Integral representation of renormalized self-intersection local times

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Abstract

In this paper we apply Clark-Ocone formula to deduce an explicit integral representation for the renormalized self-intersection local time of the d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. As a consequence, we derive the existence of some exponential moments for this random variable.

1 Introduction

The purpose of this paper is to apply Clark-Ocone's formula to the renormalized self-intersection local time of the d-dimensional fractional Brownian motion. As a consequence, we derive the existence of some exponential moments for this local time.

A well-known result in Itô's stochastic calculus asserts that any square integrable random variable in the filtration generated by a d-dimensional Brownian motion $W = \{W_t, t \geq 0\}$ can be expressed as the sum of its expectation plus the stochastic integral of a square integrable adapted process:

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i.$$

The process u is determined by F, except on sets of measure zero. In this context, Clark-Ocone formula provides an explicit representation of u in

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terms of the derivative operator in the sense of Malliavin calculus. More precisely, if F belongs to the Sobolev space $\mathbb{D}^{1,2}$, then $u^i(t) = E(D^i_t F|\mathcal{F}_t)$, where D^i denotes the derivative with respect to the ith component of the Brownian motion and $\{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by the Brownian motion. Extensions of this formula have been developed by Üstünel in [17], and by Karatzas, Ocone and Li in [12]. Clark-Ocone formula has proved to be a useful tool in finding hedging portfolios in mathematical finance (see, for instance, [11]).

The fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a d-dimensional Gaussian process $B^H = \{B^H_t, t \geq 0\}$ with zero mean and covariance function given by

$$E(B^{H,i}_t B^{H,j}_s) = \frac{\delta_{ij}}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \tag{1.1}$$

where $i, j = 1, \ldots, d$, $s, t \geq 0$, and

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \quad\quad i \neq j \end{cases}$$

is the Kronecker symbol. Assume $d \geq 2$. The *self-intersection local time* of B^H is formally defined as

$$L = \int_0^T \int_0^t \delta_0(B^H_t - B^H_s) ds,$$

where δ_0 is the Dirac delta function. It measures the amount of time that the process spends intersecting itself on the time interval $[0, T]$. Rigorously, L is defined as the limit in L^2, if it exists, of $L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B^H_t - B^H_s) ds dt$, as ε tends to zero, where p_ε denotes the heat kernel.

For $H = \frac{1}{2}$, the process B^H is a classical Brownian motion and its self-intersection local time has been studied by many authors (see Albeverio et al. [1], Calais and Yor [4], He et al. [6], Hu [7], Imkeller et al. [10], Varadhan [18], Yor[20], and the references therein). In this case, if $d = 2$, Varadhan [18] has proved that L_ε does not converge in L^2, but it can be renormalized so that $L_\varepsilon - E(L_\varepsilon)$ converges in L^2 as ε tends to zero to a random variable that we denote by $\widetilde{L}$. This result has been extended by Rosen [16] to the case $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$ (still when $d = 2$), and by Hu and Nualart in [9], where they have obtained the following complete result on the existence of the self-intersection local time of the fractional Brownian motion:

(i) The self-intersection local time L exists if and only if $Hd < 1$.

(ii) If $Hd \geq 1$, the renormalized self-intersection local time $\widetilde{L}$ exists if and only if $Hd < \frac{3}{2}$.

An important question is the existence of moments and exponential moments for the (renormalized) self-intersection local time. Along this direction, Le Gall [13] proved that for the planar Brownian motion, there is a critical exponent λ_0, such that $E\left(\exp \lambda \widetilde{L}\right) < \infty$ for all $\lambda < \lambda_0$, and $E\left(\exp \lambda \widetilde{L}\right) = \infty$ if $\lambda > \lambda_0$. Using the theory of large deviations, Bass and Chen proved in [2] that the critical exponent λ_0 coincides with A^{-4}, where A is the best constant in the Gagliardo-Nirenberg inequality.

Clark-Ocone formula seems to be a suitable tool to analyze the renormalized self-intersection local time, because in this formula we do not take into account the expectation of the random variable. The fractional Brownian motion can be expressed as the stochastic integral

$$B_t^H = \int_0^t K_H(t,s) dW_s$$

of a square integrable kernel $K_H(t,s)$ with respect to an underlying Brownian motion W. In this way the renormalized self-intersection local time $\widetilde{L}$ is a functional of the Brownian motion W, and we can obtain an explicit integral representation $\widetilde{L}$, in the general case $Hd < \frac{3}{2}$. This formula allows us to obtain some exponential moments for the renormalized self-intersection local time, using the method of moments.

The paper is organized as follows. In Section 2 we present some preliminaries on Malliavin calculus and Clark-Ocone formula. Section 3 is devoted to derive estimates for the moments of the self-intersection local time in the case of a general d-dimensional Gaussian process, using the method of moments. In the case of the fractional Brownian motion, this provides the existence of exponential moments in the case $Hd < 1$. Section 4 contains the main result, which is the integral representation of the renormalized self-intersection local time of the fractional Brownian motion in the case $H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. As an application we show that $E\left(\exp\left|\widetilde{L}\right|^p\right) < \infty$ if $p < \frac{1}{2}\left[\left(\frac{1}{2} + H\right)\left(\frac{d}{2} - \frac{1}{4H}\right)\right]^{-1}$. A crucial tool is the local nondeterminism property introduced by Berman in [3] and developed by many authors (see Xiao [19] and the references therein).

2 Preliminaries on Malliavin calculus and Clark-Ocone formula

We need some preliminaries on the Malliavin calculus for the d-dimensional Brownian motion $W = \{W_t, t \geq 0\}$. We refer to Malliavin [14] and Nualart [15] for a more detailed presentation of this theory.

We assume that W is defined in a complete probability space $(\Omega, \mathcal{F}, P)$, and the σ-field $\mathcal{F}$ is generated by W. Let us denote by H the Hilbert space $L^2(\mathbb{R}_+; \mathbb{R}^d)$, and for any function $h \in H$ we set

$$W(h) = \sum_{i=1}^{d} \int_0^\infty h^i(t) dW_t^i.$$

Let $\mathcal{S}$ be the class of smooth and cylindrical random variables of the form

$$F = f(W(h_1), \ldots, W(h_n)),$$

where $n \geq 1$, $h_1, \ldots, h_n \in H$, and f is an infinitely differentiable function such that together with all its partial derivatives has at most polynomial growth order. The derivative operator of the random variable F is defined as

$$D_t^i F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(W(h_1), \ldots, W(h_n)) h_j^i(t),$$

where $i = 1, \ldots, d$ and $t \geq 0$. In this way, we interpret DF as a random variable with values in the Hilbert space H. The derivative is a closable operator on $L^2(\Omega)$ with values in $L^2(\Omega; H)$. We denote by $\mathbb{D}^{1,2}$ the Hilbert spaced defined as the completion of $\mathcal{S}$ with respect to the scalar product

$$\langle F, G \rangle_{1,2} = E(FG) + E\left(\sum_{i=1}^{d} \int_0^\infty D_t^i F D_t^i G dt \right).$$

The divergence operator δ is the adjoint of the derivative operator D. The operator δ is an unbounded operator from $L^2(\Omega; H)$ into $L^2(\Omega)$, and is determined by the duality relationship

$$E(\delta(u)F) = E(\langle u, DF \rangle_H),$$

for any u in the domain of δ, and F in $\mathbb{D}^{1,2}$. Gaveau and Trauber [5] proved that δ is an extension of the classical Itô integral in the sense that any d-dimensional square integrable adapted process belongs to the domain of δ, and $\delta(u)$ coincides with the Itô integral of u:

$$\delta(u) = \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i.$$

It is well-known that any random variable $F \in L^2(\Omega)$, possesses a stochastic integral representation of the form

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty u^i(t) dW_t^i,$$

for some d-dimensional square integrable adapted process u. Clark-Ocone formula says that if $F \in \mathbb{D}^{1,2}$, then

$$F = E(F) + \sum_{i=1}^{d} \int_0^\infty E(D_t^i F|\mathcal{F}_t) dW_t^i. \tag{2.1}$$

3 Exponential integrability of the self-intersection local time

Suppose that $W = \{W_t, t \geq 0\}$ is a d-dimensional standard Brownian motion, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. Suppose that $\mathcal{F}$ is generated by W. We denote by $\{\mathcal{F}_t, t \geq 0\}$ the filtration generated by W and the sets of probability zero. Consider a d-dimensional Gaussian processs of the form

$$B_t = \int_0^t K(t,s) dW_s, \tag{3.1}$$

where $K(t,s)$ is a measurable kernel satisfying $\int_0^t K(t,s)^2 ds < \infty$ for all $t \geq 0$. We will assume that $K(t,s) = 0$ if $s > t$.

