Monodromy

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Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday.

Abstract

Let (X, x) be an isolated complete intersection singularity and let $f : (X, x) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singularity at x. An important topological invariant in this situation is the Picard-Lefschetz monodromy operator associated to f. We give a survey on what is known about this operator. In particular, we review methods of computation of the monodromy and its eigenvalues (zeta function), results on the Jordan normal form of it, definition and properties of the spectrum, and the relation between the monodromy and the topology of the singularity.

Introduction

The word 'monodromy' comes from the greek word $\mu o \nu o - \delta \rho o \mu \psi$ and means something like 'uniformly running' or 'uniquely running'. According to [99, 3.4.4], it was first used by B. Riemann [135]. It arose in keeping track of the solutions of the hypergeometric differential equation going once around a singular point on a closed path (cf. [30]). The group of linear substitutions which the solutions are subject to after this process is called the *monodromy group*.

Since then, monodromy groups have played a substantial rôle in many areas of mathematics. As is indicated on the webside 'www.monodromy.com' of N. M. Katz, there are several incarnations, classical and l-adic, local and global, arithmetic and geometric. Here we concentrate on the classical local geometric monodromy in singularity theory. More precisely we focus on the monodromy operator of an isolated hypersurface or complete intersection singularity. The investigation of this operator started in 1967 with the proof of the famous monodromy theorem (see §1). This theorem can be proved

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using the theory of the Gauß-Manin connection which was introduced by E. Brieskorn for isolated hypersurface singularities [28]. The study of this connection for isolated complete intersection singularities was started by G.-M. Greuel in his thesis [76].

We try to review the results of 37 years of investigation of the monodromy operator. The results include results on the zeta function of the monodromy and the spectrum of a singularity. The monodromy contains a lot of information about the topology of the singularity. This was one motivation to study the monodromy. We review the known facts in the last section.

For a basic introduction to the subject for non-specialists see [55].

Aspects which are not mentioned or only touched in this survey are

- Monodromy of a polynomial function. For a survey on this topic see [40] and [87].
- Generalizations to non-isolated singularities. Here we refer to the survey of D. Siersma [151].
- Monodromy groups. We do not talk about monodromy groups of isolated complete intersection singularities. We refer to our book [51] for this topic.
- Braid monodromy. Recently, M. Lönne introduced a notion of braid monodromy of singularities. He computed the braid monodromy and the fundamental group of the complement of the discriminant of a Brieskorn-Pham singularity [116], making a substantial contribution to the last problem [29, Problème 20] of Brieskorn's list of problems on monodromy.

1 The monodromy operator

Let $(Y, 0) \subset (\mathbb{C}^N, 0)$ be an isolated complete intersection singularity (abbreviated ICIS in the sequel) of dimension $n+1$, i.e. $(Y, 0)$ is the germ of an analytic variety of pure dimension $n+1$ with an isolated singularity at the origin given by $Y = F^{-1}(0)$, where $F = (f_1, \ldots, f_{N-n-1}) : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n-1}, 0)$ is the germ of an analytic mapping. Let $f : \mathbb{C}^N \to \mathbb{C}$ be an analytic function such that the restriction $f : Y \to \mathbb{C}$ which we denote by the same symbol has an isolated singularity at the origin. We assume that $f(0) = 0$. Let $\varepsilon > 0$ be small enough such that the closed ball $B_\varepsilon \subset \mathbb{C}^N$ of radius ε around the origin in $\mathbb{C}^N$ intersects the fibre $f^{-1}(0)$ transversely. Let $0 < \delta \ll \varepsilon$ be such that for t in the disc $D_\delta \subset \mathbb{C}$ around the origin, the fibre $f^{-1}(t) \cap Y$ intersects the ball B_ε transversely. Let

$$\begin{aligned} X_t &:= f^{-1}(t) \cap B_\varepsilon \cap Y \text{ for } t \in D_\delta, \\ X &:= f^{-1}(D_\delta) \cap B_\varepsilon \cap Y, \\ X' &:= X \setminus X_0, \\ D' &:= D \setminus \{0\}. \end{aligned}$$

Then $(X_0, 0)$ is an ICIS of dimension n. In the important special case when Y is smooth, $(Y, 0) = (\mathbb{C}^{n+1}, 0)$, then $(X_0, 0)$ is an isolated hypersurface singularity. By a result of J. Milnor [125] in the case when Y is smooth and H. Hamm [89] in the general case, the mapping $f|_{X'} : X' \to D'$ is the projection of a locally trivial C^∞ fibre bundle. A fibre X_t of this bundle is called *Milnor fibre*. It has the homotopy type of a bouquet of μ n-spheres where μ is the Milnor number. Therefore its only interesting homology group is the group $H_n(X_t, \mathbb{Z})$. It is of rank μ. Parallel translation along the path

$$\gamma : [0, 1] \to D_\delta, \quad t \mapsto \delta e^{2\pi i t},$$

yields a diffeomorphism $h : X_\delta \to X_\delta$ called the *geometric monodromy* of the singularity. It is determined up to isotopy.

