Equivalence of anchored and ANOVA spaces via interpolation

Aicke Hinrichs
Institut für Analysis, Johannes Kepler Universität Linz
Altenberger Str. 69, 4040 Linz, Austria
email: aicke.hinrichs@jku.at

Jan Schneider
Institut für Mathematik, Friedrich Schiller Universität Jena
Ernst-Abbe-Platz 2, 07743 Jena, Germany
email: jan.schneider@uni-jena.de

October 22, 2015

Abstract

We consider weighted anchored and ANOVA spaces of functions with first order mixed derivatives bounded in L_p. Recently, Hefter, Ritter and Wasilkowski established conditions on the weights in the cases $p = 1$ and $p = \infty$ which ensure equivalence of the corresponding norms uniformly in the dimension or only polynomially dependent on the dimension. We extend these results to the whole range of $p \in [1, \infty]$. It is shown how this can be achieved via interpolation.

MSC: 65D30,65Y20,41A63,41A55

Keywords: ANOVA decomposition, anchored decomposition, norm equivalences

1 Introduction

In this paper we continue the study of norm equivalences of anchored and ANOVA spaces of functions with bounded order one mixed partial derivatives. Equivalence of these norms in the Hilbert space case was recently shown by M. Hefter and K. Ritter in [2]. They prove very general norm equivalences for reproducing kernel Hilbert spaces. The main point is the possibility of transferring error bounds for algorithms and complexity bounds from one setting to another.

One example is the multivariate decomposition method or changing dimension algorithm introduced in [5]. This method has been shown to be very effective for the anchored decomposition for spaces of functions with mixed partial derivatives bounded in L_p norms in [8]. To transfer these results to the ANOVA setting, a norm equivalence is needed. In the case $p = 2$ this follows from the results in [2]. In the cases $p = 1$ and $p = \infty$, such norm equivalences where recently established by M. Hefter, K. Ritter and G. W. Wasilkowski in [3]. Naturally, the question arises what happens for $1 < p < 2$ and $2 < p < \infty$. It is the purpose of this note to derive the corresponding results via interpolation methods.

In Section 2, we introduce the considered spaces of functions with mixed derivatives and state our main general result. In Section 3, we characterize the spaces as ℓ_p-sums of certain L_p-spaces, which is a suitable characterization for interpolation. In Section 4, we proof the main result. In Section 5, we apply the result to different classes of special weights.

2 Spaces of multivariate functions

In this section we introduce the ANOVA and anchored spaces as completions of algebraic tensor products. This approach is slightly different from the one taken in [3] but the resulting spaces are the same. Our approach is better suited to the direct application of complex interpolation, which is the main technical tool later.

Let $1 \leq p \leq \infty$. In the case $d = 1$ we consider the space F_p of complex valued absolutely continuous functions on the interval $[0, 1]$ with first derivatives bounded in L_p. We work with complex valued functions since the direct application of the complex interpolation method needs complex scalars. The following considerations can be done in both the real and complex case. For complex scalars, we just need to consider a complex valued function $f : [0, 1]^d \to \mathbb{C}$ as a sum $f = g + \mathrm{i}h$ of real and imaginary part and apply derivatives and integrals to both parts. Now we consider the algebraic tensor product

$$F_{d,p} = \bigotimes_{j=1}^{d} F_p$$

of functions on $[0, 1]^d$. These functions have mixed derivatives of first order in L_p. We denote by

$$f^{(u)} = \prod_{j \in u} \frac{\partial}{\partial x_j} f$$

the mixed derivative of f with respect to the variables in the subset $u \subset [d] = \{1, \ldots, d\}$.