Fix a time interval $[0,T]$. We will make use of the following property on the kernel $K(t,s)$:

(H1) *For any $s, t \in [0,T]$, $s < t$ we have*

$$\int_s^t K(t,\theta)^2 d\theta \geq k_1 (t-s)^{2H} \tag{3.2}$$

for some constants $k_1 > 0$, and $H \in (0,1)$.

Notice that $\operatorname{Var}\left(B_t^i|\mathcal{F}_s\right) = \int_s^t K(t,\theta)^2 d\theta$, so condition **(H1)** is equivalent to say that $\operatorname{Var}\left(B_t^i|\mathcal{F}_s\right) \geq k_1(t-s)^{2H}$, for each component $i = 1, \ldots, d$. This property is satisfied, for instance, in the following two examples:

Example 1 Suppose that $K(t,s) = (t-s)^{H-\frac{1}{2}}$. Then, we have equality in (3.2) with $k_1 = \frac{1}{2H}$.

Example 2 Condition **(H1)** is satisfied by the kernel of the fractional Brownian motion, as a consequence of the local nondeterminism property (see (4.1) below).

We will denote by C a generic constant depending on T, the dimension d, and the constants appearing in the hypothesis such as H and k_1.

The *self-intersection local time* of the process B in the time interval $[0,T]$, denoted by L, is defined as the limit in L^2 as ε tends to zero of

$$L_\varepsilon = \int_0^T \int_0^t p_\varepsilon(B_t - B_s) ds, \tag{3.3}$$

where p_ε denotes the heat kernel

$$p_\varepsilon(x) = (2\pi\varepsilon)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2\varepsilon}\right).$$

The next theorem asserts that L exists if $Hd < 1$, and it has exponential moments of order $\frac{1}{Hd}$.

Theorem 1 *Suppose that $Hd < 1$. Then, the self-intersection local time L exists as the limit in L^2 of L_ε, as ε tends to zero, and for all integers $n \geq 1$ we have*

$$E(L^n) \leq C^n \left(n!\right)^{Hd},$$

for some constant C. As a consequence,

$$E(e^{L^p}) < \infty,$$

for any $p < \frac{1}{Hd}$, and there exists a constant $\lambda_0 > 0$ such that $E(e^{\lambda L^{\frac{1}{Hd}}}) < \infty$ for all $\lambda < \lambda_0$.

Proof. From the equality

$$p_\varepsilon(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp\left(i\left\langle \xi, x\right\rangle - \frac{\varepsilon|\xi|^2}{2}\right) d\xi$$

and the definition of L_ε, we obtain

$$L_\varepsilon = \frac{1}{(2\pi)^d} \int_0^T \int_0^t \int_{\mathbb{R}^d} \exp\left(i\left\langle \xi, B_t - B_s\right\rangle - \frac{\varepsilon|\xi|^2}{2}\right) d\xi ds dt.$$

This expression allows us to compute the moments of L_ε. Fix an integer $n \geq 1$. Denote by T_n the set $\{0 < s < t < T\}^n$. Then

$$\begin{aligned} E(L_\varepsilon^n) &= \frac{1}{(2\pi)^{nd}} \int_{T_n} \int_{\mathbb{R}^{nd}} E\left[\exp\left(i \langle \xi_1, B_{t_1} - B_{s_1}\rangle + \cdots + i \langle \xi_n, B_{t_n} - B_{s_n}\rangle\right)\right] \\ &\quad \times \exp\left(-\frac{\varepsilon}{2} \sum_{j=1}^n |\xi_j|^2\right) d\xi_1 \cdots d\xi_n ds dt, \end{aligned} \tag{3.4}$$

where $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$. Notice that

$$\begin{aligned} &\int_{\mathbb{R}^{nd}} E\left[\exp\left(i \langle \xi_1, B_{t_1} - B_{s_1}\rangle + \cdots + i \langle \xi_n, B_{t_n} - B_{s_n}\rangle\right)\right] \\ &\times e^{-\frac{\varepsilon}{2}\sum_{j=1}^n |\xi_j|^2} d\xi_1 \cdots d\xi_n \\ = &\int_{\mathbb{R}^{nd}} \exp\left(-\frac{1}{2} E\left[\left(\langle \xi_1, B_{t_1} - B_{s_1}\rangle + \cdots + \langle \xi_n, B_{t_n} - B_{s_n}\rangle\right)^2\right]\right) \\ &\times e^{-\frac{\varepsilon}{2}\sum_{j=1}^n |\xi_j|^2} d\xi_1 \cdots d\xi_n \\ = &\left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\xi^T Q \xi\right) e^{-\frac{\varepsilon}{2}|\xi|^2} d\xi\right)^d, \end{aligned} \tag{3.5}$$

where Q is the covariance matrix of the n-dimensional random vector $(B_{t_1}^1 - B_{s_1}^1, \ldots, B_{t_n}^1 - B_{s_n}^1)$. Substituting (3.5) into (3.4) yields

$$E(L_\varepsilon^n) = \frac{1}{(2\pi)^{nd}} \int_{T_n} \left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\xi^T Q \xi\right) e^{-\frac{\varepsilon}{2}|\xi|^2} d\xi\right)^d ds dt,$$

and $E(L_\varepsilon^n)$ converges as ε tends to zero to

$$\begin{aligned} \alpha_n &= \frac{1}{(2\pi)^{nd}} \int_{T_n} \left(\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}\xi^T Q \xi\right) d\xi\right)^d ds dt \\ &= \frac{1}{(2\pi)^{\frac{nd}{2}}} \int_{T_n} (\det Q)^{-\frac{d}{2}} ds dt, \end{aligned}$$

provided α_n is finite.

If $\alpha_2 < \infty$, then in the same way as before we obtain

$$\lim_{\varepsilon, \delta \downarrow 0} E(L_\varepsilon L_\delta) = \alpha_2,$$

which implies that L_ε converges in L^2 as ε tends to zero. Furthermore, if α_n is finite for all $n \geq 1$, then we deduce the convergence in L^p for any $p \geq 2$

of L_ε as ε tends to zero. The limit, denoted by L, will be, by definition, the self-intersection local time of the process B in the time interval $[0, T]$. To complete the proof of the theorem it suffices to show that α_n is bounded by $C^n (n!)^{Hd}$, for some constant C.

We can write

$$\alpha_n = \frac{n!}{(2\pi)^{\frac{nd}{2}}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} (\det Q)^{-\frac{d}{2}} \, dsdt.$$

For each $i = 1, \ldots, n$ we denote by τ_i the point in the set $\{s_i, s_{i+1}, \ldots, s_n, t_{i-1}\}$ which is closer to t_i from the left. Then, by **(H1)** and the fact that $s_i < t_i$, $i = 1, \ldots, n$, we obtain, using Lemma 5 in the Appendix,

$$\begin{aligned} \det Q &= \mathrm{Var}(B^1_{t_1} - B^1_{s_1}) \mathrm{Var}(B^1_{t_2} - B^1_{s_2} | B^1_{t_1} - B^1_{s_1}) \\ &\quad \times \cdots \times \mathrm{Var}(B^1_{t_n} - B^1_{s_n} | B^1_{t_1} - B^1_{s_1}, \ldots, B^1_{t_{n-1}} - B^1_{s_{n-1}}) \\ &\geq \mathrm{Var}(B^1_{t_1} | B^1_{s_1}) \mathrm{Var}(B^1_{t_2} | B^1_{t_1}, B^1_{s_1}, B^1_{s_2}) \\ &\quad \times \cdots \times \mathrm{Var}(B^1_{t_n} | B^1_{t_1}, B^1_{s_1}, \ldots, B^1_{t_{n-1}}, B^1_{s_{n-1}}, B^1_{s_n}) \\ &\geq \mathrm{Var}(B^1_{t_1} | \mathcal{F}_{\tau_1}) \mathrm{Var}(B^1_{t_2} | \mathcal{F}_{\tau_2}) \cdots \mathrm{Var}(B^1_{t_n} | \mathcal{F}_{\tau_n}) \\ &\geq k_1^n (t_1 - \tau_1)^{2H} (t_2 - \tau_2)^{2H} \cdots (t_n - \tau_n)^{2H}. \end{aligned}$$

