiction suppose that $x \in S$ is a periodic point. Take a transversal l trough x and consider the first return map $f\colon l' \subset l \to l$ defined in a smaller section l'. Expansive implies that the map f can not have fixed points close to x. Therefore there exist wandering points, contradicting Proposition 2.4. $\qquad\square$

An embedded circle $\gamma \subset S$ is said to be a *transversal circle* if for all $x \in \gamma$ there exists a neighborhood U of x such that $U \cap \gamma$ is a local cross section.

Lemma 4.2. *Suppose that ϕ do not have periodic points, Sing is a finite set, the union of the stable separatrices is not dense in the surface and $\Omega(\phi) = S$. Then S is the torus and ϕ is an irrational flow.*

Proof. Since the union of the stable separatrices is not dense and Sing is a finite set there exists a cross section l without points in separatrices. There exist $x \in l$ and $t > 0$ such that $\phi_t(x) \in l$ because $\Omega(\phi) = S$. As there are no periodic points we have that $\phi_t(x) \neq x$. Let $l^* = \{y \in l : \phi_{\mathbb{R}+}(y) \cap l \neq \emptyset\}$ and consider $f\colon l^* \to l$ the first return map. If the surface is not orientable then f may reverse orientation in some connected component of l^*. But applying Lemma 2.2 we can easily show that there exist a periodic point. So we can suppose that f preserves orientation and with standard techniques it can be shown that there exists a transversal circle γ. Moreover we can suppose that $\gamma \subset \phi_{\mathbb{R}}(l)$. Therefore there is no separatrix meeting γ. Now we consider γ^* as the set of points of γ that returns to γ. Since there are no points in separatrices in γ we can apply Lemma 2.2 to conclude that $\gamma^* = \gamma$. So the first return map of $g\colon \gamma \to \gamma$ is an homeomorphism. Since $\Omega(\phi) = S$ and there are no periodic points we have that g is conjugated to an irrational rotation on the circle γ. Then ϕ is conjugated to a suspension of g and S is the torus. $\qquad\square$

Singularities play an important role in the subject of expansive flows on surfaces. It is shown in the following result.

Proposition 4.3. *If ϕ is an expansive flow on a compact surface S then the union of the stable separatrices is dense in S. In particular $Sing \neq \emptyset$.*

Proof. Arguing by contradiction, suppose that the union of the stable separatrices is not dense in S. By Proposition 4.1 ϕ has no periodic points and by Proposition 2.4 we have that $\Omega(\phi) = S$. So we can apply Lemma 4.2 and conclude that ϕ is a suspension of an irrational rotation of the circle. In particular there are no singularities and then ϕ is expansive in the sense of [1]. But this is a contradiction because there are no expansive homeomorphisms on the circle (Theorem 6 in [1] shows that a homeomorphism is expansive if and only if its suspensions are expansive flows). $\square$

In [4] it is shown that Cherry flows are topologically equivalent to C^∞ flows. The definition of Cherry flows given in [4] is the following. A continuous flow $\phi\colon \mathbb{R} \times S \to S$ is a *Cherry flow* if the following conditions hold.

(1) ϕ has only finitely many singular points.
(2) Let $p_1, p_2, \ldots, p_m$ be the source singular points of ϕ and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be their basins of repulsion. Then each λ_j contains a unique trajectory θ_j connecting p_j to another (unique) fixed point $q_j \in \partial\lambda_j$.
(3) There are finitely many points $x_1, x_2, \ldots, x_n$ such that

$$(\cup_{i=1}^m \lambda_i) \cup (\cup_{j=1}^n \phi_{\mathbb{R}^+}(x_j))$$

is dense in S.

Proposition 4.4. *Expansive flows are Cherry flows.*

Proof. Expansiveness easily implies that Sing is a finite set. By Proposition 2.4 we have that there are no source singular points, so item 2 of the definition of Cherry flow need not to be verified. In order to check item 3, notice that Proposition 3.1 implies that there is a finite number of stable separatrices and by Proposition 4.3 their union is dense in S. $\square$

Theorem 4.5. *Expansive flows on compact surfaces are topologically equivalent to C^∞ flows.*

Proof. Just apply the results in [4] and Proposition 4.4. $\square$

5. Removable singularities

In that section we deal with singularities of index zero. We show that they can be *removed* or *added* without loss of expansiveness.

Definition 5.1. Consider two flows ϕ and ψ defined on the same surface S and let $p \in \mathrm{Sing}$ be a singular points of index 0 of ϕ. Suppose that:

  (1) p is non-singular for ψ,
  (2) for all $x \notin \psi_{\mathbb{R}}(p)$ it holds $\phi_{\mathbb{R}}(x) = \psi_{\mathbb{R}}(x)$ and
  (3) the direction of both flows coincide on each orbit.

In that case we say that ψ *removes* the singularity p of ϕ or equivalently ϕ *adds* a singularity to ψ.

