NATURALLY REDUCTIVE PSEUDO-RIEMANNIAN SPACES

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Abstract. A family of naturally reductive pseudo-Riemannian spaces is constructed out of the representations of Lie algebras with ad-invariant metrics. We exhibit peculiar examples, study their geometry and characterize the corresponding naturally reductive homogeneous structure.

1. Introduction

Recent advances in mathematics and mathematical physics have renovated the interest in the geometry of pseudo-Riemannian metrics of general signature. For example, such metrics constitute the basis for certain sigma models, supergravities, braneworld cosmology, etc.(see [5, 7, 13, 14, 35, 37, 40] and references therein).

Some of this work leads to the investigation of homogeneous geodesics (see for instance [10, 27, 28]) and manifolds in which every geodesic is homogeneous, called *g.o. spaces* (see the survey in [8]). For a time, it was believed that every Riemannian g.o. space is naturally reductive, until Kaplan [19] found the first counterexample, extended to the pseudo-Riemannian case recently in [11]. There are significant differences between the definite situation, started by Vinberg [41] and followed by others ([22, 23], etc.), and the indefinite situation in which advances were done in [38, 17].

Ambrose and Singer [1] achieved an infinitesimal characterization of connected simply connected and complete homogeneous Riemannian manifolds in terms of a (1,2) tensor, called a homogeneous structure. This was generalized to the pseudo-Riemannian case by Gadea and Oubiña [16]. For instance some homogeneous structures of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ characterize homogeneous Lorentzian spaces for which every null-geodesic is canonically homogeneous [30]. Naturally reductive pseudo-Riemannian spaces are defined by the existence of a homogeneous structure of type $\mathcal{T}_3$ ([39, 17]). Particular examples are the pseudo-Riemannian symmetric spaces (homogeneous structures of type $\{0\}$), which in the Lorentzian situation emerge as supersymmetric supergravity backgrounds [13]. Furthermore, it is conjectured that a pseudo-Riemannian g.o. space with non-compact isotropy group should be naturally reductive [9].

In [21] Kostant proved that if M denotes a naturally reductive Riemannian space, such that the action of the isometry group G on M is transitive and almost effective, then G can be provided with a bi-invariant metric. Our first result in this work

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(Theorem (2.2) below) extends Kostant's Theorem to the pseudo-Riemannian case. As consequence, naturally reductive metrics can be produced in the following way. Let $\mathfrak{g}$ denote a Lie algebra equipped with an ad-invariant metric Q and let $\mathfrak{h} \subset \mathfrak{g}$ be a nondegenerate Lie subalgebra. Thus one has the following reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \text{with} \qquad [\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m} \quad \text{and} \quad \mathfrak{m} = \mathfrak{h}^{\perp}.$$

Let G denote a Lie group with Lie algebra $\mathfrak{g}$ and endowed with the bi-invariant metric induced by Q and let $H \subset G$ denote a closed Lie subgroup with Lie algebra $\mathfrak{h}$. Then the coset space G/H becomes a naturally reductive pseudo-Riemannian space.

Here we focus on the construction of a family of simply connected naturally reductive pseudo-Riemannian Lie groups $\mathcal{G}(D)$, which can be obtained as follows. For a class of Lie algebras $\mathfrak{g}$ which can be provided with an ad-invariant metric, one has the following decomposition as a semidirect sum of vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathcal{G}(\mathfrak{d}),$$

where $\mathcal{G}(\mathfrak{d})$ denotes the Lie algebra of $\mathcal{G}(D)$ and $\mathfrak{g}$ is an isometry Lie algebra acting on $\mathcal{G}(\mathfrak{d})$ with stability Lie algebra $\mathfrak{h}$. In general $\mathfrak{m} \neq \mathcal{G}(\mathfrak{d})$, however as vector spaces they are isomorphic via the map $\lambda : \mathcal{G}(\mathfrak{d}) \to \mathfrak{m}$ which induces the metric of $\mathfrak{m}$ to $\mathcal{G}(\mathfrak{d})$ making both spaces linearly isometric. The metric $\langle\, , \rangle$ induced on $\mathcal{G}(\mathfrak{d})$ is defined on the Lie group $\mathcal{G}(D)$ by translations on the left.

The algebraic structure of $\mathfrak{g}$ can be specified. The Lie algebra $\mathcal{G}(\mathfrak{d})$ is an ideal in $\mathfrak{g}$ defined on the underlying vector space

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{d} \oplus \mathfrak{h}^* \qquad \text{(direct sum as vector spaces)}$$

where $\mathfrak{h}^*$ denotes the dual space of $\mathfrak{h}$, which together with $\mathfrak{d}$, obeys the following data:

- a Lie algebra $\mathfrak{h}$ with ad-invariant metric $\langle\, , \rangle_{\mathfrak{h}}$
- a Lie algebra $\mathfrak{d}$ with ad-invariant metric $\langle\, , \rangle_{\mathfrak{d}}$,
- a Lie algebra homomorphism $\pi : \mathfrak{h} \to \mathrm{Dera}(\mathfrak{d}, \langle\, , \rangle_{\mathfrak{d}})$ from $\mathfrak{h}$ to the Lie algebra of skew-symmetric derivations of $(\mathfrak{d}, \langle\, , \rangle_{\mathfrak{d}})$.

Another proof for the naturally reductivity property of the Lie groups $(\mathcal{G}(D), \langle\, , \rangle)$ is done in terms of homogeneous structures. For example,

$$T_x y = \frac{1}{2}\lambda^{-1}[\lambda x, \lambda y]_{\mathfrak{m}} \qquad \text{for all } x, y \in \mathcal{G}(\mathfrak{d})$$

gives rise to a naturally reductive structure on $\mathcal{G}(D)$, with $\lambda : (\mathcal{G}(\mathfrak{d}), \langle\, , \rangle) \to (\mathfrak{m}, Q)$ as already mentioned. This formula generalizes that one of Riemannian nilmanifolds, where such a naturally reductive homogeneous structure is unique if the nilmanifold has no Euclidean factor [26].

We study the geometry and in particular the group of orthogonal automorphisms of $\mathcal{G}(D)$, which allows the production of examples of naturally reductive pseudo-Riemannian spaces with compact or noncompact isotropy group.

Finally we analyze some special cases. Starting with $\mathfrak{d}$ which is k-step nilpotent (resp. solvable) then $\mathcal{G}(\mathfrak{d})$ turns out to be at most $k+1$-step nilpotent (resp.

solvable). This gives rise to the first examples (known to us) of naturally reductive metrics on non semisimple Lie groups, which are neither symmetric nor 2-step nilpotent.

2. Isometries on naturally reductive pseudo-Riemannian manifolds

Let M denote a connected manifold with homogeneous pseudo-Riemannian metric $\langle\, ,\, \rangle$. Let G be any transitive group of isometries with Lie algebra $\mathfrak{g}$ and let H be the isotropy subgroup at some point $o \in M$. Then $M \simeq G/H$ is called a homogeneous space which is *reductive* if there exists a subspace $\mathfrak{m} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m} \tag{1}$$

where $\mathfrak{h}$ denotes the Lie algebra of H. If $\langle\, ,\, \rangle$ is positive definite then M is always reductive, that is, a subspace $\mathfrak{m}$ satisfying (1) always exists.

The action is said *(almost) effective* if the set of elements in G acting as the identity transformation is (discrete) trivial. Almost effectiveness is equivalent to the requirement that the Lie algebra $\mathfrak{h}$ contains no non trivial ideal of $\mathfrak{g}$. Thus the linear representation of the isotropy group H in T_oM, the tangent space to the manifold at the point $o = eH \in M$, is faithful. One may identify $\mathfrak{m}$ with T_oM by the map $x \to x^\bullet_o$, with

$$x^\bullet_o = \frac{d}{dt}|_{t=0} \exp tx \cdot o. \tag{2}$$

The map $x \to x^\bullet$ is induced by an antihomomorphism of Lie algebras, so $[x^\bullet, y^\bullet]_o = -[x,y]^\bullet_o$. One may pull back $\langle\, ,\, \rangle_o$ on T_oM to a (non necessarily definite) metric on $\mathfrak{m}$. Let $x_{\mathfrak{h}}$ and $x_{\mathfrak{m}}$ denote the $\mathfrak{h}$, resp. $\mathfrak{m}$ component of $x \in \mathfrak{g}$.

The canonical connection ∇^c on the reductive space G/H is the unique G-invariant affine connection on M such that for any vector $x \in \mathfrak{m}$ and any frame u at the point o, the curve $(\exp tx)u$ in the bundle of frames over M is horizontal. The canonical connection is complete and the set of its geodesics through o coincides with the set of curves of the type $\exp(tx) \cdot o$, with $x \in \mathfrak{m}$. Explicitly

$$(\nabla^c_{x^\bullet} y^\bullet)_o := [x^\bullet, y^\bullet]_o = -[x,y]^\bullet_o \quad \longleftrightarrow \quad -[x,y]_{\mathfrak{m}} \quad \text{for } x, y \in \mathfrak{m}. \tag{3}$$

This is an affine connection whose torsion is

$$T^c(x^\bullet, y^\bullet) = -[x^\bullet, y^\bullet],$$

thus via the identifications, $T^c(x,y) = -[x,y]_{\mathfrak{m}}$ for $x, y \in \mathfrak{m}$ and as in [20] Ch. X, the curvature is given by

$$R^c(x,y)z = -[[x,y]_{\mathfrak{h}}, z] \qquad x, y, z \in \mathfrak{m}.$$

The tensor fields T and R are parallel with respect to the canonical connection. In the reductive space $M = G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as in (1) there is a unique G-invariant affine connection with zero torsion having the same geodesics as the canonical connection. This connection is called the *natural torsion-free connection* on M (with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$).

Definition 2.1. A homogeneous pseudo-Riemannian manifold $(M = G/H, \langle\,,\,\rangle)$ is said to be *naturally reductive* if it is reductive, i.e. there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad \text{with} \qquad \mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$$

and

$$\langle [x, y]_{\mathfrak{m}}, z\rangle + \langle y, [x, z]_{\mathfrak{m}}\rangle = 0 \qquad \text{for all } x, y, z \in \mathfrak{m}.$$

Frequently we will say that a metric on a homogeneous space M is naturally reductive even though it is not naturally reductive with respect to a particular transitive group of isometries (see Lemma 2.3 in [18]).

Indeed pseudo-Riemannian symmetric spaces are naturally reductive. Other examples arise from Lie groups equipped with a bi-invariant metric.

If M is naturally reductive the natural torsion-free connection coincides with the corresponding Levi Civita connection on M, which is given by

$$(\nabla_{x^\bullet} y^\bullet)_o = \frac{1}{2}[x^\bullet, y^\bullet]_o \quad \longleftrightarrow \quad -\frac{1}{2}[x, y]_{\mathfrak{m}} \qquad \text{for } x, y \in \mathfrak{m}.$$

The geodesics passing through $o \in M$ are of the form

$$\gamma(t) = \exp(tx) \cdot o \qquad \text{for some } x \in \mathfrak{m}.$$

The canonical connection on a naturally reductive pseudo-Riemannian space $M = G/H$ is metric and the curvature tensor satisfies the following identity

$$\langle R(x, y)z, w\rangle = \langle R(z, w)x, y\rangle \qquad x, y, z, w \in \mathfrak{m} \tag{4}$$

which follows after applications of the Jacobi identity and the naturally reductive condition (see [6]).

