# ON MEASURING UNBOUNDEDNESS OF THE $H^\infty$-CALCULUS FOR GENERATORS OF ANALYTIC SEMIGROUPS

FELIX L. SCHWENNINGER

Abstract. We investigate the boundedness of the $H^\infty$-calculus by estimating the bound $b(\varepsilon)$ of the mapping $H^\infty \to \mathcal{B}(X)$: $f \mapsto f(A)T(\varepsilon)$ for $\varepsilon$ near zero. Here, $-A$ generates the analytic semigroup $T$ and $H^\infty$ is the space of bounded analytic functions on a domain strictly containing the spectrum of $A$. We show that $b(\varepsilon) = \mathcal{O}(|\log\varepsilon|)$ in general, whereas $b(\varepsilon) = \mathcal{O}(1)$ for bounded calculi. This generalizes a result by Vitse and complements work by Haase and Rozendaal for non-analytic semigroups. We discuss the sharpness of our bounds and show that single square function estimates yield $b(\varepsilon) = \mathcal{O}(\sqrt{|\log\varepsilon|})$.

## 1. Introduction

Functional calculus, the procedure to define a new operator as evaluation of an intial operator in a (scalar-valued) function, had its beginnings with von Neumann's work [42] more than 80 years ago. Typically, the aim is to preserve the algebraic structures of the set of functions for the operators, such as linearity and multiplicativity. Therefore, an ultimate goal is to get a homomorphism from a functions algebra to an operator algebra, e.g. the Banach algebra of bounded operators on a Banach space. However, sometimes such a mapping is not possible for the chosen pair of algebras and we are forced to weaken the homomorphism property. This can be done by considering a subclass of functions first, on which a homomorphism is possible, and extend this mapping (algebraically), see e.g. [18, Chapter 1] and the references therein.

In the case of the $H^\infty$-calculus this means that we may get unbounded operators. Here, we consider the pair of sectorial operators $A$ and functions $f$ which are bounded and analytic on a sector that is containing the spectrum of $A$, see Section 1.1 for a brief introduction. From the very beginnings of this calculus 30 years ago, [27], it has been known that we cannot expect the $H^\infty$-calculus to be bounded, i.e., that $f(A)$ is a bounded operator for every $f \in H^\infty$, [28]. Starting with the work by McIntosh, [27], for sectorial operators on Hilbert spaces, the $H^\infty$-calculus turned out to be very useful in various situations, in particular studying maximal regularity, see [18, Chapter 9], [24] and the references therein. For a recent survey and open problems of the $H^\infty$-calculus for sectorial operators we refer to [11].

The question of boundedness of the calculus in a particular situation remains crucial in the applications and has been subject to research over the last decades, see e.g. [7, 23, 24] and [18, Chapter 5] for an overview. The main goal of this work is to investigate and 'measure' the (un)boundedness of the $H^\infty$-calculus.

Functional calculus for subalgebras of $H^\infty$ are of own interest. For instance, in [41] Vitse proves estimates for a Besov space functional calculus for analytic semigroups, (see [20] for the case of $C_0$-semigroup generators on Hilbert spaces). We

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will discuss this result in Section 5 and give a slight improvement. Furthermore, the corresponding framework of $H^\infty$-calculus for $C_0$-semigroup generators was recently developed in [4, 19, 29] where *half-plane operators* take over the role of sectorial operators.

Let us state a first observation which can be seen as the starting point for the results to come. For the precise definition of the used notions and a proof we refer to Sections 1.1, 2.

**Proposition 1.1.** *Let $A$ be a densely defined, invertible, sectorial operator of angle $\omega < \frac{\pi}{2}$ on the Banach space $X$. Then, for $\phi \in (\omega, \pi)$ the $H^\infty(\Sigma_\phi)$-calculus is bounded if and only if*

$$\forall f \in H^\infty(\Sigma_\phi) \qquad \limsup_{\varepsilon \to 0^+} \|(f \cdot e_\varepsilon)(A)\| =: C_f < \infty, \tag{1.1}$$

*where $e_\varepsilon(z) = e^{-\varepsilon z}$ and $\Sigma_\phi := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \phi\}$.*

In Example 2.1 we show that the assumption of $A$ being invertible cannot be weakened in general. Neither can we allow for $\omega = \frac{\pi}{2}$. In fact, $(fe_\varepsilon)(A)$ need not be a bounded operator otherwise, since $e_\varepsilon$ cannot control the behavior of $f$ along the imaginary axis. However, it is a remarkable result that by incorporating the geometry of the Banach space, one indeed gets that $(fe_\varepsilon)(A)$ is bounded for, not necessarily analytic, $C_0$-semigroup generators $-A$ (which are sectorial operators of angle $\frac{\pi}{2}$). Precisely, on Hilbert spaces $(fe_\varepsilon)(A)$ always defines a bounded operator if $-A$ generates an exponentially stable semigroup and if $f$ is bounded and analytic on the right half-plane. This was first proved by Zwart in [43, Theorem 2.5]. Using powerful *transference principles* from [20], Haase and Rozendaal generalized this to arbitrary Banach spaces for $f$ in the *analytic multiplier algebra* $\mathcal{AM}_p(X) \subset H^\infty(\mathbb{C}_+)$, $p \geq 1$, see in [21]. Note that the latter inclusion is even a strict embedding unless $p = 2$ and $X$ is a Hilbert space (in which case equality holds by Parseval's theorem). They also showed that, alternatively, one can make additional assumptions on the semigroup rather than on the function space. Namely, by requiring that the (shifted) semigroup is $\gamma$-bounded, see [21, Theorem 6.2]. Again, this result generalizes the Hilbert space case as $\gamma$-boundedness coincides with classical boundedness then. Moreover, although norm bounds in terms of $\varepsilon$ were already present in [43], they were significantly improved in [21], see also below.

We remark that the definition of functional calculus for non-analytic $C_0$-semigroups differs by nature from the one for sectorial operators. Using the axiomatics of holomorphic calculus in [18, Chapter 1], this can be done by either directly extending the well-known Hille-Phillips calculus, see [21], or the above mentioned calculus for half-plane operators, [4, 19, 29]. In [33, 43] an alternative definition using notions from systems theory is used. However, as all these techniques are extensions of the Hille-Phillips calculus, the notions are consistent in the considered situation.

From Proposition 1.1 we see that the behavior of the norm $\|(fe_\varepsilon)(A)\|$ for $\varepsilon$ near zero characterizes the boundedness of the $H^\infty$-calculus for the sectorial operator $A$ of angle less than $\frac{\pi}{2}$ that has 0 in its resolvent set. The negative, $-A$, of such an operator corresponds precisely to the generator of an analytic and exponentially stable $C_0$-semigroup. Denoting this semigroup by $T(t) = e^{-tA}$, we have $(fe_\varepsilon)(A) = f(A)T(\varepsilon)$. As the $H^\infty$-calculus need not be bounded, in general, we cannot bound $\|(fe_\varepsilon)(A)\|$ uniformly in $\varepsilon$. Therefore, it is our goal to establish estimates of the form

$$\|(f \cdot e_\varepsilon)(A)\| \leq b(\varepsilon) \cdot \|f\|_\infty, \tag{1.2}$$

