Abstract. In this paper, we show that astrophysical accretion disks are dynamically unstable to non-axisymmetric disturbances. This instability is present in any stably stratified anticyclonically sheared flow as soon as the angular velocity decreases outwards. In the large Froude number limit, the maximal growth rate is proportional to the angular rotation velocity, and is independent of the stratification. In the low Froude number limit, it decreases like the inverse of the Froude number, thereby vanishing for unstratified, centrigugally stable flows. The instability is not sensitive to disk boundaries. We discuss the possible significance of our result, and its implications on the turbulent state achieved by the disks. We conclude that this linear instability is one of the best candidates for the source of turbulence in geometrically thin disks, and that magnetic fields can be safely ignored when studying their turbulent state. The relevance of the instability for thick disks or nearly neutrally stratified disks remains to be explored.

Key words: Accretion disks – hydrodynamic instabilities – turbulence

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A powerful local shear instability in stratified disks

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1. Introduction

The simplest model of an accretion disk is that of an axisymmetric rotating shear flow in hydrostatic vertical equilibrium, with a Keplerian velocity law. The hydrostatic state generates a vertical stratification. If this stratification is unstable, it leads to turbulence via convective instability. When the stratification is stable, it is generally ignored or thought to be unimportant in the stability analysis, under the rationale that it can only *stabilize* the flow. Ignoring the stratification makes the accretion disk look like a simple differentially rotating shear flow, with an azimuthal keplerian velocity profile $V(r) \sim r^{-1/2}$. Its linear stability with respect to axisymmetric disturbances is governed by the Rayleigh criterion in the inviscid limit:

$$\frac{d(rV)^2}{dr} > 0, \tag{1}$$

for stability. Flows obeying this criterion are called *centrifugally stable*. The keplerian flow, in which angular momentum increases outwards, falls into this category. Yet, there are observational evidences that astrophysical (putatively keplerian) disks are turbulent, and thus that a source of instability exists in these flows.

Various mechanism have been found able to destabilize centrifugally stable flows. They may or may not apply to astrophysical disks.

i) Centrifugally stable flows can be destabilized by non-axisymmetric instabilities (Papaloizou & Pringle 1984), but these instabilities generally involve reflecting boundary conditions over sharp edges, which appear unrealistic in the context of astrophysical disks.

ii) Centrifugally stable flows can also be destabilized by finite amplitude disturbances involving a non-linear mechanism not captured by the Rayleigh criterion (Dubrulle 1993; Richard & Zahn 1999). Laboratory experiments have shown that there is a critical Reynolds number above which the flow becomes turbulent (Wendt 1933; Taylor 1936):

$$R = \frac{r\Omega\Delta r}{\nu} > R_c. \tag{2}$$

Here, Δr is the radial extent of the flow (the gap), $\Omega = V/r$ is the rotation, and ν is the viscosity. The critical Reynolds number increases with gap width. For $\Delta r/r \longrightarrow 0$, it tends to that of plane Couette flow: $R_c \approx 2,000$, and for large enough gap ($\Delta r/r > 1/20$), the instability criterion becomes :

$$\frac{r^3}{\nu}\frac{\Delta\Omega}{\Delta r} > R_c^*; \tag{3}$$

this critical gradient Reynolds number R_c^* is of order 6×10^5 when the inner cylinder is at rest, which is the only case which has been thoroughly explored.

iii) Centrifugally stable flows can be further destabilized via a physical *catalyzer*, an additional ingredient making the flow unstable at smaller Reynolds numbers. A first instance is a vertical magnetic field (Chandrasekhar 1961). This magnetic field provides a linear axisymmetric instability mechanism for centrifugally stable flows via an interchange instability if $d(V/r)^2 < 0$, i.e. for anticyclonic sheared flows. The application of this mechanism to disks was first discussed by Balbus and Hawley (1991); they showed that the stratification of the disk does not modify the result, and that the maximal growth rate of instability in that case is proportional to the angular rotation velocity in the inviscid limit. The influence of viscosity and magnetic diffusivity η has been recently numerically studied by Rüdiger and Zhang (2001); they found that instability occurs above a critical Reynolds number, which depends on the magnetic Prandtl number $P_m = \nu/\eta$. From their numerical simulations, they fit $R_c = 51 Pm^{-0.65}$ down to $Pm = 0.01$. In cold disks, $Pm = 10^{-5}$, and so R_c is of the order 10^5 (in the inner part of the disk) (Rudiger & Zhang 2001). For anyone familiar with MHD flows, the instability of a centrifugally stable flow subject to vertical magnetic field is a surprise: magnetic fields generally inhibit instabilities, and thus, have a *stabilizing* influence. In the present case, it appears that a stabilizing factor acts upon a stable flow so as to generate instability!