In view of the definition above if one wants to determine if a given metric is or is not naturally reductive, one would first have to find all Lie groups G acting transitive on M and in presence of a naturally reductive metric, study all H-invariant complements $\mathfrak{m}$ of $\mathfrak{h}$. The latter task can be simplified by the next theorem, proved in its Riemannian version originally by Kostant [21] (see also [6]). A modification of the proof adapted to the indefinite case validates the result for the pseudo-Riemannian situation.

Theorem 2.2. *Let $(M = G/H, \langle\,,\,\rangle)$ denote a pseudo-Riemannian naturally reductive space on which G acts almost effectively. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive decomposition. Then $\bar{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is an ideal in $\mathfrak{g}$ whose corresponding analytic subgroup $\bar{G}$ is transitive on M and there exists a unique $\mathrm{Ad}(\bar{G})$-invariant symmetric nondegenerate bilinear form Q on $\bar{\mathfrak{g}}$ such that*

$$Q(\mathfrak{h} \cap \bar{\mathfrak{g}}, \mathfrak{m}) = 0 \qquad Q|_{\mathfrak{m}\times\mathfrak{m}} = \langle\,,\,\rangle$$

where $\mathfrak{h} \cap \bar{\mathfrak{g}}$ will be the isotropy algebra in $\bar{\mathfrak{g}}$.

Conversely if G is connected, then for any $\mathrm{Ad}(G)$-invariant symmetric nondegenerate bilinear form Q on $\mathfrak{g}$, which is also nondegenerate on $\mathfrak{h}$ (and on $\mathfrak{m} = \mathfrak{h}^\perp$), the metric on M defined by $\langle\,,\,\rangle = Q|_{\mathfrak{m}\times\mathfrak{m}}$ is naturally reductive. In this case $\mathfrak{g} = \bar{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$.

Proof. Let x denote an element of $\mathfrak{g}$. If $x \in \mathfrak{h}$ then $[x, y] \in \mathfrak{m}$ for all $y \in \mathfrak{m}$ and clearly $[x, y] \in [\mathfrak{m}, \mathfrak{m}]$ if $x \in \mathfrak{m}$. This proves that $\bar{\mathfrak{g}}$ is an ideal in $\mathfrak{g}$ and its Lie group is transitive on M, since $\bar{\mathfrak{g}}$ contains $\mathfrak{m}$ and any point in M can be reached by a geodesic $\exp tx \cdot o$ for some $x \in \mathfrak{m}$. Thus by replacing $\mathfrak{g}$ by $\bar{\mathfrak{g}}$, assume $\mathfrak{g} = \bar{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$. Since one also has $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the set

$$S = \{[y, y']_{\mathfrak{h}} \, : \, y, y' \in \mathfrak{m}\}$$

generates $\mathfrak{h}$ as a vector space. The relation in (4) establishes that

$$\langle [y, [z, z']_{\mathfrak{h}}], y' \rangle = \langle [z, [y, y']_{\mathfrak{h}}], z' \rangle \qquad y, y', z, z' \in \mathfrak{m}. \tag{5}$$

Each side of (5) depends only on $[y, y']$, $[z, z']$ in S and not on the choice of $y, y', z, z' \in \mathfrak{m}$. Thus define Q as

$$Q_{|\mathfrak{m}} := Q_{|\mathfrak{m}\times\mathfrak{m}} = \langle \, , \, \rangle \qquad\qquad Q(\mathfrak{h}, \mathfrak{m}) = 0$$

$$\begin{aligned} Q([y, y']_{\mathfrak{h}}, [z, z']_{\mathfrak{h}}) &= -\langle [y, [z, z']_{\mathfrak{h}}], y' \rangle \\ &= -\langle [z, [y, y']_{\mathfrak{h}}], z' \rangle \qquad y, y', z, z' \in \mathfrak{m}. \end{aligned} \tag{6}$$

The first equation in (6) shows that Q extends uniquely to a linear function on $\mathfrak{h}$ in the second variable for fixed $[y, y']_{\mathfrak{h}} \in S$ and the second equation in (6) shows that Q extends uniquely to a symmetric bilinear form on $\mathfrak{h} \times \mathfrak{h}$. Now we check that Q is ad-invariant

$$Q([x, y], z) = -Q(y, [x, z]) \qquad \text{for all } x, y, z \in \mathfrak{g}. \tag{7}$$

For $x, y, z \in \mathfrak{m}$, (7) follows from the definition of Q and the naturally reductivity condition. For $x, y \in \mathfrak{h}$, $z \in \mathfrak{m}$ every side in (7) vanishes, while for $x \in \mathfrak{h}$ and $y, z \in \mathfrak{m}$, (7) follows from the $\mathrm{Ad}(H)$-invariant condition of $\langle \, , \, \rangle$.

Finally take $x, y \in \mathfrak{h}$ and $z = [w, w']_{\mathfrak{h}}$ where $w, w' \in \mathfrak{m}$. Thus

$$\begin{aligned} Q([x, y], [w, w']_{\mathfrak{h}}) &= -\langle w, [[x, y], w'] \rangle \\ &= \langle w, [[w', x], y]] \rangle + \langle w, [[y, w'], x] \rangle \\ &= Q([w, [w', x]], y) + Q([[w, x], w'], y) \\ &= Q([x, [w, w']], y) = -Q([x, [w, w']_{\mathfrak{h}}], y) \end{aligned}$$

which proves (7) for $x, y, z \in \mathfrak{h}$. We now show that $Q_{|\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate. Let $x \in \mathfrak{h}$ and assume that $Q(x, y) = 0$ for all $y \in \mathfrak{h}$. In particular

$$Q(x, [z, z']_{\mathfrak{h}}) = 0 \qquad\qquad \text{for all } z, z' \in \mathfrak{m},$$

thus $Q(x, [z, z']) = 0$ which after the ad-invariance condition already proved, says that

$$Q([x, z], z') = 0 \qquad \forall\, z' \in \mathfrak{m} \quad \Longrightarrow \quad [x, z] = 0 \qquad \forall\, z \in \mathfrak{m}$$

and since the action is effective one gets $x = 0$.

To prove the converse, notice that Q is nondegenerate on $\bar{\mathfrak{g}}$ and since it is an ideal, its orthogonal space

$$\bar{\mathfrak{g}}^{\perp} = \{x \in \mathfrak{g} \, : \, Q(x, y) = 0 \text{ for all } y \in \bar{\mathfrak{g}}\}$$

is also a nondegenerate ideal of $\mathfrak{g}$ which is contained in $\mathfrak{h}$ and thus $\bar{\mathfrak{g}}^{\perp} = 0$ since the action of G is almost effective.

Let $\mathfrak{m} = \mathfrak{h}^{\perp}$ and let $\langle \, , \, \rangle := Q_{|\mathfrak{m}\times\mathfrak{m}}$. Then $M := G/H$ endowed with the G-invariant metric induced by $\langle \, , \, \rangle$ is a naturally reductive space. Since $Q([h,x],\tilde{h}) = -Q(x,[h,\tilde{h}]) = 0$ for all $h,\tilde{h} \in \mathfrak{h}$ and $x \in \mathfrak{m} = \mathfrak{h}^{\perp}$, the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition. Clearly

$$\langle [x,y]_{\mathfrak{m}}, z\rangle + \langle y, [x,z]_{\mathfrak{m}}\rangle = 0$$

follows from the ad-invariant property of Q. □

Remark. The action of the Lie group $\bar{G}$ with Lie algebra $\bar{\mathfrak{g}}$ is also transitive whenever G/H is a Lorentzian space admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ and it plays an important role in the results in [31].

Lie algebras with ad**-invariant metrics.** An example of a Lie algebra $\mathfrak{g}$ with an ad-invariant metric is a semisimple Lie algebra together with its Killing form. The corresponding Lie group G is a pseudo-Riemannian Einstein space with negative scalar curvature. In the other extreme, the abelian Lie group $\mathbb{T}^r \times \mathbb{R}^s$ endowed with any pseudo-Riemannian invariant metric is flat.

Among others, Lie algebras with ad-invariant metrics can be constructed out from the following data:

- a Lie algebra $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$ with an ad-invariant metric $\langle \, , \, \rangle_{\mathfrak{d}}$,
- a Lie algebra $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ with ad-invariant symmetric bilinear (possibly degenerate) form $\langle \, , \, \rangle_{\mathfrak{h}}$,
- a Lie algebra homomorphism $\pi : (\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}) \to \mathrm{Dera}(\mathfrak{d}, \langle \, , \, \rangle_{\mathfrak{d}})$ from $\mathfrak{h}$ to the Lie algebra of skew-symmetric derivations of $\mathfrak{d}$.

Consider the following vector space direct sum

$$\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{d} \oplus \mathfrak{h}^*. \tag{8}$$

Let Q be the symmetric bilinear map on $\mathfrak{g}$, which for $x_i \in \mathfrak{d}$, $\alpha_i \in \mathfrak{h}^*$, $h_i \in \mathfrak{h}$, $i = 1, 2$, is defined by

$$Q((h_1,x_1,\alpha_1),(h_2,x_2,\alpha_2)) := \langle h_1,h_2\rangle_{\mathfrak{h}} + \langle x_1,x_2\rangle_{\mathfrak{d}} + \alpha_1(h_2) + \alpha_2(h_1); \tag{9}$$

it is nondegenerate and of signature $sgn(Q) = sgn(\langle \, , \, \rangle_{\mathfrak{d}}) + (\dim \mathfrak{h}, \dim \mathfrak{h})$.

Let $\mathrm{ad}_{\mathfrak{h}}$ denote the adjoint action of $\mathfrak{h}$ to itself. The Lie bracket on $\mathfrak{g}$ is given by

$$\begin{aligned} &[(h_1,x_1,\alpha_1),(h_2,x_2,\alpha_2)] \\ :=\; &([h_1,h_2]_{\mathfrak{h}}, [x_1,x_2]_{\mathfrak{d}} + \pi(h_1)x_2 - \pi(h_2)x_1, \beta(x_1,x_2) + \mathrm{ad}^*_{\mathfrak{h}}(h_1)\alpha_2 - \mathrm{ad}^*_{\mathfrak{h}}(h_2)\alpha_1) \end{aligned} \tag{10}$$

$$\text{where} \quad \beta(x_1,x_2)(h) := \langle \pi(h)x_1, x_2\rangle_{\mathfrak{d}} \quad \text{and} \quad \mathrm{ad}^*_{\mathfrak{h}}(h)\alpha = -\alpha \circ \mathrm{ad}_{\mathfrak{h}}(h).$$

Some straightforward computations show that the metric Q in (9) is ad-invariant with respect to this bracket.