for all $f \in H^\infty$ on a sector larger than the sector of sectorality of $A$. In general, $b(\varepsilon)$ will become unbounded for $\varepsilon \to 0^+$.
In Theorem 2.10 we show that $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ as $\varepsilon \to 0^+$ on general Banach spaces. For $0 \notin \rho(A)$, we derive a similar result for functions $f \in H^\infty$ which are holomorphic at 0, see Theorem 2.3. It turns out that the latter result generalizes a result by Vitse in [41] and improves the dependence on the sectorality constant $M(A, \phi)$ significantly, see Section 2.2. Moreover, our techniques seem to be more elementary as we do not employ the Hille-Phillips calculus.
For Hilbert spaces and general exponentially stable $C_0$-semigroup generators $-A$ an estimate of the form (1.2) $b(\varepsilon) = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$ was derived in [43]. It was subsequently improved to $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ by Haase and Rozendaal, [21, Theorem 3.3], using an adaption of a lemma due to Haase, Hytönen, [20, Lemma A.1]. As mentioned in the lines following Proposition 1.1 above, the techniques rely on the geometry of the Hilbert space and cannot be extended to general Banach spaces without either changing to another function space, [21, Theorems 3.3 and 5.1], or strengthening the assumption on the semigroup using $\gamma$-boundedness, [21, Theorem 6.2]. Hence, our results can be seen as additionally requiring analyticity of the semigroup, but dropping any additional assumption on the Banach space. As will be visible in the proofs of Theorems 2.3 and 2.10, in our case the logarithmic dependence on $\varepsilon$ is much easier to derive than for general semigroups.
Let us remark that estimates of the form (1.2) reveal information about the domain of $f(A)$. In particular, $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ implies that $D(A^\alpha) \subset D(f(A))$ for $\alpha > 0$, see [21, Theorem 3.7]. For instance, this can be used to derive convergence results for numerical schemes, see, e.g., [8].
In Section 3.1 it is shown that the logarithmic behavior is essentially optimal on Hilbert spaces by means of a scale of examples of Schauder basis multipliers. More precisely, in Theorem 3.6, we see that for any $\gamma < 1$, we find a suitable sectorial operator on $L^2(-\pi, \pi)$ such that $b(\varepsilon)$ grows like $|\log(\varepsilon)|^\gamma$. Moreover, in the examples we also focus on tracking the dependence on the sectorality constant.
*Square function estimates* or *quadratic estimates* play a crucial role in characterizing bounded $H^\infty$-calculi for sectorial operators, see [7, 12, 23, 24, 27]. On Hilbert spaces this means that an estimate of the form

$$\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \leq K^2 \|x\|^2, \quad \forall x \in X,$$

has to hold and an analogous one for the adjoint $A^*$. Whereas it is known that such an estimate for only one of $A$ or $A^*$ is not sufficient for a bounded calculus, as shown by Le Merdy in [25], we show in Section 4 that a single estimate does improve the situation in the way that $b(\varepsilon) = \mathcal{O}(\sqrt{|\log \varepsilon|})$ then. Again, by means of an example it is shown that this behavior is essentially sharp.
In Section 5 we compare our result with the one by Haase, Rozendaal in the case of an analytic semigroup on a Hilbert space. Furthermore, using the results of Section 2, we derive a slightly improved estimate for the Besov space functional calculus introduced by Vitse in [41]. We conclude by mentioning the relation to *Tadmor-Ritt* or *Ritt* operators which can be seen as the discrete analog for analytic semigroups.

1.1. **Semigroups, sectorial operators and functional calculus.** In the following let $X$ denote a complex Banach space. If $X$ is a Hilbert space the inner product will be denoted by $\langle \cdot, \cdot \rangle$. $\mathcal{B}(X, Y)$ is the Banach algebra of bounded linear operators from $X$ to $Y$, where $Y$ is another Banach space, and $\mathcal{B}(X) := \mathcal{B}(X, X)$.
If we write $E(z) \lesssim F(z)$, where $E, F$ are expressions depending on the variable $z$, we mean that there exists a universal constant $K$ not depending on $z$ such

that $E(z) \leq KF(z)$ for all $z$. By $E(z) \sim F(z)$, we mean that $F(z) \lesssim E(z)$ and $E(z) \lesssim F(z)$.
For a $C_0$-semigroup $T$ on $X$, $-A$ denotes its generator. By $X_1$ we denote the domain $D(A)$ of $A$ equipped with the graph norm. The resolvent set of $A$ will be denoted by $\rho(A)$ and $\sigma(A)$ refers to its spectrum. For $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda \mathrm{I} - A)^{-1}$. $T$ is called an *analytic* $C_0$-semigroup if it can be extended to a sector in the complex plane, see e.g. [10, Definition II.4.5].
For $\delta \in (0, \pi)$ define the sector $\Sigma_\delta = \left\{re^{i\phi} \in \mathbb{C} : r > 0, |\phi| < \delta\right\}$ and set $\Sigma_0 = (0, \infty)$. A linear operator $A$ is called *sectorial of angle* $\omega \in [0, \pi)$, if $\sigma(A) \subset \overline{\Sigma_\omega}$ and for all $\delta \in (\omega, \pi)$

$$M(A, \delta) := \sup\left\{\|\lambda(\lambda - A)^{-1}\| : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\delta}\right\} < \infty. \tag{1.3}$$

The minimal $\omega$ such that $A$ is sectorial of angle $\omega$ will be denoted by $\omega_A$. By $\mathrm{Sect}(\omega)$ we denote the set of sectorial operators of angle $\omega$. We recall that there is a one-to-one correspondence between densely-defined sectorial operators of angle strictly less than $\frac{\pi}{2}$ and generators of bounded analytic $C_0$-semigroups, namely, $A \in \mathrm{Sect}\,(\omega)$ with $\omega < \frac{\pi}{2}$ and $\overline{D(A)} = X$ if and only if $-A$ generates a bounded analytic $C_0$-semigroup, see e.g. [10, Theorem II.4.6].
We now briefly introduce the (holomorphic) functional calculus for sectorial operators. For a detailed treatment we refer the reader to the book of Haase, [18]. Let $\Omega \subset \mathbb{C}$ be an open set and let $H(\Omega)$ be the analytic functions on $\Omega$. The Banach algebra of bounded analytic functions on $\Omega$, equipped with $\|f\|_{\infty,\Omega} := \sup_{z\in\Omega} |f(z)|$, is denoted by $H^\infty(\Omega)$. As we will mainly use sectors $\Omega = \Sigma_\delta$, we abbreviate $\|f\|_{\infty,\Sigma_\delta}$ by $\|f\|_{\infty,\delta}$ or write $\|f\|_\infty$ if the set is clear from the context. For $\delta = \frac{\pi}{2}$ we will write $H^\infty(\Sigma_\delta) = H^\infty(\mathbb{C}_+)$. Furthermore, let us define

$$H^\infty_{(0)}(\Sigma_\delta) = \left\{f \in H^\infty(\Sigma_\delta) : |f(z)| \leq C|z|^{-s} \text{ for some } C, s > 0\right\},$$
$$H^\infty_0(\Sigma_\delta) = \left\{f \in H^\infty(\Sigma_\delta) : |f(z)| \leq C\tfrac{|z|^s}{1+|z|^{2s}} \text{ for some } C, s > 0\right\},$$

which are the bounded analytic functions which decay polynomially at $\infty$ (and 0). Let $A$ be a sectorial operator of angle $\omega$. Then, the Riesz-Dunford integral

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z, A)\,dz, \tag{1.4}$$

is well-defined in $\mathcal{B}(X)$ in each of the following situations, with $\omega < \delta' < \delta < \pi$,

(1) $f \in H^\infty_0(\Sigma_\delta)$ and $\Gamma = \partial\Sigma_{\delta'}$, where $\partial\Sigma_\delta$ denotes the boundary of $\Sigma_\delta$,
(2) $f \in H^\infty_{(0)}(\Sigma_\delta) \cap H(B_{r'}(0))$ for some $r > 0$ and $\Gamma = \partial\left(B_{r'}(0) \cup \Sigma_{\delta'}\right)$ for $r \in (0, r')$ ,
(3) $f \in H^\infty_{(0)}(\Sigma_\delta)$, $0 \in \rho(A)$ and $\Gamma = \partial\left(B_r(0)^c \cap \Sigma_{\delta'}\right)$ for $r > 0$ sufficiently small,

where $B_r(0) = \{z \in \mathbb{C} : |z| < r\}$. The above paths $\Gamma$ are orientated positively and by Cauchy's theorem it follows that the definitions are consistent and independent of the choice of $\delta'$ and $r'$.
The mapping $f \mapsto f(A)$ is an algebra homomorphism from $H^\infty_0(\Sigma_\delta)$ to $\mathcal{B}(X)$. It is straight-forward to extend it to a homomorphism $\Phi$ from $\mathcal{E} = H^\infty_0(\Sigma_\delta) \oplus \langle 1\rangle \oplus \langle\frac{1}{1+z}\rangle$ to $\mathcal{B}(X)$. The tupel $(\mathcal{E}, \mathcal{H}(\Sigma_\delta), \Phi)$ is called a *primary calculus* which, by a *regularization argument*, can be extended to more general $f \in H(\Sigma_\delta)$. This algebraic procedure yields an, in general unbounded, calculus of closed operators. The regularization argument can be sketched as follows. The set of *regularizers* is defined as

$$\mathrm{Reg}_A = \{e \in H^\infty_0(\Sigma_\delta) : e(A) \text{ is injective}\}$$

and the functions that can be *regularized* by elements in $\mathrm{Reg}_A$ are

$$\mathcal{M}_A = \{f \in H(\Sigma_\delta) : \exists e \in \mathrm{Reg} \text{ with } (ef) \in H_0^\infty(\Sigma_\delta)\},$$

where $H(\Omega)$ denotes the analytic functions on $\Omega$. Then, for any $f \in \mathcal{M}_A$, we can define $f(A) = e(A)^{-1}(ef)(A)$ which turns out to be independent of the choice of $e$. If $A$ is injective, it holds that $H^\infty(\Sigma_\delta) \subset \mathcal{M}_A$. One can show that the extension procedure is in conformity with the Riesz-Dunford integral definition in items 2 and 3 above. Clearly, for invertible $A$ one can do the analogous construction with a primary calculus on $H^\infty_{(0)}(\Sigma_\delta)$, which extends the previous calculus. For detailed and more general axiomatic treatment of the construction of the calculus we refer to Chapter 1 and 2 in [18].