As a consequence,

$$\alpha_n \leq \frac{n!}{(2\pi)^{\frac{nd}{2}}} k_1^{-\frac{nd}{2}} \int_{T_n \cap \{t_1 < \cdots < t_n\}} \prod_{i=1}^n (t_i - \tau_i)^{-Hd} dsdt.$$

If we fix the points $t_1 < \cdots < t_n$, there are $3 \times 5 \times \cdots \times (2n-1) = (2n-1)!!$ posible ways to place the points $s_1, \ldots, s_n$. In fact, s_1 must be in $(0, t_1)$. For s_2 we have three choices: $(0, s_1)$, (s_1, t_1) and (t_1, t_2). By a recursive argument it is clear that we have $(2i - 1)$ possible choices for s_i, given $s_1, \ldots, s_{i-1}$. In this way, up to a set of measure zero, we can decompose the set $T_n \cap \{t_1 < \cdots < t_n\}$ into the union of $(2n-1)!!$ disjoint subsets. The integral of $\prod_{i=1}^n (t_i - \tau_i)^{-Hd}$ on each one of these subset can be expressed as

$$\Phi_\sigma = \int_{\{0 < z_1 < \cdots < z_{2n} < T\}} \prod_{i=1}^n (z_{\sigma(i)} - z_{\sigma(i)-1})^{-Hd} dz,$$

where $\sigma(1) < \cdots < \sigma(n)$ are n elements in $\{1, 2, \ldots, 2n\}$, and $z = (z_1, \ldots, z_{2n})$. Making the change of variables $y_i = z_i - z_{i-1}$, $i = 1, \ldots, 2n$ (with the convention $z_0 = 0$) we obtain

$$\begin{aligned}\Phi_\sigma &= \int_{\{0<y_1+\cdots+y_{2n}<T\}} \prod_{i=1}^{n} y_{\sigma(i)}^{-Hd} dy \leq \frac{T^n}{n!} \int_{\{0<y_1+\cdots+y_n<T\}} \prod_{i=1}^{n} y_i^{-Hd} dy \\ &= \frac{1}{n!} T^{n(2-Hd)+Hd} \frac{\Gamma(1-Hd)^{n-1}}{\Gamma(n(1-Hd)+Hd+1)}.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_n &\leq \frac{k_1^{-\frac{nd}{2}} (2n-1)!! T^{n(2-Hd)+Hd} \Gamma(1-Hd)^{n-1}}{(2\pi)^{\frac{nd}{2}} \Gamma(n(1-Hd)+Hd+1)} \\ &= C_1 C_2^n \frac{(2n-1)!!}{\Gamma(n(1-Hd)+Hd+1)},\end{aligned}$$

with $C_1 = T^{Hd}\Gamma(1-Hd)^{-1}$, and $C_2 = \frac{k_1^{-\frac{d}{2}}\Gamma(1-Hd)T^{2-Hd}}{(2\pi)^{\frac{d}{2}}}$. Taking into account that $(2n-1)!! \leq 2^{n-1}n!$, and that

$$\Gamma(n(1-Hd)+Hd+1) \geq C^n (n!)^{1-Hd},$$

for some constant C, we obtain the desired estimate. ■

If $Hd \geq 1$, the above result is no longer true. In that case the expectation of L_ε blows up as ε tends to zero. In fact, if we denote $\sigma^2(s,t) = \mathrm{Var}(B_t^1 - B_s^1)$, for $s < t$, then

$$E(L_\varepsilon) = \int_0^T \int_0^t p_{\varepsilon+\sigma^2(s,t)}(0) dsdt = (2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t (\varepsilon + \sigma^2(s,t))^{-\frac{d}{2}} dsdt,$$

which converges to

$$(2\pi)^{-\frac{d}{2}} \int_0^T \int_0^t \sigma^2(s,t)^{-\frac{d}{2}} dsdt \geq (2\pi)^{-\frac{d}{2}} k_1^{-\frac{d}{2}} \int_0^T \int_0^t (t-s)^{-Hd} dsdt = \infty.$$

In this case, one can study the existence of the renormalized self-intersection local time defined as the limit as ε tends to zero of $L_\varepsilon - E(L_\varepsilon)$. In the next section we discuss the existence and exponential moments of the renormalized self-intersection local time, using Clark-Ocone formula, in the case of the fractional Brownian motion.

4 Renormalized self-intersection local time of the fBm

The fractional Brownian motion on $\mathbb{R}^d$ with Hurst parameter $H \in (0,1)$ is a d-dimensional Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with zero mean and covariance function given by (1.1). We will assume that $d \geq 2$.

It is well-known that B^H possesses the following integral representation

$$B_t^H = \int_0^t K_H(t,s) dW_s,$$

where $W = \{W_t, t \geq 0\}$ is a d-dimensional Brownian motion, and $K_H(s,t)$ is the square integrable kernel given by

$$K_H(t,s) = C_{H,1} s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

if $H > \frac{1}{2}$, and by

$$K_H(t,s) = C_{H,2} \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right],$$

if $H < \frac{1}{2}$, for any $s < t$, where the constants are $C_{H,1} = \left[\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}$, and $C_{H,2} = \left[\frac{2H}{(1-2H)b(1-2H, H+\frac{1}{2})}\right]^{\frac{1}{2}}$, where $B(\alpha,\beta)$ denotes th beta function.

The processes B^H and W generate the same filtration, that is, $\mathcal{F}_t = \sigma\{W_s, 0 \leq s \leq t\} = \sigma\{B_s^H, 0 \leq s \leq t\}$.

The fractional Brownian motion satisfies the following local nondeterminism property:

(LND) *There exists a constant* $k_2 > 0$*, depending only on* H *and* T*, such that for any* $t \in [0,T]$, $0 < r < t \wedge (T-t)$ and for $i = 1, \dots, d$,

$$\mathrm{Var}(B_t^{H,i} | B_s^{H,i} : |s-t| \geq r) \geq k_2 \, r^{2H}. \tag{4.1}$$

Consider the approximated self-intersection local time L_ε introduced in (3.3). From the general result proved in Section 2 it follows that if $Hd < 1$, then L_ε converges in L^2 to the self-intersection local time L, and the random variable L has exponential moments. If $Hd \geq 1$, this result is no longer true, and one considers the renormalization of the self-intersection local time, introduced by Varadhan.

The purpose of this section is to apply the Clark-Ocone formula to provide a stochastic integral representation for the renormalized self-intersection local time $\widetilde{L}$. As a consequence, we will prove the existence of some exponential moments for the random variable $\widetilde{L}$.

Theorem 2 *Suppose that $H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. Then the renormalized self-intersection local time of the d-dimensional fractional Brownian motion B^H exists in L^2 and it has the following integral representation*

$$\widetilde{L} = -\sum_{i=1}^{d} \int_0^T \left(\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\sigma^2_{r,s,t}} p_{\sigma^2_{r,s,t}}(A^i_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt \right) dW^i_r, \tag{4.2}$$

where

$$A_{r,t,s} = E(B^H_t - B^H_s | \mathcal{F}_r)$$

and

$$\sigma^2_{r,s,t} = \mathrm{Var}(B^{H,i}_t - B^{H,i}_s | \mathcal{F}_r).$$

Proof. The proof will be done in several steps.

Step 1 We are going to apply Clark-Ocone formula to the random variable L_ε. It is clear that L_ε belongs to $\mathbb{D}^{1,2}$, and its derivative can be computed as follows

$$D^i_r L_\varepsilon = \int_0^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i}(B^H_t - B^H_s) D^i_r \left(B^{H,i}_t - B^{H,i}_s\right) dsdt,$$

where $r \in [0,T]$, and $i = 1, \ldots, d$. Using

$$D^i_r \left(B^{H,i}_t - B^{H,i}_s\right) = \left[K_H(t,r) - K_H(s,r)\right] \mathbf{1}_{[0,t]}(r),$$

we obtain

$$D^i_r L_\varepsilon = \int_r^T \int_0^t \frac{\partial p_\varepsilon}{\partial x_i}(B^H_t - B^H_s) \left[K_H(t,r) - K_H(s,r)\right] dsdt. \tag{4.3}$$

The next step is to compute the conditional expectation $E(D^i_r L_\varepsilon | \mathcal{F}_r)$. The conditional law of $B^H_t - B^H_s$ given $\mathcal{F}_r$ is normal with mean $A_{r,t,s}$ and covariance matrix $\sigma^2_{r,s,t} I_d$, where I_d is the d-dimensional identity matrix. Hence, the conditional expectation $E\left(\frac{\partial p_\varepsilon}{\partial x_i}(B_t^H - B_s^H)|\mathcal{F}_r\right)$ is given by

$$\begin{aligned} E\left(\frac{\partial p_\varepsilon}{\partial x_i}(B_t^H - B_s^H)|\mathcal{F}_r\right) &= \int_{\mathbb{R}^d} \frac{\partial p_\varepsilon}{\partial x_i}(y) p_{\sigma^2_{r,s,t}}(y - A_{r,t,s}) dy \\ &= \frac{\partial p_{\varepsilon+\sigma^2_{r,s,t}}}{\partial x_i}(A_{r,t,s}) \\ &= -\frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}). \end{aligned}$$