Definition 1.1. The induced homomorphism $h_*^{\mathbb{C}} : H_n(X_\delta, \mathbb{C}) \to H_n(X_\delta, \mathbb{C})$ (resp. $h_*^{\mathbb{Z}} : H_n(X_\delta, \mathbb{Z}) \to H_n(X_\delta, \mathbb{Z})$) is called the *complex* (resp. *integral*) *monodromy (operator)* of the singularity.

This operator is also sometimes called the *Picard-Lefschetz monodromy operator* since the consideration of this operator goes back to E. Picard [133] and S. Lefschetz [107] (see also [98], [132]).

Theorem 1.2 (Monodromy theorem). (a) *The eigenvalues of h_* are roots of unity.*

(b) *The size of the blocks in the Jordan normal form of h_* is at most $(n+1) \times (n+1)$.*

(c) *If $(Y, 0)$ is smooth, then the size of the Jordan blocks for the eigenvalue 1 is at most $n \times n$.*

There are many different proofs of this theorem: by A. Borel (unpublished), E. Brieskorn [28] (for $(Y, 0)$ smooth, generalized by G.-M. Greuel [76]), C. H. Clemens [33], P. Deligne [35, 36], P. A. Griffiths [79], A. Grothendieck [80], N. M. Katz [95], A. Landman [100, 101], Lê Dũng Tráng [106] (of (a), for (b) see the book of E. Looijenga [117]), B. Malgrange [120], and W. Schmid [146] (see also the survey [79]).

Examples of B. Malgrange [118] show that the bounds on the size of the Jordan blocks are sharp.

For weighted homogeneous singularities with $(Y, 0)$ smooth, Milnor [125] has shown that the complex monodromy $h_*^{\mathbb{C}}$ is diagonalizable. For weighted homogeneous ICIS this was shown by A. Dimca [39]. For irreducible plane curve singularities, Lê [104] has shown that the monodromy is of finite order. N. A'Campo [2] has shown that for isolated plane curve singularities with more than one branch the monodromy is in general not of finite order. A. H. Durfee [47] has given a necessary and sufficient condition for the monodromy of a degenerating family of curves to be of finite order.

We now mention several results which are only valid in the case when $(Y, 0) = (\mathbb{C}^{n+1}, 0)$. J. Scherk [144] has shown that if f^{r+1} belongs to the ideal $(\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$ of the ring $\mathcal{O}_{n+1}$ of germs of holomorphic functions on $\mathbb{C}^{n+1}$, then the size of the Jordan blocks of $h_*^{\mathbb{C}}$ is at most $(r+1) \times (r+1)$. By a theorem of J. Briançon and H. Skoda [27], $f^{n+1} \in (\partial f/\partial x_0, \ldots, \partial f/\partial x_n)$. Therefore Scherk's theorem implies the Monodromy theorem. Generalizations of Scherk's theorem can be found in [145].

M. G. M. van Doorn and J. H. M. Steenbrink [41] have proved the following supplement to the Monodromy theorem: If there exists a Jordan block of size $(n+1) \times (n+1)$, then there exists a Jordan block of size $n \times n$ for the eigenvalue 1. Since a plane curve singularity is reducible if and only if $h_*^{\mathbb{C}}$ has an eigenvalue 1, this implies Lê's theorem.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $g : (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$ be two germs of analytic functions with an isolated singularity at 0. Denote by c_f and c_g the complex monodromy operators of f and g respectively. Denote by c_{f+g} the complex monodromy operator of the germ $f + g$. The famous theorem of M. Sébastiani and R. Thom [150] states that

$$c_{f+g} = c_f \otimes c_g.$$

The author and Steenbrink [64] have proved a generalization of this theorem for a suspension of an ICIS.

In the case when $Y = \mathbb{C}^{n+1}$ one can associate to $f \in \mathbb{C}\{x_0, \ldots, x_n\}$ its *Bernstein-Sato polynomial*. This is defined as follows. Let s be a new variable. Then there exists a differential operator $P = P(x, s, \partial/\partial x)$ whose coefficients are convergent power series in s and $x_0, \ldots, x_n$ and a nonzero polynomial $b(s) \in \mathbb{C}[s]$ satisfying the formal identity

$$Pf^s = b(s)f^{s-1}.$$

The set of all polynomials $b(s) \in \mathbb{C}[s]$ for which such an identity holds (for some operator P) forms an ideal, and the unique monic generator for this ideal is called the *Bernstein-Sato polynomial* of f. It is denoted by $b_f(s)$. According to Malgrange [119, 121] (see also [22]) there is the following relation to the monodromy of f: The zeros s_1, s_2, ... of $\widetilde{b}_f(s) := b_f(s)/s$ are rational and less than 1, the minimal polynomial of the monodromy divides the polynomial $p(t) := \prod(t - \exp(-2\pi i s_j))$, and on the other hand, $p(t)$ divides the characteristic polynomial of the monodromy. D. Barlet [18] has shown that if there exists a $k \times k$ Jordan block for the eigenvalue $\exp(-2\pi i u)$ of the monodromy, then there exist at least k (counted with multiplicity) roots of $b_f(s)$ of the form $-q - u$ with $q \in [0, n]$ $(0 \leq u < 1)$.

2 Computation of monodromy

We now review methods to compute the monodromy operator.