Let $(\mathcal{F}, \|\cdot\|_\mathcal{F})$ be a Banach algebra such that $\mathcal{F}$ is a subalgebra of $H^\infty(\Sigma_\delta)$ and that $f(A)$ is defined by the above calculus for all $f \in \mathcal{F}$. Following Haase [18, Chapter 5.3], we say that the $\mathcal{F}$-calculus is *bounded* if $f(A)$ is bounded for all $f \in \mathcal{F}$ and

$$\exists C > 0 : \quad \|f(A)\| \leq C\|f\|_\mathcal{F}, \quad \forall f \in \mathcal{F}. \tag{1.5}$$

The infimum over all possible $C$ is called the bound of the calculus. Note that for $\mathcal{F}$ closed in $H^\infty(\Sigma_\delta)$ with $\|\cdot\|_\mathcal{F} = \|\cdot\|_{\infty,\delta}$ and $A$ injective, (1.5) follows already if $f(A)$ is bounded for all $\mathcal{F}$ by the Convergence Lemma, [18, Proposition 5.1.4] and the Closed Graph Theorem.

By $e_\varepsilon$ we denote the function $z \mapsto e^{-\varepsilon z}$ which lies in $H^\infty_{(0)}(\Sigma_\delta)$ for $\delta < \frac{\pi}{2}$ and $\varepsilon > 0$. In the following the *exponential integral* function

$$\mathrm{Ei}(x) = \int_1^\infty \frac{e^{-xt}}{t}\, dt, \quad x > 0, \tag{1.6}$$

will be used several times. It is clear that $\mathrm{Ei}(x)$ is decreasing. The asymptotic behavior of $\mathrm{Ei}(x)$ is reflected in the estimates

$$\tfrac{1}{2}e^{-x}\log\left(1 + \tfrac{2}{x}\right) < \mathrm{Ei}(x) < e^{-x}\log\left(1 + \tfrac{1}{x}\right), \quad x > 0, \tag{1.7}$$

which go back to Gautschi [14] and can also be found in [1, 5.1.20]. Moreover, it is easy to prove that

$$\mathrm{Ei}(x) < \log\left(\tfrac{1}{x}\right), \quad x \in \left(0, \tfrac{1}{2}\right). \tag{1.8}$$

Thus, by (1.7), $\mathrm{Ei}(x) \sim |\log x|$ for $x < \frac{1}{2}$.

## 2. Main results

Unless explicitely stated, the space $X$ will always denote a general Banach space.

### 2.1. Sectorial operators and functions holomorphic at 0.

We first give a proof for Proposition 1.1

*Proof of Proposition 1.1.* Since $A$ is invertible, $f(A)$ is defined as a closed operator for every $f \in H^\infty(\Sigma_\phi)$ and $D(A) \subset D(f(A))$ because $\frac{z}{(1+z)^2}$ is a regularizer for $f$. Since for every $\delta < \pi/2$, $e_\varepsilon \in H^\infty_{(0)}(\Sigma_\delta)$ we have that $(fe_\varepsilon) \in H^\infty_{(0)}(\Sigma_\delta)$ for some $\delta < \pi/2$. Hence $e_\varepsilon(A)$ and $(fe_\varepsilon)(A)$ are bounded operators. If the calculus is bounded, (1.5) holds with $\mathcal{F} = H^\infty(\Sigma_\phi)$. Thus,

$$\|(fe_\varepsilon)(A)\| = \|f(A)e_\varepsilon(A)\| \leq C\|e_\varepsilon(A)\| \cdot \|f\|_{\infty,\phi} \leq \tilde{C}\|f\|_{\infty,\phi},$$

where the last inequality follows by [18, Proposition 3.4.1c] (or Corollary 2.12 of this paper) and $\tilde{C}$ does not depend on $\varepsilon$. Therefore, (1.1) holds.Conversely, let (1.1) be satisfied. Since $e_\varepsilon(A) \in \mathcal{B}(X)$ we have for $x \in D(A)$ that

$$\|f(A)x\| \leq \|f(A)x - e_\varepsilon(A)f(A)x\| + \|e_\varepsilon(A)f(A)x\|.$$

For $\varepsilon \to 0^+$, the first term on the right-hand-side goes to zero as $(e^{-\varepsilon\cdot})(A)$ converges to $I$ strongly on $\overline{D(A)} = X$, see [18, Proposition 3.4.1.f)]. Since $e_\varepsilon(A)f(A)x =$

$(e_\varepsilon f)(A)x$ for $x \in D(A)$, see [18, Theorem 1.3.2.c)], the second term can be estimated by the assumption of (1.1). Thus, we get $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(\Sigma_\phi)$ because $D(A)$ is dense. Hence, the calculus is bounded. The norm inequality then follows automatically, see the remark after (1.5). □

The following example shows that the assumption on the invertibility of $A$ cannot be neglected, if we restrict to functions on $H^\infty(\mathbb{C}_+)$.

***Example* 2.1.** Let $-B$ be the generator of the bounded analytic semigroup $S$ with $0 \in \rho(B)$. Assume that the $H^\infty(\mathbb{C}_+)$-caclulus is not bounded, thus, there exists $f \in H^\infty(\mathbb{C}_+)$ such that $f(B)$ is unbounded. Such examples exist even on Hilbert spaces, see e.g. [3] or Section 3.1. Then, $A = B^{-1}$ is bounded, sectorial of the same angle as $B$, see [18], and has dense range. Thus $g(A)$ is defined by the $H^\infty$-calculus for sectorial operators for $g \in H^\infty$ in some sector. Furthermore, by the *composition rule*, see [18, Proposition 2.4.1], we have that for $h = (z \mapsto z^{-1})$,

$$(f \circ h)(A) = f(B),$$

where $(f \circ h) \in H^\infty(\mathbb{C}_+)$. Since $A$ is bounded, it even generates a group $T$. Hence, $(f \circ h)(A)T(t) = f(B)T(t)$ cannot be bounded for any $t \geq 0$.

The reason why we cannot expect $(f \cdot e_\varepsilon)(A)$ to be a bounded operator if $0 \notin \rho(A)$ is that the integrand in (1.4) may have a singularity at 0. However, instead of making the resolvent exist at 0, we can pass over to functions that are holomorphic at 0.

**Proposition 2.2.** *Let $A$ be a densely defined, sectorial operator of angle $\omega < \frac{\pi}{2}$ on the Banach space $X$ with dense range. Then, for $\phi \in (\omega, \pi)$ the $H^\infty(\Sigma_\phi)$-calculus is bounded if and only if*

$$\exists C > 0\ \forall g \in H^\infty(\Sigma_\phi), g \text{ hol. at } 0: \quad \limsup_{\varepsilon \to 0^+} \|(g e_\varepsilon)(A)\| < C\|g\|_{\infty,\phi}. \tag{2.1}$$

*Proof.* The proof is essentially the same as for Proposition 1.1 with the following adaptions: Note that $A$ is injective as it is a sectorial operator with dense range, see [18, Proposition 2.1.1]. Thus, the calculus is defined for $H^\infty(\Sigma_\phi)$. For $g$ holomorphic at 0, $(\frac{g(z)}{1+z})(A)$ is defined by (1.4), and hence bounded. Thus, $D(A) \subset D(g(A))$. Because $D(A)$ is dense, it follows analogously to the proof of Proposition 1.1 that

$$\|g(A)\| \leq \limsup_{\varepsilon \to 0^+} \|(g e_\varepsilon)(A)\| \leq C\|g\|_{\infty,\phi}, \tag{2.2}$$

where the last inequality holds if (2.1) holds. For arbitrary $f \in H^\infty(\Sigma_\phi)$ take a sequence $g_n \in H^\infty(\Sigma_\phi)$ which are holomorphic at 0 and converge to $f$ pointwise in $\Sigma_\phi$ with $\sup_n \|g_n\|_{\infty,\phi} < \infty$. Applying the Convergence Lemma [18, Proposition 5.1.4b)], using (2.2) and the fact that $D(A) \cap R(A)$ is dense, yields that $f(A)$ is bounded. □

In the following theorem we estimate $\|(f \cdot e_\varepsilon)(A)\|$. In Section 3 we show that this estimate is sharp.