To denote function values of d-variate functions emphasizing different subsets $u \subset [d]$ of coordinates we use the convention to write

$$f(x_u; t_{u^c}) := f(x_u; t) := f(y) \qquad \text{with } y_j = x_j \text{ for } j \in u \text{ and } y_j = t_j \text{ for } j \notin u.$$

Given a sequence $(\gamma_u)_{u \subset [d]}$ of nonnegative weights, we consider two norms on $F_{d,p}$, the anchored norm

$$\|f\|_{\pitchfork,d,p} = \left(\sum_{u \subset [d]} \gamma_u^{-p} \left\| f^{(u)}(\cdot_u; 0) \right\|_p^p \right)^{1/p} = \left(\sum_{u \subset [d]} \gamma_u^{-p} \int_{[0,1]^u} \left| f^{(u)}(x_u; 0) \right|^p \, \mathrm{d}x_u \right)^{1/p} \tag{1}$$

and the ANOVA norm

$$\begin{aligned} \|f\|_{A,d,p} &= \left(\sum_{u \subset [d]} \gamma_u^{-p} \left\| \int_{[0,1]^{[d]\setminus u}} f^{(u)}(\cdot_u; t) \, \mathrm{d}t \right\|_p^p \right)^{1/p} \\ &= \left(\sum_{u \subset [d]} \gamma_u^{-p} \int_{[0,1]^u} \left| \int_{[0,1]^{[d]\setminus u}} f^{(u)}(x_u; t) \, \mathrm{d}t \right|^p \, \mathrm{d}x_u \right)^{1/p} . \end{aligned} \tag{2}$$

Here and in the following we do not explicitly state the standard modification required in the case $p = \infty$. Moreover, if the weight γ_u is zero, we consider only the subspace $F_{d,p}$ of functions f for which the corresponding term in the norm vanishes.

The space $F_{d,p}$ with respect to these norms will be denoted by $G_{\pitchfork,d,p}$ and $G_{A,d,p}$ and its completions by $F_{\pitchfork,d,p}$ and $F_{A,d,p}$, respectively. It was shown in [3, Proposition 13] that the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ coincide independently of $p \in [1, \infty]$ if and only if the weights satisfy the compatibility condition

$$\gamma_u > 0 \qquad \text{implies} \qquad \gamma_v > 0 \text{ for all } v \subset u.$$

From this point on we always assume that this condition is fulfilled. Moreover, it was also observed in [3] that $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ can be identified with Banach function spaces with continuous point evaluations. The closed graph theorem then ensures that the identity mappings

$$J_{d,p}^{\pitchfork,A} : F_{\pitchfork,d,p} \to F_{A,d,p} \qquad \text{and} \qquad J_{d,p}^{A,\pitchfork} : F_{A,d,p} \to F_{\pitchfork,d,p}$$

are bounded. Our main interest in this note is to estimate the corresponding operator norms.

The cases $p = 1$ and $p = \infty$ were considered in [3]. There the following constants were introduced

$$C_{d,1} = \max_{u \subset [d]} \sum_{v \subset u} \frac{\gamma_u}{\gamma_v} \quad \text{and} \quad C_{d,\infty} = \max_{u \subset [d]} \sum_{v \subset u^c} 2^{-|v|} \frac{\gamma_{u \cup v}}{\gamma_u}. \tag{3}$$

The main result in [3] formulated as Theorem 14 there reads as

Theorem 1. Let $p = 1$ or $p = \infty$. Then

$$\|J_{d,p}^{\pitchfork,A} : F_{\pitchfork,d,p} \to F_{A,d,p}\| = \|J_{d,p}^{A,\pitchfork} : F_{A,d,p} \to F_{\pitchfork,d,p}\| = C_{d,p}.$$

Here we should observe that in [3] real valued functions are treated. But the lower bounds for the norms proved in [3] of course also hold for complex valued functions. Moreover, the proofs of the upper bounds remain valid also in the complex case. This is due to the fact that the inequalities used are triangle inequalities also valid for complex scalars.

Our main result is the following interpolation theorem which extends the upper bounds on the operator norm to $1 < p < \infty$.