While $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, in general $\mathfrak{d}$ is not a subalgebra. The subspace $\mathcal{G}(\mathfrak{d}) := \mathfrak{d} \oplus \mathfrak{h}^*$ is always an ideal in $\mathfrak{g}$, which is obtained as a central extension of $\mathfrak{d}$ by the 2-cocycle β and $\mathfrak{g}$ is the semidirect sum of $\mathcal{G}(\mathfrak{d})$ and $\mathfrak{h}$.

The resulting Lie algebra $\mathfrak{g}$ is called the *double extension* of $\mathfrak{d}$ with respect to $(\mathfrak{h}, \pi)$.

One says that a Lie algebra $(\mathfrak{g}, Q)$ equipped with an ad-invariant metric is *indecomposable* whenever it has no nondegenerate ideal. [1]

The existence of an ad-invariant metric Q on $\mathfrak{g}$ impose restrictions in the algebraic structure of such a $\mathfrak{g}$: if $(\mathfrak{g}, Q)$ is indecomposable, then $\mathfrak{g}$ is either

- one-dimensional or
- simple or
- it is a double extension of $(\mathfrak{d}, \langle\, ,\, \rangle_{\mathfrak{d}})$ with respect to $(\mathfrak{h}, \pi)$ where $\mathfrak{h}$ is either one-dimensional or simple.

See the proof in [32], and also [15] for more details in the solvable case.

Example 2.3. If G denotes the corresponding simply connected Lie group G with Lie algebra $(\mathfrak{g}, Q)$ and we equip G with the corresponding pseudo-Riemannian metric induced by Q and invariant by left and right translations, then G/H is a naturally reductive pseudo-Riemannian manifold whenever H is trivial or H is a cocompact discrete subgroup, therefore in the last situation G/H is a naturally reductive pseudo-Riemannian compact space.

3. The naturally reductive pseudo-Riemannian Lie groups $\mathcal{G}(D)$

In this section we produce naturally reductive pseudo-Riemannian spaces. We introduce them as an application of Theorem (2.2), but we also find a naturally reductive homogeneous structure for them.

Consider the following data

- $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$ a Lie algebra with ad-invariant metric $\langle\, ,\, \rangle_{\mathfrak{h}}$,
- $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}})$ a Lie algebra with ad-invariant metric $\langle\, ,\, \rangle_{\mathfrak{d}}$,
- $\pi : \mathfrak{h} \to \mathrm{Dera}(\mathfrak{d}, \langle\, ,\, \rangle_{\mathfrak{d}})$ a Lie algebra homomorphism from $\mathfrak{h}$ to the Lie algebra of skew-symmetric derivations of $\mathfrak{d}$.

This data enables the construction of a Lie algebra $\mathfrak{g}$ with an ad-invariant metric as explained in the previous section. The vector space underlying $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{d} \oplus \mathfrak{h}^*$ and which is equipped with the Lie bracket showed in (10). Let Q_- denote the next metric on $\mathfrak{g}$:

$$Q_-((h_1, x_1, \alpha_1), (h_2, x_2, \alpha_2)) := -\langle h_1, h_2\rangle_{\mathfrak{h}} + \langle x_1, x_2\rangle_{\mathfrak{d}} + \alpha_1(h_2) + \alpha_2(h_1). \tag{11}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let H be a closed Lie subgroup of G with Lie algebra $\mathfrak{h}$. Since Q_- is an ad-invariant metric on $\mathfrak{g}$ and $\mathfrak{h}$ is nondegenerate for Q_-, Theorem 2.2 says that the homogeneous manifold G/H **becomes a naturally reductive pseudo Riemannian space**. In fact one has the following decomposition as direct sum as vector spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \qquad \mathfrak{m} = \mathfrak{h}^{\perp},$$

[1] Notice that if $\mathfrak{j}$ is a ideal in $\mathfrak{g}$ then $\mathfrak{j}^{\perp}$ is also a ideal of $\mathfrak{g}$. Hence if $\mathfrak{j}$ is non degenerate then $\mathfrak{g}$ admits a direct sum as vector spaces decomposition: $\mathfrak{g} = \mathfrak{j} \oplus \mathfrak{j}^{\perp}$. This shows that the definition of indecomposibility given here is equivalent to that one given for instance in [2].

where $\mathfrak{m}$ is the vector subspace of $\mathfrak{g}$ Q_--orthogonal to $\mathfrak{h}$:

$$\begin{aligned}\mathfrak{m} := \mathfrak{h}^\perp &= \{x \in \mathfrak{g} \,:\, Q_-(x, \mathfrak{h}) = 0\} \\ &= \mathfrak{d} \oplus \{\alpha \in \mathfrak{h}^*,\, h \in \mathfrak{h} \,:\, \alpha(\tilde h) - \langle h, \tilde h\rangle_{\mathfrak{h}} = 0 \quad \forall\, \tilde h \in \mathfrak{h}\} \\ &= \mathfrak{d} \oplus \{(h, 0, \alpha) \,:\, \alpha(\cdot) = \langle h, \cdot\rangle_{\mathfrak{h}}\} \\ &= \{(h, x, h^*) \,/\, h \in \mathfrak{h}, x \in \mathfrak{d}\},\end{aligned}$$

where h^* denotes the image of $h \in \mathfrak{h}$ by the linear isomorphism $\ell : \mathfrak{h} \to \mathfrak{h}^*$

$$\ell : h \to h^* := \langle h, \cdot\rangle_{\mathfrak{h}}.$$

With this notation the coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^*$ is

$$\mathrm{ad}^*_{\mathfrak{h}}(h_1) h_2^* = [h_1, h_2]^*_{\mathfrak{h}}$$

and the adjoint action of $\mathfrak{g}$ restricted to $\mathfrak{h}$ acts on $\mathfrak{m}$ in the next way

$$[h_1, (h_2, x, h_2^*)] = ([h_1, h_2]_{\mathfrak{h}}, \pi(h_1)x, [h_1, h_2]^*_{\mathfrak{h}}) \qquad h_1, h_2 \in \mathfrak{h}, x \in \mathfrak{d},$$

which shows that $\mathfrak{m}$ is $\mathrm{ad}(\mathfrak{h})$-invariant. Since $\mathfrak{h}$ is nondegenerate, the restriction of Q_- to $\mathfrak{m}$ (also nondegenerate) is given by

$$Q_-((h_1, x, h_1^*), (h_2, x, h_2^*)) = \langle x, y\rangle_{\mathfrak{d}} + \langle h_1, h_2\rangle_{\mathfrak{h}} \tag{12}$$

and it makes G/H a naturally reductive space. To define completely the metric on G/H, as usual identify $T_o(G/H) \simeq \mathfrak{m}$ and define the metric on $g \cdot H$ for $g \in G$ in such way that the map $\tau(g) : x \cdot H \to gx \cdot H$ becomes an isometry.

Our goal now is to prove that $(\mathfrak{m}, Q_-)$ is the subspace of Definition (2.1) corresponding to a Lie group provided with a naturally reductive pseudo-Riemannian metric for which left-translations by elements of the group are isometries, sometimes called a left-invariant pseudo-Riemannian metric. Such a metric is determined by its values at the identity, hence at the Lie algebra level.

Consider the simply connected Lie group $\mathcal{G}(D)$ with Lie algebra

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{d} \oplus \mathfrak{h}^*$$

where $\mathfrak{h}$ and $\mathfrak{d}$ are taken as in the data in the beginning of this section and $\mathfrak{h}^*$ is the dual space of $\mathfrak{h}$. The Lie bracket in $\mathcal{G}(\mathfrak{d})$ makes of the canonical inclusion $\iota : \mathcal{G}(\mathfrak{d}) \to \mathfrak{g}$, $\iota(x + h^*) = (0, x, h^*) \in \mathfrak{g}$ a Lie algebra monomorphism:

$$[x_1 + h_1^*, x_2 + h_2^*] = [x_1, x_2]_{\mathfrak{d}} + \beta(x_1, x_2). \tag{13}$$

The vector space underlying the Lie algebra $\mathcal{G}(\mathfrak{d})$ is isomorphic to $\mathfrak{m}$ via the linear map $\lambda : \mathcal{G}(\mathfrak{d}) \to \mathfrak{m}$ given by

$$\lambda : x + h^* \quad \to \quad (h, x, h^*). \tag{14}$$

Provide $\mathcal{G}(D)$ with the pseudo-Riemannian metric $\langle\, ,\, \rangle$ obtained by left-translation of the following metric on $\mathfrak{d} \oplus \mathfrak{h}^*$:

$$\langle x_1 + h_1^*, x_2 + h_2^*\rangle = \langle h_1, h_2\rangle_{\mathfrak{h}} + \langle x_1, x_2\rangle_{\mathfrak{d}}. \tag{15}$$

This is the only metric on $\mathcal{G}(\mathfrak{d})$ which makes λ a linear isometry identifying $\mathfrak{m}$ with $\mathcal{G}(\mathfrak{d})$,

$$\langle x_1 + h_1^*, x_2 + h_2^*\rangle = Q_-((h_1, x_1, h_1^*), (h_2, x_2, h_2^*)) \qquad h_1, h_2 \in \mathfrak{h},\, x_1, x_2 \in \mathfrak{d}.$$

With respect to this metric $\mathfrak{h}^*$ and $\mathfrak{d}$ are nondegenerate subspaces of $\mathcal{G}(\mathfrak{d})$ which are at the same time, orthogonal and complementary to each other. Furthermore since

$$Q_-((h,0,h^*),(\beta^*(x_1,x_2),0,\beta(x_1,x_2)) = Q_-(h,\beta(x_1,x_2)) = Q_-(\pi(h)x_1,x_2)$$

via application of λ one gets the following relation on $\mathcal{G}(\mathfrak{d})$

$$\langle h^*,[x_1,x_2]\rangle = \langle h^*,\beta(x_1,x_2)\rangle_{\mathfrak{h}^*} = \langle \pi(h)x_1,x_2\rangle_{\mathfrak{d}} \qquad \text{for } h\in\mathfrak{h},\ x_1,x_2\in\mathfrak{d} \tag{16}$$

where $\langle\,,\,\rangle_{\mathfrak{h}^*}$ denotes the restriction of $\langle\,,\,\rangle$ to $\mathfrak{h}^*$.

Since $\mathfrak{h}^*\subseteq(\mathcal{G}(\mathfrak{d}),\langle\,,\,\rangle)$ is nondegenerate and $\mathfrak{h}^*\subseteq(\mathfrak{g},Q_-)$ is degenerate, the mapping ι can **not** be an isometry from $\mathcal{G}(\mathfrak{d})\to\iota(\mathcal{G}(\mathfrak{d}))\subseteq\mathfrak{g}$. However the group $\mathcal{G}(D)$ acts as a group of isometries in both pseudo-Riemannian homogeneous spaces (M,Q_-) and $(\mathcal{G}(D),\langle\,,\,\rangle)$.

We shall see that the Lie group H acts by orthogonal automorphisms of $(\mathcal{G}(D),\langle\,,\,\rangle)$. Since $\mathcal{G}(D)$ is simply connected we make no distintion between automorphisms of $\mathcal{G}(D)$ and $\mathcal{G}(\mathfrak{d})$.