We first consider the complex monodromy operator. In the case when $(Y, 0)$ is smooth, Brieskorn [28] has indicated a method to compute the complex monodromy operator. This method has been implemented by M. Schulze to Singular [147] (see also [148]).

For plane curve singularities there is an algorithm in the book of D. Eisenbud and W. Neumann [68] to compute the Jordan normal form of the complex monodromy operator from a splicing diagram of the singularity.

For superisolated surface singularities (see the article of E. Artal-Bartolo, I. Luengo and A. Melle-Hernández in this volume [17]), Artal-Bartolo [12, 14] has determined the Jordan normal form of the complex monodromy operator.

We now consider the integral monodromy operator. Let $Y_\eta := F^{-1}(\eta) \cap B_\varepsilon$ where η is a regular value of F sufficiently close to 0. Let $c := h_*^{\mathbb{Z}}$ be the integral monodromy operator. The above path γ also induces a map $\widehat{c} : H_{n+1}(Y_\eta, X_\delta) \to H_{n+1}(Y_\eta, X_\delta)$ on the relative homology groups. (Unless otherwise stated, we consider homology with integral coefficients). Then we have the following diagram with exact rows and commutative squares (cf. [51]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_{n+1}(Y_\eta) & \longrightarrow & H_{n+1}(Y_\eta, X_\delta) & \longrightarrow & H_n(X_\delta) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \widehat{c}} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & H_{n+1}(Y_\eta) & \longrightarrow & H_{n+1}(Y_\eta, X_\delta) & \longrightarrow & H_n(X_\delta) & \longrightarrow & 0 \end{array}$$

Let A be the intersection matrix on $H_{n+1}(Y_\eta, X_\delta)$ with respect to a distinguished basis of thimbles (cf. [51]). It is a $\nu \times \nu$-matrix where ν is the number of thimbles in a distinguished basis of thimbles. One has $\nu = \mu + \mu'$ where μ' is the Milnor number of the singularity $(Y, 0)$. The matrix A is encoded in the *Coxeter-Dynkin diagram* of the singularity. This matrix is of the form

$A = V + (-1)^n V^t$ for some upper triangular matrix

$$V = \begin{pmatrix} (-1)^{\frac{n(n+1)}{2}} & * & \cdots & * & * \\ 0 & (-1)^{\frac{n(n+1)}{2}} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & (-1)^{\frac{n(n+1)}{2}} & * \\ 0 & \cdots & \cdots & 0 & (-1)^{\frac{n(n+1)}{2}} \end{pmatrix}.$$

If $Y = \mathbb{C}^{n+1}$ then V is the matrix of the *(integral) Seifert form* or of the inverse of the *variation operator* of the singularity (see [11, 45, 98]). For general $(Y, 0)$, the author and S. M. Gusein-Zade defined in [61] a variation operator the inverse of which has the matrix V. The operator $\widehat{c}$ is the product of the Picard-Lefschetz transformations corresponding to the elements of a distinguished basis of thimbles (cf. [51]). In the case when $Y = \mathbb{C}^3$ and f defines a simple singularity, the Coxeter-Dynkin diagram is the classical Coxeter-Dynkin diagram of a root system of type A_μ, D_μ, E_6, E_7, or E_8 and $\widehat{c} = c$ is the corresponding *Coxeter element.* It follows from [23, Chap. V, §6, Exercice 3] (see also [108]) that the matrix $\widehat{C}$ of the operator $\widehat{c}$ is given by

$$\widehat{C} = (-1)^{n+1} V^{-1} V^t.$$

F. Lazzeri [102] (see also [103]) and independently A. M. Gabrielov [71] showed that in the case when $Y = \mathbb{C}^{n+1}$ the Coxeter-Dynkin diagram is connected thus extending an earlier result of C. H. Bey [20, 21] for curves. A. Hefez and Lazzeri [91] computed the intersection matrix of Brieskorn-Pham singularities solving in this way an open problem stated by Brieskorn [29] and F. Pham [132].

Gabrielov [70, 72] has given methods to compute the intersection matrix A for some special singularities. N. A'Campo [4, 5] and independently Gusein-Zade [81, 82] have found a beautiful method to compute the intersection matrix for isolated plane curve singularities using real morsifications. This method was generalized in [59, 60] to suspensions of fat points.

A rather general method to compute an intersection matrix for isolated hypersurface singularities using polar curves was found by Gabrielov [73]. This method was generalized to ICIS in [51]. The author [48, 49] has computed the characteristic polynomial of the monodromy for the uni- and bimodal hypersurface singularities in Arnold's classification [8] and the intersection matrix A for the elliptic hypersurface singularities [50]. Gusein-Zade [83] gave a recursive formula for the characteristic polynomials of the monodromy for the singularities of Arnold's series of singularities [8].

In [51] the matrices A were computed for the simple space curve singularities classified by M. Giusti [74] except Z_9 and Z_{10} and many of the

$\mathcal{K}$-unimodal isolated singularities of complete intersection surfaces classified by C. T. C. Wall [165]. The missing cases Z_9 and Z_{10} were studied in [58] and the case $I_{1,0}$ was treated in [53].

P. Orlik and R. Randell [130] computed the integral monodromy for some classes of weighted homogeneous singularities.

If $Y = \mathbb{C}^{n+1}$ and f is a real analytic function, i.e. takes real values on $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$, then Gusein-Zade [84] showed that the integral monodromy is the product of two involutions (see also [6]).