Theorem 2. Let $1 \leq p \leq \infty$. Then

$$\|J_{d,p}^{\pitchfork,A} : F_{\pitchfork,d,p} \to F_{A,d,p}\| \leq C_{d,1}^{1/p} C_{d,\infty}^{1-1/p} \quad \text{and} \quad \|J_{d,p}^{A,\pitchfork} : F_{A,d,p} \to F_{\pitchfork,d,p}\| \leq C_{d,1}^{1/p} C_{d,\infty}^{1-1/p}. \tag{4}$$

We now recall the notions of uniform and polynomial equivalence from [3, Definition 14]. The spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are called uniformly equivalent (in d), if there is $c > 0$ not depending on d, such that

$$\max\left(\|J_{d,p}^{\pitchfork,A}\|, \|J_{d,p}^{A,\pitchfork}\|\right) \leq c,$$

which means that

$$c^{-1} \|f\|_{\pitchfork,d,p} \leq \|f\|_{A,d,p} \leq c \|f\|_{\pitchfork,d,p} \text{ for all } f \in F_{\pitchfork,d,p}.$$

The spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are called polynomially equivalent (in d), if there is $\tau > 0$ not depending on d, such that

$$\max\left(\|J_{d,p}^{\pitchfork,A}\|, \|J_{d,p}^{A,\pitchfork}\|\right) = O(d^\tau),$$

which means that

$$c^{-1} d^{-\tau} \|f\|_{\pitchfork,d,p} \leq \|f\|_{A,d,p} \leq c d^\tau \|f\|_{\pitchfork,d,p} \text{ for all } f \in F_{\pitchfork,d,p}$$

with a constant $c > 0$ independent of d. The infimum over all such τ is called the exponent of polynomial equivalence. Of course, these concepts can be formulated more general for sequences of spaces indexed by the dimension, where each element is equipped with two norms.

As a corollary of Theorem 2, we obtain immediately from [3, Proposition 17] the sufficiency in the following corollary. The necessity will be shown in Section 5.

Corollary 3. Let $1 \le p \le \infty$. For product weights $\gamma_u = \prod_{j \in u} \gamma_j$, a necessary and sufficient condition for the uniform equivalence of $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ is that $(\gamma_j)_{j \in \mathbb{N}}$ is summable.

Further applications to different classes of weights will be considered in Section 5.

3 Characterization of anchored and ANOVA spaces as $\ell_p(A_j)$-sums

Given normed spaces $(A_u)_{u \subset [d]}$, a weight sequence $(\gamma_u)_{u \subset [d]}$ and $p \in [1, \infty]$, the weighted ℓ_p-sum $\ell_p(A_u)$ of the spaces A_u is the set of 2^d-tuples $a = (a_u)_{u \subset [d]}$ with $a_u \in A_u$ and norm

$$\|a\| = \left(\sum_{u \subset [d]} \gamma_u^{-p} \|a_u\|^p \right)^{1/p} .$$

If $\gamma_u = 0$, the corresponding A_u has to be the trivial normed space.

In this section we explain how the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are characterized isometrically as spaces $\ell_p\left(L_p([0,1]^u)\right)$. Since the completion of $\ell_p(A_u)$ is $\ell_p(\overline{A_u})$, where $\overline{A_u}$ is the completion of A_u we first work in the algebraic tensor products $F_{d,p} = \bigotimes_{j=1}^d F_p$ equipped with the corresponding norms, i.e. in $G_{\pitchfork,d,p}$ and $G_{A,d,p}$. For each function $f \in F_{d,p}$ we consider the operators

$$R_\pitchfork f = \left(f^{(u)}(x_u; 0) \right)_{u \subset [d]} \qquad \text{and} \qquad R_A f = \left(\int_{[0,1]^{[d] \setminus u}} f^{(u)}(\cdot_u; t) \, \mathrm{d}t \right)_{u \subset [d]}$$

mapping f to 2^d-tuples $(g_u^\pitchfork)$ and (g_u^A), respectively, where the functions g_u depend only on the coordinates in u. Note that $g_\emptyset^\pitchfork = f(0)$ and $g_\emptyset^A = \int_{[0,1]^{[d]}} f(t) \, \mathrm{d}t$. Hence the norm of f in (1) and (2) can be written as

$$\|f\|_{\pitchfork,d,p} = \left(\sum_{u \subset [d]} \gamma_u^{-p} \|g_u^\pitchfork\|_p^p \right)^{1/p} \qquad \text{and} \qquad \|f\|_{A,d,p} = \left(\sum_{u \subset [d]} \gamma_u^{-p} \|g_u^A\|_p^p \right)^{1/p} \tag{5}$$

where the norms $\|g_u^\pitchfork\|_p$ and $\|g_u^A\|_p$ are L_p-norms on the domain $[0,1]^u$.