Consider the mapping μ from $\mathfrak{h}$ to $\operatorname{End}(\mathcal{G}(\mathfrak{d}))$ induced by the action on $\mathcal{G}(\mathfrak{d}) = \mathfrak{d}\oplus\mathfrak{h}^*$ taken as π in the first summand and the coadjoint action in the second; that is for all $\tilde h,h\in\mathfrak{h},x\in\mathfrak{d}$, the action is given by

$$\tilde h\cdot(x+h^*) := \pi(\tilde h)x + \operatorname{ad}^*_{\mathfrak{h}}(\tilde h)(h^*) = \pi(\tilde h)x + [\tilde h,h]^*_{\mathfrak{h}}. \tag{17}$$

This is an action which provides a Lie algebra homomorphim from $\mathfrak{h}$ to $\operatorname{End}(\mathcal{G}(\mathfrak{d}))$ acting by skew-symmetric derivations with respect to $\langle\,,\,\rangle$. In fact on the one hand

$$h\cdot[x_1+h_1^*,x_2+h_2^*] = \pi(h)[x_1,x_2]_{\mathfrak{d}} + \operatorname{ad}^*_{\mathfrak{h}}(h)\cdot\beta(x_1,x_2)$$

and $\operatorname{ad}^*_{\mathfrak{h}}(h)\cdot\beta(x_1,x_2)(\tilde h) = -\beta(x_1,x_2)[h,\tilde h]_{\mathfrak{h}} = \langle\pi([\tilde h,h]_{\mathfrak{h}})x_1,x_2\rangle_{\mathfrak{d}}$ (see (10)).

On the other hand

$$\begin{aligned}[][h\cdot(x_1+h_1^*),x_2+h_2^*] &= [\pi(h)x_1+\operatorname{ad}^*_{\mathfrak{h}}(h)(h_1^*),x_2+h_2^*]\\ &= [\pi(h)x_1+[h,h_1]^*_{\mathfrak{h}},x_2+h_2^*]\\ &= [\pi(h)x_1,x_2]_{\mathfrak{d}}+\beta(\pi(h)x_1,x_2)\end{aligned}$$

and similarly

$$[x_1+h_1^*,h\cdot(x_2+h_2^*)] = [x_1,\pi(h)x_2]_{\mathfrak{d}} + \beta(x_1,\pi(h)x_2).$$

By (10), $\beta(\pi(h)x_1,x_2)(\tilde h) = \langle\pi(\tilde h)\pi(h)x_1,x_2\rangle_{\mathfrak{d}}$ therefore

$$(\beta(\pi(h)x_1,x_2)+\beta(x_1,\pi(h)x_2))(\tilde h) = \langle\pi([\tilde h,h])x_1,x_2\rangle.$$

Both the coadjoint representation and π are homomorphisms from $\mathfrak{h}$ to $\operatorname{End}(\mathfrak{h}^*)$ and $\operatorname{Dera}(\mathfrak{d},\langle\,,\,\rangle_{\mathfrak{d}})$ respectively, therefore the map of $\mathfrak{h}$ to $\operatorname{Der}(\mathcal{G}(\mathfrak{d}))$, denoted μ, is a Lie algebra homomorphism.

The paragraphs above show that we achieved the next result.

Theorem 3.1. *The pseudo-Riemannian metric on the Lie group $\mathcal{G}(D)$ induced by left-translations of $\langle\,,\,\rangle$ in (15), makes of it a naturally reductive pseudo-Riemannian manifold.*

Specifically it admits a transitive action by isometries of the Lie group whose Lie algebra is the semidirect sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathcal{G}(\mathfrak{d}) \qquad \textit{with } \mathfrak{h} \subseteq \mathrm{Der}(\mathcal{G}(\mathfrak{d})) \cap \mathfrak{so}(\mathcal{G}(\mathfrak{d})) := \mathrm{Dera}(\mathcal{G}(\mathfrak{d})).$$

Proof. We have already proved that $\mathfrak{h}$ acts by derivations on $\mathcal{G}(\mathfrak{d})$ and it is not hard to prove that $\mu(h)$ is skew-symmetric with respect to $\langle\, ,\, \rangle$ for every $h \in \mathfrak{h}$. Therefore $\mathfrak{h} \subseteq \mathrm{Dera}(\mathcal{G}(\mathfrak{d}))$ which is a subalgebra of the Lie algebra of the isotropy group of isometries of $(\mathcal{G}(\mathfrak{d}), \langle\, ,\, \rangle)$. Since $\mu : \mathfrak{h} \to \mathrm{Der}(\mathfrak{d})$ is a homomorphism, actually $\mathcal{G}(\mathfrak{d})$ is an ideal in $\mathfrak{g}$ so that one gets the semidirect sum structure $\mathfrak{g} = \mathfrak{h} \oplus \mathcal{G}(\mathfrak{d})$. Finally the construction of the metric $\langle\, ,\, \rangle$ on $\mathcal{G}(\mathfrak{d})$ via λ shows that the conditions of (2.1) hold, and therefore $(\mathcal{G}(D), \langle\, ,\, \rangle)$ is a naturally reductive pseudo-Riemannian homogeneous space. $\square$

Example 3.2. The low dimensional non-abelian examples of the the construction in Theorem (3.1) arise by starting with the abelian Lie algebra $\mathfrak{a}$ spanned by the vectors e_1, e_2. It can be equipped with the metrics:

- $\langle e_1, e_1\rangle_+ = 1 = \langle e_2, e_2\rangle_+, \quad \langle e_1, e_2\rangle_+ = 0.$
- $-\langle e_1, e_1\rangle_- = 1 = \langle e_2, e_2\rangle_-, \quad \langle e_1, e_2\rangle_- = 0.$

For the metric $\langle\, ,\, \rangle_+$ the space of skew-symmetric maps is generated by

$$t_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

while for $\langle\, ,\, \rangle_-$ the space of skew-symmetric maps is generated by

$$t_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\mathbb{R}^{p,q}$ denote the abelian Lie algebra equipped with the metric of signature (p, q). In each case considered above, the Lie algebra $\mathcal{G}(\mathbb{R}^{p,q})$ has dimension three; one has $\mathcal{G}(\mathbb{R}^{p,q}) = \mathfrak{a} \oplus \mathbb{R}e_3$ and the Lie bracket

$$[e_1, e_2] = e_3$$

identifies the Heisenberg Lie algebra of dimension three $\mathfrak{h}_3$. The metric on $\mathfrak{h}_3$ is defined in such way that e_3 is orthogonal to $\mathfrak{a}$ giving rise to the next possibilities:

- $\langle e_i, e_i\rangle_0 = 1$, for all i,
- $\langle e_i, e_i\rangle_1 = 1 = -\langle e_3, e_3\rangle_1 \quad$ for i=1,2,
- $-\langle e_1, e_1\rangle_2 = 1 = \langle e_2, e_2\rangle_2 = \langle e_3, e_3\rangle_2,$
- $-\langle e_1, e_1\rangle_3 = 1 = \langle e_2, e_2\rangle_3 = -\langle e_3, e_3\rangle_3.$

The corresponding pseudo-Riemannian metrics invariant by left-translations, on the Heisenberg Lie group H_3 are naturally reductive. The Lie algebra of the isometry group is $\mathbb{R} \ltimes \mathfrak{h}_3$.

A naturally reductive homogeneous structure on $\mathcal{G}(\mathfrak{d})$

Our goal now is to prove the existence of a homogeneous structure of type $\mathcal{T}_3$ on the simply connected Lie group $\mathcal{G}(D)$.

Recall that a *homogeneous structure* on a connected simply connected pseudo-Riemannian manifold (M, g) is a (1,2) tensor field on M satisfying the conditions of the next theorem, first proved in [1].

Theorem 3.3. *A connected complete and simply connected Riemannian manifold* (M, g) *is homogeneous if and only if there exists a tensor field* T *of type (1,2) on* M *such that*

(i) $g(T_x y, z) + g(y, T_x z) = 0$
(ii) $(\nabla_x R)(y, z) = [T_x, R(y, z)] - R(T_x y, z) - R(y, T_x z)$
(iii) $(\nabla_x)T_y = [T_x, T_y] - T_{T_x y}$

for $x, y, z \in \xi(M)$*, where* ∇ *denotes the Levi-Civita connection of* (M, g) *and* R *the corresponding curvature tensor.*

If, in addition, T satisfies the following condition (iv), then M is a naturally reductive homogeneous space:

(iv) $T_x x = 0$ for all $x \in \xi(M)$.

And the converse holds, that is, if (M, g) is naturally reductive, then there is a (1,2) tensor on M satisfying (i)-(iv). This was proved in the Riemannian case by Tricerri and Vanhecke [38] (see also [39]) and generalized to the pseudo-Riemannian case by Gadea and Oubiña (Proposition 4.1 in [17]). The tensor T is called a *naturally reductive homogeneous structure* on M or a *homogeneous structure of type* $\mathcal{T}_3$.

Let $\tilde{\nabla}$ define on (M, g) by $\tilde{\nabla} := T - \nabla$, then the conditions (i)-(iii) above can be rewriten in the following way:

(i)' $\tilde{\nabla} g = 0$
(ii)' $\tilde{\nabla} R = 0$
(iii)' $\tilde{\nabla} T = 0$.

Since the metric $\langle\, ,\, \rangle$ on $\mathcal{G}(D)$ is invariant by translations on the left, to give a naturally reductive homogeneous structure for the groups $\mathcal{G}(D)$, it suffices to define such a T for left-invariant vector fields, that is, on $\mathcal{G}(\mathfrak{d})$.

In the next section we shall study in more detail some geometric properties of $\mathcal{G}(D)$. For now, we make use of the Levi Civita connection ∇ and the curvature tensor, which for elements in $\mathcal{G}(\mathfrak{d})$ are given by

$$\nabla_{x_1 + h_1^*} x_2 + h_2^* = \frac{1}{2}([x_1, x_2] - \pi(h_1)x_2 - \pi(h_2)x_1) \qquad x_i \in \mathfrak{d}, h_i^* \in \mathfrak{h}^*,\ i = 1, 2.$$

and $R(w_1, w_2) = [\nabla_{w_1}, \nabla_{w_2}] - \nabla_{[w_1, w_2]}$, respectively.

Theorem 3.4. *Let* $\mathcal{G}(D)$ *be the simply connected Lie group endowed with the pseudo-Riemannian metric obtained by left-translations of* $\langle\, ,\, \rangle$ *on its Lie algebra* $\mathcal{G}(\mathfrak{d})$ *(15). The following tensor* T *defines a naturally reductive structure on* $\mathcal{G}(D)$*:*

$$T_x y = \frac{1}{2}\lambda^{-1}([\lambda x, \lambda y]_{\mathfrak{m}}) \tag{18}$$

explicitly

$$T_{x_1+h_1^*}x_2+h_2^* = \frac{1}{2}([x_1,x_2]+\pi(h_1)x_2-\pi(h_2)x_1)+[h_1,h_2]_{\mathfrak h}^*. \tag{19}$$

Proof. The equality (18) proves (i). In fact via the linear isometry $\lambda : \mathcal{G}(\mathfrak{d}) \to \mathfrak{m}$ condition (i) is equivalent to

$$Q_-([\lambda x,\lambda y]_{\mathfrak m},\lambda z)+Q_-(\lambda y,[\lambda x,\lambda z]_{\mathfrak m})=0 \qquad \text{for all } x,y,z\in\mathcal{G}(\mathfrak{d}).$$

One clearly has $T_x x=0$ for all $x\in\mathcal{G}(\mathfrak{d})$.