**Theorem 2.3.** *Let $A \in Sect(\omega)$, $0 < \omega < \phi < \frac{\pi}{2}$ and $\varepsilon, r_0 > 0$. Further, let $f \in H^\infty(\Omega_{\phi,r_0})$ with $\Omega_{\phi,r_0} := \Sigma_\phi \cup B_{r_0}(0)$. Then $(f e_\varepsilon)(A)$ is bounded and*

$$\|(f \cdot e_\varepsilon)(A)\| \leq \frac{M(A,\phi)}{\pi} \cdot b(\varepsilon, r_0, \phi) \cdot \|f\|_{\infty,\Omega_{\phi,r_0}}, \tag{2.3}$$

*with*

$$b(\varepsilon, r_0, \phi) = \begin{cases} \mathrm{Ei}(\varepsilon r_0 \cos\phi) + e^{\varepsilon r_0}(\pi - \phi), & 2\varepsilon r_0 \le 1, \\ \mathrm{Ei}\left(\frac{\cos\phi}{2}\right) + \sqrt{e}(\pi - \phi), & 2\varepsilon r_0 > 1. \end{cases} \tag{2.4}$$

*Here,* $\mathrm{Ei}(x)$ *is the the exponential integral, see (1.6)–(1.8), therefore,*

$$b(\varepsilon, r_0, \phi) \sim \begin{cases} |\log(\varepsilon r_0 \cos\phi)|, & \varepsilon r_0 < \frac{1}{2}, \\ |\log \frac{\cos\phi}{2}|, & \varepsilon r_0 \ge \frac{1}{2}. \end{cases} \tag{2.5}$$

*Proof.* Since $fe_\varepsilon \in H^\infty_{(0)}(\Sigma_\phi) \cap H^\infty(\Omega_{\phi,r_0})$, we get (see (1.4))

$$(fe_\varepsilon)(A) = \frac{1}{2\pi i}\int_{\Gamma_r} f(z)e^{-\varepsilon z}R(z,A)\,dz \in \mathcal{B}(X), \tag{2.6}$$

where the integration path is $\Gamma_r = \Gamma_{1,r} \cup \Gamma_{2,r} \cup \Gamma_{3,r}$ with

$$\Gamma_{1,r} = \left\{\tilde{r}e^{i\delta}, \tilde{r} > r\right\}, \Gamma_{2,r} = \left\{re^{i\psi}, |\psi| \ge \delta\right\}, \Gamma_{3,r} = \left\{\tilde{r}e^{-i\delta}, \tilde{r} > r\right\},$$

$r \in (0, r_0)$, $\delta \in (\omega, \phi)$, orientated counter-clockwise. Since $f \in H^\infty(\Omega_{\phi,r_0})$, we can estimate

$$\|(fe_\varepsilon)(A)\| \le \frac{\|f\|_{\infty,\Omega_{\phi,r_0}}}{2\pi}\int_{\Gamma_r} \|e^{-\varepsilon z}R(z,A)\|\,|dz|. \tag{2.7}$$

The rest of the proof is similar to a standard argument to show that sectorial operators of angle $< \frac{\pi}{2}$ are generators of bounded analytic semigroup, see e.g. [10, 31, 41]. Splitting up the integral, for $z \in \Gamma_{1,r}$,

$$\|e^{-\varepsilon z}R(z,A)\| \le e^{-\varepsilon \Re z} \cdot \frac{M(A,\delta)}{|z|} = \frac{e^{-\varepsilon|z|\cos\delta}}{|z|}M(A,\delta).$$

On $\Gamma_{3,r}$ the same estimate holds. For $z \in \Gamma_{2,r}$,

$$\|e^{-\varepsilon z}R(z,A)\| \le e^{\varepsilon r} \cdot \frac{M(A,\delta)}{r}.$$

Therefore,

$$\begin{aligned} \int_{\Gamma r} \|e^{-\varepsilon z}R(z,A)\||dz| &\le M(A,\delta)\left(2\int_r^\infty \frac{e^{-\varepsilon\tilde{r}\cos\delta}}{\tilde{r}}d\tilde{r} + \frac{e^{\varepsilon r}}{r}\int_{\Gamma_{2,r}}|dz|\right) \\ &\le 2M(A,\delta)(\mathrm{Ei}(\varepsilon r\cos\delta) + e^{\varepsilon r}(\pi - \delta)). \end{aligned} \tag{2.8}$$

Next we choose $r$ as

$$r = \begin{cases} r_n = r_0(1 - 2^{-n}), & 2\varepsilon r_0 \le 1, \\ \frac{1}{2\varepsilon}, & 2\varepsilon r_0 > 1. \end{cases}$$

Clearly, $r$ lies within $(0, r_0)$. Hence, by (2.7) and (2.8),

$$\|(fe_\varepsilon)(A)\| \le \frac{M(A,\delta)}{\pi}\left\{\begin{array}{ll} \mathrm{Ei}(\varepsilon r_n\cos\delta) + e^{\varepsilon r_n}(\pi - \delta), & 2\varepsilon r_0 \le 1, \\ \mathrm{Ei}\left(\frac{\cos\delta}{2}\right) + \sqrt{e}(\pi - \delta), & 2\varepsilon r_0 > 1 \end{array}\right\}\|f\|_{\infty,\Omega_{\phi,r_0}}.$$

Letting $n \to \infty$ and $\delta \to \phi^-$ shows the assertion. □

As $H^\infty(\Omega_{\frac{\pi}{2},r_0})$ is continuously embedded in $H^\infty(\mathbb{C}_+)$, and since $\|f\|_{\infty,\mathbb{C}_+} = \|e_\varepsilon f\|_{\infty,\mathbb{C}_+}$ we have the following direct consequence of Theorem 2.3.

**Corollary 2.4.** *Let* $A \in \mathrm{Sect}(\omega)$ *on the Banach space* $X$ *and* $\omega < \frac{\pi}{2}$*. Then, for any* $r > 0$ *and* $\varepsilon > 0$*,* $A$ *has a bounded* $e_\varepsilon H^\infty(\Omega_{\frac{\pi}{2},r})$*-calculus with* $\Omega_{\frac{\pi}{2},r} = \mathbb{C}_+ \cup B_r(0)$.

Note that $e_\varepsilon H^\infty(\Omega_{\frac{\pi}{2}}, r)$ is a closed ideal in $H^\infty(\mathbb{C}_+)$.

2.2. **The space $H^\infty[\varepsilon,\sigma]$ and Vitse's result.** In this subsection we show that the result in Theorem 2.3 generalizes Theorem 1.6 in [41].

For $\varepsilon,\sigma\in\mathbb{R}$ with $0\le\varepsilon<\sigma\le\infty$ let $H^\infty[\varepsilon,\sigma]$ denote the space of functions which are in $H^\infty(\mathbb{C}_+)$ and are the Laplace-Fourier transform of a distribution supported in $[\varepsilon,\sigma]$. Recall that an entire function $g$ is of *exponential type* $\sigma>0$ if for any $\epsilon>0$ there exists $C_\epsilon>0$ such that $|g(z)|\le C_\epsilon e^{(\sigma+\epsilon)|z|}$ for all $z\in\mathbb{C}$.

For $\sigma<\infty$, the following Paley-Wiener-Schwartz type result holds,

$$g\in H^\infty[\varepsilon,\sigma] \iff g \text{ is entire of exponential type } \sigma \text{ and } ge^{\varepsilon\cdot}\in H^\infty(\mathbb{C}_+). \tag{2.9}$$

For $\sigma=\infty$, we get $H^\infty[\varepsilon,\infty]=e^{-\varepsilon z}H^\infty(\mathbb{C}_+)$. For more details about $H^\infty[\varepsilon,\sigma]$, we refer to [41] and the references therein.