As a consequence, from (4.3) we obtain

$$E\left(D_r^i L_\varepsilon|\mathcal{F}_r\right) = -\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt,$$

and this leads to the following integral representation for $L_\varepsilon - E(L_\varepsilon)$

$$\begin{aligned} & L_\varepsilon - E(L_\varepsilon) \\ = & -\sum_{i=1}^d \int_0^T \left(\int_r^T \int_0^t \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right] dsdt\right) dW_r^i. \end{aligned}$$

Step 2 In order to pass to the limit as ε tends to zero we proceed as follows. Set

$$\Sigma^i_\varepsilon(r,t,s) = \frac{A^i_{r,t,s}}{\varepsilon + \sigma^2_{r,s,t}} p_{\varepsilon+\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right]. \tag{4.4}$$

Clearly, $\Sigma^i_\varepsilon(r,t,s)$ converges pointwise as ε tends to zero to

$$\Sigma^i(r,t,s) = \frac{A^i_{r,t,s}}{\sigma^2_{r,s,t}} p_{\sigma^2_{r,s,t}}(A_{r,t,s}) \left[K_H(t,r) - K_H(s,r)\right].$$

In order to establish the convergence of the integrals in the variables s and t, we will first decompose the interval $[0,t]$ into the disjoint union of $[r,t]$ and $[0,r)$. In this way we obtain

$$L_\varepsilon - E(L_\varepsilon) = L_\varepsilon^{(1)} + L_\varepsilon^{(2)},$$

where

$$L_\varepsilon^{(1)} = -\sum_{i=1}^d \int_0^T \left(\int_r^T \int_r^t \Sigma^i_\varepsilon(r,t,s) dsdt\right) dW_r^i,$$

and

$$L_\varepsilon^{(2)} = -\sum_{i=1}^d \int_0^T \left(\int_r^T \int_0^r \Sigma_\varepsilon^i(r,t,s) ds dt \right) dW_r^i.$$

Step 3 We claim that the random field $\Sigma_\varepsilon^i(r,t,s)$ is uniformly bounded on the set $0 < r < s < t$ by an integrable function not depending on ε. In fact, using the local nondeterminism property **(LND)**, and Lemma 5 in the Appendix, we obtain the following lower bound for the conditional variance $\sigma_{r,s,t}^2 = \mathrm{Var}(B_t^{H,i} - B_s^{H,i} | \mathcal{F}_r)$:

$$\sigma_{r,s,t}^2 \geq \mathrm{Var}(B_t^{H,i} - B_s^{H,i} | \mathcal{F}_s) = \mathrm{Var}(B_t^{H,i} | \mathcal{F}_s) \geq k_2 (t-s)^{2H}. \tag{4.5}$$

We can get rid off the factor $A_{r,t,s}^i$ in the expression (4.4) of $\Sigma_\varepsilon^i(r,t,s)$ using the inequality

$$p_t(x) \leq C \frac{t^{-\frac{d}{2}+\frac{1}{2}}}{|x|} e^{-\frac{|x|^2}{4t}} \leq C \frac{t^{-\frac{d}{2}+\frac{1}{2}}}{|x|}, \tag{4.6}$$

for some constant $C > 0$. In this way we obtain, using (4.5) and (4.6)

$$\left| \Sigma_\varepsilon^i(r,t,s) \right| \leq C (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right|, \tag{4.7}$$

for some constant $C > 0$, and by Lemma 7 in the Appendix we obtain that

$$\int_r^T \int_r^t (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right| ds dt \leq C (r^{\frac{1}{2}-H} \vee 1). \tag{4.8}$$

By dominated convergence we deduce the convergence of the integrals

$$\lim_{\varepsilon \downarrow 0} \int_r^T \int_r^t \Sigma_\varepsilon^i(r,t,s) ds dt = \int_r^T \int_r^t \Sigma^i(r,t,s) ds dt$$

for all $(r,\omega) \in [0,T] \times \Omega$, and a second application of the dominated convergence theorem yields that $\int_r^T \int_r^t \Sigma_\varepsilon^i(r,t,s) ds dt$ converges in $L^2([0,T] \times \Omega)$ to $\int_r^T \int_r^t \Sigma^i(r,t,s) ds dt$. This implies the convergence of $L_\varepsilon^{(1)}$ to

$$-\sum_{i=1}^d \int_0^T \left(\int_r^T \int_r^t \Sigma^i(r,t,s) ds dt \right) dW_r^i$$

in $L^2(\Omega)$ as ε tends to zero.

Step 4 Consider now the case $s < r < t$. In this case the integral of the term $\Sigma_\varepsilon^i(r,t,s)$ is not necessarily bounded, and in order to show the convergence of $L_\varepsilon^{(2)}$ we will prove uniform bounds in ε for the expectation

$E\left(\int_r^T \int_r^t \left|\Sigma_\varepsilon^i(r,t,s)\right|^p dsdt\right)$, for some $p>1$. We can write for $s<r<t$, using the first inequality in (4.6)

$$
\begin{aligned}
\left|\Sigma_\varepsilon^i(r,t,s)\right| &\leq \frac{|A_{r,t,s}|}{(\varepsilon+\sigma_{r,s,t}^2)} p_{\varepsilon+\sigma_{r,s,t}^2}(A_{r,t,s})\,|K_H(t,r)| \\
&= (2\pi)^{-\frac{d}{2}} \frac{|A_{r,t,s}|}{\left(\varepsilon+\sigma_{r,s,t}^2\right)^{1+\frac{d}{2}}} \exp\left(-\frac{|A_{r,t,s}|^2}{2(\varepsilon+\sigma_{r,s,t}^2)}\right) |K_H(t,r)| \\
&\leq C\left(\varepsilon+\sigma_{r,s,t}^2\right)^{-\frac{d+1}{2}} \exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+\sigma_{r,s,t}^2)}\right) |K_H(t,r)|\,, \quad (4.9)
\end{aligned}
$$

for some constant $C>0$. If $s<r<t$, using the local nondeterminism property **(LND)** we obtain the following lower bound for the conditional variance $\sigma_{r,s,t}^2$:

$$
\sigma_{r,s,t}^2 = \mathrm{Var}(B_t^{H,i}-B_s^{H,i}|\mathcal{F}_r) = \mathrm{Var}(B_t^{H,i}|\mathcal{F}_r) \geq k_2(t-r)^{2H}. \qquad (4.10)
$$

On the other hand, if $s<r<t$

$$
\begin{aligned}
\sigma_{r,s,t}^2 &= \mathrm{Var}(B_t^{H,i}-B_s^{H,i}|\mathcal{F}_r) = \mathrm{Var}(B_t^{H,i}-B_r^{H,i}|\mathcal{F}_r) \\
&\leq \mathrm{Var}(B_t^{H,i}-B_r^{H,i}) = (t-r)^{2H}. \qquad (4.11)
\end{aligned}
$$

Also we will make use of the estimate (see [8])

$$
|K_H(t,r)| \leq k_3(t-r)^{H-\frac{1}{2}} r^{\frac{1}{2}-H}. \qquad (4.12)
$$

Substituting the estimates (4.10), (4.11) and (4.12) into (4.9) yields

$$
\left|\Sigma_\varepsilon^i(r,t,s)\right| \leq C r^{\frac{1}{2}-H} \Psi_\varepsilon(r,t,s), \qquad (4.13)
$$

for some constant C, where

$$
\Psi_\varepsilon(r,t,s) = \left(\varepsilon + k_2\,(t-r)^{2H}\right)^{-\frac{d+1}{2}} (t-r)^{H-\frac{1}{2}} \exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right). \qquad (4.14)
$$

Notice that if $Hd<\frac{1}{2}$, then $\left|\Sigma_\varepsilon^i(r,t,s)\right|$ is uniformly bounded by the integrable function $Cr^{\frac{3}{2}-H}\ (t-r)^{-Hd-\frac{1}{2}}$, and we can conclude as in Step 3. For this reason, we can assume that $Hd\geq\frac{1}{2}$.