By taking $\tilde\nabla = T-\nabla$ gives

$$\tilde\nabla_{x_1+h_1^*}x_2+h_2^* = \pi(h_1)x_2+[h_1,h_2]_{\mathfrak h}^*. \tag{20}$$

We show that $\tilde\nabla T=0$. On the one hand one has

(21)
$$\begin{array}{rcl}\tilde\nabla_{x_1+h_1^*}(T_{x_2+h_2^*}x_3+h_3^*) & = & \tilde\nabla_{h_1^*}(\frac12\pi(h_2)x_3-\frac12\pi(h_3)x_2+\frac12[x_2,x_3]+[h_1,h_2]_{\mathfrak h}^*)\\ & = & \frac12\pi(h_1)\pi(h_2)x_3-\frac12\pi(h_1)\pi(h_3)x_2+\frac12\pi(h_1)[x_2,x_3]_{\mathfrak d}\\ & & +\frac12[h_1,\beta^*(x_2,x_3)]_{\mathfrak h}^*+[h_1,[h_2,h_3]_{\mathfrak h}]_{\mathfrak h}^*\end{array}$$

and on the other hand

(22)
$$\begin{array}{rcl}T_{\tilde\nabla_{x_1+h_1^*}x_2+h_2^*}x_3+h_3^* & = & T_{\pi(h_1)x_2+[h_1,h_2]_{\mathfrak h}^*}x_3+h_3^*\\ & = & \frac12[\pi(h_1)x_2,x_3]-\frac12\pi(h_3)\pi(h_1)x_2+\frac12\pi([h_1,h_2]_{\mathfrak h})x_3\\ & & +[[h_1,h_2]_{\mathfrak h},h_3]_{\mathfrak h}^*\end{array}$$

(23)
$$\begin{array}{rcl}T_{x_2+h_2^*}(\tilde\nabla_{x_1+h_1^*}x_3+h_3^*) & = & T_{x_2+h_2^*}(\pi(h_1)x_3+[h_1,h_3]_{\mathfrak h}^*)\\ & = & \frac12[x_2,\pi(h_1)x_3]-\frac12\pi([h_1,h_3]_{\mathfrak h})x_2+\frac12\pi(h_2)\pi(h_1)x_3\\ & & +[h_2,[h_1,h_3]_{\mathfrak h}]_{\mathfrak h}^*\end{array}$$

Hence by computing (21)-(22)-(23) we see that $\tilde\nabla T$ vanishes: by using the Jacobi identity in $\mathfrak h$, the fact that π is a representation of $\mathfrak h$ to $\mathrm{End}(\mathfrak d)$ acting by skew-symmetric derivations of $\mathfrak d$ and the relation

$$[h_1,\beta^*(x,y)]_{\mathfrak h}^* = \beta(\pi(h_1)x,y)+\beta(x,\pi(h_1)y) \tag{24}$$

which derives from the definition of β. All these facts should be also used to prove $\tilde\nabla R=0$. We exemplify here one case, looking at the formulas for the curvature tensor in the next section.

$$\begin{array}{rcl}\tilde\nabla_{x_1+h_1^*}(R(x_2,x_3)h^*) & = & -\frac14[h_1,\beta^*(x_2,\pi(h)x_3)]_{\mathfrak h}^*-\frac14[h_1,\beta^*(\pi(h)x_2,x_3)]_{\mathfrak h}^*\\ & & +\frac14\pi(h_1)\pi(h)[x_2,x_3]_{\mathfrak d}\end{array} \tag{25}$$

$$\begin{array}{rcl}R(\tilde\nabla_{x_1+h_1^*}x_2,x_3)h^* & = & -\frac14\beta(\pi(h_1)x_2,\pi(h)x_3)-\frac14\beta(\pi(h)\pi(h_1)x_2,x_3)\\ & & +\frac14\pi(h)[\pi(h_1)x_2,x_3]_{\mathfrak d}\end{array} \tag{26}$$

$$\begin{array}{rcl}R(x_2,\tilde\nabla_{x_1+h_1^*}x_3)h^* & = & -\frac14\beta(x_2,\pi(h)\pi(h_1)x_3)-\frac14\beta(\pi(h)x_2,\pi(h_1)x_3)\\ & & +\frac14\pi(h)[x_2,\pi(h_1)x_3]_{\mathfrak d}\end{array} \tag{27}$$

$$R(x_2,x_3)\tilde{\nabla}_{x_1+h_1^*}h^* = -\tfrac{1}{4}\beta(\pi([h_1,h]_{\mathfrak{h}})x_2,x_3) - \tfrac{1}{4}\beta(x_2,\pi([h_1,h]_{\mathfrak{h}})x_3) + \tfrac{1}{4}\pi([h_1,h]_{\mathfrak{h}})[x_2,x_3]_{\mathfrak{d}} \tag{28}$$

The expression (25)-(26)-(27)-(28) vanishes, which follows from similar arguments as above, and also from the next relations

$$[h_1,\beta^*(x_2,\pi(h)x_3)]^*_{\mathfrak{h}} = \beta(\pi(h_1)x_2,\pi(h)x_3)+\beta(x_2,\pi(h_1)\pi(h)x_3)$$

$$[h_1,\beta^*(\pi(h)x_2,x_3)]^*_{\mathfrak{h}} = \beta(\pi(h_1)\pi(h)x_2,x_3)+\beta(\pi(h)x_2,\pi(h_1)x_3)$$

□

3.1. **Naturally reductive Riemannian nilmanifolds.** In the next paragraphs we explain the construction of Riemannian nilmanifolds according to Theorem (3.1). The translation of our model to that one given in [25] is basically due to the equivalence between the adjoint and the coadjoint representations on a compact Lie algebra. We also summarize some known results concerning these spaces. See [18, 25] and references therein for more details.

The simply connected naturally reductive Riemannian nilmanifolds without Euclidean factor arise from the following data set:

- a compact Lie algebra $\mathfrak{k} = \bar{\mathfrak{k}} \oplus \mathfrak{c}$ with $\bar{\mathfrak{k}} = [\mathfrak{k},\mathfrak{k}]$ and $\mathfrak{c}$ the center of $\mathfrak{k}$;
- a faithful representation (π, V) of $\mathfrak{k}$ without trivial subrepresentations;
- an $\mathrm{ad}_{\mathfrak{k}}$-invariant inner product $\langle\, ,\,\rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ and a $\pi(\mathfrak{k})$-invariant inner product $\langle\, ,\,\rangle_V$ on V.

Let $\mathfrak{n}$ denote the vector space direct sum $\mathfrak{n} = V \oplus \mathfrak{k}^*$ and equip $\mathfrak{n}$ with the metric $\langle\, ,\,\rangle$ obtained as in (15). The Lie bracket on $\mathfrak{n} = V \oplus \mathfrak{k}^*$ satisfies $[\mathfrak{k}^*,\mathfrak{n}]_{\mathfrak{n}} = 0$, $[V,V] \subseteq \mathfrak{k}$ and

$$\langle [u,v],k^*\rangle_{\mathfrak{k}^*} = \langle \pi(k)u,v\rangle_V \qquad \text{for } k\in\mathfrak{k}, u,v\in V. \tag{29}$$

Since (π, V) has no trivial subrepresentations, the center of $\mathfrak{n}$ coincides with $\mathfrak{k}^*$ and since (π, V) is faithful, the commutator of $\mathfrak{n}$ is $\mathfrak{k}^*$: $C(\mathfrak{n}) = \mathfrak{k}^*$. Take N the simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ and endow it with the Riemannian metric invariant by a left-action determined by $\langle\, ,\,\rangle$.

The Lie group $(N,\langle\, ,\,\rangle)$ is naturally reductive and with the notations of this section it coincides with the group $\mathcal{G}(V)$ (by identifying V with an abelian Lie algebra).

In the case of Riemannian nilmanifolds, the isotropy group of isometries coincides with the group of orthogonal automorphisms (see [43]). Its Lie algebra $\mathfrak{k} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n},\langle\, ,\,\rangle)$ for naturally reductive simply connected nilmanifolds, is given by

$$\mathfrak{k} = \bar{\mathfrak{k}} \oplus \mathfrak{u}, \qquad [\bar{\mathfrak{k}},\mathfrak{u}] = 0, \tag{30}$$

where $\mathfrak{u} = \mathrm{End}_{\mathfrak{k}}(V) \cap \mathfrak{so}(V,\langle\, ,\,\rangle_V)$ and $\mathrm{End}_{\mathfrak{k}}(V)$ denotes the set of intertwinning operators of the representation (π, V) of $\mathfrak{k}$.

- $\bar{\mathfrak{k}}$ acts on $\mathfrak{n} = V \oplus \mathfrak{k}^*$ by $(\pi(k), \mathrm{ad}^*(k))$ for all $k \in \bar{\mathfrak{k}}$,
- $\mathfrak{u}$ acts trivially on $\mathfrak{k}^*$.

On the simply connected naturally reductive nilmanifold $(N, \langle\, ,\, \rangle)$ a naturally reductive homogeneous structure is given by

$$T_{v_1+k_1^*} v_2 + k_2^* = \frac{1}{2}(\pi(k_1)v_2 - \pi(k_2)v_1) + \frac{1}{2}\beta(v_1, v_2) + [k_1, k_2]_{\mathfrak{k}}^*$$

as shown in [26]. Moreover whenever N has no Euclidean factor, such a naturally reductive homogeneous structure is unique.