It is easy to see that every singular point of index 0 can be removed. Conversely, given any regular point $x \in S$ there exist a flow that adds a singularity of index 0 at x. In fact the only singular points that can be added or removed are those of saddle type of index 0. That kind of singular points are also called *impassable grains* or *fake saddles* in the literature.

On surfaces one can remove or add singularities without loosing expansiveness. The following proof works on compact metric spaces. The converse will be shown on surfaces in Theorem 5.4

Proposition 5.2. *If ψ is expansive and removes a singular point of ϕ then ϕ is expansive.*

Proof. Let $p \in X$ be the singular point that ψ removes from ϕ. Consider a local cross section l through p and take $t^* > 0$ such that $\psi|_{(-t^*,t^*) \times l}$ is a homeomorphism. Let $U = \psi_{(-t^*,t^*)}(l)$ be a flow box. For any $\beta > 0$ consider an expansive constant $\delta > 0$ of ψ with $\delta < \frac{1}{2} \mathrm{dist}(p, \partial U)$. Suppose that $\mathrm{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ for some $h \in \mathcal{H}_0^+(\mathbb{R})$. Now we study the only three possible cases.

Case 1. Suppose that $x, y \notin \psi_{\mathbb{R}}(p)$. In that case there exist $g_x, g_y \in \mathcal{H}_0^+(\mathbb{R})$ such that $\phi_t(x) = \psi_{g_x(t)}(x)$ and $\phi_t(y) = \psi_{g_y(t)}(y)$ for all $t \in \mathbb{R}$. Then $\mathrm{dist}(\psi_t(x), \psi_{g_x^{-1} \circ h \circ g_y(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ being $g_x^{-1} \circ h \circ g_y \in \mathcal{H}_0^+(\mathbb{R})$. Therefore $\mathrm{dist}_\psi(x, y) < \beta$ by hypothesis. Since x and y are not in $\psi_{\mathbb{R}}(p)$ we have that $\mathrm{dist}_\psi(x, y) = \mathrm{dist}_\phi(x, y) < \beta$.

Case 2. Consider $x \in \psi_{\mathbb{R}}(p)$ and $y \notin \psi_{\mathbb{R}}(p)$. Without loss of generality we can suppose that $\phi_t(x) \to p$ as $t \to +\infty$. Notice that for all $t' > 0$ there exist $t > t'$ such that $\phi_t(y) \notin U$. This contradicts that $\mathrm{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta < \frac{1}{2} \mathrm{dist}(p, \partial U)$ for all $t \in \mathbb{R}$.

Case 3. Assume that $x, y \in \psi_{\mathbb{R}}(p)$. It easily implies that $\mathrm{dist}_\psi(x, y) = \mathrm{dist}_\phi(x, y) < \beta$. $\square$

Lemma 5.3. *Suppose that the singularities of ϕ are of saddle type. Then for all $\beta > 0$ there exists $\delta > 0$ such that if $\mathrm{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta$ for all $t \in \mathbb{R}$ with $\mathrm{dist}_\phi(x, y) \geq \beta$ and $h \in \mathcal{H}_0^+(\mathbb{R})$ then there exists a local cross section l containing x and y such that every separatrix meeting l between x and y is associated to a singularity of index 0.*

Proof. Consider any value of $\beta > 0$ given. Since singularities of saddle type are isolated by definition, Sing is a finite set and

$$\beta_0 = \inf\{\operatorname{diam}(\phi_\mathbb{R}(x)) : x \notin \operatorname{Sing}\} > 0.$$

So we can apply the result stated in Remark 1.4 and we have the constants ε and δ'' given there. For each singular point p take an adapted neighborhood D_p. Suppose that $\operatorname{diam}(D_p) < \beta$ for all $p \in$ Sing. For each singular point p take the value δ_p given by Lemma 3.3 (considering the value of ε already fixed). Consider the flow boxes U_i given by Lemma 2.1. Suppose that $\mathcal{C}$ is a finite covering of S made with D_p, $p \in$ Sing, and a finite number of flow boxes U_i, $i = 1, \ldots, N$. Eventually subdividing the flow boxes we can assume that $\operatorname{diam}(U_i) < \beta$ for all $i = 1, \ldots, N$. We can also suppose that the intersection of any two open sets in $\mathcal{C}$ is connected or empty. Also take δ' such that if the diameter of $X \subset S$ is less than δ', then X is contained in some open set of $\mathcal{C}$. Finally define $\delta = \min\{\delta', \delta'', \delta_p\}_{p \in \operatorname{Sing}}$.

To show that δ works suppose that

$$\operatorname{dist}(\phi_t(x), \phi_{h(t)}(y)) < \delta$$

for all $t \in \mathbb{R}$ and some $h \in \mathcal{H}_0^+(\mathbb{R})$.