We now observe that $R_\pitchfork f$ and $R_A f$ are actually elements of the spaces $\ell_p(B_u)$ where $B_u \subset A_u = L_p([0,1]^u)$ is the algebraic tensor product $B_u = \bigotimes_{j\in u} L_p[0,1]$ (with $B_\emptyset = A_\emptyset = \mathbb{R}$). It also follows from (5) that the mappings $R_\pitchfork$ and R_A considered as mappings from $F_{d,p} = \bigotimes_{j=1}^d F_p$ to $\ell_p(B_u)$ are isometric embeddings, i.e.

$$\|f\|_{\pitchfork,d,p} = \|R_\pitchfork f | l_p(B_u)\| \quad \text{and} \quad \|f\|_{A,d,p} = \|R_A f | l_p(B_u)\|.$$

We now show that they actually are isometric isomorphisms. In the anchored case, given $g = (g_u) \in \ell_p(B_u)$, we let

$$f(x) = (S_\pitchfork g)(x) = \sum_{u\subset[d]} \int_{[0,x]^u} g_u(t;0)\,\mathrm{d}t = \sum_{u\subset[d]} f_u(x).$$

If $g_u(x) = \prod_{j\in u} h_j(x_j)$ is an elementary tensor in $B_u = \bigotimes_{j\in u} L_p[0,1]$, then

$$f_u(x) = \int_{[0,x]^u} g_u(t;0)\,\mathrm{d}t = \prod_{j\in u} \int_0^{x_j} h_j(t_j)\,\mathrm{d}t_j$$

shows that

$$f_u \in \bigotimes_{j\in u} F_p \subset \bigotimes_{j=1}^{d} F_p = F_{d,p}.$$

Consequently, $S_\pitchfork$ indeed maps $\ell_p(B_u)$ into $F_{d,p}$.

Now f_u depends only on the coordinates in u, hence $f_u^{(v)} = 0$ if $v \not\subset u$. Moreover, if $v \subset u$ and $v \neq u$, then $f_u^{(v)}(\cdot_v;0) = 0$, since for the coordinates in $u \setminus v$ the integration in

$$f_u(x) = \int_{[0,x]^u} g_u(t;0)\,\mathrm{d}t$$

is over the trivial interval $[0,0]$. Hence $f_u^{(v)}(\cdot_v;0) = 0$ unless $u = v$. Consequently,

$$f^{(u)}(\cdot_u;0) = f_u^{(u)}(\cdot_u;0) = g_u,$$

which shows that $\|S_\pitchfork g\|_{\pitchfork,d,p} = \|g|l_p(B_u)\|$. Therefore $S_\pitchfork$ is also an isometric embedding. Using the above notation together with (1) in [3, Lemma 1], we find that

$$S_\pitchfork[R_\pitchfork f](x) = S_\pitchfork[(g_u^\pitchfork)_u](x) = \sum_{u\subset[d]} \int_{[0,x]^u} f^{(u)}(t;0)\,\mathrm{d}t = f(x).$$

Observe that Lemma 1 in [3] is formulated only for f in the algebraic tensor product of C^∞-functions, but the proof immediately extends to $f \in F_{d,p}$. This shows that $S_\pitchfork R_\pitchfork = Id_{F_{d,p}}$. Since injectivity of $S_\pitchfork$ together with $S_\pitchfork R_\pitchfork = Id_{F_{d,p}}$ automatically implies $R_\pitchfork S_\pitchfork = Id_{\ell_p(B_u)}$, we have shown that $S_\pitchfork$ and $R_\pitchfork$ are isometric isomorphisms.

