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SOME GRADIENT ESTIMATES FOR A DIFFUSION EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this note we present some gradient estimates for the diffusion equation $\partial_t u = \Delta u - \nabla\phi \cdot \nabla u$ on Riemannian manifolds, where ϕ is a C^2 function, which generalize estimates of R. Hamilton's and Qi S. Zhang's on the heat equation.

1. Introduction

Using ideas from Li-Yau [LY], P. Souplet and Qi S. Zhang [SZ] give an elliptic estimate for the heat equation on Riemannian manifolds which is analogous to the Cheng-Yau estimate for the harmonic functions [CY] and the Hamilton estimates for the heat equation [H]. Later Qi S. Zhang [Z] gives a sharpened local Li-Yau gradient estimates for the heat equation using ideas from [H] and [LY].

In this note we present some gradient estimates for the diffusion equation $\partial_t u = \Delta u - \nabla\phi \cdot \nabla u$ on Riemannian manifolds, where ϕ is a C^2 function, which generalize estimates of R. Hamilton's and Qi S. Zhang's on the heat equation. Note that similar generalizations of the Cheng-Yau and Li-Yau estimates have already appeared in X.-D. Li [L], see also Q. Ruan [R] for a similar generalization (and for a more general equation) of the Souplet-Zhang estimate.

Let $\mathcal{M}$ be a Riemannian manifold of dimension n, and $L = \Delta - \nabla\phi\cdot\nabla$, where ϕ is a C^2 function on $\mathcal{M}$. Recall the Bakry-Emery Ricci curvature ([BE]) $Ric_{m,n}(L) := Ric + \nabla^2\phi - \frac{\nabla\phi \bigotimes \nabla\phi}{m-n}$, where $m \geq n$ is a constant. (Here we use the convention that $m = n$ if and only if $L = \Delta$.) Let $B(x_0, R)$ be a geodesic ball in $\mathcal{M}$, and $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$. Then we have the following generalization of the theorem of Zhang [Z] mentioned above,

Theorem 1.1 Assume that the Bakry-Emery Ricci curvature $Ric_{m,n}(L) \geq -k$ in $B(x_0, R)$, $k \geq 0$. Suppose u is any positive solution to the diffusion equation $\partial_t u = Lu$ in $Q_{R,T}$. Then there exists a dimension constant c such that
$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq c(\frac{1}{R^2} + \frac{1}{T} + k + (\sqrt{k} + \frac{1}{R})sup\frac{|\nabla u|}{u})$
in $Q_{R/2,T/2}$.

The following is a generalization of a theorem due to Hamilton mentioned above,

Theorem 1.2 Assume that $\mathcal{M}$ is compact, and that the Bakry-Emery Ricci curvature $Ric_{m,n}(L) \geq -k$, $k \geq 0$. Suppose u is any positive solution to the diffusion equation $\partial_t u = Lu$ in $\mathcal{M} \times (0, T]$ $(0 < T \leq \infty)$ with $u \leq M$. Then

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$\frac{|\nabla u|^2}{u^2} \leq (\frac{1}{t}+2k)ln\frac{M}{u}$.

2. Proof of theorems

Proof of theorem 1.1

As in [Z], we use idea from [H] and [LY].

By Bakry's generalized Bochner-Weitzenböck formula ([B]),

$\frac{1}{2}L|\nabla f|^2 = |\nabla^2 f|^2 + (\nabla Lf)\cdot\nabla f + (Ric+\nabla^2\phi)(\nabla f,\nabla f)$,

we compute

$(L-\partial_t)(\frac{|\nabla u|^2}{u}) = \frac{2}{u}|\nabla^2 u - \frac{\nabla u \bigotimes \nabla u}{u}|^2 + 2\frac{(Ric+\nabla^2\phi)(\nabla u,\nabla u)}{u}$

$\geq \frac{2}{nu}(\Delta u - \frac{|\nabla u|^2}{u})^2 + 2\frac{(Ric+\nabla^2\phi)(\nabla u,\nabla u)}{u}$.

Note that using the simple inequality $(a+b)^2 \geq \frac{a^2}{1+\theta} - \frac{b^2}{\theta}$ for any $\theta > 0$, we have

$(\Delta u - \frac{|\nabla u|^2}{u})^2 = (Lu - \frac{|\nabla u|^2}{u} + \nabla\phi\cdot\nabla u)^2$

$\geq \frac{n}{m}(Lu - \frac{|\nabla u|^2}{u})^2 - n\frac{\nabla\phi\bigotimes\nabla\phi}{m-n}(\nabla u,\nabla u)$.