First we will show that there exists a local cross section through x and y. Since $\operatorname{dist}(x, y) < \delta \leq \delta'$ we have that $x, y \in U$ for some $U \in \mathcal{C}$. Since $\operatorname{diam}(U) < \beta$ and $\operatorname{dist}_\phi(x, y) \geq \beta$ we have that x and y are not in an orbit segment contained in U. If U is a flow box or an adapted neighborhood of a singular point of index 0 the prove is trivial. Suppose that $x, y \in D_p$ for some $p \in$ Sing of negative index. Again, if x and y are in the same hyperbolic sector the proof is easy. Supposing that this is not the case we will arrive to a contradiction. Applying Lemma 3.3 we have that there exist $t_0, s \in \mathbb{R}$ such that $\phi_{h(t_0)}(y) = \phi_{t_0+s}(x)$ and $|s| < \varepsilon$. Then we can apply the result stated in Remark 1.4 to conclude that $\operatorname{dist}_\phi(x, y) < \beta$ arriving to a contradiction.

The previous argument also shows that if $\phi_t(x), \phi_{h(t)}(y) \in D_p$ for some $t \in \mathbb{R}$ and $p \in$ Sing of negative index then $\phi_t(x)$ and $\phi_{h(t)}(y)$ belongs to the same hyperbolic sector in D_p. In particular x and y are not in separatrices. Moreover $\phi_t(x)$ and $\phi_{h(t)}(y)$ can be connected with a local cross section for all $t \in \mathbb{R}$.

Now consider an increasing divergent sequence $\{t_n\}_{n \in \mathbb{N}}$, $t_0 = 0$, such that for all $n \in \mathbb{N}$ there exist $i(n)$ such that

$$\phi_{[t_n, t_{n+1}]}(x) \cup \phi_{[h(t_n), h(t_{n+1})]}(y) \subset V_{i(n)} \in \mathcal{C},$$

where $V_{i(n)}$ may be a flow box or an adapted neighborhood D_p. Consider in $V_{i(n)} \cap V_{i(n+1)}$ a local cross section l_n that connects $x_n = \phi_{t_n}(x)$ and $y_n = \phi_{h(t_n)}(y)$. Now take any point $z \in l_0 = [x, y] \subset l$. Suppose that z do not belong to a separatrix of a singularity of index 0. Consider a sequence $s_n \in \mathbb{R}$ such that $z_n = \phi_{s_n}(z) \in l_n$ for all $n \geq 0$. It

exist because if x_n and y_n are in D_p both points are in the same hyperbolic sector. If z_n is convergent then x_n and y_n should be convergent too. But this is a contradiction because x and y are not in separatrices. Then z_n is not convergent and z do not belong to any separatrix. $\square$

Theorem 5.4. *Suppose that ψ removes a singular point of ϕ. Then ψ is expansive if and only if ϕ is expansive.*

Proof. By Proposition 5.2 it only rest to show the converse. By contradiction suppose that ψ is not expansive and take $\beta > 0$, $x_n, y_n \in S$, $h_n \in \mathcal{H}_0^+(\mathbb{R})$ and $\delta_n \to 0$ such that $\mathrm{dist}(\psi_t(x_n), \psi_{h_n(t)}(y_n)) < \delta_n$ for all $t \in \mathbb{R}$ and $\mathrm{dist}_\psi(x_n, y_n) > \beta$. We study the possible cases.

Case 1: $x_n, y_n \notin \psi_\mathbb{R}(p)$ for infinite values of n. It contradicts the expansiveness of ϕ as was explained in case (1) of Proposition 5.2.

Case 2: $x_n \in \psi_\mathbb{R}(p)$ and $y_n \notin \psi_\mathbb{R}(p)$ for infinite values of n. Let $t_n \in \mathbb{R}$ be such that $\psi_{t_n}(x_n) = p$ and define $z_n = \psi_{h_n(t_n)}(y_n)$. Now consider $s = t - t_n$ and $g_n \in \mathcal{H}_0^+(\mathbb{R})$ defined as $g_n(s) = h_n(s + t_n) - h_n(t_n)$. It is easy to check that $\mathrm{dist}(\psi_s(p), \psi_{g_n(s)}(z_n)) < \delta_n$ for all $s \in \mathbb{R}$. Taking $s = 0$ we see that $z_n \to p$ since $\delta_n \to 0$. Therefore we can assume that $z_n \in U$ for all n and it is easy to see that, eventually taking a subsequence of z_n, there exist $\beta' > 0$ such that $\mathrm{dist}_\phi(z_{n_1}, z_{n_2}) > \beta'$ if $n_1 \neq n_2$. So $\mathrm{dist}(\psi_{g_{n_1}(s)}(z_{n_1}), \psi_{g_{n_2}(s)}(z_{n_2})) < \delta_{n_1} + \delta_{n_2}$ for all $s \in \mathbb{R}$. Now it contradicts the expansiveness of ϕ because z_{n_1} and z_{n_2} are not in $\psi_\mathbb{R}(p)$.