3 Zeta function

The *zeta function* of the monodromy is defined to be

$$\zeta(t) := \prod_{q \geq 0} \{\det(\mathrm{id}_* - t h_*; H_q(X_\delta, \mathbb{C}))\}^{(-1)^{q+1}}.$$

The relation with the characteristic polynomial $\Delta(t)$ of the monodromy is

$$\Delta(t) := \det(t\mathrm{id}_* - h_*) = t^\mu \left[\frac{t-1}{t}\zeta\left(\frac{1}{t}\right)\right]^{(-1)^{n+1}}.$$

The *Lefschetz numbers* are defined by

$$\Lambda_k := \Lambda(h_*^k) = \sum_{q \geq 0} (-1)^q \mathrm{Tr}[h_*^k; H_q(X_\delta, \mathbb{C})].$$

We define rational numbers χ_m by

$$\Lambda_k = \sum_{m|k} m\chi_m.$$

Explicitely, these numbers can be defined by Möbius inversion

$$\chi_m = \frac{1}{m}\left(\sum_{k|m} \mu\left(\frac{m}{k}\right)\Lambda_k\right),$$

where $\mu(\ )$ denotes the Möbius function. By A. Weil (cf. [125]) we have

$$\zeta(t) = \prod_{m \geq 1} (1 - t^m)^{-\chi_m}.$$

The following statements were explained to me by D. Zagier.

Proposition 3.1 (Zagier). (i) *The numbers χ_m are integers.*

(ii) *The following statements are equivalent:*

(a) $\Delta(t)$ *is a product of cyclotomic polynomials.*

(b) $\chi_m \neq 0$ *for only finitely many* m.

(c) *The sequence* (Λ_k) *is periodic.*

Proof. (i) is proved by induction: Assume that $\chi_m \in \mathbb{Z}$ for $m \leq \ell$. Then

$$\prod_{m=1}^{\ell} (1-t^m)^{-\chi_m}$$

is a formal power series with integer coefficients which starts with 1. Since $\zeta(t)$ is also a power series with integral coefficients, the same is true for

$$\frac{\zeta(t)}{\prod_{m=1}^{\ell}(1-t^m)^{-\chi_m}} = \prod_{m=\ell+1}^{\infty} (1-t^m)^{-\chi_m}.$$

But this power series starts with $1 + \chi_{\ell+1} t^{\ell+1}$.

The proof of (ii) is done in several steps:

The implication (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c): Let $\chi_m = 0$ for all m which do not divide a number Q. Then for positive integers d, r with $0 < r \leq Q$ we have

$$\Lambda_{dQ+r} = \sum_{m|dQ+r} m\chi_m = \sum_{m|r} m\chi_m = \Lambda_r.$$

(c) $\Rightarrow$ (a): Let the sequence (Λ_k) be periodic of period Q. Let $s_k := \operatorname{Tr} h_*^k$ and let $p(t) := \det(\mathrm{id}_* - th_*)$. Then

$$p(t) = \exp(\operatorname{Tr}(\log(\mathrm{id}_* - th_*))) = \exp\left(-\sum_{k=1}^{\infty} s_k \frac{t^k}{k}\right).$$

For the logarithmic derivative of $p(t)$ we get from this

$$\frac{p'(t)}{p(t)} = -\sum_{k=1}^{\infty} s_k t^{k-1}.$$

Since s_k is periodic of period Q, we get

$$\begin{aligned}\frac{p'(t)}{p(t)} &= -\sum_{k=1}^{Q} s_k t^{k-1}(1 + t^Q + t^{2Q} + \ldots) \\ &= -\sum_{k=1}^{Q} s_k \frac{t^{k-1}}{1-t^Q} = \frac{q(t)}{1-t^Q}\end{aligned}$$

for some polynomial $q(t)$. But each zero of $p(t)$ must be a simple pole of the logarithmic derivative and hence a zero of $1 - t^Q$. $\square$

If $(Y,0)$ is smooth and f is singular, then by [1] $\Lambda_1 = 0$. Lê [105] proved that in this case there exists a characteristic diffeomorphism h without fixed points. In a letter to A'Campo, P. Deligne showed that more generally in the case when $(Y,0)$ is smooth, $\Lambda_k = 0$ for $0 < k < \operatorname{mult}(f)$ where $\operatorname{mult}(f)$ is the multiplicity of f. G. G. Il'yuta [94] gave formulae expressing the Lefschetz numbers in terms of cycles of the Coxeter-Dynkin diagram. In [52] the author showed that $\operatorname{Tr} C^2 = (-1)^r$ where r is the corank of f.

Let $\pi : \widetilde{Y} \to Y$ be a resolution of Y and let $\widetilde{f} := f \circ \pi$ be the composition. We denote by $\widetilde{X}_0$ the proper transform of $X_0 = f^{-1}(0)$. Let $\widetilde{f}^{-1}(0) = \widetilde{X}_0 \cup E_1 \cup \cdots \cup E_s$ where E_i is irreducible. We assume that the following conditions are satisfied:

(1) $\pi : \widetilde{X}_0 \to X_0$ is a resolution of X_0.