We claim that for some $p>1$, we have

$$
\sup_{\varepsilon>0} E\left(\int_r^T \int_0^r \Psi_\varepsilon^p(r,t,s)dsdt\right) < \infty. \qquad (4.15)
$$

To show this estimate we first derive a lower bound for the expectation of $|A^1_{r,t,s}|^2 = \Big[E(B^{H,1}_t - B^{H,1}_s|\mathcal{F}_r)\Big]^2$. The main idea is to add and substract the term $B^{H,1}_r$, and then neglect the expectation $E\left(\left(\left(E(B^{H,1}_t|\mathcal{F}_r) - B^{H,1}_r\right)^2\right)\right)$. This argument will be used later to find a lower bound for the covariance matrix of the vector $\Big(E(B^{H,1}_{t_i} - B^{H,1}_{s_i}|\mathcal{F}_r), 1 \le i \le n\Big)$.

$$
\begin{aligned}
E\left(|A^1_{r,t,s}|^2\right) &= E\left(\left(E(B^{H,1}_t - B^{H,1}_s|\mathcal{F}_r)\right)^2\right) \\
&= E\left(\left(E(B^{H,1}_t|\mathcal{F}_r) - B^{H,1}_r\right)^2\right) \\
&\quad +2E\left(\left(E(B^{H,1}_t|\mathcal{F}_r) - B^{H,1}_r\right)\left(B^{H,1}_r - B^{H,1}_s\right)\right) + E\left(\left(B^{H,1}_r - B^{H,1}_s\right)^2\right) \\
&\ge 2E\left(\left(B^{H,1}_t - B^{H,1}_r\right)\left(B^{H,1}_r - B^{H,1}_s\right)\right) + E\left(\left(B^{H,1}_r - B^{H,1}_s\right)^2\right) \\
&= E\left(\left(B^{H,1}_t - B^{H,1}_s\right)^2\right) - E\left(\left(B^{H,1}_t - B^{H,1}_r\right)^2\right) \\
&= (t-s)^{2H} - (t-r)^{2H}.
\end{aligned}
$$

As a consequence, we obtain, assuming $p < 2$

$$
\begin{aligned}
&E\left(\exp\left(-\frac{p|A_{r,t,s}|^2}{4(\varepsilon + (t-r)^{2H})}\right)\right) \\
&\quad = \left(1 + \frac{p}{2}(\varepsilon + (t-r)^{2H})^{-1} E\left(|A^1_{r,t,s}|^2\right)\right)^{-\frac{d}{2}} \\
&\quad \le \left(1 + \frac{p}{2}(\varepsilon + (t-r)^{2H})^{-1}\left[(t-s)^{2H} - (t-r)^{2H}\right]\right)^{-\frac{d}{2}} \\
&\quad = (\varepsilon + (t-r)^{2H})^{\frac{d}{2}} \\
&\qquad \times \left(\varepsilon + \left(1 - \frac{p}{2}\right)(t-r)^{2H} + \frac{p}{2}(t-s)^{2H}\right)^{-\frac{d}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E\left(\exp\left(-\frac{p|A_{r,t,s}|^2}{4(\varepsilon + (t-r)^{2H})}\right)\right) \\
&\quad \le C(\varepsilon + (t-r)^{2H})^{\frac{d}{2}}(t-r)^{-2H\alpha}(t-s)^{-2H\beta}, \qquad (4.16)
\end{aligned}
$$

where $\alpha+\beta=\frac{d}{2}$. Substituting (4.16) into (4.14) yields

$$\begin{aligned} E\left(\int_r^T\int_0^r \Psi_\varepsilon^p(r,t,s)dsdt\right) &\leq C\int_r^T\int_0^r \left(\varepsilon+(t-r)^{2H}\right)^{-\frac{d+1}{2}p+\frac{d}{2}-\alpha} \\ &\quad \times(t-r)^{\left(H-\frac{1}{2}\right)p}(t-s)^{-\beta 2H}dsdt \\ &\leq C\int_r^T\int_0^r (t-r)^{-pHd-\frac{p}{2}+2H\beta}\,(t-s)^{-2H\beta}dsdt. \end{aligned}$$

If $Hd>1$, we can choose β such that $2H\beta>1$, and integrating in the variable s, the above integral is bounded by

$$C\int_r^T (t-r)^{-pHd-\frac{p}{2}+1}\,dt,$$

which is finite it $p>1$ satisfyes $\left(Hd+\frac{1}{2}\right)p<2$ (this is possible because $Hd+\frac{1}{2}<2$). If $Hd\leq 1$, we can choose β such that $2H\beta=Hd-\delta$, for any $\delta>0$, and we obtain the bound

$$C\int_r^T (t-r)^{-pHd-\frac{p}{2}+Hd-\delta}\,dt,$$

which is again finite if $p>1$ is close to one, and $\delta>0$ is small enough.

As a consequence, from (4.13) and (4.15), for any fixed $r\in[0,T]$, the family of functions $\left\{\Sigma_\varepsilon^i(r,t,s),\varepsilon>0\right\}$, is uniformly integrable in $[r,T]\times[0,r]$, so it converges in $L^1([r,T]\times[0,r])\times\Omega$ to $\Sigma^i(r,t,s)$, for $i=1,\dots,d$. This implies the convergence of the integrals

$$\lim_{\varepsilon\downarrow 0}\int_r^T\int_0^r \Sigma_\varepsilon^i(r,t,s)dsdt=\int_r^T\int_0^r \Sigma^i(r,t,s)dsdt,$$

for each fixed $r\in[0,T]$ in $L^1(\Omega)$.

Finally, we claim that this convergence also holds in $L^2([0,T]\times\Omega)$, and this implies the convergence of $L_\varepsilon^{(2)}$ to

$$-\sum_{i=1}^d\int_0^T\left(\int_r^T\int_0^r \Sigma^i(r,t,s)dsdt\right)dW_r^i$$

in $L^2(\Omega)$ as ε tends to zero. To show the convergence in $L^2([0,T]\times\Omega)$ of the integrals

$$Y_\varepsilon^i(r)=\int_r^T\int_0^r \Sigma_\varepsilon^i(r,t,s)dsdt$$

it suffices to prove that

$$\sup_{\varepsilon>0}\int_0^T E\left(\left|Y_\varepsilon^i(r)\right|^p\right)dr<\infty \tag{4.17}$$

for all $i=1,\dots,d$ and for some $p>2$. The proof of (4.17) will be the last step in the proof of this theorem.

Step 5 Suppose first that $Hd<1$. Then, from (4.13) we obtain

$$\int_0^T E\left(\left|Y_\varepsilon^i(r)\right|^p\right)dr\le C\int_0^T E\left[\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^p\right]r^{p\left(\frac{1}{2}-H\right)}dr.$$

Using (4.14) and Minkowski's inequality yields

$$\begin{aligned}\left\|\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right\|_p &\le \int_r^T\int_0^r\left(\varepsilon+k_2\left(t-r\right)^{2H}\right)^{-\frac{d+1}{2}}(t-r)^{H-\frac{1}{2}}\\ &\quad\times\left\|\exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right\|_p dsdt, \end{aligned} \tag{4.18}$$

and from (4.16), choosing $\beta=\frac{d}{2}$, we get

$$\left\|\exp\left(-\frac{|A_{r,t,s}|^2}{4(\varepsilon+(t-r)^{2H})}\right)\right\|_p\le C(\varepsilon+(t-r)^{2H})^{\frac{d}{2p}}(t-s)^{-\frac{Hd}{p}}. \tag{4.19}$$

Substituting (4.19) into (4.18) yields

$$\left\|\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right\|_p\le C\int_r^T (t-r)^{-Hd-\frac{1}{2}+\frac{Hd}{p}}dr,$$

which is finite if we choose $p>2$ such that $p<\frac{2Hd}{2Hd-1}$. Finally, if $p\left(\frac{1}{2}-H\right)>-1$ we complete the proof of (4.17) in the case $Hd<1$.

In the case $Hd\ge 1$ we cannot apply the previous arguments, and the proof of (4.17) follows from the moment estimates given in Proposition 3. ■

Remark 1 Theorem 2 also provides an alternative proof of the existence of the self-intersection local time in the case $H\in[\frac{1}{d},\min(\frac{3}{2d},\frac{2}{d+1}))$, which was proved by Hu and Nualart in [9] in the general case $Hd<\frac{3}{2}$. Notice that for $d\ge 3$, the condition $H\in[\frac{1}{d},\min(\frac{3}{2d},\frac{2}{d+1}))$ is equivalent to $1\le Hd<\frac{3}{2}$, and for $d=2$ we require $H<\frac{2}{3}$, instead of the more general condition $H<\frac{3}{4}$, that guarantees the existence of the renormalized local time (see [16] and [9]).