In the ANOVA case we use the one-dimensional identity

$$f(x) = \int_0^1 f(t)\,\mathrm{d}t + \int_0^1 \big(t - \chi_{[x,1]}(t)\big)\, f'(t)\,\mathrm{d}t$$

for $f \in F_p$ where $\chi_{[a,b]}$ is the indicator function of the interval $[a,b]$. Taking tensor products, this leads to the d-dimensional identity

$$f(x) = \sum_{u \subset [d]} \int_{[0,x]^u} \prod_{j \in u} \big(t_j - \chi_{[x_j,1]}(t_j)\big)\, g_u(t)\,\mathrm{d}t \tag{6}$$

for $f \in F_{d,p}$ where

$$g_u = \int_{[0,1]^{[d]\setminus u}} f^{(u)}(\cdot_u; t)\,\mathrm{d}t$$

is the corresponding term in the ANOVA-decomposition of f, compare (2) in [3, Lemma 1]. So, given $g = (g_u) \in \ell_p(B_u)$, we let

$$f(x) = (S_A g)(x) = \sum_{u \subset [d]} \int_{[0,x]^u} \prod_{j \in u} \big(t_j - \chi_{[x_j,1]}(t_j)\big)\, g_u(t)\,\mathrm{d}t = \sum_{u \subset [d]} f_u(x).$$

If $g_u(x) = \prod_{j\in u} h_j(x_j)$ is an elementary tensor in $B_u = \bigotimes_{j \in u} L_p[0,1]$, then

$$f_u(x) = \int_{[0,x]^u} \prod_{j \in u} \big(t_j - \chi_{[x_j,1]}(t_j)\big)\, g_u(t)\,\mathrm{d}t = \prod_{j \in u} \int_0^{x_j} \big(t_j - \chi_{[x_j,1]}(t_j)\big)\, h_j(t_j)\,\mathrm{d}t_j$$

shows that

$$f_u \in \bigotimes_{j \in u} F_p \subset \bigotimes_{j=1}^{d} F_p = F_{d,p}.$$

Consequently, S_A maps $\ell_p(B_u)$ into $F_{d,p}$.

Now f_u depends only on the coordinates in u, hence $f_u^{(v)} = 0$ if $v \not\subset u$. Moreover, if $v \subset u$ and $v \neq u$, then

$$\int_{[0,1]^{[d]\setminus v}} f_u^{(v)}(\cdot_v; t)\,\mathrm{d}t = 0$$

since the variables x_j with $j \in u \setminus v$ lead to integrals $\int_0^1 \left(t_j - \chi_{[x_j,1]}(t_j)\right) \mathrm{d}x_j = 0$. Consequently,

$$\int_{[0,1]^{[d]\setminus u}} f^{(u)}(\cdot_u; t)\, \mathrm{d}t = g_u,$$

which shows that $\|S_A g\|_{A,d,p} = \|g|l_p(B_u)\|$. Therefore S_A is also an isometric embedding. Using the above notation together with (6), we find that

$$S_A[R_A f](x) = S_A[(g_u^A)_u](x) = \sum_{u \subset [d]} \int_{[0,x]^u} \prod_{j \in u} \left(t_j - \chi_{[x_j,1]}(t_j)\right) g_u^A(t)\, \mathrm{d}t = f(x).$$

This shows that $S_A R_A = Id_{F_{d,p}}$. Since injectivity of S_A together with $S_A R_A = Id_{F_{d,p}}$ automatically implies $R_A S_A = Id_{\ell_p(B_u)}$, we have shown that S_A and R_A are also isometric isomorphisms.

Now extending $S_{\pitchfork}$, S_A, $R_{\pitchfork}$, R_A to the completions and observing that the completion of B_u is just $L_p([0,1]^u)$, we obtain isometric isomorphisms of the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ on the one side and $\ell_p\left(L_p([0,1]^u)\right)$ on the other side. Slightly abusing notation, we will denote these extensions again with $S_{\pitchfork}$, S_A, $R_{\pitchfork}$, R_A, respectively.