(2) Each exceptional divisor E_i is smooth and $\widetilde{f}^{-1}(0)$ has only normal crossings.

Let m_i be the order of the function $\widetilde{f}$ along the divisor E_i and let

$$E_i' := E_i \setminus \left(\bigcup_{j \neq i} E_j \right) \cup \widetilde{X}_0.$$

Then we have the following famous theorem of A'Campo [3]:

Theorem 3.2 (A'Campo). *Under the above assumptions, we have*

$$\zeta(t) = \prod_{i=1}^{s} (1 - t^{m_i})^{-\chi(E_i')}$$

where $\chi(E_i')$ is the topological Euler characteristic of E_i'.

The theorem was formulated by A'Campo only for the case $Y = \mathbb{C}^{n+1}$ but it can be easily generalized to this more general situation (see e.g. [128]). From Proposition 3.1 we see that A'Campo's theorem implies Part (a) of the Monodromy theorem. In [11] it is shown that Part (b) can also be derived from that theorem.

A generalization of A'Campo's theorem using partial resolutions was given by Gusein-Zade, Luengo and Melle-Hernández [86].

If $Y = \mathbb{C}^{n+1}$ and f is non-degenerate with respect to the Newton diagram, then A. Varchenko [158] (and also independently F. Ehlers [66]) have given a formula to compute $\zeta(t)$ from the Newton diagram. This formula was generalized to the general case by M. Oka [128].

A. Campillo, F. Delgado and Gusein-Zade [85] have shown that for an irreducible curve singularity, the zeta function $\zeta(t)$ coincides with the Poincaré series $P(t)$ of the natural filtration on the ring of functions of such a singularity given by the order with respect to a uniformization.

Now suppose $Y = \mathbb{C}^{n+1}$ and f is weighted homogeneous of weights $q_0, \dots, q_n$ and degree d. Here $q_0, \dots, q_n$ are assumed to be coprime. Then the geometric monodromy $h : X_1 \to X_1$ can be described as follows [125]:

$$h(z_0, \dots, z_n) = (e^{2\pi i/q_0} z_0, \dots, e^{2\pi i/q_n} z_n).$$

The monodromy operator h_* is of order d. Milnor and P. Orlik [126] have shown how to compute $\zeta(t)$ from the weights and the degree of f. Greuel and H. Hamm [77] have given a more general formula for a weighted homogeneous ICIS $(Y, 0)$.

K. Saito [137] has shown that if $Y = \mathbb{C}^3$ then all the primitive d-th roots of unity are eigenvalues of h_*.

Saito [138, 139] also introduced a duality between rational functions of the form of the zeta function. If $\phi(t)$ is a rational function of the form

$$\phi(t) = \prod_{m|d} (1 - t^m)^{\chi_m} \quad \text{for } \chi_m \in \mathbb{Z} \text{ and some } d \in \mathbb{N},$$

then he defines

$$\phi^*(t) = \prod_{k|d} (1 - t^k)^{-\chi_{d/k}}.$$

Let f be a function defining one of the 14 exceptional unimodal hypersurface singularities in the sense of V. I. Arnold [9]. Arnold has observed a strange duality between these singularities [8]. Saito has observed the following fact: If $\Delta(t)$ is the characteristic polynomial of the monodromy of f then $\Delta^*(t)$ is the characteristic polynomial of the monodromy of the dual singularity. The author and C. T. C.Wall [65] have found an extension of Arnold's strange duality embracing also ICIS. The author [53] has shown that Saito's duality also holds for this extension and he has related it to polar duality and to a duality of weight systems found by M. Kobayashi [54, 57].

If $n = 2$ and $Y = \mathbb{C}^3$ or $(Y, 0)$ is a certain special ICIS, then it was shown [56] that the Saito dual $\Delta^*(t)$ of the characteristic polynomial of the monodromy is equal to the product of the Poincaré series $P(t)$ of the coordinate algebra and some rational function $\mathrm{Or}(t)$ depending only on the orbit invariants of the natural $\mathbb{C}^*$-action on the singularity. In the case when $Y = \mathbb{C}^{n+1}$ and f is a Newton non-degenerate function, the author and Gusein-Zade [62] showed that the same holds for the Saito dual of the inverse of the reduced zeta function $\widetilde{\zeta}(t)$ (reduced means considering reduced homology). Finally, in [63] this was generalized to the case when Y is a complete intersection given as the zero set of functions $f_1, \dots, f_{k-1}$ and $\underset{\sim}{f} = f_k$ to the product of the Saito duals of the inverse reduced zeta functions $\widetilde{\zeta}_j(t)$ of the monodromy operators of f_j on $f_1 = \ldots = f_{j-1} = 0$ for $j = 1, \ldots, k$. J. Stevens [156] proved that this result implies the theorem of Campillo, Delgado and Gusein-Zade [85].