4. About the geometry of $(\mathcal{G}(D), \langle\, ,\, \rangle)$

Since $\langle\, ,\, \rangle$ is invariant by left-translations, the covariant derivative ∇ is also invariant by left-translations and one can see it as a bilinear map $\nabla : \mathcal{G}(\mathfrak{d}) \times \mathcal{G}(\mathfrak{d}) \to \mathcal{G}(\mathfrak{d})$. To get it explicitly one applies the Koszul formula for $w_1, w_2, w_3 \in \mathcal{G}(\mathfrak{d})$:

$$2\langle \nabla_{w_1} w_2, w_3\rangle = \langle [w_1, w_2], w_3\rangle - \langle [w_2, w_3], w_1\rangle + \langle [w_3, w_1], w_2\rangle.$$

By computing, if $w_i = x_i + h_i^* \in \mathfrak{d} \oplus \mathfrak{h}^*$ for i=1,2,3, one obtains that the second term in the right side of the Koszul formula is

$$\begin{aligned} \langle \beta(x_2, x_3), h_1^*\rangle + \langle [x_2, x_3]_{\mathfrak{d}}, x_1\rangle &= \langle \pi(h_1)x_2, x_3\rangle + \langle [x_2, x_3]_{\mathfrak{d}}, x_1\rangle \\ &= \langle \pi(h_1)x_2, x_3\rangle + \langle [x_1, x_2]_{\mathfrak{d}}, x_3\rangle \end{aligned}$$

and the third term is

$$\begin{aligned} \langle \beta(x_3, x_1), h_2^*\rangle + \langle [x_3, x_1]_{\mathfrak{d}}, x_2\rangle &= \langle \pi(h_2)x_3, x_1\rangle + \langle [x_3, x_1]_{\mathfrak{d}}, x_2\rangle \\ &= -\langle \pi(h_2)x_1, x_3\rangle + \langle [x_1, x_2]_{\mathfrak{d}}, x_3\rangle \end{aligned}$$

where we are making use of (16) and the fact that $\langle\, ,\, \rangle_{\mathfrak{d}}$ is $\mathrm{ad}_{\mathfrak{d}}$-invariant. Therefore the Levi Civita connection is

$$(31) \quad \nabla_{x_1+h_1^*} x_2 + h_2^* = \frac{1}{2}([x_1, x_2] - \pi(h_1)x_2 - \pi(h_2)x_1) \qquad h_i^* \in \mathfrak{h}^*, x_i \in \mathfrak{d},\ i = 1, 2.$$

Using this, the curvature tensor defined by

$$R(w_1, w_2)w_3 = [\nabla_{w_1}, \nabla_{w_2}]w_3 - \nabla_{[w_1,w_2]}w_3$$

follows

- for $x, y, z \in \mathfrak{d}$:
$$R(x, y)z = \frac{1}{2}\pi(\beta^*(x, y))z - \frac{1}{4}\pi(\beta^*(y, z))x - \frac{1}{4}\pi(\beta^*(z, x))y - \frac{1}{4}[[x, y]_{\mathfrak{d}}, z]$$
- for $x_1, x_2 \in \mathfrak{d}$, $h^* \in \mathfrak{h}^*$:
$$\begin{aligned} R(x_1, x_2)h^* &= -\tfrac{1}{4}\beta(x_1, \pi(h)x_2) - \tfrac{1}{4}\beta(\pi(h)x_1, x_2) + \tfrac{1}{4}\pi(h)[x_1, x_2]_{\mathfrak{d}} \\ R(x_1, h^*)x_2 &= -\tfrac{1}{4}[x_1, \pi(h)x_2] + \tfrac{1}{4}\pi(h)[x_1, x_2]_{\mathfrak{d}} \\ &= [\tfrac{1}{4}\pi(h)x_1, x_2]_{\mathfrak{d}} - \tfrac{1}{4}\beta(x_1, \pi(h)x_2) \end{aligned}$$
- for $x \in \mathfrak{d}$, $h_1^*, h_2^* \in \mathfrak{h}^*$:
$$\begin{aligned} R(x, h_1^*)h_2^* &= -\tfrac{1}{4}\pi(h_1)\pi(h_2)x \\ R(h_1^*, h_2^*)x &= \tfrac{1}{4}\pi([h_1, h_2]_{\mathfrak{h}})x \end{aligned}$$
- for $h_i^* \in \mathfrak{h}^*$ i=1,2,3 $\qquad R(h_1^*, h_2^*)h_3^* = 0,$

where we use the convention: if $z \in \mathfrak{h}^*$ then $z^* = \ell^{-1}(z) \in \mathfrak{h}$ denotes the image by $\ell^{-1} : \mathfrak{h}^* \to \mathfrak{h}$.

The curvature tensor of $\mathfrak{d}$ equipped with the $\mathrm{ad}_{\mathfrak{d}}$-invariant metric $\langle\, ,\, \rangle_{\mathfrak{d}}$ is given by $R^d(x,y) = -\frac{1}{4}\,\mathrm{ad}_{\mathfrak{d}}([x,y]_{\mathfrak{d}})$. Hence the curvature tensor of $\mathfrak{d}$ and $\mathcal{G}(\mathfrak{d})$ are related by the formula

$$R(x,y)z = \frac{1}{2}\pi(\beta^*(x,y))z - \frac{1}{4}\pi(\beta^*(y,z))x - \frac{1}{4}\pi(\beta^*(z,x))y + \frac{1}{4}\beta(z,[x,y]_{\mathfrak{d}}) + R^d(x,y)z.$$

Let $\Pi \subseteq \mathcal{G}(\mathfrak{d})$ denote a plane and let Q be the real number obtained by computing

$$Q(x,y) = \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2 \qquad x, y \text{ basis of } \Pi.$$

The plane Π is nondegenerate if and only if $Q(x,y) \neq 0$ for one (hence every) basis of Π [34]. Take x, y be a orthonormal basis of a plane Π, that is, a linearly independent set $\{x,y\}$ such that $\langle x,y\rangle = 0$, $\langle x,x\rangle = \varepsilon_1$, $\langle y,y\rangle = \varepsilon_2$, where $\varepsilon_i = \pm 1$ for i=1,2. The sectional curvature of Π is given by

$$K(x,y) = \varepsilon_1\varepsilon_2\langle R(x,y)y, x\rangle$$

which, after the formulas above for the curvature tensor, gets:

$$K(x,y) = \begin{cases} \varepsilon_1\varepsilon_2\left(\frac{1}{4}\langle [x,y]_{\mathfrak{d}}, [x,y]_{\mathfrak{d}}\rangle - \frac{3}{4}\langle \beta(x,y), \beta(x,y)\rangle\right) & \text{for } x, y \in \mathfrak{d} \\ \frac{\varepsilon_1\varepsilon_2}{4}\langle \pi(y^*)x, \pi(y^*)x\rangle & \text{for } x \in \mathfrak{d}, y \in \mathfrak{h}^* \\ 0 & \text{for } x, y \in \mathfrak{h}^*. \end{cases}$$

Example 4.1. Let $\langle\, ,\, \rangle_i$ i=1,2, denote the Lorentzian metrics on the Heisenberg Lie group H_3 in Example (3.2). One verifies that for $\langle\, ,\, \rangle_1$ the sectional curvature is nonpositive, while for $\langle\, ,\, \rangle_2$ it could take both positive and negative values, for instance $K(e_1, e_2) > 0$ and $K(e_1, e_3) < 0$.

The Ricci tensor, defined as $\mathrm{Ric}(x,y) = \mathrm{tr}(z \to R(z,x)y)$ for $x, y \in \mathcal{G}(\mathfrak{d})$, becomes a symmetric bilinear form on $\mathcal{G}(\mathfrak{d})$; this is due to the symmetries of the curvature tensor. Hence there exists a symmetric linear transformation $T : \mathcal{G}(\mathfrak{d}) \to \mathcal{G}(\mathfrak{d})$ satisfying

$$\mathrm{Ric}(x,y) = \langle Tx, y\rangle \qquad \text{for all } x, y \in \mathcal{G}(\mathfrak{d})$$

and which is called the *Ricci transformation*. If $\{e_k\}$ denotes a orthonormal basis of $\mathcal{G}(\mathfrak{d})$ such that $\langle e_k, e_k\rangle = \varepsilon_k$, then

$$\mathrm{Ric}(x,y) = \sum_k \varepsilon_k \langle R(e_k,x)y, e_k\rangle = \langle -\sum_k \varepsilon_k \langle R(e_k,x)e_k, y\rangle$$

and this implies

$$T(x) = -\sum_k \varepsilon_k R(e_k,x)e_k \qquad \text{for all } x \in \mathcal{G}(\mathfrak{d}).$$

Let $\{z_i^*\}$ denote a orthonormal basis of $\mathfrak{h}^*$ with $\varepsilon_i = \langle z_i^*, z_i^*\rangle$ and $\{d_j\}$ denote a orthonormal basis of $\mathfrak{d}$ with $\varepsilon_j = \langle d_j, d_j\rangle$. By computing one obtains

- for $x \in \mathfrak{d}$, $h^* \in \mathfrak{h}^*$:
$$\mathrm{Ric}(x, h^*) = \frac{1}{4}\sum_j \varepsilon_j \langle \beta(d_j, [d_j, x]_{\mathfrak{d}}), h^*\rangle = -\frac{1}{4}\,\mathrm{tr}(\pi(h)\,\mathrm{ad}_{\mathfrak{d}}(x))$$
- for $x, y \in \mathfrak{d}$:
$$\mathrm{Ric}(x, y) = \frac{1}{2}\sum_i \varepsilon_i \langle \pi^2(z_i)x, y\rangle - \frac{1}{4}\,\mathrm{tr}(\mathrm{ad}_{\mathfrak{d}}(x)\circ \mathrm{ad}_{\mathfrak{d}}(y))$$
- for $h_1^*, h_2^* \in \mathfrak{h}^*$:
$$\mathrm{Ric}(h_1^*, h_2^*) = -\frac{1}{4}\,\mathrm{tr}(\pi(h_1)\pi(h_2)).$$

To get these relations, one makes use of the formulas for the curvature tensor. For $x, y \in \mathfrak{d}$ one needs to work a little more. By writing $\beta(d_j, x) = \sum_i \varepsilon_i \langle \beta(d_j, x), z_i^*\rangle z_i^*$ one has

$$\begin{array}{rcl} \sum_j \varepsilon_j \langle \pi(\beta^*(d_j, x))d_j, y\rangle & = & \sum_j \varepsilon_j \sum_i \varepsilon_i \langle \beta(d_j, x), z_i^*\rangle \langle \pi(z_i)d_j, y\rangle \\ & = & \sum_i \varepsilon_i \langle \pi(z_i)(\sum_j \varepsilon_j \langle \pi(z_i)d_j, x\rangle d_j), y\rangle \\ & = & \sum_i \varepsilon_i \langle \sum_j \varepsilon_j \langle \pi(z_i)x, d_j\rangle d_j, \pi(z_i)y\rangle \\ & = & \sum_i \varepsilon_i \langle \pi(z_i)x, \pi(z_i)y\rangle \\ & = & -\sum_i \varepsilon_i \langle \pi^2(z_i)x, y\rangle \end{array}$$

where also $\sum_j \varepsilon_j \langle \pi(z_i)x, d_j\rangle d_j = \pi(z_i)x$. Thus the Ricci transformation is

$$\begin{array}{rcll} T(h^*) & = & \frac{1}{4}\sum_j \varepsilon_j [d_j, \pi(h)d_j] & h^* \in \mathfrak{h}^* \\ T(x) & = & \frac{1}{2}\sum_i \varepsilon_i \pi(z_i)^2 x - \frac{1}{4}\sum_j \varepsilon_j\, \mathrm{ad}^2(d_j)x & x \in \mathfrak{d}. \end{array}$$

Remark. Note that whenever $\mathfrak{d}$ is abelian, (and therefore $\mathcal{G}(\mathfrak{d})$ is 2-step nilpotent) the Ricci transformation preserves the decomposition $\mathfrak{d} \oplus \mathfrak{h}^*$, that is $T(\mathfrak{h}^*) \subseteq \mathfrak{h}^*$ and $T(\mathfrak{d}) \subseteq \mathfrak{d}$.