The next Proposition contains the basic estimates on the moments of the quadratic variation of the stochastic integral appearing in the representation of the renormalized self-intersection local time.

Proposition 3 *Assume* $1 \leq Hd < \frac{3}{2}$. *Set*

$$\Lambda_\varepsilon(r) = \int_r^T \int_0^r \Psi_\varepsilon(r,t,s)dsdt,$$

where $\Psi_\varepsilon(r,t,s)$ *has been defined in (4.14). Then, for any integer* $n \geq 1$,

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n (n!)^\gamma,$$

for some constant $C > 0$, *where*

$$\gamma > \left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right).$$

Proof. Set $g_\varepsilon(t-r) = \left(\varepsilon + k_2 (t-r)^{2H}\right)^{-\frac{d+1}{2}} (t-r)^{H-\frac{1}{2}}$. We have

$$\begin{aligned} E\left(\Lambda_\varepsilon^n(r)\right) &= E\left[\left(\int_r^T \int_0^r g_\varepsilon(t-r) \exp\left(-\frac{|A_{r,s,t}|^2}{4(\varepsilon + (t-r)^{2H})}\right) dsdt\right)^n\right] \\ &= n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^n g_\varepsilon(t_i - r) \\ &\quad \times \left(E\left(\exp\left(-\sum_{i=1}^n \frac{|A^1_{r,s_i,t_i}|^2}{4(\varepsilon + (t_i-r)^{2H})}\right)\right)\right)^d dsdt, \qquad (4.20) \end{aligned}$$

where $S_n = \{0 < s_1 < \cdots < s_n < r\}$, $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$.

We denote by Q the covariance matrix of the vector

$$\left(E(B_{t_1}^{H,1} - B_{s_1}^{H,1} | \mathcal{F}_r), \ldots, E(B_{t_n}^{H,1} - B_{s_n}^{H,1} | \mathcal{F}_r)\right).$$

Then, a well-known formula for Gaussian random variables implies that

$$\begin{aligned} E\left[\exp\left(-\sum_{i=1}^n \frac{|A^1_{r,s_i,t_i}|^2}{4(\varepsilon + (t_i-r)^{2H})}\right)\right] &= \det\left(I + \frac{1}{2} Q D^{-1}\right)^{-\frac{1}{2}} \\ &= 2^{\frac{n}{2}} \prod_{i=1}^n \sqrt{a_i} \det(2D + Q)^{-\frac{1}{2}}, \quad (4.21) \end{aligned}$$

where D denotes the $n\times n$ diagonal matrix with entries $a_i=\varepsilon+(t_i-r)^{2H}$. As in the computation of $E\left(|A^1_{r,t,s}|^2\right)$, adding and substracting the term $B^{H,1}_r$ yields

$$\begin{aligned}
Q_{ij} &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{s_i}|\mathcal{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{s_j}|\mathcal{F}_r)\right)\\
&= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathcal{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathcal{F}_r)\right)\\
&\quad +E\left((B^{H,1}_{r}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{r})\right)+E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{r}-B^{H,1}_{s_j})\right)\\
&\quad +E\left((B^{H,1}_{r}-B_{s_i})(B^{H,1}_{r}-B^{H,1}_{s_j})\right)\\
&= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathcal{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathcal{F}_r)\right)\\
&\quad -E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{t_j}-B^{H,1}_{r})\right)+E\left((B^{H,1}_{t_i}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{s_j})\right).
\end{aligned}$$

Hence, we obtain

$$Q=R-N+M,$$

where

$$\begin{aligned}
R_{ij} &= E\left(E(B^{H,1}_{t_i}-B^{H,1}_{r}|\mathcal{F}_r)E(B^{H,1}_{t_j}-B^{H,1}_{r}|\mathcal{F}_r)\right),\\
M_{ij} &= E\left((B^{H,1}_{t_i}-B^{H,1}_{s_i})(B^{H,1}_{t_j}-B^{H,1}_{s_j})\right),\\
N_{ij} &= E\left((B^{H,1}_{t_i}-B^{H,1}_{r})(B^{H,1}_{t_j}-B^{H,1}_{r})\right).
\end{aligned}$$

All these matrices are nonnegative definite. The main idea will be to get rid off the matrix R, and control the matrix N by its diagonal elements which are

$$N_{ii}=(t_i-r)^{2H}.$$

Indeed, the matrix N is nonnegative definite and, hence, it safisties the inequality

$$N\le nD_N, \tag{4.22}$$

where D_N is a diagonal matrix whose entries are N_{ii}. Therefore,

$$Q\ge -N+M\ge -nD_N+M,$$

and for any $1\le\delta<2$, we can write

$$\det(2D+Q)\ge\det(2D+\frac{2-\delta}{n}Q)\le\det(2D-(2-\delta)D_N+\frac{2-\delta}{n}M). \tag{4.23}$$

The entries of the diagonal matrix $D_1 = 2D - (2-\delta)D_N$ are the positive numbers

$$2\varepsilon + \delta(t_i - r)^{2H} > 0.$$

From (4.20), (4.21) and (4.23) we obtain

$$\begin{aligned} E\left(\Lambda_\varepsilon^n(r)\right) &\leq 2^{\frac{nd}{2}} n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^n \left(g_\varepsilon(t_i - r) a_i^{\frac{d}{2}} \right) \\ &\quad \times \det(D_1 + \frac{2-\delta}{n} M)^{-\frac{d}{2}} ds dt. \end{aligned}$$

We have

$$\det(D_1 + \frac{2-\delta}{n} M)^{-\frac{d}{2}} \leq \left(\frac{n}{2-\delta} \right)^{n\beta} \left(\det D_1\right)^{-\alpha} \left(\det M\right)^{-\beta},$$

where $\alpha + \beta = \frac{d}{2}$. Hence,

$$\begin{aligned} E\left(\Lambda_\varepsilon^n(r)\right) \leq & \left(\frac{n}{2-\delta} \right)^{n\beta} 2^{\frac{nd}{2}} n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^n \left(g_\varepsilon(t_i - r) a_i^{\frac{d}{2}} \left(2\varepsilon + \delta(t_i - r)^{2H}\right)^{-\alpha} \right) \\ & \times (\det M)^{-\beta} ds dt. \end{aligned}$$

Then,

$$\begin{aligned} & g_\varepsilon(t_i - r) a_i^{\frac{d}{2}} \left(2\varepsilon + 2(t_i - r)^{2H}\right)^{-\alpha} \\ = & \left(\varepsilon + k_2 (t_i - r)^{2H}\right)^{-\frac{d+1}{2}} (t_i - r)^{H - \frac{1}{2}} \left(\varepsilon + (t_i - r)^{2H}\right)^{\frac{d}{2}} \left(2\varepsilon + 2(t_i - r)^{2H}\right)^{-\alpha} \\ \leq & C (t_i - r)^{-\frac{1}{2} - 2H\alpha}, \end{aligned}$$

for some constant $C > 0$. Thus

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n n^{\beta n} n! \int_{[r,T]^n} \int_{S_n} \prod_{i=1}^n (t_i - r)^{-\frac{1}{2} - 2H\alpha} (\det M)^{-\beta} ds dt, \quad (4.24)$$

for some constant $C > 0$.

Applying Lemma 5 in the Appendix and the local nondeterminism property of the fractional Brownian motion we obtain

$$\begin{aligned} \det M = & \operatorname{Var}(B_{t_n} - B_{s_n}) \operatorname{Var}(B_{t_{n-1}} - B_{s_{n-1}} | B_{t_n} - B_{s_n}) \\ & \times \cdots \times \operatorname{Var}(B_{t_1} - B_{s_1} | B_{t_2} - B_{s_2}, \ldots, B_{t_n} - B_n) \\ & = (t_n - s_n)^{2H} \operatorname{Var}(B_{s_{n-1}} | B_{t_{n-1}}, B_{t_n}, B_{s_n}) \\ & \times \cdots \times \operatorname{Var}(B_{s_1} | B_{t_1}, \ldots, B_{t_n}, B_{s_1}, \ldots, B_{s_{n-1}}) \\ \geq & k_2^{n-1} (r - s_n)^{2H} \left((s_n - s_{n-1}) \wedge s_{n-1}\right)^{2H} \cdots \left((s_2 - s_1) \wedge s_1\right)^{2H}. \quad (4.25) \end{aligned}$$

Substituting (4.25) into (4.24), and choosing α such that $\alpha < \frac{1}{4H}$ (this is possible because $Hd \geq 1$) yields

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n n^{\beta n} n! \int_{S_n} \left[(r - s_n)\left((s_n - s_{n-1}) \wedge s_{n-1}\right) \cdots \left((s_2 - s_1) \wedge s_1\right)\right]^{-2\beta H} ds.$$

Finally, by Lemma 8 in the Appendix we obtain

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq \frac{C^n n^{\beta n} n!}{\Gamma(n(1 - 2H\beta) + 1)}.$$

Notice that $\beta = \frac{d}{2} - \alpha > \frac{d}{2} - \frac{1}{4H}$. And hence,

$$E\left(\Lambda_\varepsilon^n(r)\right) \leq C^n (n!^{\beta + 2H\beta},$$

where

$$\beta(1 + 2H) > \frac{d}{2} - \frac{1}{4H} + Hd - \frac{1}{2} = \left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right).$$

This concludes the proof. ■

Using the above proposition we can deduce the following integrability results for the renormalized self-intersection local time.