Now let $Y = \mathbb{C}^{n+1}$ and assume that $f \in \mathbb{Z}[x_0, \ldots, x_n]$. For a prime number p denote by $\mathbb{Z}_p$ the p-adic integers. Consider the p-adic integral

$$Z_p(s) := \int_{\mathbb{Z}_p^{n+1}} |f(x)|_p^s |dx|$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p^{n+1}$ normalized in such a way that $\mathbb{Z}_p^{n+1}$ is of volume 1. This function is called the *p-adic Igusa zeta function*. Now there is the following famous conjecture [93] (see also [37]):

Conjecture 3.3 (Igusa's monodromy conjecture). *For almost all prime numbers p, if s_0 is a pole of $Z_p(s)$ then $e^{2\pi i \mathrm{Re}(s_0)}$ is an eigenvalue of the monodromy operator h_* at some point of $\{f = 0\}$.*

J. Denef and F. Loeser [38] defined a topological zeta function $Z_{\mathrm{top}}(t)$ generalizing Igusa's zeta function. The analogous conjecture is stated for this function. Loeser [113, 114], W. Veys [162], Artal-Bartolo, P. Cassou-Noguès, Luengo and Melle-Hernández [15, 16] and B. Rodrigues and Veys [136] proved various special cases of this conjecture. See [163] for an excellent survey on this topic and the article of Artal-Bartolo, Luengo and Melle-Hernández in this volume [17].

In [88] a motivic version of the zeta function of the monodromy is discussed and compared with the motivic zeta function of Denef and Loeser.

4 Spectrum

In the case $(Y, 0)$ smooth, Steenbrink [153] showed that there exists a mixed Hodge structure on the Milnor fibre. Let $H = H^n(X_\delta, \mathbb{Z})$. Such a mixed Hodge structure consists of an increasing *weight* filtration

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2n} = H \otimes \mathbb{Q}$$

of $H \otimes \mathbb{Q}$ and a decreasing *Hodge* filtration

$$H \otimes \mathbb{C} = F^0 \supset F^1 \supset \cdots \supset F^n \subset F^{n+1} = 0.$$

It follows from a result of M. Saito [141] that in the general case, the analogue in cohomology of the short exact sequence in §2 can be considered as a sequence of mixed Hodge structures (see [64]).

The mixed Hodge structure is used to define the spectrum of a singularity. The spectrum was defined by Steenbrink [153] and Arnold [10] in the case when $(Y, 0)$ is smooth and in [64] in the general case.

Definition 4.1. The *spectrum* $\mathrm{Sp}(f)$ of f is defined as follows. Let $p \in \mathbb{Z}$, $0 \leq p \leq n$. A rational number $\alpha \in \mathbb{Q}$ with $n-p-1 < \alpha \leq n-p$ is in $\mathrm{Sp}(f)$ if and only if $e^{2\pi i \alpha}$ is an eigenvalue of the semisimple part of h^* on $F^p H / F^{p+1} H$. Here $H = H^n(X_\delta, \mathbb{C})$ if $Y = \mathbb{C}^{n+1}$ and $H = H^{n+1}(Y_\eta, X_\delta, \mathbb{C})$ in the general case. The multiplicity of α is the dimension of the corresponding eigenspace.

The spectrum is an unordered tuple of ν rational numbers $\alpha_1, \ldots, \alpha_\nu$ which lie between -1 and n. We order these numbers as follows:

$$-1 < \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_\nu < n.$$

There is a symmetry property

$$\alpha_i + \alpha_{\nu+1-i} = n - 1.$$

V. V. Goryunov [75] computed the spectra of the simple, uni- and bimodal hypersurface singularities. Steenbrink [155] compiled tables of the spectra for all $\mathcal{K}$-unimodal ICIS. If $Y = \mathbb{C}^{n+1}$ and f is Newton non-degenerate then the spectrum can be computed from the Newton diagram, see [140, 161]. Other methods to compute the spectrum in the case when $(Y, 0)$ is smooth have been given by Schulze and Steenbrink [149].

The most famous property of the spectrum is the semicontinuity conjectured by Arnold [10] and proved by Steenbrink [154] for the case when $(Y, 0)$ smooth and the author and Steenbrink in the general case [64]:

Theorem 4.2 (Semicontinuity theorem). *The spectrum behaves semicontinuously under deformation of the singularity in the following sense: If f' (with $\nu' < \nu$) appears in the semi-universal deformation of f, then*

$$\alpha_i \leq \alpha'_i.$$

The variance of the spectrum measures the distribution of the spectral numbers with respect to the central point and is defined by

$$V = \frac{1}{\nu} \sum_{i=1}^{\nu} \left(\alpha_i - \frac{n-1}{2} \right)^2.$$

C. Hertling [92] proposed the following conjecture

Conjecture 4.3 (Hertling). *If $(Y, 0)$ is smooth (so $\nu = \mu$), then*

$$V \leq \frac{\alpha_\mu - \alpha_1}{12}.$$

One has equality if f is weighted homogeneous, as shown by A. Dimca [40] and Hertling [92]. M. Saito [143] showed that Hertling's conjecture holds for all irreducible plane curve singularities. Th. Brélivet [24, 25] showed that the conjecture holds for all curve singularities. Brélivet and Hertling have stated more general conjectures involving higher moments [26].

Let $Y = \mathbb{C}^{n+1}$. We shall now give several different interpretations of the smallest exponent α_1.