Case 3: $x_n, y_n \in \psi_\mathbb{R}(p)$ for infinite values of n. We can suppose that $x_n = p$ for all n. Let l be a transversal section of ψ through p. Without loss of generality we can assume that $y_n \in l$ for all n. In order to apply Lemma 5.3 we must notice that by Proposition 3.1 the singularities of ψ are of saddle type. Then we have that if n is big enough, in the subsegment $l' \subset l$ limited by y_n and p there are no separatrices of no removable singularities. Let ψ' be a flow that removes every removable singular point of ψ. Then we have that there are not separatrices of ψ' in l'. Since ϕ is expansive we have that: 1) by Proposition 2.4, $\Omega(\phi) = S$ and so $\Omega(\psi') = S$; 2) by Proposition 4.1 ϕ do not have periodic points and then ψ' has the same property and 3) the singular set of ψ' is finite. Therefore we can apply Lemma 4.2 to conclude that ψ' is the suspension of an irrational rotation. On the other hand ϕ is obtained from ψ' adding singularities, and so it is easy to see that ϕ is not expansive. $\square$

Notice that the previous proof only used the fact that S is a surface, instead of being a compact metric space, just in case 3.

Remark 5.5. Proposition 4.1 and Theorem 5.4 shows that expansive flows presents at least one singularity of negative index on each boundary component.

6. Characterizations

In this section S denotes a compact surface and $\phi\colon \mathbb{R} \times S \to S$ a continuous flow. The main result of this section is Theorem 6.1 that gives a characterization of expansive flows on surfaces. Theorem 5.4 explains the generality lost supposing that the flow do not has singularities of index 0.

Theorem 6.1. *Let ϕ be a flow without singularities of index 0 on a compact surface S. Then the following statements are equivalent:*

(1) *ϕ is expansive,*
(2) *Sing is a finite and non empty set, $\Omega(\phi) = S$ and ϕ has no periodic points and*
(3) *the singularities are of saddle type and the union of its separatrices is dense in the surface.*

Proof. $(1) \Rightarrow (2)$ It is a consequence of Propositions 4.3, 2.4 and 4.1.

$(2) \Rightarrow (3)$ By Lemma 2.3 we have that the singularities are of saddle type because by hypothesis there are no wandering points. The union of the separatrices is dense because Proposition 4.3.

$(3) \Rightarrow (1)$ Given any $\beta > 0$, by Lemma 5.3 we have the constant δ given there. Such δ is an expansive constant by Lemma 5.3 because the union of the separatrices is dense and there are no singularities of index 0. $\qquad\square$

Notice that expansiveness implies (2) and (3) with removable singularities too. But if one adds a finite number of singularities of index 0 to an irrational flow on the torus we get a flow that satisfies (2) and (3) but it is not expansive. In fact this is the only possible case.

To state a characterization without restrictions we introduce the following concept. Let $p \in S$ be a singular point of negative index of ϕ. Consider ϕ' that removes all the singularities of index 0 from ϕ. A ϕ'-separatrix of p is called an *extended separatrix* of p for the flow ϕ.

Theorem 6.2. *A flow ϕ on a compact surface is expansive if and only if the singularities are of saddle type and the union of the extended separatrices of the singularities of negative index is dense in the surface.*

Proof. Consider ϕ' the flow obtained from ϕ by removing every singularity of index 0. The proof follows easily from Theorem 6.1, Theorem 5.4 and the following fact: the union of the separatrices of ϕ' is equal to the union of the extended separatrices of the singularities of negative index. $\qquad\square$

Now we are going to give a characterization of the surfaces admitting expansive flows. For this we will introduce some surgery tools. Let S and S' be two compact surfaces and $\phi\colon S \times \mathbb{R} \to S$ and $\phi'\colon S' \times \mathbb{R} \to S'$ be two continuous flows. A *semi-conjugacy* from ϕ' to ϕ is a surjective

and continuous map $h \colon S' \to S$ such that $\phi_t \circ h = h \circ \phi'_t$ for all $t \in \mathbb{R}$. If h is a homeomorphism then it is called a *conjugacy*.

If $\gamma \subset S \backslash \partial S$ and $\gamma'_1, \gamma'_2 \subset \partial S'$ are saddle connections then we say that ϕ' *makes a cut along* γ if there exists a semi-conjugacy $h \colon S' \to S$ such that $h \colon S' \backslash \mathrm{clos}(\gamma'_1 \cup \gamma'_2) \to S \backslash \mathrm{clos}(\gamma)$ is a conjugacy from $\phi'|_{S' \backslash \mathrm{clos}(\gamma'_1 \cup \gamma'_2)}$ to $\phi|_{S \backslash \mathrm{clos}(\gamma)}$. In that case we also say that ϕ' *glues* γ'_1 with γ'_2.

Definition 6.3. Given a flow ϕ on S we say that another flow ϕ' on S' is obtained from ϕ by a *basic operation* if it is obtained by adding o removing a singularity of index 0, by gluing saddle connections or cutting along a saddle connection.