iv) Very recently, Molemaker et al. (2001) discovered that a similar catalyzing effect could be provided by a vertical stable stratification, which induces a linear non-axisymmetric instability for all anticyclonically sheared flow. Astrophysical disks are subject to both vertical and radial stratification. Our primary goal here is to show that this mechanism may apply to any accretion disk and that the inclusion of the radial stratification essentially does not modify the instability. Our second goal is to discuss the importance of this mechanism in turbulence generation in disks, and compare it to other proposed mechanisms, namely ii) and iii).

2. A shearing instability in a stratified disk

2.1. Basic equations

We consider a rotating compressible stratified disk of finite vertical extent, with velocity $\mathbf{U} = r\Omega(r)e_\theta$ and density and pressure ρ_0, P. We consider perturbation of the basic state $\mathbf{u}, \rho, p$. In order to eliminate the acoustic waves from the problem, we shall work in the Boussinesq approximation. The basic dynamical equations ruling the perturbation are in this approximation:

$$\begin{aligned} \nabla \cdot (\rho \mathbf{u}) &= 0, \\ D_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U} + \frac{1}{\rho_0} \nabla p + \frac{3h}{5\rho_0} \nabla P &= 0, \\ D_t h + \mathbf{u} \cdot \nabla H &= 0. \end{aligned} \tag{4}$$

Here, $D_t = \partial_t + \mathbf{U} \cdot \nabla$ is the Lagrangian derivative, $H = \ln(P\rho_0^{-5/3})$ and $h = -5\rho/3\rho_0$ are the entropy and the entropy perturbation in the Boussinesq approximation, and assuming for simplicity the perfect gas law. Note that the incompressibility condition can be used to eliminate the pressure via the Poisson equation:

$$\frac{1}{\rho_0} \Delta p = -\nabla \cdot \left(D_t u + u \nabla U + \frac{3h}{5\rho_0} \nabla P \right). \tag{5}$$

We work in the local approximation, and decompose the perturbation over Fourier modes with wavenumber (k_r, k_θ, k_z). Because of the mean velocity, the perturbation azimuthal coordinate increases with time like:

$$\theta(t) = \theta(0) + k_\theta r \Omega t. \tag{6}$$

Because of the shear, the radial wavenumber increases with time according to (Dubrulle & Knobloch 1993)

$$k_r(t) = k_r(0) - k_\theta St \tag{7}$$

where $S = r\partial_r\Omega$ is the shear rate. To take these two effects into account, we then decompose our perturbations along a "travelling" basis $\exp[\omega t + irk_\theta(\Omega - S)t + i\mathbf{k}\cdot\mathbf{r}]$, with $\mathbf{k}\cdot\mathbf{r} << 1$. To leading order in $1/(kr)$ (i.e. considering that S is independent of r), and using cylindrical coordinates the Poisson equation becomes:

$$-ik^2\frac{\partial_r P}{\partial_z P}p = 2\Omega(k_\theta u - k_r v) + s\left(k_r N_r N_z + k_z N_z^2\right), \tag{8}$$

where $k^2 = k_r^2 + k_\theta^2 + k_z^2$, $\mathbf{u} = \partial_z P/\partial_r P(u, v, w)$ and $s = -h/\partial_z H$. Here, we have introduced the two components of the Brunt-Vaissala frequency:

$$N_r^2 = -\frac{3}{5\rho_0}\partial_r P\partial_r H,$$

$$N_z^2 = -\frac{3}{5\rho_0}\partial_z P\partial_z H, \tag{9}$$

and used the assumption of rotation on cylinders $\Omega(r)$:

$$\partial_r P\partial_z H = \partial_z P\partial_r H. \tag{10}$$

Note that in deriving (8) we have taken care to first taking the divergence, and then replace the derivative in front of (u, v, w, p) by $ik(u, v, w, p)$. Using (8), we obtain the dynamic equation under the local approximation as:

$$\begin{aligned}
k^2\omega u &= 2\Omega v\left(k^2 - k_r^2\right) + 2\Omega u k_r k_\theta + s\left(N_r N_z(k^2 - k_r^2) + k_r k_z N_z^2\right),\\
k^2\omega v &= -2\Omega v k_r k_\theta + 2\Omega u(k_\theta^2 - k^2) - Suk^2 + s\left(-N_r N_z k_r k_\theta + k_\theta k_z N_z^2\right),\\
k^2\omega w &= 2\Omega v k_r k_z + 2\Omega u k_z k_\theta + s\left(-N_r N_z k_r k_z + (k_z^2 - k^2)N_z^2\right),\\
\omega s &= -\frac{N_r}{N_z}u + w.
\end{aligned} \tag{11}$$