Let ω be a holomorphic $(n+1)$-form on $\mathbb{C}^{n+1}$. For $0 < |t| < \delta$ let $\eta(t)$ be a continuously varying homology class of dimension n on X_t and consider the function

$$I(t) = \int_{\eta(t)} \frac{\omega}{df}.$$

This function admits an asymptotic expansion as t tends to zero:

$$I(t) = \sum_{\alpha,q} \frac{1}{q!} C^{\omega,\eta}_{\alpha,q} t^{\alpha} (\log t)^q,$$

such that $q \in \mathbb{Z}$, $0 \leq q \leq n$, $\alpha \in \mathbb{Q}$, $\alpha > -1$ and $e^{2\pi i \alpha}$ is an eigenvalue of the semisimple part of the monodromy operator. By [159] we have

$$\alpha_1 = \beta_{\mathbb{C}} - 1 := \min\{\alpha \mid \exists \omega, \eta, q \ C^{\omega,\eta}_{\alpha,q} \neq 0\}.$$

The number $\beta_{\mathbb{C}}$ is the *complex singularity index* (cf. [7], where in fact the number $\frac{n+1}{2} - \beta_{\mathbb{C}}$ is called the complex singularity index). For a simple singularity in $\mathbb{C}^3$, one has $\beta_{\mathbb{C}} = 1 + \frac{1}{N}$ where N is the Coxeter number of the singularity (cf. [7]).

With the notations of §3, let k_i be the multiplicity of $\pi^*(dx_0 \wedge \ldots \wedge dx_n)$ along the divisor E_i, $i = 1, \ldots, s$. Let m_0 and k_0 be the order of $\widetilde{f}$ and the multiplicity of $\pi^*(dx_0 \wedge \ldots \wedge dx_n)$ respectively along the divisor $\widetilde{X}_0$. So $m_0 = 1$ and $k_0 = 0$. Let

$$\lambda := \min\left\{ \frac{k_i + 1}{m_i} \,\middle|\, i = 0, \ldots, s \right\}.$$

K.-Ch. Lo [112] has shown that

$$\beta_{\mathbb{C}} \geq \lambda.$$

T. Yano [167] and B. Lichtin [111] have shown that if $\lambda < 1$ then

$$\beta_{\mathbb{C}} = \lambda.$$

Ehlers and Lo [67] have shown that for a Newton non-degenerate function, $\beta_{\mathbb{C}} = 1/t_0$ where $(t_0, \ldots, t_0)$ is the intersection point of the diagonal $t \mapsto (t, \ldots, t)$ with the Newton diagram of f.

J. Kollar [97] has shown that λ is equal to the *log canonical threshold*.

Moreover, we have the following relations which were recently brought back into attention in the framework of multiplier ideals (see e.g. [69]).

For a rational number α we define the following ideal $\mathcal{A}_\alpha$ in the ring $\mathcal{O}_{Y,0}$ of analytic functions on $(Y,0)$:

$$\mathcal{A}_\alpha := \left\{ \phi \in \mathcal{O}_{Y,0} \;\middle|\; \inf_{1\leq i\leq s} \left(\frac{1+k_i+\nu_i(\phi)}{m_i} - 1 \right) > \alpha \right\}$$

where $\nu_i(\phi)$ denotes the order of ϕ along the divisor E_i. This is a *multiplier ideal* in the sense of Y.-T. Siu and A. Nadel (see [69]).

Definition 4.4. We define a sequence of numbers

$$\xi_0 = 0 < \xi_1 < \xi_2 < \dots$$

as follows: $\mathcal{A}_\alpha = \mathcal{A}_{\xi_i}$ for $\alpha \in [\xi_i, \xi_{i+1})$ and $\mathcal{A}_{\xi_{i+1}} \neq \mathcal{A}_{\xi_i}$ for $i = 0, 1, \dots$. These numbers are called *jumping numbers*.

These numbers first appeared implicitly in a paper of A. Libgober [110]. The above definition is due to Loeser and M. Vaquié [115, 157].

Varchenko [160] (see also [31]) proved the following statement:

$$\alpha \in (-1, 0], \quad \alpha \in \mathrm{Sp}(f) \Leftrightarrow \alpha + 1 = \xi_i \text{ for some } i.$$

M. Saito [142] showed that the Bernstein-Sato polynomial $b_f(s)$ (see §1) has roots in $[0,1)$ which do not come from the spectrum of f.

5 Monodromy and the topology of the singularity

Let B_ε be a closed ball as in §1 and let $K := f^{-1}(0) \cap \partial B_\varepsilon \cap Y$ be the link of the singularity $(X_0, 0)$.

First assume that $Y = \mathbb{C}^{n+1}$. Milnor [125] has shown that the manifold K is a homology sphere (and when $n \neq 2$ actually a topological sphere) if and only if the integer

$$\Delta(1) = \det(\mathrm{id}_* - h_*)$$

is equal to ± 1. Let $n \neq 2$ and assume that K is a topological sphere. The differentiable structure of K is completely determined by the Kervaire invariant $c(X_\delta) \in \mathbb{Z}_2$ if n is odd, or by the signature of the intersection matrix A if n is even (cf. [125]). If n is odd, then by a theorem of J. Levine [108] the Kervaire invariant is given by

$$c(X_\delta) = \begin{cases} 0 & \text{if } \Delta(-1) \equiv \pm 1 \pmod 8, \\ 1 & \text{if } \Delta(-1) \equiv \pm 3 \pmod 8. \end{cases}$$

If n is even and f is weighted homogeneous, then the signature of the intersection matrix A is determined by the eigenvalues of the monodromy, see [152]. Hence in many cases the complex monodromy operator determines the differentiable structure of K.