The following lemma is a consequence of the Phragmén-Lindelöf principle and can be found in Boas [6, Theorem 6.2.4, p.82].

**Lemma 2.5.** *Let $g$ be an entire function of exponential type $\sigma$ such that $\|g\|_{\infty,i\mathbb{R}}:=\sup_{y\in\mathbb{R}}|g(iy)|<\infty$. Then, for all $x,y\in\mathbb{R}$,*

$$|g(x+iy)|\le e^{\sigma|y|}\,\|g\|_{\infty,i\mathbb{R}}.$$

Using Lemma 2.5, Theorem 2.3 yields an estimate in the $H^\infty(\mathbb{C}_+)$-norm.

**Theorem 2.6.** *Let $A\in Sect(\omega)$ with $\omega<\frac{\pi}{2}$, and let $0<\varepsilon<\sigma<\infty$. For $g\in H^\infty[\varepsilon,\sigma]$,*

$$\|g(A)\|\le\|g\|_{\infty,\mathbb{C}_+}\cdot\inf_{\phi\in(\omega,\frac{\pi}{2}),k\ge1}\frac{M(A,\phi)}{\pi}\,b\left(\varepsilon,\tfrac{1}{k\sigma},\phi\right)e^{\frac{\sigma-\varepsilon}{k\sigma}}. \tag{2.10}$$

*where $b(\varepsilon,r,\phi)$ is defined in (2.4).*

*Proof.* Let $f(z)=e^{\varepsilon z}g(z)$. By (2.9), $f$ lies in $H^\infty(\mathbb{C}_+)$ and is entire of exponential type $\sigma-\varepsilon$. Let $k\ge1$. Since $f$ is entire, and bounded on $\mathbb{C}_+$, we can apply Theorem 2.3 with $r_0=\frac{1}{k\sigma}$. Thus, for $\phi\in(\omega,\frac{\pi}{2})$,

$$\|g(A)\|=\|(fe_\varepsilon)(A)\|\le\inf_{\phi\in(\omega,\frac{\pi}{2})}\frac{M(A,\phi)}{\pi}\cdot b\left(\varepsilon,\tfrac{1}{k\sigma},\phi\right)\cdot\|f\|_{\infty,\Omega_{\phi,\frac{1}{k\sigma}}},$$

where $\Omega_{\phi,\frac{1}{k\sigma}}=\Sigma_\phi\cup B_{\frac{1}{k\sigma}}(0)$. Clearly, $\|f\|_{\infty,\Omega_{\phi,\frac{1}{k\sigma}}}\le\|f\|_{\infty,\mathbb{C}_+\cup B_{\frac{1}{k\sigma}}(0)}$. Moreover, as $f$ is entire of exponential type $\sigma-\varepsilon$ and $\sup_{y\in\mathbb{R}}|f(iy)|=\|f\|_{\infty,\mathbb{C}_+}$, we can apply Lemma 2.5 to conclude that

$$\|f\|_{\infty,\Omega_{\phi,\frac{1}{k\sigma}}}\le e^{\frac{\sigma-\varepsilon}{k\sigma}}\|f\|_{\infty,\mathbb{C}_+}.$$

Since $\|g\|_{\infty,\mathbb{C}_+}=\|f\|_{\infty,\mathbb{C}_+}$, the assertion follows. □

Now we write Theorem 2.6 in the terminology used in [41]. This will reveal that the dependence on $M(A,\phi)$ of our approach is improving the corresponding estimate in [41].

In [41], for $\theta\in(0,\pi]$, a densely defined closed operator is called *$\theta$-sectorial*, if $\sigma(A)$ is contained in $\Sigma_\theta\cup\{0\}$ (note that in our definition of $\mathrm{Sect}(\theta)$, $\sigma(A)$ is contained in $\overline{\Sigma_\theta}$) and

$$\tilde{M}(A,\theta)=\sup_{z\in\mathbb{C}\setminus(\Sigma_\theta\cup\{0\})}\|zR(z,A)\|<\infty.$$

By $S(\theta)$ let us denote the $\theta$-sectorial operators on $X$. As pointed out in [41, Section 1.1], $S(\theta)\subset\mathrm{Sect}(\theta)\subset S(\theta+\epsilon)$ for all $\epsilon>0$ and $S(\theta)=\bigcup_{0<\theta'<\theta}\mathrm{Sect}(\theta')$. Moreover, for $A \in S(\frac{\pi}{2})$ there exists a $\theta < \frac{\pi}{2}$ such that $A \in S(\theta)$, see Lemma 2.7 below. Hence, $A \in \mathrm{Sect}(\theta)$ for some $\theta < \frac{\pi}{2}$ if and only if $A \in S(\frac{\pi}{2})$. Furthermore, for $A \in S(\theta)$, it is a simple consequence of the Phragmén-Lindelöf principle that

$$\tilde{M}(A,\theta) = \sup_{z \in \mathbb{C}\setminus(\Sigma_\theta \cup \{0\})} \|zR(z,A)\| = \sup_{z\in\mathbb{C}\setminus\overline{\Sigma_\theta}} \|zR(z,A)\| = M(A,\theta). \tag{2.11}$$

The following lemma can be found in [41, Lemma 1.1].

**Lemma 2.7.** *Let $A \in S(\frac{\pi}{2})$ and $M = \tilde{M}(A, \frac{\pi}{2})$. Then, $A \in S(\theta)$ for*

$$\theta = \arccos \frac{1}{2M} \quad \text{and} \quad \tilde{M}(A,\theta) = M(A,\theta) \leq 2M. \tag{2.12}$$

*Note that $S(\theta) = \bigcup_{0<\theta'<\theta} \mathrm{Sect}(\theta')$. Since $M \geq 1$ ([18, Prop.2.1.1]), $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$.*

**Theorem 2.8.** *Let $A \in \mathrm{Sect}(\omega)$ with $\omega < \frac{\pi}{2}$ which is equivalent to $A \in S(\frac{\pi}{2})$ (see above). Then, with $M = M(A, \frac{\pi}{2})$,*

*(i) For all $t \geq 0$,*

$$\|e^{-tA}\| \leq \frac{2M}{\pi}(\log(M) + 5). \tag{2.13}$$

*(ii) For $0 < \varepsilon < \sigma < \infty$ and $g \in H^\infty[\varepsilon, \sigma]$,*

$$\|g(A)\| \leq \left(C_1 + C_2 \log\left(\frac{\sigma}{\varepsilon}\right)\right) \|g\|_{\infty,\mathbb{C}_+} \leq C_3 \log\left(\frac{\sigma e}{\varepsilon}\right) \|g\|_{\infty,\mathbb{C}_+}, \tag{2.14}$$

*with $C_1 = c_1 M + c_2 M \log(M)$, $C_2 = c_2 M$ and $C_3 = c_1 M + c_2 M \log(M)$ and*

$$c_1 = \tfrac{2e^{\frac{1}{5}}}{\pi}\left(\log(10) + \tfrac{2\pi}{3}\right) \approx 3.42, \;\; c_2 = \tfrac{2e^{\frac{1}{5}}}{\pi} \approx 0.78.$$

*Proof.* Let $\theta$ be the defined as in Lemma 2.7, hence, $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, $\cos\theta = \frac{1}{2M}$, and $M(A,\theta) \leq 2M$. Using Theorem 2.6, we get

$$\|g(A)\| \leq \tfrac{2M}{\pi} \cdot \|g\|_{\infty,\mathbb{C}_+} \cdot \inf_{k\geq 1} b(\varepsilon, \tfrac{1}{k\sigma}, \theta) e^{\frac{\sigma-\varepsilon}{k\sigma}}. \tag{2.15}$$

It remains to estimate the infimum. For $k \geq 2$, $\frac{\varepsilon}{2Mk\sigma} < \frac{\varepsilon}{k\sigma} < \frac{1}{2}$ and thus, by (2.4) and (1.8), we get for $b = b(\varepsilon, \frac{1}{k\sigma}, \theta)$ that

$$b \cdot e^{\frac{\sigma-\varepsilon}{k\sigma}} = \left[\mathrm{Ei}\left(\frac{\varepsilon}{2Mk\sigma}\right) e^{\frac{\sigma-\varepsilon}{k\sigma}} + e^{\frac{1}{k}} \frac{2\pi}{3}\right] \leq \left[\log\left(\frac{2Mk\sigma}{\varepsilon}\right) e^{\frac{\sigma-\varepsilon}{k\sigma}} + e^{\frac{1}{k}} \frac{2\pi}{3}\right]. \tag{2.16}$$

To prove (ii), let $t = \varepsilon$ and set $g = e_t \in H^\infty[t,\sigma]$. Then, let $\sigma \to \varepsilon^+ = t^+$ in (2.16) and choose $k = 5$ (alternatively, apply Theorem 2.3 with $f(z) = 1$, $\varepsilon = t$ and $r_0 = \frac{1}{5\varepsilon}$).