Theorem 4 *Assume $\frac{1}{d} \leq H < \min\left(\frac{3}{2d}, \frac{2}{d+1}\right)$. For any integer $p < \frac{1}{2}\left[\left(\frac{1}{2} + H\right)\left(d - \frac{1}{2H}\right)\right]^{-1}$ we have*

$$E(\exp|\widetilde{L}|^p) < \infty.$$

Proof. Taking into account Lemma 6 in the Appendix, it suffices to show that

$$E\left(\exp\left\langle \widetilde{L}\right\rangle^p\right) < \infty,$$

where

$$\left\langle \widetilde{L}\right\rangle = \sum_{i=1}^d \int_0^T \left(\int_r^T \int_0^t \Sigma^i(r, t, s) ds dt\right)^2 dr.$$

As in the proof of Theorem 2 we make the decomposition

$$\int_r^T \int_0^t \Sigma^i(r, t, s) ds dt = \int_r^T \int_r^t \Sigma^i(r, t, s) ds dt + \int_r^T \int_0^r \Sigma^i(r, t, s) ds dt.$$

From (4.7) and (4.8) we know that

$$\left|\int_r^T \int_r^t \Sigma^i(r, t, s) ds dt\right| \leq C(r^{\frac{1}{2} - H} \vee 1).$$

Therefore, applying Fatou's lemma and the estimate (4.13) yields

$$\begin{aligned}
E(\exp\left\langle \widetilde{L}\right\rangle^{p}) &\leq CE\left(\exp\left(\left|\sum_{i=1}^{d}\int_0^T\left(\int_r^T\int_0^r \Sigma^i(r,t,s)dsdt\right)^2 dr\right|^p\right)\right) \\
&\leq C\liminf_{\varepsilon\downarrow 0} E\left(\exp\left(\left|\sum_{i=1}^{d}\int_0^T\left(\int_r^T\int_0^r \Sigma^i_\varepsilon(r,t,s)dsdt\right)^2 dr\right|^p\right)\right) \\
&\leq C\liminf_{\varepsilon\downarrow 0} E\left(\exp\left(C\left|\int_0^T r^{1-2H}\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^2 dr\right|^p\right)\right).
\end{aligned}$$

Applying Hölder and Jensen inequalities we obtain

$$\begin{aligned}
E(\exp\left\langle \widetilde{L}\right\rangle^{p}) &\leq C\liminf_{\varepsilon\downarrow 0} E\left(\exp\left(C\int_0^T r^{1-2H}\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2p} dr\right)\right) \\
&\leq C\liminf_{\varepsilon\downarrow 0}\int_0^T r^{1-2H} E\left(\exp\left(C\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2p}\right)\right) dr.
\end{aligned}$$

Finally,

$$\begin{aligned}
& E\left(\exp\left(C\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2p}\right)\right) \\
= & \sum_{n=1}^{\infty}\frac{C^n}{n!}E\left(\left(\int_r^T\int_0^r \Psi_\varepsilon(r,t,s)dsdt\right)^{2np}\right) \\
\leq & \sum_{n=1}^{\infty}\frac{C^n}{n!}(([2np]+1)!)^{\gamma},
\end{aligned}$$

and it suffices to apply Proposition 3 to conclude the proof. ■

Remark 2 The exponent $p_0 = \frac{1}{2}\left[\left(\frac{1}{2}+H\right)\left(d-\frac{1}{2H}\right)\right]^{-1}$ is not optimal. For instance, if $Hd = 1$, then $p_0 = \frac{2H}{1+2H}$ and we know that for $Hd < 1$, then $p_0 = \frac{1}{Hd}$. In particular, if $H = \frac{1}{2}$ and $d = 2$ we obtain $p_0 = \frac{1}{2}$, and we know that in this case the critical exponent is $p_0 = 1$. The lack of optimality is due to the factor n in the estimation of the positive definite matrix N by its diagonal elements given in (4.22). Without this factor n we would get the critical exponent $\frac{1}{2Hd-1}$, but our method does not allow to get this value.

Remark 3 In the case of the planar Brownian motion $B = \{B_t, t \geq 0\}$ (that is, $d = 2$, and $H = \frac{1}{2}$), formula (4.2) yields

$$\widetilde{L} = -\frac{1}{2\pi} \sum_{i=1}^{2} \int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2} \exp\left(-\frac{|B_r - B_s|^2}{2(t-r)} \right) dsdt \right) dB_r^i. \tag{4.26}$$

The quadratic variation of this stochastic integral is

$$\begin{aligned} \left\langle \widetilde{L} \right\rangle &= \frac{1}{4\pi^2} \sum_{i=1}^{2} \int_0^T \left(\int_r^T \int_0^r \frac{B_r^i - B_s^i}{(t-r)^2} \exp\left(-\frac{|B_r - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr \\ &\leq \frac{1}{4\pi^2} \int_0^T \left(\int_r^T \int_0^r \frac{|B_r - B_s|}{(t-r)^2} \exp\left(-\frac{|B_r - B_s|^2}{2(t-r)} \right) dsdt \right)^2 dr \\ &= \frac{1}{\pi^2} \int_0^T \left(\int_0^r \frac{1}{|B_r - B_s|} \exp\left(-\frac{|B_r - B_s|^2}{2(T-r)} \right) ds \right)^2 dr \\ &\leq \frac{1}{\pi^2} \int_0^T \left(\int_0^r \frac{ds}{|B_r - B_s|} \right)^2 dr. \end{aligned}$$

From Itô's calculus we know that

$$\int_0^r \frac{ds}{|B_r - B_s|} = \frac{1}{d-1} \left(X_r - b_r \right),$$

where X_r has the law of the modulus of a d-dimensional Brownian motion at time r (Bessel process), and b_r has a normal $N(0, r)$ law. We can write

$$\exp\left(\lambda \left\langle \widetilde{L} \right\rangle \right) \leq \frac{1}{T} \int_0^T \exp\left(\frac{T\lambda}{\pi^2} \left(\int_0^r \frac{ds}{|B_r - B_s|} \right)^2 \right) dr,$$

which clearly imply the existence of some λ_0 such that $E\left(\exp\left(\lambda \left\langle \widetilde{L} \right\rangle \right) \right) < \infty$ for all $\lambda < \lambda_0$. From Lemma 6 we get that there exists β_0 such that $E\left(\exp\left(\beta \left| \widetilde{L} \right| \right) \right) < \infty$ for all $\beta < \beta_0$. This method does not allows us to obtain the critical exponent, just the existence of exponential moments.

Remark 4 The above results remain true if we replace the fractional Brownian motion with Hurst paramter H, by an arbitrary centered Gaussian process of the form (3.1) satisfying the local nondeterminism property **(LND)** and following properties:

(C1) For any $s, t \in [0, T]$, $s < t$, there exist constants k_3 and k_4 such that

$$k_3 (t-s)^{2H} \leq E(|B_t^i - B_s^i|^2) \leq k_4 (t-s)^{2H}.$$

(C2) The kernel $K(t,s)$ satisfies the estimates

$$|K(t,s)| \le k_5 (t-s)^{H-\frac{1}{2}} s^{\frac{1}{2}-H},$$

for all $s<t$, and

$$\int_r^T \int_r^t (t-s)^{-Hd-H} \left| K(t,r) - K(s,r) \right| ds dt \le \ \psi(r),$$

where $\int_0^T \ \psi(r)^2 dr < \infty$.

5 Appendix

In this Appendix we will first state and prove some elementary lemmas. The first one is well-known.