Let $\mathcal{S}(D)$ denote a connected subgroup of $\mathcal{G}(D)$ with Lie algebra $\mathcal{S}(\mathfrak{d}) \subseteq \mathcal{G}(\mathfrak{d})$. Since translations on the left by elements of $\mathcal{S}(D)$ are isometries of $\mathcal{G}(D)$ then $\mathcal{S}(D)$ is totally geodesic if and only if it is totally geodesic at the identity. Therefore $\mathcal{S}(D) \subseteq \mathcal{G}(D)$ is a totally geodesic submanifold of $\mathcal{G}(D)$ if and only if $\nabla_u v \in \mathcal{S}(\mathfrak{d})$ whenever $u, v \in \mathcal{S}(\mathfrak{d})$.

Definition 4.2. A Lie algebra $\mathcal{S}(\mathfrak{d}) \subseteq \mathcal{G}(\mathfrak{d})$ is totally geodesic if $\nabla_u v \in \mathcal{S}(\mathfrak{d})$ for all $u, v \in \mathcal{S}(\mathfrak{d})$.

Example 4.3. The Lie subalgebra $\mathfrak{h}^*$ is a flat totally real Lie algebra.

Example 4.4. Let $\xi = x + h^* \in \mathcal{G}(\mathfrak{d})$ be arbitrary. The 1-parameter subgroup $\exp(t\xi)$ is a geodesic if and only if $\pi(h)x = 0$. In particular if $\xi \in \mathfrak{h}^*$ or $\xi \in \mathfrak{d}$ the curve $t \to \exp(t\xi)$ is a geodesic starting at the identity.

Now we would like to investigate some groups acting by isometries on $(\mathcal{G}(D), \langle\,,\,\rangle)$. Let $\mathrm{I}(\mathcal{G}(D))$ denote the group of all isometries of $\mathcal{G}(D)$. It is well known that the group $\mathrm{I}(\mathcal{G}(D))$, endowed with the compact-open topology, is a Lie group, even a Lie transformation group of $\mathcal{G}(D)$. For $u \in \mathcal{G}(D)$ let L_u denote the left-translation by $u \in \mathcal{G}(D)$ and let $\mathcal{L}$ denote the subgroup of all left-translations. Let $\mathcal{F}$ denote the stability subgroup of the identity element e (see [33]).

Lemma 4.5. *$\mathcal{F}$ is a closed subgroup of $\mathcal{G}(D)$, $\mathcal{L}$ is a closed connected and simply connected Lie subgroup isomorphic to $\mathcal{G}(D)$ and $\mathrm{I}(\mathcal{G}(D)) = \mathcal{F}\mathcal{L}$, where $\mathcal{L}\cap\mathcal{F} = \{id\}$. Each element of $\mathcal{F}$ is determined by its differential at e.*

Let $\mathsf{O}_{aut}(\mathcal{G}(D))$ denote the group of isometric automorphisms and let $\mathrm{I}_{spl}(\mathcal{G}(D))$ denote the subgroup of $\mathrm{I}(\mathcal{G}(D))$ preserving the splitting $T\mathcal{G}(D) = \mathfrak{h}^*\mathcal{G}(D) \oplus \mathfrak{d}\mathcal{G}(D)$.

Then $\mathrm{I}_{aut}(\mathcal{G}(D)) = \mathsf{O}_{aut}(\mathcal{G}(D)) \oplus \mathcal{G}(D)$ semidirect sum and it holds

$$\mathrm{I}_{aut}(\mathcal{G}(D)) \subseteq \mathrm{I}_{spl}(\mathcal{G}(D)) \subseteq \mathrm{I}(\mathcal{G}(D)).$$

We shall study here the case of isometries preserving the subspaces $\mathfrak{h}^*$ and $\mathfrak{d}$.

Let φ denote an isometry which leaves invariant the subspaces $\mathfrak{h}^*$ and $\mathfrak{d}$; write $\varphi = \psi + \phi$ where $\psi \in \mathsf{O}(\mathfrak{h}^*, \langle\, ,\, \rangle_{\mathfrak{h}^*})$ and $\phi \in \mathsf{O}(\mathfrak{d}, \langle\, ,\, \rangle_{\mathfrak{d}})$. It is clear that the restriction of φ to $\mathfrak{d}$ must be an isometry of $\mathfrak{d}$, hence according to [33], ϕ must satisfy

$$\phi[x,[y,z]]_{\mathfrak{d}} = [\phi x,[\phi y,\phi z]]_{\mathfrak{d}} \qquad \text{for all } x,y,z \in \mathfrak{d}.$$

By using this, whenever $\psi[x,[y,z]]_{\mathfrak{h}^*} = [\phi x,[\phi y,\phi z]]_{\mathfrak{h}^*}$ for all $x,y,z \in \mathfrak{d}$, by computing $\langle \psi[x,[y,z]]_{\mathfrak{h}^*}, \psi h^* \rangle = \langle [x,[y,z]], h^* \rangle$ one gets

$\langle \pi(\psi h^*)\phi x, [\phi y, \phi z]_{\mathfrak{d}} \rangle = \langle \pi(h)x, [y,z]_{\mathfrak{d}} \rangle$ and therefore

$\mathrm{ad}_{\mathfrak{d}}(\phi z) \circ \pi(\psi h) = \phi \circ \pi(h) \circ \psi^{-1} \quad$ for all $h \in \mathfrak{h}, z \in \mathfrak{d}$.

Now we compute the group of orthogonal automorphisms of $\mathcal{G}(D)$. Since $\mathcal{G}(D)$ is simply connected we see it at the Lie algebra level. Let $\varphi : \mathcal{G}(\mathfrak{d}) \to \mathcal{G}(\mathfrak{d})$ be a orthogonal automorphism. Since φ preserves the center, as above we can write it as $\varphi = \psi + \phi$. One computes that $\langle [\phi x_1, \phi x_2], \psi h^* \rangle = \langle [x_1, x_2], h^* \rangle$ so one has

$$\langle \beta(\phi x_1, \phi x_2), \psi h^* \rangle \;=\; \langle \beta(x_1, x_2), h^* \rangle$$

getting

$$\pi(\psi h) = \phi \pi(h) \phi^{-1} \qquad \text{for all } h \in \mathfrak{h}, \tag{32}$$

while

$$\langle [\phi x_1, \phi x_2], \phi x_3 \rangle_{\mathfrak{d}} = \langle [x_1, x_2], x_3 \rangle_{\mathfrak{d}}$$

implies that $\phi \in \mathsf{O}(\mathfrak{d}, \langle\, ,\, \rangle_{\mathfrak{d}}) \cap \mathrm{Aut}(\mathfrak{d})$.

Proposition 4.6. *The group of orthogonal automorphisms of $\mathcal{G}(\mathfrak{d})$ consists of elements*

$$\psi + \phi \in \mathsf{O}(\mathfrak{h}^*, \langle\, ,\, \rangle_{\mathfrak{h}^*}) \times (\mathsf{O}(\mathfrak{d}, \langle\, ,\, \rangle_{\mathfrak{d}}) \cap \mathrm{Aut}(\mathfrak{d})) \,:\, \phi\pi(h)\phi^{-1} = \pi(\psi h) \text{ for all } h \in \mathfrak{h}.$$

Its Lie algebra $\mathfrak{so}_{aut}(\mathcal{G}(\mathfrak{d}))$ is therefore

$$A + B \in \mathfrak{so}(\mathfrak{h}^*, \langle\, ,\, \rangle_{\mathfrak{h}^*}) \times \mathrm{Dera}(\mathfrak{d}) \quad : \quad [B, \pi(h)] = \pi(Ah) \quad \text{for all } h \in \mathfrak{h}.$$

In particular $\pi(\mathfrak{h}) \subseteq \mathfrak{so}_{aut}(\mathcal{G}(\mathfrak{d}))$.

Remark. The isometry group of a bi-invariant metric was studied in [33]. Isotropy groups of compact and non compact type can be obtained starting with an abelian Lie algebra provided with different metrics. The resulting Lie group is 2-step nilpotent. Moreover whenever the center is nondegenerate the stability group coincides with the group of orthogonal automorphisms [4] (see [36] for examples).

5. Examples and applications

The examples considered in (3.2) appeared in [36] while naturally reductive Riemannian nilmanifolds were extended studied in the past. For those cases (3.1) and (3.4) offer a new point of view. An original contribution is the production of several Lie groups equipped with naturally reductive metrics (and the corresponding homogeneous structures) which have no analogous in the Riemannian case. Explicitly we shall construct examples in dimension six which are 4-step nilpotent. We shall also study the algebraic structure of these objects.

Let $\mathfrak{a}(p,q)$ denote the double extension Lie algebra of $\mathbb{R}^{p,q}$ via the skew-symmetric map $A \in \mathfrak{so}(p,q)$. Thus $\mathfrak{a}(p,q)$ is solvable and scalar flat (see [2]). It is indecomposable if $\ker A$ is totally isotropic.

Let $\mathcal{G}(\mathbb{R}^{p,q})$ denote the naturally reductive Lie algebra constructed fron $\mathbb{R}^{p,q}$ via A after Theorem (3.1). Let z be an element generating the one-dimensional vector space complementary to $\mathbb{R}^{p,q}$. The Lie bracket on $\mathcal{G}(\mathbb{R}^{p,q})$ is given by

$$[x,y] = \langle Ax, y\rangle z \qquad \text{for all } x,y \in \mathbb{R}^{p,q}.$$

The center of $\mathcal{G}(\mathbb{R}^{p,q})$ is

$$\mathfrak{z}(\mathcal{G}(\mathbb{R}^{p,q})) = \mathbb{R}z \oplus \ker A$$

and if $\tilde{\mathfrak{z}} \subseteq \ker A$ is a nondegenerate subspace in $\mathbb{R}^{p,q}$ then $\mathcal{G}(\mathbb{R}^{p,q})$ is decomposable.

Proposition 5.1. *Any Lie algebra $\mathcal{G}(\mathbb{R}^{p,q})$ is 2-step nilpotent. Moreover*

- *If A is nonsingular then $\mathcal{G}(\mathbb{R}^{p,q})$ is the Heisenberg Lie algebra $\mathfrak{h}_{2s+1}$ with $2s = p+q$.*
- *If A is singular then $\mathcal{G}(\mathbb{R}^{p,q})$ is a central extension of a Heisenberg Lie algebra $\mathfrak{h}_{2s+1}$ with $2s \leq p+q$, and $\mathcal{G}(\mathbb{R}^{p,q})$ is indecomposable if $\ker A$ is totally isotropic.*

The signature of the metric on $\mathcal{G}(\mathbb{R}^{p,q})$ is $(p, q+1)$, or $(p, q+1)$, depending on the sign of z.