Remark 4. If the weights γ_u are all positive, then we obtain that the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are actually the classical spaces W_p^{mix} of functions in $f \in L_p$ whose mixed derivative $\frac{\partial^d f}{\partial x_1 \ldots \partial x_d}$ also belongs to L_p, see [3]. If some of the weights γ_u are zero, we obtain subspaces of W_p^{mix} where the corresponding derivatives $f^{(u)}$ satisfy

$$f^{(u)}(x_u; 0) = 0 \qquad \text{and} \qquad \int_{[0,1]^{[d]\setminus u}} f^{(u)}(\cdot_u; t)\, \mathrm{d}t = 0$$

in the anchored case or in the ANOVA case, respectively.

4 Proof of Theorem 2

In the previous section, we have shown that the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are isometrically isomorhic to $\ell_p\left(L_p([0,1]^u)\right)$. Now we shortly state known results for this type of spaces for complex interpolation. The corresponding theory is described in detail in [7], Section 1.9. The relevant results for spaces of type $\ell_p(A_u)$ with $A_u = L_p([0,1]^u)$ are Theorem 1.18.1 (formula (4)) in [7], i.e.

$$[\ell_{p_0}(A_j), \ell_{p_1}(B_j)]_\theta = \ell_p\left([A_j, B_j]_\theta\right), \tag{7}$$

complemented by the following Remark 2, and Theorem 1.18.6.2 (formula (15)) in [7], i.e.

$$[L_{p_0}(A), L_{p_1}(A)]_\theta = L_p(A), \tag{8}$$

for Banach spaces A, A_j, B_j and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$. We want to apply (7) and (8) to the situation corresponding to Figure 1 to prove Theorem 2.

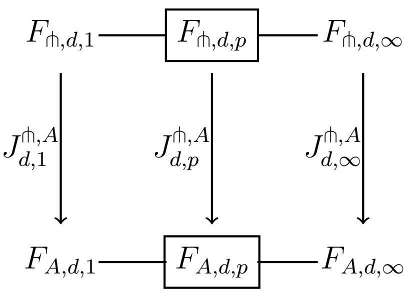

Figure 1: Interpolation scheme

For the operators $J_{d,p}^{\pitchfork,A}$ and $J_{d,p}^{A,\pitchfork}$ (with arrows in Figure 1 pointing in the opposite direction), we know from [3], see Theorem 1, that

$$\left\| J_{d,1}^{\pitchfork,A} \right\| = \left\| J_{d,1}^{A,\pitchfork} \right\| = C_{d,1} \quad \text{and} \quad \left\| J_{d,\infty}^{\pitchfork,A} \right\| = \left\| J_{d,\infty}^{A,\pitchfork} \right\| = C_{d,\infty},$$

with $C_{d,1}, C_{d,\infty}$ from (3). Now, (7) and (8) give

$$\left[\ell_1 \left(L_1([0,1]^u)\right), \ell_\infty \left(L_\infty([0,1]^u)\right)\right]_\theta = \ell_p \left(L_p([0,1]^u)\right) \quad \text{with} \quad p = \frac{1}{1-\theta}, \quad 0 < \theta < 1.$$

Using the isometric isomorphisms $S_\pitchfork$, S_A, $R_\pitchfork$, R_A, this characterizes the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ as complex interpolation spaces

$$F_{\pitchfork,d,p} = [F_{\pitchfork,d,1}, F_{\pitchfork,d,\infty}]_\theta \qquad \text{and} \qquad F_{A,d,p} = [F_{A,d,1}, F_{A,d,\infty}]_\theta$$

justifying the above diagram.