To show (i), observe that, using $e^{\frac{\sigma-\varepsilon}{k\sigma}} < e^{\frac{1}{k}}$, the right-hand-side of (2.16) can be further estimated,

$$b \cdot e^{\frac{\sigma-\varepsilon}{k\sigma}} \leq \left[\log(M) + \log\left(\frac{\sigma}{\varepsilon}\right) + \log(2k) + \frac{2\pi}{3}\right] \cdot e^{\frac{1}{k}}.$$

Setting $k = 5$, we get the result. □

*Remark* 2.9. (1) In [41, Lemma 1.2 and Theorem 1.6] Vitse derives similar estimates as in Theorem 2.8. However, she uses the Hille-Phillips calculus and considers elements of $H^\infty[\varepsilon,\sigma]$ that are Laplace transforms of $L^1(\varepsilon,\sigma)$-functions first. The approach moreover relies on estimates of derivatives of the (analytic) semigroup. This results in a similar estimate as in (2.14), but with the following constants

$$\tilde{C}_1 = \frac{30}{\pi} M^2, \quad \tilde{C}_2 = \frac{16}{\pi} M^3, \quad \tilde{C}_3 = \frac{30}{\pi} M^3.$$

The dependence on $M$ is strongly improved by our approach, as $M^3$ gets replaced by $M(1+\log M)$. Moreover, a more careful study even shows that $C_i \leq \tilde{C}_i$, $i \in \{1,2,3\}$, for every $M \geq 1$.

(2) We point out that Vitse uses an estimate for the semigroup, [41, Lemma 1.2] (which is slightly improved by (2.13)), to get an estimate for $H^\infty[\varepsilon,\sigma]$ functions. Whereas our estimates all follow directly from Theorem 2.6. In other words, (the estimate for) the dependence on $M$ is the same for any $H^\infty[\varepsilon,\sigma]$ function, including $e_\varepsilon$.

(3) The constants $c_1$ and $c_2$ in Theorem 2.8 can possibly be further improved by optimizing the choice of $k$ in the proof.

2.3. **Invertible $A$ - exponential stable semigroups.** Theorems 2.3 and 2.6 deal with the situation of *bounded* analytic semigroups and functions $f$ which are holomorphic at 0. As might be expected, a similar result holds for functions $f$ not necessarily holomorphic at 0, but with a sectorial operator $A$ having $0 \in \rho(A)$.

**Theorem 2.10.** *Let $A \in \mathrm{Sect}(\omega)$, $\omega < \phi < \pi/2$, and $0 \in \rho(A)$. Then, for $\varepsilon > 0$, $f \in H^\infty(\Sigma_\phi)$ the operator $(fe_\varepsilon)(A)$ is bounded and for all $\kappa \in (0,1)$,*

$$\|(f \cdot e_\varepsilon)(A)\| \leq \frac{M(A,\phi)}{\pi} \cdot b_\kappa\left(\varepsilon, \tfrac{1}{\|A^{-1}\|}, \phi\right) \cdot \|f\|_{\infty,\phi}. \tag{2.17}$$

*Here,*

$$b_\kappa(\varepsilon, R, \phi) = \mathrm{Ei}\left(\varepsilon\kappa R\cos\phi\right) + \frac{\kappa}{1-\kappa} e^{-\varepsilon\kappa R\cos\phi}, \tag{2.18}$$

*Hence, $b_\kappa(\varepsilon,R,\phi) \sim C_\kappa |\log(\varepsilon R\cos\phi)|$ for $\varepsilon R < \frac{1}{2}$ and $\|(fe_\varepsilon)(A)\|$ goes to zero exponentially as $\varepsilon \to \infty$ by the properties of* Ei*, see (1.7) and (1.8).*

*Proof.* Since $0 \in \rho(A)$ and $fe_\varepsilon \in H^\infty_{(0)}(\Sigma_\phi)$, $(fe_\varepsilon)(A)$ is well-defined by (1.4),

$$(f \cdot e_\varepsilon)(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_\theta} f(z) e^{\varepsilon z} R(z,A)\, dz,$$

for $\theta \in (\omega,\phi)$ and where $\partial\Sigma_\theta$ denotes the boundary (orientated positively) of $\Sigma_\theta$. Because $0 \in \rho(A)$, we have that the ball $B_{\frac{1}{\|A^{-1}\|}}(0)$ lies in $\rho(A)$. For $\kappa \in (0,1)$ set $r = \frac{\kappa}{\|A^{-1}\|}$. By Cauchy's theorem, we can replace the integration path $\partial\Sigma_\theta$ by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1 = \left\{se^{i\theta}, s \geq r\right\}, \Gamma_2 = \left\{re^{i\theta} - it, t \in (0, 2\Im(re^{i\theta}))\right\}, \Gamma_3 = \left\{-se^{-i\theta}, s \leq -r\right\}.$$

Thus,

$$\|(fe_\varepsilon)(A)\| \leq \frac{\|f\|_{\infty,\phi}}{2\pi} \int_\Gamma e^{-\varepsilon\Re z} \|R(z,A)\|\, |dz|. \tag{2.19}$$

By the resolvent identity, $\|R(z,A)\| \leq \frac{\|A^{-1}\|}{1-|z|\|A^{-1}\|}$, and thus, for $\kappa \in (0,1)$,

$$\|R(z,A)\| \leq \frac{\|A^{-1}\|}{1-\kappa} \quad \text{for} \quad |z| \leq r = \frac{\kappa}{\|A^{-1}\|}.$$

This yields, since $\Gamma_2 \subset B_r(0)$,

$$\begin{aligned}\int_\Gamma e^{-\varepsilon \Re z}\|R(z,A)\|\,|dz| &\le \tfrac{\|A^{-1}\|}{1-\kappa}\int_{\Gamma_2} e^{-\varepsilon r\cos\theta}\,dt + 2M(A,\theta)\int_r^\infty \frac{e^{-\varepsilon s\cos\theta}}{s}\,ds \\ &= \tfrac{2\|A^{-1}\|}{1-\kappa}\, r\sin\theta\; e^{-\varepsilon r\cos\theta} + 2M(A,\theta)\mathrm{Ei}(\varepsilon r\cos\theta), \\ &\le 2M(A,\theta)\left(\tfrac{\kappa}{1-\kappa}e^{-\varepsilon r\cos\theta} + \mathrm{Ei}(\varepsilon r\cos\theta)\right),\end{aligned}$$

as $M(A,\theta) \ge 1$, see e.g. [18, Proposition 2.1.1]. Letting $\theta \to \phi^-$ yields the assertion. □

*Remark* 2.11. If $A$ is sectorial and $R > 0$, then clearly $RA$ is sectorial of the same angle. Since $f \mapsto f_R = f(R\cdot)$ is an isometric isomorphism on $H^\infty(\Sigma_\phi)$, and $(fe_\varepsilon)(RA) = (f_R e_{\varepsilon R})(A)$ by the composition rule of holomorphic functional calculus [18, Theorem 2.4.2], we see that it is sufficient to consider $\frac{1}{\|A^{-1}\|} = 1$ in the proof of Theorem 2.10.

Applying Theorem 2.10 to $f \equiv 1$ shows that $\|e_\varepsilon(A)\|$ decays exponentially for $\varepsilon \to \infty$. This behavior is natural as the condition that $0 \in \rho(A)$ implies that the analytic semigroup is exponentially stable. However, for $\varepsilon \to 0$, the theorem gives no bound for the norm. This can be derived by Theorem 2.3 as we will see in the following result.