It is easy to see that every saddle connection not contained in the boundary of the surface, can be cut. Also, every pair of saddle connections in the boundary can be glued.

Remark 6.4. We will show a special way to add a boundary with basic operations. That construction will be useful in Theorem 6.5. Take two points p and $q = \phi_t(p)$, $t > 0$, in a regular orbit and put singularities of index 0 in them. Make a cut on the saddle connection determined by p and q. Let γ and γ' be the saddle connections in the boundary connecting p and q. Put two singular points r and s of index 0 in γ. Now glue the saddle connections that starts in p. See figure 2.
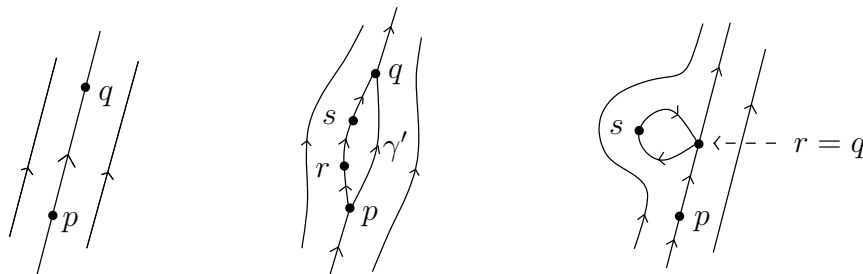

FIGURE 2. Adding a boundary.

Recall that every compact connected surface S is obtained from the sphere attaching $h \geq 0$ handles, $b \geq 0$ boundaries and $c \geq 0$ cross-cups. Denote by $S^{h,b,c}$ such surface. It is well known that if $c \geq 3$ then $S^{h,b,c}$ is homeomorphic to $S^{h+1,b,c-2}$. So we will assume that $c = 0, 1, 2$. See for example [8]

Theorem 6.5. *A compact connected surface S admits an expansive flow if and only if $h > 0$ and $h + b + c > 1$.*

Proof. (*Direct.*) Suppose that S admits an expansive flow. By Theorem 5.4 we can assume that this flow do not have removable singularities. So the index of each singular point is negative. Then the Euler's characteristic of $\mathcal{X}(S)$ is negative. Theorem 6.1 implies that expansive

flows has recurrent non-trivial orbits. If $c = 0$, i.e. S is orientable, then by the results in [14] (see Theorem 2 in [15]) we have that the genus of S is positive and then $h > 0$. In the orientable case $\mathcal{X}(S) = 2 - 2h - b$ and then $\mathcal{X}(S) < 0$ implies $2 < 2h + b$. Therefore $h + b > 1$. If $c > 0$, the non-orientable case, we can apply Theorem 4 of [15] to conclude that the genus of S is greater than 2. In the non-orientable case the genus of S equals $c + 2h$, then $c + 2h > 2$ and since $c \leq 2$ we have that $h > 0$. Now $c > 0$ and $h > 0$ implies $c + h > 1$.

(*Converse.*) The conditions $h > 0$ and $h + b + c > 0$ are equivalent with saying that S is a torus with b boundaries, h' handles and c' cross-cups attached such that $h' + b + c' > 0$. So we will show that such surfaces admits expansive flows. Given $k >$ we consider an expansive flow on the torus with k boundaries. It can be done in the following way: take an irrational flow on the torus, add k disjoint boundaries as in Remark 6.4. In this way we get an expansive flow ϕ on $S = S^{1,k,0}$ such that if γ is a component of ∂S then $\mathrm{Sing} \cap \gamma = \{p, q\}$ and there exist two regular points $a, b \in \gamma$ such that $\omega(a) = \alpha(b) = \{q\}$ and $\omega(b) = \alpha(a) = \{p\}$.

Given $h', b, c' \geq 0$ with $h' + b + c' > 0$ consider $k = 2h' + c' + b$. So $\partial S = \cup_{i=1}^{i=k} \gamma_i$ being each γ_i homeomorphic to a circle. It is easy to see that two boundaries can be glued to obtain a handle without loosing expansiveness. Also each boundary can make a cross-cup in the following way. Take $a, b, p, q \in \gamma_i$ as explained above. Identify p with q and $\phi_t(a)$ with $\phi_t(b)$ for all $t \in \mathbb{R}$.

$\square$

Remark 6.6. The compact connected surfaces that do not admit expansive flows are: the torus, the sphere with b boundaries, the projective plane with b boundaries and the Klein's bottle with b boundaries (with $0 \leq b < \infty$ in the three cases).

Theorem 6.7. *Suppose that S is a compact surface different of the torus. A flow ϕ on S is expansive if and only if $\Omega(\phi) = S$, ϕ do not has periodic orbits and Sing is finite.*

Proof. Direct. It is a consequence of Propositions 2.4 and 4.1 and the fact that expansive flow on compact spaces has a finite number of singularities.