Because the corresponding interpolation functor is exact of type θ (see Theorem 1.9.3 in [7], we get

$$\left\| J_{d,p}^{\pitchfork,A} \right\| \le \left\| J_{d,1}^{\pitchfork,A} \right\|^{1-\theta} \left\| J_{d,\infty}^{\pitchfork,A} \right\|^{\theta} = C_{d,1}^{1/p} C_{d,\infty}^{1-1/p} =: C_{d,p},$$

which is the left side of (4). Since the arguments are identical for $J_{d,p}^{A,\pitchfork}$, we also get

$$\left\| J_{d,p}^{A,\pitchfork} \right\| \le \left\| J_{d,1}^{A,\pitchfork} \right\|^{1-\theta} \left\| J_{d,\infty}^{A,\pitchfork} \right\|^{\theta} = C_{d,p},$$

which completes the proof.

Remark 5. For some potential applications it might be useful to use different weight sequences for different p. This is possible without much difficulties. Let the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ be defined in (1) and (2), respectively, with a weight sequence $(\gamma_{u,p})_{u\subset[d]}$ depending on p. Then Theorem 1.8.10.5 in [7] shows that the complex interpolation spaces with index θ between $p=1$ and $p=\infty$ are just given by the interpolated weights $\gamma_{u,p}=\gamma_{u,1}^{1-\theta}\cdot\gamma_{u,\infty}^{\theta}$. Then Theorem 2 holds verbatim.

5 Applications to different classes of weights

We now consider special classes of weights as in [3] and extend the corresponding results from $p=1,\infty$ to the whole range $p\in[1,\infty]$, if that is possible.

We start with *product weights* introduced in [6], for which

$$\gamma_u=\prod_{j\in u}\gamma_j,\qquad \text{for } u\subset[d]$$

where $(\gamma_j)_{j\in\mathbb{N}}$ is a given sequence of positive numbers. Product weights were already mentioned in Corollary 3. To prove that corollary, it remains to show that the summability of $(\gamma_j)_{j\in\mathbb{N}}$ is necessary for the uniform equivalence of $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ in case $p\in[1,\infty]$. To this end, we consider the function

$$f(x)=f(x_1,\ldots,x_d)=\prod_{j=1}^{d}(1+\gamma_j x_j).$$

Then we have

$$f^{(u)}(\cdot_u,0)=\gamma_u\qquad\text{and}\qquad \int_{[0,1]^{[d]\setminus u}}f^{(u)}(\cdot_u;t)\,\mathrm{d}t=\gamma_u\prod_{j\notin u}(1+\gamma_j/2)$$

and calculate

$$\|f\|_{\pitchfork,d,p}^p=\sum_{u\subset[d]}\gamma_u^{-p}\int_{[0,1]^u}|f^{(u)}(x_u;0)|^p\,\mathrm{d}x_u=2^d$$

as well as

$$\begin{aligned}\|f\|_{A,d,p}^p &= \sum_{u\subset[d]}\gamma_u^{-p}\int_{[0,1]^u}\left|\int_{[0,1]^{[d]\setminus u}}f^{(u)}(x_u;t)\,\mathrm{d}t\right|^p\mathrm{d}x_u=\sum_{u\subset[d]}\prod_{j\notin u}(1+\gamma_j/2)^p\\ &= \prod_{j=1}^{d}(1+(1+\gamma_j/2)^p).\end{aligned}$$

Now we see

$$\left\| J_{d,p}^{\pitchfork,A} \right\|^p \geq \frac{\|f\|_{A,d,p}^p}{\|f\|_{\pitchfork,d,p}^p} = \prod_{j=1}^d \frac{1+(1+\gamma_j/2)^p}{2} \geq \prod_{j=1}^d (1+\gamma_j/4),$$

which shows that, if the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are uniformly equivalent, then $(\gamma_j)_{j\in\mathbb{N}}$ must be summable. This finishes the proof of Corollary 3.