**Corollary 2.12.** *Let $A \in \mathrm{Sect}(\omega)$ and $0 < \omega < \phi < \frac{\pi}{2}$. If $A$ is invertible, then we define $R = \frac{1}{\|A^{-1}\|}$, otherwise we set $R$ to be zero. Then, for any $\kappa \in [0,1)$, there exists a $C > 0$ such that*

$$\|e_\varepsilon(A)\| \le Ce^{-\varepsilon\kappa R\cos\phi}, \quad \varepsilon > 0, \tag{2.20}$$

*with $C \le C_\kappa M(A,\phi)\mathrm{Ei}(\cos\phi)$.*

*Proof.* Let $f \equiv 1$. If $\varepsilon\kappa R > 1$, by (1.7),

$$\mathrm{Ei}(\varepsilon\kappa R\cos\phi) < e^{-\varepsilon\kappa R\cos\phi}\log\left(1 + \tfrac{1}{\cos\phi}\right) < 2e^2 e^{-\varepsilon\kappa R\cos\phi}\mathrm{Ei}(\cos\phi),$$

where we used that $\mathrm{Ei}(2\cos\phi) < \mathrm{Ei}(\cos\phi)$ in the last inequality. Using this, Theorem 2.10 yields

$$\|e_\varepsilon(A)\| \le \tilde{C}_\kappa M(A,\phi)\mathrm{Ei}(\cos\phi)e^{-\varepsilon\kappa R\cos\phi}, \quad \varepsilon\kappa R > 1, \tag{2.21}$$

where $\tilde{C}_\kappa > 0$ only depends on $\kappa$.
Now, let $\varepsilon\kappa R \le 1$. We apply Theorem 2.3 with $r_0 = \frac{1}{\varepsilon}$. It implies that there exists an absolute constant $C_2$ such that $\|e_\varepsilon(A)\| \le C_2 M(A,\phi)\mathrm{Ei}(\cos\phi)$. Together with (2.21) the assertion follows. □

Let us point out that the corollary is interesting in terms of the dependence on the constants $M(A,\phi)$, $\|A^{-1}\|$ and $\phi$, whereas the exponential decay is clear for exponentially stable semigroups.
Further note that the use of the scaling variable $\kappa$ is not so artificial as it might seem: By $B_{\frac{1}{\|A^{-1}\|}}(0) \subset \rho(A)$, we have that the growth bound $\omega_0$ of the semigroup satisfies $\omega_0 \le -\frac{\cos\phi}{\|A^{-1}\|}$. It is well-known that, even in the case of a *spectrum-determined* growth bound, as we have it for analytic semigroups, this rate need not be attained, see e.g. [10, Example I.5.7]. The $\kappa$ encodes that we can achieve any exponential decay of rate $\tilde{\omega} \in (-\frac{\cos\phi}{\|A^{-1}\|}, 0]$.

## 3. Sharpness of the result

3.1. **Diagonal operators on Schauder bases (Schauder multiplier).** A typical construction of an unbounded calculus goes back to Baillon and Clement [3] and has been used in extensively since then, see [11] and the references therein. The situation is as follows.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Schauder basis of the Banach space $X$. For the sequence $\mu = (\mu_n)_{n\in\mathbb{N}}$ define the multiplication operator $\mathcal{M}_\mu$ by its action on the basis, i.e. $\mathcal{M}_\mu\phi_n = \mu_n\phi_n$, $n \in \mathbb{N}$, with maximal domain. The choice $\lambda_n = 2^n$ yields a
sectorial operator $A = \mathcal{M}_\lambda \in \mathrm{Sect}(0)$ with $0 \in \rho(\mathcal{M}_\lambda)$, and for $f \in H^\infty(\mathbb{C}_+)$,

$$\begin{aligned} &f(A) = f(\mathcal{M}_\lambda) = \mathcal{M}_{f(\lambda)}, \\ &D(\mathcal{M}_{f(\lambda)}) = \left\{x = \textstyle\sum_{n\in\mathbb{N}} x_n\phi_n \in X : \sum_{n\in\mathbb{N}} f(\lambda_n)x_n\phi_n \text{ converges}\right\}. \end{aligned} \tag{3.1}$$

See e.g. [18, Chapter 9] and [11].

Because of (3.1), a way of constructing unbounded calculi consists of the following two steps:

(1) Find a sequence $\mu \in \ell^\infty(\mathbb{N}, \mathbb{C})$ such that $\mathcal{M}_\mu \notin \mathcal{B}(X)$.
(2) Find $f \in H^\infty(\mathbb{C}_+)$ such that $f(\lambda_n) = \mu_n$ for all $n \in \mathbb{N}$.

Since $\{\lambda_n\}$ is interpolating, see [13], the second step is always possible. Note that the first step follows if we can

$$\text{find an } x \in X \text{ such that } x = \sum_{n\in\mathbb{N}} x_n\phi_n \text{ does NOT converge unconditionally.} \tag{3.2}$$

In fact, then there exists a sequence $\mu_n \subset \{-1, 1\}$ such that $\sum_{n\in\mathbb{N}} \mu_n x_n\phi_n$ does not converge. Thus, $x \notin D(\mathcal{M}_\mu)$, and so $\mathcal{M}_\mu \notin \mathcal{B}(X)$.
Conversely, this indicates that *a bounded $H^\infty$-calculus implies a large amount unconditionality*, [18, p.124], which can be made rigorous, see [18, Section 5.6] and [24]. For more information about unbounded $H^\infty$-calculi via diagonal operators, see [18, Chapter 9].
Let $\{\phi_n\}_{n\in I}$, $I \subset \mathbb{N}$, be a Schauder basis of a Banach space $X$. For finite $\sigma \subset I$, $P_\sigma$ denotes the projection onto $X_\sigma := \{\phi_n\}_{n\in\sigma}$. Let us introduce the following constants,

$$m_\phi = \sup_{n\in I} \|P_{\{n\}}\|, \quad \kappa_\phi = \sup_{k\le\ell} \left\|P_{[k,\ell]\cap I}\right\|, \quad ub_\phi = \sup_{\sigma\subset I, |\sigma|<\infty} \|P_\sigma\|. \tag{3.3}$$

The constant $\kappa_\phi$ is called the *basis constant* of $\{\phi_n\}_{n\in\mathbb{N}}$ and $ub_\phi$ the *uniform basis constant*. Clearly,

$$m_\phi \le \kappa_\phi \le ub_\phi. \tag{3.4}$$

**Theorem 3.1.** *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Schauder basis on a Banach space $X$ with $m_\phi < \infty$. Let $\lambda_n = c^n$, $n \in \mathbb{N}$ for $c > 1$. Then $A := \mathcal{M}_\lambda \in \mathrm{Sect}(0)$ and*

*(i) $M(A, \psi) \le \kappa_\phi M(\psi)$ for all $\psi \in (0, \pi]$, where $M(\psi)$ only depends on $\psi$.*
*(ii) $0 \in \rho(A)$ and $\mathrm{dist}(\sigma(A), 0) = c$.*
*(iii) For $\varepsilon > 0$ and $N_\varepsilon = \lfloor \frac{2\mathrm{Ei}(\varepsilon)}{\log c} \rfloor$, there holds*

$$\|(f \cdot e_\varepsilon)(A)\| \le \left(\pi \cdot ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}} + m_\phi e^{-k_\varepsilon} \left(\frac{K_1}{\log c} + 1\right)\right) \|f\|_{\infty,\psi}, \tag{3.5}$$

*for all* $f \in H^\infty(\Sigma_\psi)$, $\psi \in (0, \frac{\pi}{2})$ *and* $\varepsilon > 0$ *and*

$$k_\varepsilon = \begin{cases} K_0 & \varepsilon \le \varepsilon_c, \\ \max\{K_0, c\varepsilon\} & \varepsilon > \varepsilon_c, \end{cases} \tag{3.6}$$

*with absolute constants* $K_0$, $K_1 > 0$ *and* $\varepsilon_c$ *such that* $2\mathrm{Ei}(\varepsilon_c) < \log c$. *Here,* $m_\phi$, $\kappa_\phi$ *and* $ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$ *are defined in (3.3).*