Since Lu is also a solution of the diffusion equation $\partial_t u = Lu$, by using the assumption $Ric_{m,n}(L) \geq -k$ we have

$(L-\partial_t)(-Lu + \frac{|\nabla u|^2}{u}) \geq \frac{2}{mu}(Lu - \frac{|\nabla u|^2}{u})^2 + \frac{2}{u}(Ric + \nabla^2\phi - \frac{\nabla\phi\bigotimes\nabla\phi}{m-n})(\nabla u,\nabla u)$

$\geq \frac{2}{mu}(Lu - \frac{|\nabla u|^2}{u})^2 - 2k\frac{|\nabla u|^2}{u}$.

Then, let $H = \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u}$, we get the inequality

$(L-\partial_t)H \geq \frac{2}{m}H^2 - 2k\frac{|\nabla u|^2}{u^2} - 2\nabla H\cdot\nabla lnu$.

Now we use the cut-off function argument as in [LY], [SZ], [Z]. Let $\psi = \psi(d(x,x_0),t) = \psi(r,t)$ be a smooth cut-off function in $Q_{R,T}$, such that (i) $0 \leq \psi \leq 1$, and $\psi = 1$ in $Q_{R/2,T/4}$; (ii) ψ is decreasing as a radial function in the spatial variables; (iii)$\frac{|\partial_r\psi|}{\psi^a} \leq \frac{C_a}{R}, \frac{|\partial_r{}^2\psi|}{\psi^a} \leq \frac{C_a}{R^2}$ when $0 < a < 1$; and (iv)$\frac{|\partial_t\psi|}{\psi^{1/2}} \leq \frac{C}{T}$.

We have

$(L-\partial_t)(\psi H)$

$= \psi(L-\partial_t)H + H(L-\partial_t)\psi + 2\nabla\psi\cdot\nabla H$

$\geq \frac{2}{m}\psi H^2 - 2k\psi\frac{|\nabla u|^2}{u^2} - 2\nabla(\psi H)\cdot\nabla lnu + 2H\nabla\psi\cdot\nabla lnu + H(L-\partial_t)\psi + 2\frac{\nabla\psi}{\psi}\cdot\nabla(\psi H) - 2\frac{|\nabla\psi|^2}{\psi}H$.

Since $Ric_{m,n}(L) \geq -k$, by the generalized Laplacian comparison theorem (cf. for example [BQ]) $Lr \leq (m-1)\sqrt{k}coth(\sqrt{k}r)$, then we have that

$L\psi = (\partial_r\psi)Lr + (\partial_r{}^2\psi)|\nabla r|^2$

$\geq (\partial_r\psi)(m-1)\sqrt{k}coth(\sqrt{k}r) + \partial_r{}^2\psi$

$\geq (\partial_r\psi)(m-1)(\frac{1}{r}+\sqrt{k}) + \partial_r{}^2\psi$.

Then noticing the properties of ψ, we get

$L\psi \geq -c\psi^{1/2}(\frac{1}{R^2} + \frac{\sqrt{k}}{R})$.

Suppose the maximum of ψH is non-negative, otherwise we are done. Then at the maximum (space-time) point (y,s) of ψH we have

0

$\geq \frac{2}{m}\psi H^2 - 2k\psi\frac{|\nabla u|^2}{u^2} + 2H\nabla\psi\cdot\nabla lnu - c\psi^{1/2}H(\frac{1}{R^2}+\frac{\sqrt{k}}{R}) - H\partial_t\psi - 2\frac{|\nabla\psi|^2}{\psi}H$.

(Note that by Calabi's trick we may assume that y is not in the cut locus of x_0.)

Then using the inequalities (compare with [SZ], [Z])

$-2H\nabla\psi\cdot\nabla lnu$

$\leq \frac{\psi H^2}{8m} + c|\frac{\nabla\psi}{\sqrt{\psi}}\cdot\nabla lnu|^2$

$\leq \frac{\psi H^2}{8m} + c(\frac{|\nabla lnu|}{R})^2$,
$H\partial_t\psi$
$\leq \frac{\psi H^2}{8m} + c(\frac{\partial_t\psi}{\psi^{1/2}})^2$
$\leq \frac{\psi H^2}{8m} + c\frac{1}{T^2}$,
and
$2\frac{|\nabla\psi|^2}{\psi}H$
$\leq \frac{\psi H^2}{8m} + c(\frac{|\nabla\psi|^2}{\psi^{3/2}})^2$
$\leq \frac{\psi H^2}{8m} + c\frac{1}{R^4}$,
we arrive at
$\psi H^2 \leq c(\frac{1}{R^4} + \frac{1}{T^2} + k^2 + k(\frac{|\nabla u|}{u})^2 + (\frac{|\nabla lnu|}{R})^2)$,
and the result follows.

Proof of theorem 1.2

As in [H], we let $P = \frac{t}{1+2kt}\frac{|\nabla u|^2}{u} - uln\frac{M}{u}$, then using Bakry's generalized Bochner-Weitzenböck formula, we get

$(L - \partial_t)P \geq 0$,

then the result follows as in [H].

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