Lemma 5 *Suppose that $\mathcal{G}_1 \subset \mathcal{G}_2$ are two σ-fields contained in $\mathcal{F}$. Then, for any square integrable random variable F we have*

$$\mathrm{Var}(F|\mathcal{G}_1) \ge \mathrm{Var}(F|\mathcal{G}_2).$$

Let $M = \{M_t, t \ge 0\}$ be a continuous local martingale such that $M_0 = 0$. Then, the following maximal exponential inequality is well-known

$$P\left(\sup_{0\le t\le T} |M_t| \ge \delta, \langle M\rangle_T < \rho \right) \le 2\exp\left(-\frac{\delta^2}{2\rho} \right).$$

As a consequence of this inequality we can obtain exponential moments for M_T from exponential moments of the quadratic variation $\langle M\rangle_T$

Lemma 6 *Suppose that for some $\alpha > 0$ and $p \in (0,1]$ we have $E(e^{\alpha \langle M\rangle_T^p}) < \infty$. Then,*

(i) if $p=1$, for any $\lambda < \sqrt{\frac{\alpha}{2}}$, $E(e^{\lambda |M_T|}) < \infty$, and

(ii) if $p<1$, $E(e^{\lambda |M_T|^p}) < \infty$ for all $\lambda > 0$.

Proof. Set $X = |M_T|^p$. For any constant $c > 0$ we can write

$$\begin{aligned} E(e^{\lambda X}) &= \int_0^\infty P(X \geq y) \lambda e^{\lambda y} dy \\ &= \int_0^\infty \left[P(X \geq y, \langle M \rangle_T^p < cy) + P(X \geq y, \langle M \rangle_T^p \geq cy) \right] \lambda e^{\lambda y} dy \\ &\leq \int_0^\infty 2 \exp\left(-\frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}} \right) \lambda e^{\lambda y} dy + \int_0^\infty P\left(\frac{\langle M \rangle_T^p}{c} \geq y \right) \lambda e^{\lambda y} dy \\ &= \int_0^\infty 2\lambda \exp\left(\lambda y - \frac{y^{\frac{1}{p}}}{2c^{\frac{1}{p}}} \right) dy + E(e^{\frac{\lambda}{c} \langle M \rangle_T^p}). \end{aligned}$$

Then it suffices to choose $c = \frac{\lambda}{\alpha}$ to complete the proof. ■

The next two results are technical lemmas used in the paper.

Lemma 7 *Suppose that $H < \min(\frac{2}{d+1}, \frac{3}{2d})$. Then, we have*

$$\int_r^T \int_r^t (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right| dsdt \leq C \left(r^{\frac{1}{2}-H} \vee 1 \right),$$

for some constant C.

Proof. We know that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(H - \frac{1}{2} \right) \left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}}.$$

Then

$$\begin{aligned} I &:= \int_r^T \int_r^t (t-s)^{-Hd-H} \left| K_H(t,r) - K_H(s,r) \right| dsdt \\ &\leq C \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} \left(\frac{\theta}{r} \right)^{H-\frac{1}{2}} (\theta - r)^{H-\frac{3}{2}} d\theta dsdt. \end{aligned}$$

If $H < \frac{1}{2}$, then, $\left(\frac{\theta}{r}\right)^{H-\frac{1}{2}} \leq 1$, and if $H > \frac{1}{2}$, then $\left(\frac{\theta}{r}\right)^{H-\frac{1}{2}} \leq C r^{\frac{1}{2}-H}$. Hence, the above integral is bounded by

$$C(r^{\frac{1}{2}-H} \vee 1) \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta - r)^{H-\frac{3}{2}} d\theta dsdt.$$

From the decomposition

$$\begin{aligned}\frac{3}{2}-H &= \alpha+\beta, \\ Hd+H &= \gamma+\delta,\end{aligned}$$

we obtain

$$\begin{aligned}& \int_r^T \int_r^t \int_s^t (t-s)^{-Hd-H} (\theta-r)^{H-\frac{3}{2}} d\theta ds dt \\ = & \int_r^T \int_r^t \int_s^t (s-r)^{-\alpha} (\theta-s)^{-\beta-\gamma} (t-\theta)^{-\delta} d\theta ds dt.\end{aligned}$$

Finally, it suffices to show the parameters α, β, γ and δ in such a way that $\alpha < 1$, $\delta < 1$ and $\beta + \gamma < 1$. This leads to the condition

$$\frac{1}{2} + Hd < \min(1, \frac{3}{2} - H) + \min(1, Hd + H),$$

which is satisfied if $H < \min(\frac{2}{d+1}, \frac{3}{2d})$. ■

Lemma 8 *Let $a < 1$. Fix an interval $[0,T]$. For each integer $n \geq 1$ we have*

$$\begin{aligned}& \int_{\Delta_n(T)} [((T-s_n) \wedge s_n)((s_n - s_{n-1}) \wedge s_{n-1}) \cdots ((s_2 - s_1) \wedge s_1)]^{-a} ds \\ & \leq \frac{T^{n(1-a)}}{\Gamma(n(1-a)+1)} C^n, \end{aligned} \tag{5.1}$$

where $\Delta_n(T) = \{0 < s_1 < \cdots < s_n < T\}$

Proof. We proceed by induction on n. For $n = 1$ we can write

$$\begin{aligned}\int_0^T ((T-s_1) \wedge s_1)^{-a} ds_1 &= \int_0^{\frac{T}{2}} s_1^{-a} ds_1 + \int_{\frac{T}{2}}^T (T-s_1)^{-a} ds_1 \\ &= \frac{2}{1-a} \left(\frac{T}{2}\right)^{1-a},\end{aligned}$$

which implies (5.1) with $C = \frac{\Gamma(2-a)}{1-a} 2^a$.

Suppose that the result holds for $n-1$. Then,

$$\begin{aligned}
I_n &= \int_{\Delta_n(T)} \left[\left((T-s_n)\wedge s_n\right)\left((s_n-s_{n-1})\wedge s_{n-1}\right)\cdots\left((s_2-s_1)\wedge s_1\right)\right]^{-a} ds \\
&= \int_0^T \left((T-s_n)\wedge s_n\right)^{-a} \\
&\quad \times \left(\int_{\Delta_{n-1}(s_n)} \left[\left((s_n-s_{n-1})\wedge s_{n-1}\right)\cdots\left((s_2-s_1)\wedge s_1\right)\right]^{-a} ds_1\cdots ds_{n-1}\right) ds_n.
\end{aligned}$$

By the induction hypothesis we can write

$$\begin{aligned}
I_n &\leq \frac{C^{n-1}}{\Gamma(n-a)} \int_0^T \left((T-s_n)\wedge s_n\right)^{-a} s_n^{(n-1)(1-a)} ds_n \\
&= \frac{C^{n-1}}{\Gamma((n-1)(1-a)+1)} \\
&\quad \times \left(\int_0^{\frac{T}{2}} s_n^{(n-1)(1-a)-a} ds_n + \int_{\frac{T}{2}}^T (T-s_n)^{-a} s_n^{(n-1)(1-a)} ds_n\right) \\
&\leq \frac{C^{n-1}}{\Gamma(n(1-a)+a)} \\
&\quad \times \left(\frac{1}{n(1-a)}\left(\frac{T}{2}\right)^{n(1-a)} + T^{n(1-a)} \int_0^1 (1-x)^{-a} x^{(n-1)(1-a)} dx\right) \\
&\leq \frac{T^{n(1-a)} C^{n-1}}{\Gamma(n(1-a)+a)} \left(\frac{1}{n(1-a)} + \frac{\Gamma(1-a)\Gamma((n-1)(1-a)+1)}{\Gamma(n(1-a)+1)}\right) \\
&= T^{n(1-a)} C^{n-1} \left(\frac{1}{n(1-a)\Gamma(n(1-a)+a)} + \frac{\Gamma(1-a)}{\Gamma(n(1-a)+1)}\right).
\end{aligned}$$

Using the relation $\Gamma(n+1) = n\Gamma(n)$ we obtain

$$n(1-a)\Gamma(n(1-a)+a) \geq n(1-a)\Gamma(n(1-a)) = \Gamma(n(1-a)+1),$$

and, as a consequence

$$I_n \leq T^{n(1-a)} C^{n-1} \left(1+\Gamma(1-a)\right) \frac{1}{\Gamma(n(1-a)+1)},$$

and it suffices to take $C \geq \max\left(\frac{\Gamma(2-a)}{1-a} 2^a, 1+\Gamma(1-a)\right)$. ■

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