Converse. Consider a flow ϕ' removing the singularities of index 0 from ϕ. By Theorem 5.4 we just have to prove that ϕ' is expansive. Notice that $\Omega(\phi) = \Omega(\phi')$, $\mathrm{Sing}(\phi')$ is finite and ϕ' do not have periodic orbits. Since S is not the torus we can apply Lemma 4.2 to conclude that the union of the separatrices of ϕ' is dense in the surface. Now since ϕ' has not singularities of index 0 we have that it is expansive. $\square$

Corollary 6.8. *If ϕ is an expansive flow on S and ϕ' is a flow on S' obtained from ϕ by a basic operation then ϕ' is expansive if and only if S' is not the torus.*

Proof. Direct. By Theorem 6.5, S' can not be the torus if ϕ' is expansive. *Converse.* By Theorem 5.4 expansiveness is invariant under adding or removing singularities. It is easy to see that the non-wandering set do no change if one cuts or glue saddle connections. Also, periodic orbits can not be created with basic operations if ϕ is expansive and the singular set of ϕ' is finite. So we conclude by Theorem 6.7. □

7. Interval exchange maps

In this section we will introduce the definition of *expansive interval exchange map* and show its relation with the expansiveness of it *suspension* flow.

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and consider $A, B \subset S^1$ finite sets, we say that $f: S^1 \setminus A \to S^1 \setminus B$ is an *interval exchange map* if it is an homeomorphism that preserves the Lebesgue measure of S^1. A point $a \in A$ is said to be *singular* if $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ and let Sing_f be the set of singular points of f. Denote by Sing_f^* the set of points $x \in S^1$ such that there exists $n \geq 0$ with $f^n(x) \in \text{Sing}_f$. Defining $f^0(a) = a$ for a point $a \in A$ we have that $A \subset \text{Sing}_f^*$.

Definition 7.1. We say that an interval exchange map is *expansive* if there exist $\delta > 0$ such that if $x, y \notin \text{Sing}_f^*$, $x \neq y$, then there exist $n \geq 0$ such that $\text{dist}(f^n(x), f^n(y)) > \delta$.

Now consider a compact surface S. An embedded circle $\gamma \subset S$ is said to be a *quasi-global cross section* for a flow ϕ if it is transversal to ϕ and intersects every regular orbit. If a flow ϕ on S with a finite number of singularities has a quasi-global cross section such that its first return map is conjugated to an interval exchange map f then ϕ it is said to be a *suspension* of f. It is easy to see that the surface S can not have boundary, that is because γ intersects every non-singular orbit.

Remark 7.2. Consider a suspension ϕ of an interval exchange map f. Then $A = \text{Sing}_f$ if and only if ϕ has no singular point of index 0. For simplicity we will assume that every point of A is singular, so $A = \text{Sing}_f$. Consequently the suspensions considered will not have singularities of index 0.

In [2] (Lemma 8) it is shown how interval exchange maps can be suspended.

Theorem 7.3. *Let f be an interval exchange map and ϕ a suspension of f. The following statements are equivalent.*
  (1) *ϕ is expansive.*
  (2) *f is expansive.*
  (3) *Sing_f^* is dense in S^1.*

(4) f *has no periodic orbits and* $Sing_f \neq \emptyset$.

Proof. $(1 \to 3)$ It follows by Theorem 6.1 item (3).

$(3 \to 2)$ Let $I_1, \ldots, I_n$ be the intervals exchanged by f. We will show that if $\delta > 0$ and 3δ is less than the length of each I_i then δ is an expansive constant for f. If $\text{dist}(x, y) < \delta$ we denote by (x, y) the smallest interval determined by x and y. In this case there is at most one singular point in (x, y). Now fix $x, y \notin \text{Sing}^*$ such that $\text{dist}(x, y) < \delta$. By hypothesis there exists a smallest $n \geq 0$ and a point $z' \in (x, y)$ such that $z = f^n(z') \in \text{Sing}_f$. Let $f(z^{\pm}) = \lim_{u \to z^{\pm}} f(u)$. Without loss of generality suppose that $\text{dist}(f^{n+1}(x), f(z^-)), \text{dist}(f^{n+1}(y), f(z^+)) < \delta$. Since $z \in \text{Sing}_f$ we have that $\text{dist}(f(z^-), f(z^+)) \geq 3\delta$. Then it follows that $\text{dist}(f^{n+1}(x), f^{n+1}(y)) \geq \delta$.

$(2 \to 4)$ By contradiction suppose that x is a periodic point of f. It is easy to see that there exists a neighborhood U of x such that $U \cap \text{Sing}_f^* = \emptyset$, moreover every point in U is periodic. This easily contradicts the expansiveness of f. If $\text{Sing}_f = \emptyset$ then f is an homeomorphism of the circle since we are assuming that $A = \text{Sing}_f$. This gives a contradiction too, because there are no expansive homeomorphisms on the circle, as proved in [9].

$(4 \to 1)$ It follows by Theorem 6.1 item (2). $\square$

Now we shall study *quasi-minimal* flows.