To illustrate the basic mechanism of instability, it is convenient to study first the case where $N_r^2 = 0$ (no radial stratification). We then return to treat the more general case by building on the results of this artificial example.

2.2. Dispersion relation for $N_r^2 = 0$

Setting the determinant of (11) to zero, with $N_r = 0$ and $N_z \equiv N$, we find after some algebra and rearrangement:

$$\omega^4 k^2 + \omega^2\left(N^2(k_r^2 + k_\theta^2) + \kappa^2 k_z^2 + 2\Omega S k_\theta^2\right) + 2\Omega S N^2 k_\theta^2 = 0. \tag{12}$$

Here, $\kappa^2 = 4\Omega^2 + 2\Omega S$ is the epicyclic frequency. In the axi-symmetric limit: $k_\theta = 0$, and the dispersion relation becomes:

$$(k_r^2 + k_z^2)\omega^4 + (N^2 k_r^2 + k_z^2\kappa^2)\omega^2 = 0. \tag{13}$$

There are two branches of solution: one with $\omega = 0$ corresponding to neutral modes propagating at the mean rotation speed. In the other branch, $(k_r^2 + k_z^2)\omega^2 = -(k_r^2 N^2 + k_z^2\kappa^2)$. In the stably stratified, centrifugally stable case we consider, both N^2 and κ^2 are positive, and ω is imaginary. This branch corresponds to gravito-inertial waves. As we shall now see, the linear non-axisymmetric instability starts from the neutral branch of solution. In the general case (12) is a quadratic equation in ω^2, which admits two roots, which may be real and distinct, real and equal, or complex conjugated. A sufficient condition for instability is therefore that one of the root has a positive real part. The root properties are determined by the sign of $2\Omega S = \partial_r\Omega^2$. If this quantity is negative, their product is negative, and the two roots are real of opposite sign, implying instability. In the other case, the two roots are equal or complex conjugated, and their real part is negative. A sufficient condition for instability is therefore that

$$\partial_r \Omega^2 < 0, \tag{14}$$

which is satisfied by Keplerian rotation. The growth rate $(\omega^2)^{1/2}$ can be found near the neutral branch, for vanishing ω^2:

$$\omega^2 = -\frac{2k_\theta^2 N^2 \Omega S}{k_\theta^2 (N^2 + 2\Omega S) + k_z^2 \kappa^2}. \tag{15}$$

The behavior depends on the Froude number $F = \Omega/N$. For strong stratification ($F \ll 1$), the growth rate is $\omega_r = \sqrt{-2S\Omega}$, i.e. independent of the stratification. The azimuthal wavenumbers in that case scales like $k_r \sim 1/\Delta r$, while the vertical wavenumber scales like k_r/F (Yavneh et al. 2001). For large Froude number, the growth rate is

$$\omega_r = \sqrt{\frac{-2k_\theta^2 N^2 \Omega S}{2k_\theta^2 \Omega S + k_z^2 \kappa^2}}. \tag{16}$$

Since in that case $k_\theta \sim k_z \sim k_r$ (Yavneh et al. 2001), the growth rate decays like $1/F$, in agreement with the numerical computations of Molemaker et al. (2001).

In summary, we have found that for non-axi-symmetric disturbances, the flow is linearly unstable, while for axisymmetric disturbances, the flow remains stable.

2.3. The general case: $N_r^2 \neq 0$

We consider now the general case. The general dispersion formula is then:

$$k^2\omega^4 + \omega^2 \left((k_z N_r + k_r N_z)^2 + k_z^2 \kappa^2 + k_\theta^2 (N_r^2 + N_z^2 + 2\Omega S) \right) + 2\Omega S N_z^2 k_\theta^2 = 0. \tag{17}$$

In the axisymmetric limit, one obtains a slightly different form of (13):

$$(k_r^2 + k_z^2)\omega^4 + \omega^2 \left((N_r k_z + k_r N_z)^2 + k_z^2 \kappa^2 \right) = 0. \tag{18}$$

Here we find again a neutral branch of solutions, plus a solution of stable waves in the centrifugally stable case.