The corresponding result for polynomial equivalence is

Corollary 6. Let $1 \leq p \leq \infty$. For product weights $\gamma_u = \prod_{j\in u} \gamma_j$, a necessary and sufficient condition for the polynomial equivalence of $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ is that

$$\tau_0 = \sup_{d\in\mathbb{N}} \frac{\sum_{j=1}^d \gamma_j}{\ln(d+1)} < \infty. \tag{9}$$

If this holds, then the exponent τ of polynomial equivalence is at most

$$\frac{\tau_0}{2}\left(1+\frac{1}{p}\right).$$

Proof. For the sufficiency, we now use [3, Proposition 17, (ii)] showing that polynomial equivalence for $p \in \{1,\infty\}$ holds if and only if $\tau_0 < \infty$. Moreover, it is also shown there that the exponents of polynomial equivalence are $\tau_0/2$ for $p=\infty$ and τ_0 for $p=1$, so the upper bound for the exponent of polynomial equivalence for $1<p<\infty$ follows from the interpolation inequalities

$$\left\| J_{d,p}^{\pitchfork,A} \right\| \leq \left\| J_{d,1}^{\pitchfork,A} \right\|^{1-\theta} \left\| J_{d,\infty}^{\pitchfork,A} \right\|^{\theta} = C_{d,1}^{1/p} C_{d,\infty}^{1-1/p} = C_{d,p}$$

and

$$\left\| J_{d,p}^{A,\pitchfork} \right\| \leq \left\| J_{d,1}^{A,\pitchfork} \right\|^{1-\theta} \left\| J_{d,\infty}^{A,\pitchfork} \right\|^{\theta} = C_{d,1}^{1/p} C_{d,\infty}^{1-1/p} = C_{d,p}.$$

The necessity of (9) follows by the same arguments (for the same function) as used above for the uniform equivalence. □

Now we discuss the class of so-called *finite order weights* first considered in [1] and having the property that

$$\gamma_u = 0 \quad \text{if } |u| > q \in \mathbb{N} \quad (q \text{ is called order}).$$

Using [3, Proposition 19] we can state

Corollary 7. Let $1 \le p \le \infty$. If there exist numbers $c, \omega > 0$, such that

$$\gamma_u = c\,\omega^{|u|} \quad \text{for all } |u| \le q,$$

then there is polynomial equivalence of the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ equipped with finite order weights. The corresponding exponent of polynomial equivalence is $\tau = q$.

Remark 8. The original condition on the weights γ_u in [3] looks a bit different because the cases $p = 1$ and $p = \infty$ are treated separately. There one can also see that the only known case where uniform equivalence holds, is $p = 1$ for these weights. So here we can not use our interpolation technique to establish the corresponding result for $p \in [1, \infty]$.

As a last application we consider here special *dimension-dependent weights*, introduced in [4], where $\gamma_u = d^{-|u|}$. Now [3, Proposition 20] gives immediately

Corollary 9. For $p \in [1, \infty]$ and weights $\gamma_u = d^{-|u|}$ the spaces $F_{\pitchfork,d,p}$ and $F_{A,d,p}$ are uniformly equivalent.

Acknowledgment: We thank two anonymous referees for valuable remarks that improved the presentation of our paper.

References

[1] J. Dick, I. H. Sloan, X. Wang, and H. Wozniakowski, Good lattice rules in weighted Korobov spaces with general weights, Numer. Math. 103 (2006), 63–97.

[2] M. Hefter and K. Ritter, On embeddings of weighted tensor product Hilbert spaces, *J. Complexity* , 31 (2015), 405–423.

[3] M. Hefter, K. Ritter and G. W. Wasilkowski, On equivalence of weighted anchored and ANOVA spaces of functions with mixed smoothness of order one in L_1 and L_∞, *J. Complexity* , in press.

[4] M. Hegland and G. W. Wasilkowski, On tractability of approximation in special function spaces, *J. Complexity* 29 (2013), 76–91.

[5] F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski, Liberating the dimension, *J. Complexity* 26 (2010), 422–454.

[6] I. H. Sloan and H. Wozniakowski, When are quasi-Monte Carlo algorithms efficient for high dimensional integrals?, *J. Complexity* 14 (1998), 1–33.

[7] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators. VEB Deutsch. Verl. Wissenschaften, Berlin 1978, North-Holland, Amsterdam 1978.

[8] G. W. Wasilkowski, Tractability of approximation of ∞-variate functions with bounded mixed partial derivatives, *J. Complexity* 30 (2014), 325–346.