Definition 7.4. A flow is said to be *quasi-minimal* if there exist a finite set $X \subset S$ such that the orbit of each $x \in S \setminus X$ is dense.

Remark 7.5. It is easy to see that ϕ is quasi-minimal if and only if every regular orbit is dense and Sing is a finite set.

A separatrix γ whose ω-limit and α-limit sets are singular points (may be the same) is called a *saddle connection*.

Lemma 7.6. *Suppose that* $\Omega(\phi) = S$, $x \in S$ *is not periodic and* $\omega(x)$ *is not a singular point. Then if* l *is an open transversal section with an extreme point* x *there exist* $t > 0$ *such that* $\phi_t(x) \in l$.

Proof. Let $l^* = \{y \in l : \phi_{\mathbb{R}+}(y) \cap l \neq \emptyset\}$ be the set of points of l returning to l and consider the first return map $f \colon l^* \to l$. Suppose that z is the extreme point of l different of x. Notice that if $f(y) \neq x$ and $f(y) \neq z$ then y is an interior point of l^*. Therefore l^* has at least two non-interior points. Let $(a, b) \subset l^*$ be a connected component of l^*. If $a \notin l^*$ then by Lemma 2.2, we have that $\omega(a)$ is a singular point and then $a \neq x$. Since $\Omega(\phi) = S$ we have by Lemma 2.3 that there is just a finite number of separatrices. So there is just a finite number of points in $l \setminus l^*$ whose ω-limit set is a singularity. On the other hand if $a \in l^*$ then either $f(a)$ is an extreme point of l or a is an extreme point of. The same consideration can be made for b. So l^* has finite number of connected components. Also, since $\Omega(\phi) = S$, we have that l^* is dense

in l. Then x belongs to the closure of some connected component of l^*. Now we can apply Lemma 2.2 to conclude that the positive orbit of x meets the closure of l. Since x is not periodic the only possible problem is that x returns to z, the other extreme of l. It can be solved considering a subsegment of l from the beginning of the proof. $\square$

Proposition 7.7. *The following statements are equivalent.*

(1) *ϕ is quasi-minimal.*
(2) *ϕ is a suspension of a minimal interval exchange map f.*
(3) *Sing is finite, $\Omega(\phi) = S$, ϕ has neither periodic orbits nor saddle connections.*

Proof. $(1 \to 2)$ It is easy to see that quasi-minimal flows are Cherry flows. Moreover, there exist $x \in S$ such that $\phi_{\mathbb{R}^+}(x)$ is dense. By [4] (section 4) we have that ϕ is a suspension of a minimal interval exchange map.

$(2 \to 3)$ Since f preserves the Lebesgue measure, $\Omega(\phi) = S$. The flow ϕ can not have periodic orbits because f is minimal. The singular set of ϕ is finite by definition of suspension. In [10] it is shown that every orbit of a minimal interval exchange map is infinite, then ϕ do not have saddle connections.

$(3 \to 1)$ Suppose there exist a regular point $x \in S$ such that $\omega(x)$ is not the whole S. Since S is connected we have that $\partial\omega(x) \neq \emptyset$. Therefore there exist a regular point $y \in \partial\omega(x)$. Take a flow box U around y. Since $U \setminus \omega(x) \neq \emptyset$ we can suppose that y belongs to the frontier of a connected component of $U \setminus \omega(x)$. Since $\omega(x)$ and its complement on S are invariant sets by the flow, we have that $\omega(x)$ is a singular point. If this were not the cases we can apply Lemma 7.6 concluding that y returns to the complement of $\omega(x)$ which is an absurd.

So we have shown that if x is a regular point and $\omega(x) \neq S$ then $\omega(x)$ is a singular point. Arguing the same for $\alpha(x)$ and using the fact that there are no saddle connections, we have that the orbit of every regular point is dense. $\square$

Corollary 7.8. *Expansive flows on connected surfaces without saddle connections are quasi-minimal.*

Proof. It follows by Theorem 6.7 and Proposition 7.7. $\square$

8. Global Structure of Expansive Flows Of Surfaces

A union of saddle connections can separate the dynamics of a flow on a surface. So if one cuts every saddle connection the surface may be disconnected. In this section we will study this decomposition for expansive flows, obtaining irreducible sub-dynamics. Similar structure theorems are given in [2, 14]. First we give an example showing the main ideas of Theorem 8.3.

Example 8.1. Consider an irrational flow on the torus. Take any orbit segment with endpoints p and q. Put singularities in p and q and make a cut along this saddle connection. Now take a copy and glue the saddle connections in the boundary getting a bi-torus. Let us show that the flow is expansive. There are two singular points and each one has index -1. And since the orbits of the irrational flow on the torus are dense, we have that the union of the separatrices is dense in the bi-torus. Now applying Theorem 6.1 we conclude that the flow is expansive. That example shows how the saddle connections separates the surface.

Remark 8.2. In the proof of Theorem 8.3 we will need to *collapse* a boundary component of the surface. Now we will show that this can be done using just basic operations.