In the non-axisymmetric limit, the dispersion relation has exactly the same structure as in the case with no radial stratification (with the product of the root depending only on $S\Omega$), and therefore the necessary condition for instability is unchanged. Moreover, in a thin disk disk $N_r \sim (H/r) N_z << N_z$, so that the expression of the growth rates is amost unaffected by the radial stratification.

2.4. Singular limits

In the present letter, we have used a crude local approximation to derive the stability criterion. In fact, these kinds of local approximations have known pitfalls (Dubrulle & Knobloch 1993). In the present case, it suggests for example that the growth rate of the perturbation vanishes algebraically fast in S and is independent on the stratification. A more careful analysis of the normal mode problem however shows that in the small gap limit, and for strong stratification ($F << 1$), the growth rate of the most unstable mode is (Molemaker et al. 2001)

$$\omega \approx -2S(k_\theta r) e^{4\Omega/S}. \tag{19}$$

Our analysis and the numerical results of Molemaker et al. (2001) also shows that for large values of F, the growth rate behaves linearly with $1/F$, thereby vanishing with vanishing stratification. This shows that there is no singular limit in this problem.

2.5. Non-linear saturation and implications for turbulence

The present analysis was performed in the inviscid limit. The influence of viscosity and the non-linear saturation of the instability have been studied numerically by Molemaker et al. (2001) and by Yavneh et al. (2001). They found that a small viscosity essentially does not change the results, and only introduces a critical Reynolds number, above which the flow is unstable. For example, at $F = 0.01$, $Ro = -2/3$, the critical Reynolds number is of order 1200, and from Section 2.2 we may extrapolate this result to higher Froude number: $R_c = 1200\sqrt{1+F^2}$. In thin astrophysical disks, we have $N_z^2 \sim \Omega^2(\nabla_{\rm ad} - \nabla)$, meaning that the Froude number is of order 10, if we assume a subadiabatic gradient of 1/10; therefore $R_c \sim 10^4$.

The transition to turbulence then proceeds via successive period doubling bifurcations (Molemaker et al. 2001). Without stratification (and without magnetic field), the turbulence would set in via finite amplitude disturbances and non-linear mechanisms, a very intermittent scenario, in which turbulent domains progressively grow in size with increasing Reynolds number, until they finally invade the whole flow, as observed in plane Couette flow (Dauchot & Daviaud 1994). Therefore, an interesting open question is what happens in the highly turbulent regime, and especially whether this turbulence, triggered by a *catalyzer*, resembles the turbulence generated by non-linear instabilities.

This question is important, because it is generally believed that turbulent transport properties depend on the source of the turbulence. The main argument is that to reach a stationary state, turbulence must organize itself so as to suppress the cause of instability. A good example is given by centrifugally unstable flows, where the turbulent state is characterized by a flat angular momentum distribution (outside boundary layers). But this holds only for moderate Reynolds number flow (say up to $R = 10^5$). For larger Reynolds number, there is experimental evidence in the rotating shear flow that a transition occurs above which the flow presents all the characterisitics of classical shear (boundary layer) turbulence (Lathrop et al. 1992). In astrophysical disks, the Reynolds number tends to be much larger than this critical value, and one may wonder whether the turbulent transport in disks is not universal. It would therefore be very interesting to conduct laboratory and numerically experiments of centrifugally stable, stratified flows to study that large Reynolds number turbulent regime.

3. Conclusion

We have shown that all accretion disks are subject to a powerful, non-axisymmetric instability. This instability lurks in any astrophysical disk, because of the stratification induced by the vertical component of gravity. It is a purely hydrodynamical instability and does not require the presence of any magnetic field, whatever small. Therefore there is no reason to invoke a “dead zone” in insufficiently ionized disks, as was done by Gammie (1996).

In astrophysical disks, the critical Reynolds number to trigger this instability is of the order of 10^4, which is less than the critical Reynolds number for both the magneto-rotational instability and the finite-amplitude hydrodynamic instability. But the question of which instability occurs first has little relevance: what one should ask is what kind of turbulence develops at high Reynolds number. In particular, it is not clear that magnetic field plays an important role in that regime, because in most turbulent flows studied so far, the level of magnetic energy turned out to be an order of magnitude less than that of kinetic energy. Of course, this does not rule out the transport of angular momentum by a large scale magnetic field.

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