In fact it is shown in [125] that K is $(n-2)$-connected, that the rank of $H_{n-1}(K)$ is equal to the dimension of the eigenspace of h_* corresponding to the eigenvalue 1, and that, if the rank is equal to zero, the order of $H_{n-1}(K)$ is equal to $|\Delta(1)|$. Here we use reduced homology if $n = 1$. If f is weighted homogeneous, a formula for the rank of $H_{n-1}(K)$ and for $\Delta(1)$ in terms of weights and degree is given in [126].

If $n = 1$, Durfee [46] relates the topology of a branched cyclic cover of the link K to the characteristic polynomial of the monodromy. B. G. Cooper [34] has calculated the homology of the link K for some special weighted homogeneous polynomials f.

Let C be the matrix of h_* with respect to a basis of $H_n(X_\delta, \mathbb{C})$ and let I be the $\mu \times \mu$ identity matrix. In the case when f is weighted homogeneous, Orlik [129] has stated the following conjecture:

Conjecture 5.1 (Orlik). *The matrix C can be diagonalized over the integers, i.e. there exist unimodular matrices $U(t)$ and $V(t)$ with entries in the ring $\mathbb{Z}[t]$ so that*

$$U(t)(tI - C)V(t) = \operatorname{diag}(m_1(t), \ldots, m_\mu(t))$$

where $m_i(t)$ divides $m_{i+1}(t)$ for $i = 1, \ldots, \mu - 1$.

Since the ring $\mathbb{C}[t]$ is a principal ideal domain, such matrices exist over $\mathbb{C}[t]$. The conjecture implies that

$$H_{n-1}(K) = \mathbb{Z}_{m_1(1)} \oplus \ldots \oplus \mathbb{Z}_{m_\mu(1)}$$

where $\mathbb{Z}_1$ is the trivial group and $\mathbb{Z}_0$ is the infinite cyclic group.

The conjecture holds for f weighted homogeneous and $n = 2$ as follows from [131].

Sometimes the conjecture is extended to germs f with finite monodromy. Then the following is known about the more general conjecture. From [2] one can derive that Orlik's conjecture is true for irreducible plane curve singularities. F. Michel and C. Weber [123, 124] have shown that Orlik's conjecture is false for plane curve singularities with more than one branch.

Let $n = 2$ and f be weighted homogeneous with weights q_1, q_2, q_3 and degree d. Then Y. Xu and S.-T. Yau [166] have shown that the characteristic polynomial $\Delta(t)$ of the monodromy and the fundamental group $\pi_1(K)$ of the link determine the embedded topological type of $(X_0, 0)$. Let K be in addition a rational homology sphere. Then R. Mendris and A. Némethi [122] have observed that it follows from [56] that $\Delta(t)$ is already determined by $\pi_1(K)$. Define

$$R := d - q_1 - q_2 - q_3.$$

Némethi and L. I. Nicolaescu [127] have derived from [56] that

$$\frac{\Delta(t)}{\Delta(1)} = 1 + \frac{\mu}{2}(t-1) + \ldots \quad \text{and} \quad \frac{\Delta^*(t)}{\Delta^*(1)} = 1 + \frac{R}{2}(1 - e_{\mathrm{st}})(t-1) + \ldots$$

where e_{st} is Batyrev's stringy Euler characteristic of $(X_0, 0)$ (cf. [19]) as generalized by Veys in [164].

Now let $Y = \mathbb{C}^{n+1}$ and let f be again general. The matrix V of §2 is the matrix of the (integral) Seifert form of the singularity $(X_0, 0)$. If $n \geq 3$ then results of M. Kervaire [96] and J. Levine [109] show that the Seifert form determines the (embedded) topological type of the singularity, see also [45]. If $n = 1$ and f defines an irreducible curve singularity, then it follows from [32] and [168] that the integral monodromy and even the rational monodromy determines the topology of the singularity. M.-C. Grima [78] has given examples of plane curve singularities with two branches of different topological types with the same rational monodromy, but different integral monodromy. Ph. Du Bois and Michel [42, 44] have shown that the integral Seifert form does not always determine the topology of the singularity in the case $n = 1$. Using suspensions of the examples of Du Bois and Michel, Artal-Bartolo [13] has shown that the same applies to the case $n = 2$.

Let $(Y, 0)$ be a weighted homogeneous ICIS and let f be weighted homogeneous. Let $L := \partial B_\varepsilon \cap Y$. Dimca [39] has shown that for $n \geq 2$ one has the following formula for the Betti numbers of L and K:

$$b_{n+1}(L) + b_n(K) = \dim \ker(\mathrm{Id}_* - h_*).$$

Hamm [90] computed the characteristic polynomial of the monodromy for some ICIS which are generalizations of the Brieskorn-Pham singularities, the Brieskorn-Hamm-Pham singularities. The homology torsion of the link of a Brieskorn-Hamm-Pham singularity was computed by Randell [134].

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