Moreover if the center of $\mathcal{G}(\mathbb{R}^{p,q})$ is nondegenerate the stability-group of isometries consists of the orthogonal automorphisms [4]*, hence they can be computed as in (4.6).*

Example 5.2. The isometry groups for the naturally reductive spaces $(H_3, \langle\,,\,\rangle_i)$ i=0,1,2 in (3.2) are

$\mathsf{O}(2) \oplus H_3$ for the metrics $\langle\,,\,\rangle_i$ i= 0,1;

$\mathsf{O}(1,1) \oplus H_3$ for the metric $\langle\,,\,\rangle_2$

where $\mathsf{O}(2)$ denote the orthogonal group of $\mathbb{R}^2$ and $\mathsf{O}(p,q)$ the orthogonal group for $\mathbb{R}^{p,q}$. On each isometry Lie algebra $\mathbb{R} \oplus \mathfrak{h}_3$ equipped with an ad-invariant metric, every skew-symmetric derivation is an inner derivation.

Example 5.3. The free 3-step nilpotent Lie algebra in two generators can be obtained by a double extension procedure: take $\mathbb{R}^{1,2}$ and the skew-symmetric map A with matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In this way we get the Lie algebra $\mathfrak{a}(1,2)$ spanned by the vectors e_0, e_1, e_2, e_3, e_4 which satisfy the Lie bracket relations

$$[e_4, e_1] = e_2 \qquad [e_4, e_2] = e_1 - e_3 \qquad [e_4, e_3] = e_2$$

$$[e_1, e_2] = e_0 = [e_3, e_2].$$

An ad-invariant metric can be defined on $\mathfrak{a}(1,2)$ obeying the rules

$$\langle e_1, e_1\rangle = -1 \qquad \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = \langle e_4, e_4\rangle = \langle e_0, e_4\rangle = 1.$$

The kernel of A is the vector space spanned by $e_1 - e_3$ which is totally isotropic. Thus

$$\mathcal{G}(\mathbb{R}^{1,2}) = \mathbb{R}z \oplus \mathfrak{h}_3$$

is indecomposable.

Note that the corresponding simply connected Lie group whose Lie algebra is $\mathcal{G}(\mathbb{R}^{1,2})$, admits two nilpotent non-isomorphic Lie groups acting transitively on it.

The resulting metric on $\mathcal{G}(\mathbb{R}^{1,2})$ is Lorentzian if the metric on z is considered positive, while it is neutral if the metric on z is chosen negative.

To get other kind of Lie algebras $\mathcal{G}(\mathfrak{d})$ which are not 2-step nilpotent, we should need nonabelian Lie algebras $\mathfrak{d}$. But moreover one also needs that the set of skew-symmetric derivations of such a $\mathfrak{d}$ consists not only of inner derivations. According to investigations in [15], if A, B are skew-symmetric derivations of $\mathfrak{d}$, then the double extension Lie algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ getting from A and B respectively are isomorphic and isometric if and only if there exist $\lambda \in \mathbb{R} - \{0\}$, $T \in \mathfrak{d}$ and $\varphi \in \mathrm{Aut}(\mathfrak{d})$ such that

$$\varphi B \varphi^{-1} = \lambda A + \mathrm{ad}(T).$$

The proof of the next lemma follows by computations following the definitions. Notice that we choose a different ad-invariant metric from that in (5.3).

Lemma 5.4. *Let* $\mathfrak{a}(1,2)$ *denote the free 3-step nilpotent Lie algebra in two generators as in (5.3). Then the Lie algebra of skew-symmetric derivations is the semidirect sum*

$$\mathrm{Dera}(\mathfrak{a}(1,2)) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{h}_3$$

where the Lie algebra of inner derivations of $\mathfrak{a}(1,2)$ *is isomorphic to* $\mathfrak{h}_3$*, the Heisenberg Lie algebra of dimension three.*

Proof. In the ordered basis $\{e_0, e_1 - e_3, e_2, e_1, e_4\}$ any skew-symmetric derivation has a matrix of the form

$$\begin{pmatrix} a & b & x & z & 0 \\ c & -a & y & 0 & z \\ 0 & 0 & 0 & y & -x \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & -a \end{pmatrix} \qquad a, b, c, x, y, z \in \mathbb{R}.$$

Denote by e_{ij} the 5×5 matrix whose entries are all 0 except for the entry 1 at the position (ij). Consider the matrices

$$\begin{array}{rcl} H & = & e_{11} - e_{22} + e_{44} - e_{55} \\ E & = & e_{12} + e_{45} \\ F & = & e_{21} + e_{54} \\ & & \\ X & = & e_{13} - e_{35} \\ Y & = & e_{23} + e_{34} \\ Z & = & e_{14} + e_{25}, \end{array}$$

then provided with the usual Lie bracket of matrices, one has $\operatorname{span}\{H, E, F\} = \mathfrak{sl}(2, \mathbb{R})$, $\operatorname{span}\{X, Y, Z\} = \mathfrak{h}_3$ and $[H, X] = X$, $[H, Y] = -Y$, $[E, Y] = X$, $[F, X] = Y$. Therefore the Lie algebra of skew-symmetric derivations is

$$\operatorname{Dera}(\mathfrak{a}(1,2)) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{h}_3 \qquad \text{semidirect sum.}$$

It is not hard to prove that the Lie algebra of inner derivations is $\operatorname{Deri}(\mathfrak{a}(1,2)) = \operatorname{span}\{X, Y, Z\}$. □

Each of the Lie algebras resulting from the construction of Theorem (3.1) by H, E, F is 4-step nilpotent. We shall study this in more detail.

For every Lie algebra $\mathfrak{g}$, recall that the derived $\{C^i(\mathfrak{g})\}$ and descending central $\{D^i(\mathfrak{g})\}$ series are inductively defined by

$$C^0(\mathfrak{g}) = \mathfrak{g} = D^0(\mathfrak{g})$$

$$C^i(\mathfrak{g}) = [C^{i-1}\mathfrak{g}, C^{i-1}\mathfrak{g}] \qquad D^i(\mathfrak{g}) = [\mathfrak{g}, D^{i-1}\mathfrak{g}] \qquad i \geq 1.$$

A Lie algebra is called *k-step* solvable if there exists k such that $C^k(\mathfrak{g}) = 0$ but $C^{k-1}(\mathfrak{g}) \neq 0$.

Let $\mathcal{G}(\mathfrak{d})$ denote the Lie algebra arising from the construction in (3.1). From the definitions above one has

$$C^0(\mathcal{G}(\mathfrak{d})) = \mathcal{G}(\mathfrak{d}) \qquad C^1(\mathcal{G}(\mathfrak{d})) = \{[x, y] \,:\, x, y \in \mathfrak{d}\} \subseteq C^1(\mathfrak{d}) \oplus \mathfrak{h}^*,$$

and inductively one verifies

$$C^i(\mathcal{G}(\mathfrak{d})) \subseteq C^i(\mathfrak{d}) \oplus \mathfrak{h}^* \qquad \text{for } i \geq 1.$$

Therefore if $\mathfrak{d}$ is k-step solvable, $C^k(\mathfrak{d}) = 0$ and hence $C^k(\mathcal{G}(\mathfrak{d})) \subseteq \mathfrak{h}^*$ giving

$$C^{k+1}(\mathcal{G}(\mathfrak{d})) = 0$$

so that $\mathcal{G}(\mathfrak{d})$ is at most $k+1$-step solvable. Notice that $\mathcal{G}(\mathfrak{d})$ could be k-step solvable if for all $x, y \in C^{k-1}(\mathfrak{d})$ it holds $\beta(x,y) = 0$, that is

$$\langle \pi(h)x, y\rangle_{\mathfrak{d}} = 0 \qquad \text{for all } h \in \mathfrak{h},\ x, y \in C^{k-1}(\mathfrak{d}). \tag{33}$$

A Lie algebra $\mathfrak{g}$ is said to be *k-step* nilpotent if $D^k(\mathfrak{g}) = 0$ but $D^{k-1}(\mathfrak{g}) \neq 0$. By computing one has

$$D^0(\mathcal{G}(\mathfrak{d})) = \mathcal{G}(\mathfrak{d}) \qquad D^1(\mathcal{G}(\mathfrak{d})) = \{[x,y] \,:\, x, y \in \mathcal{G}(\mathfrak{d})\} \subseteq D^1(\mathfrak{d}) \oplus \mathfrak{h}^*,$$

and in general by induction one gets

$$D^i(\mathcal{G}(\mathfrak{d})) \subseteq D^i(\mathfrak{d}) \oplus \mathfrak{h}^* \qquad \text{for } i \geq 1.$$

So if $\mathfrak{d}$ is k-step nilpotent, $\mathcal{G}(\mathfrak{d})$ is at most $k+1$-step nilpotent. Moreover, $\mathcal{G}(\mathfrak{d})$ is k-step nilpotent if and only if

$$\langle \pi(h)x, y\rangle_{\mathfrak{d}} = 0 \qquad \text{for all } h \in \mathfrak{h}, x \in D^k(\mathfrak{d}), y \in \mathfrak{d},$$

equivalently

$$D^k(\mathfrak{d}) \subseteq \bigcap_{h \in \mathfrak{h}} \ker \pi(h). \tag{34}$$

Proposition 5.5. *Let $\mathfrak{d}$ denote a Lie algebra with* ad*-invariant metric $\langle\, ,\, \rangle_{\mathfrak{d}}$ and let $\mathcal{G}(\mathfrak{d})$ denote the Lie algebra $\mathcal{G}(\mathfrak{d}) = \mathfrak{d} \oplus \mathfrak{h}^*$ with the Lie bracket as in (3.1). Then*

- *If $\mathfrak{d}$ is k-step solvable, $\mathcal{G}(\mathfrak{d})$ is either k- or $k+1$-step solvable. It is k-step solvable if and only if (33) holds.*
- *If $\mathfrak{d}$ is k-step nilpotent, $\mathcal{G}(\mathfrak{d})$ is either k- or $k+1$-step nilpotent. It is k-step nilpotent if and only if (34) holds.*

Example 5.6. Let $\{z, e_0, e_1 - e_3, e_2, e_1, e_4\}$ denote a basis of the central extension of $\mathfrak{a}(1,2)$ as in (3.1) by the cocycle induced by one of the skew-symmetric derivations H, E, F in Lemma (5.4). For each extension one has

$$[e_4, e_2] = e_2 \qquad [e_4, e_2] = e_1 - e_3 \qquad [e_1, e_2] = e_0,$$

and the additional Lie brackets follow

$$\begin{array}{ll} \mathcal{G}_H(\mathfrak{a}(1,2)) & -[e_4, e_0] = z = [e_1, e_1 - e_3] \\[2ex] \mathcal{G}_E(\mathfrak{a}(1,2)) & \qquad [e_4, e_1 - e_3] = z \\[2ex] \mathcal{G}_F(\mathfrak{a}(1,2)) & \qquad [e_1, e_1 - e_3] = z \end{array}$$

giving rise to 4-step nilpotent Lie algebras. The corresponding Lie groups provided with suitable pseudo Riemannian metrics as in Theorem (3.1) are naturally reductive but nonsymmetric.

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