First remove every singularity of index 0. Take a singular point p in a component of the boundary γ (by Remark 5.5 each boundary component contains at least one singular point of negative index). Consider a separatrix γ of p such that γ with another separatrix of p in the boundary they determine a hyperbolic sector of p. Put a singularity q of index 0 in s. Make a cut along the saddle connection with extreme points p and q. In this way p splits in r and s. Now either r or s is of index 0. Suppose that it is r, as in figure 3, and remove it. Now glue the saddle connections that connects q and s. Notice that q can be removed.

Repeating this procedure we remove the boundary components. The figure 3 shows this procedure in the special case where there is only one saddle connection in a boundary component, but it is work in any case. Also, in the figure we see that finally s can be removed. This fact depends on how many interior separatrices had p in the beginning.
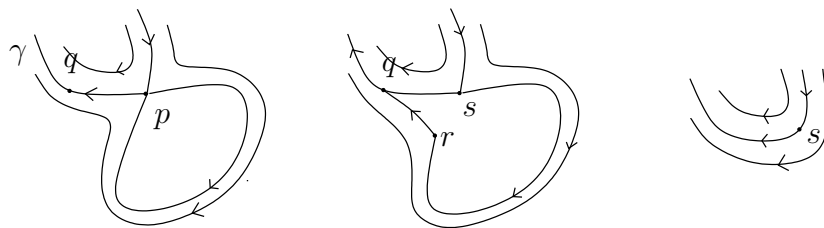

FIGURE 3. Removing a saddle connection in the boundary.

Theorem 8.3. *Every expansive flow on a compact surface can be obtained in the following way:*

(1) *Take $f_1, \ldots, f_n$ minimal interval exchange maps.*
(2) *Consider their suspensions $\phi_1, \ldots, \phi_n$ on the surfaces $S_1, \ldots, S_n$.*
(3) *Do a finite number of basic operations on $\cup_{i=1}^n S_i$.*

Proof. Take each interior saddle connection and make a cut on the surface through this orbit. What we get is a surface with boundary that may not be connected. Let $S_1, \ldots, S_N$ be such connected components and let ϕ_i be the flow on S_i induced by ϕ. By construction the saddle connections are in the boundary of S_i. Now collapsing each boundary component to a singular point we get flows without wandering points, without periodic orbits and without saddle connections. This are suspensions of minimal interval exchange maps by Proposition 7.7. $\square$

Conversely, if the procedure of Theorem 8.3 gives you a connected surface different from the torus, then the flow is expansive by Theorem 6.7.

9. Rational Billiards

Consider a polygon $P \subset \mathbb{R}^2$ with angles $\theta_i \in \mathbb{Q}\pi$, $i = 1, \ldots, n$. In the literature it is called a *rational polygon*. Let $V_1, ..., V_n$ be its corners and denote by $l_1, ..., l_n$ its sides. Consider n lines $r_1, ..., r_n$, $0 \in r_i$, parallel to each l_i respectively and let $S_1, ..., S_n$ be the reflections associated with the lines r_i. Let G be the group of isometries generated by $\{S_i\}$. On the set $P \times G$ we define an equivalence relation generated by the following property: if $x \in l_i \subset \partial P$ then $(x, g) \simeq (x, S_i g)$. Let S be the quotient space $P \times G / \simeq$. Denote by p_i the class of the corner V_i for all $i = 1, \ldots, n$ and define $\text{Sing} = \{p_1, \ldots, p_n\}$. Endowed with the quotient topology it is known that S is a connected closed surface (see [18] Proposition 1). Also, the planar metric of P induces a flat metric in $S \setminus \{V_1, \ldots, V_n\}$ and the parallel transport do not depends on the curve (see [18] Proposition 2).

With this properties we can construct a flow on S. Fix a direction $v \in \mathbb{R}^2$, $||v|| = 1$. Define a vector field Y on $S \setminus \text{Sing}$, with the parallel transport of v. Consider a non negative smooth function $\rho \colon S \to \mathbb{R}$ vanishing only in the singular points. Define the vector field X in the whole surface S as $X(x) = \rho(x)Y(x)$. Let ϕ_v be the flow on S associated with the vector field X.

Theorem 9.1. *The associated flow ϕ_v is expansive if and only if S is not the torus and there are no periodic orbit in the polygon with initial direction v.*

Proof. We have that $\Omega(\phi_v) = S$ because there is an invariant measure that is positive on open sets (see [18]). Hence we can apply Theorem 6.7 to conclude. $\square$

It is known that the surface S is the torus if and only if the polygon has all of its angles of the form π/n with $n = 2, 3, 4, ...$ This polygons are: rectangles and triangles $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ and $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$.

The previous result gives us a family of examples of expansive flows. Fix a polygon P and notice that the set of direction v such that there

exist a periodic orbit with initial direction v, is countable. Therefore, in order to get an expansive flow, just take v without periodic orbits.

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