*Proof.* By [18, Lemma 9.1.2 and its proof], $A \in \mathrm{Sect}(0)$ with $M(A, \phi) \le \kappa_\phi M(\psi)$, where $M(\psi)$ only depends on $\psi \in (0, \pi]$. Clearly, $\sigma(A) \subset [\lambda_1, \infty)$. This shows (i) and (ii).
To show (iii), note that for $N_\varepsilon = \lfloor \frac{2\mathrm{Ei}(\varepsilon)}{\log c} \rfloor$,

$$h(\varepsilon) := c^{N_\varepsilon + 1}\varepsilon \ge c^{\frac{2\mathrm{Ei}(\varepsilon)}{\log c}}\varepsilon = e^{2\mathrm{Ei}(\varepsilon)}\varepsilon \overset{(1.7)}{\ge} \left(1 + \tfrac{1}{\varepsilon}\right)^{e^{-\varepsilon}} \varepsilon > K_0,$$

for some constant $K_0 \in (0, 1)$ and all $\varepsilon > 0$. If $N_\varepsilon = 0$, which means that $2\mathrm{Ei}(\varepsilon) < \log c$, then $h(\varepsilon) = c\varepsilon$. Since Ei is bijective and decreasing on $(0, \infty)$, this yields that there exists an $\varepsilon_c > 0$ such that $h(\varepsilon) \ge k_\varepsilon$, with $k_\varepsilon$ defined in (3.6).
Now,

$$\begin{aligned} \left\| \sum_{n\in\mathbb{N}} f(\lambda_n) e^{-c^n \varepsilon} P_{\{n\}} \right\| &\le \left\| \sum_{n=1}^{N_\varepsilon} f(c^n) e^{-c^n\varepsilon} P_{\{n\}} \right\| + \left\| \sum_{n=N_\varepsilon+1}^{\infty} f(c^n) e^{-c^n\varepsilon} P_{\{n\}} \right\| \\ &\le \pi \cdot ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \cdot \|f e_\varepsilon\|_\infty + \sum_{k=0}^{\infty} \left| f(c^{k+N_\varepsilon+1}) e^{-h(\varepsilon)c^k} \right| \left\| P_{\{k+N_\varepsilon+1\}} \right\| \\ &\le \pi \cdot ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \cdot \|f\|_\infty + m_\phi \|f\|_\infty \sum_{k=0}^{\infty} e^{-k_\varepsilon c^k}, \end{aligned}$$

where we used [30, Lemma 2.9.1] to estimate the first term in the second line. It remains to estimate the sum. By (A.2),

$$\sum_{k=0}^{\infty} e^{-k_\varepsilon c^k} \le e^{-k_\varepsilon} + \frac{\mathrm{Ei}(k_\varepsilon)}{\log c} \overset{(1.7)}{\le} e^{-k_\varepsilon} \left( 1 + \frac{\log(1 + \frac{1}{k_\varepsilon})}{\log c} \right).$$

Since $k_\varepsilon \ge K_0$, we can bound $\log(1 + \frac{1}{k_\varepsilon})$ by $K_1 = \log\left(1 + \frac{1}{K_0}\right)$. □

*Remark* 3.2.
- We point out that (3.5) shows that for $\varepsilon \to \infty$, $\|(fe_\varepsilon)(A)\|$ goes to 0 exponentially.
- Using (1.7) it is easy to show that in Theorem 3.1, $\varepsilon_c$ can be chosen to be $\frac{1}{\sqrt{c}-1}$.

In (3.5) the $\varepsilon$-dependence for small $\varepsilon$ of the right hand side appears only in the term $ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$. The following result shows that this indeed exhibits a logarithmic behavior for $\varepsilon \to 0$, which confirms the result from Theorem 2.3. We also show that on Hilbert spaces the behavior is slightly better.

**Theorem 3.3.** *Let* $\{\phi_n\}_{n\in\mathbb{N}}$, $X$, $c$, $A$ *be as in Theorem 3.1. Then, the following assertions hold for all* $\psi \in (0, \pi)$, $f \in H^\infty(\Sigma_\psi)$, $\varepsilon > 0$.
*If* $X$ *is a Banach space, then*

$$\|(f \cdot e_\varepsilon)(A)\| \le \left( \frac{K_2}{\log c} + 1 \right) \cdot m_\phi \cdot \mathrm{Ei}(\varepsilon) \cdot \|f\|_{\infty,\psi}. \tag{3.7}$$

*If $X$ is a Hilbert space, then*

$$\|(f\cdot e_\varepsilon)(A)\| \le \left(\frac{K_3}{\log c}+1\right)\cdot m_\phi \cdot \mathrm{Ei}(\varepsilon)^{1-\frac{0.32}{\kappa_\phi^2}}\cdot \|f\|_{\infty,\psi}. \tag{3.8}$$

*Here the $K_2$ and $K_3$ are absolute constants.*

*Proof.* By (3.5), it remains to estimate $ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$. For a basis $\tilde\phi$ of a general $N$-dimensional Banach space, it is easy to see that $ub_{\tilde\phi}\le N m_{\tilde\phi}$. Since $N_\varepsilon = \lfloor\frac{2\mathrm{Ei}(\varepsilon)}{\log c}\rfloor$, and $m_{\{\phi_n\}_{n=1}^{N_\varepsilon}}\le m_\phi$, this implies (3.7).
For a basis $\tilde\phi$ of an $N$-dimensional Hilbert space, we have that

$$ub_{\tilde\phi}\le 2m_{\tilde\phi}\cdot N^{1-\frac{0.32}{\kappa_{\tilde\phi}^2}}. \tag{3.9}$$

This is due to a recent result by Nikolski, [30, Theorem 3.1], which is slight generalization of a classic theorem by McCarthy-Schwartz, [26]. Hence, because $m_{\{\phi_n\}_{n=1}^{N_\varepsilon}}\le m_\phi$ and $\kappa_{\{\phi_n\}_{n=1}^{N_\varepsilon}}\le\kappa_\phi$,

$$ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}\le 2m_\phi N_\varepsilon^{1-\frac{0.32}{\kappa_\phi^2}}.$$

By the definition of $N_\varepsilon$, this yields (3.8). □

*Remark* 3.4. The key ingredient of the proof of (3.8) in Theorem 3.3 is the McCarthy-Schwartz-type result, (3.9). For general Banach spaces this does not hold. However, there exists a version of McCarthy-Schwartz's result for uniformly convex spaces by Gurarii and Gurarii [15], see also [30, Theorem 3.6.1 and Corollary 3.6.8]. In particular, this enables us to deduce an estimate similar to (3.8) for $L^p$-spaces with $p>1$.

3.2. **A particular example.** Apart from functional calculus, the following type of example has been used to construct Schauder multilpliers in various situations, e.g., [5, 9, 17, 22, 44].

**Definition 3.5.** Let $X=L^2=L^2(-\pi,\pi)$, $\beta\in(\frac14,\frac12)$. Define $\{\phi_n\}_{n\in\mathbb{N}}$ by

$$\phi_{2k}(t)=w_\beta(t)e^{ikt},\quad \phi_{2k+1}(t)=w_\beta(t)e^{-ikt},$$

where $k\in\mathbb{N}\cup\{0\}$, $t\in(-\pi,\pi)$ and

$$w_\beta(t)=\begin{cases}|t|^\beta, & |t|\in(0,\frac{\pi}{2}),\\ (\pi-|t|)^{-\beta}, & |t|\in[\frac{\pi}{2},\pi).\end{cases}$$

$\{\phi_n\}_{n\in\mathbb{N}}$ forms a Schauder basis of $L^2$, see Lemma A.3.

**Theorem 3.6.** *There exists a $g\in H^\infty(\mathbb{C}_+)$ such that the following holds. For every $\delta\in\left(0,\frac12\right)$ there exists $A\in\mathrm{Sect}(0)$ on $H=L^2(-\pi,\pi)$ with*

*(i)* $0\in\rho(A)$ *and* $\mathrm{dist}(\sigma(A),0)=2$,

*(ii)* $M(A,\phi)\le\frac{1}{\delta}M(\phi)$ *for all $\phi\in(0,\pi]$, where $M(\phi)$ only depends on $\phi$.*

*(iii) For all $\varepsilon>0$, $f\in H^\infty(\mathbb{C}_+)$, and some absolute constant $K_0$,*

$$\|(f\cdot e_\varepsilon)(A)\|\lesssim\tfrac{1}{\delta}\cdot\mathrm{Ei}(\varepsilon)^{1-K_0\delta^2}\cdot\|f\|_\infty. \tag{3.10}$$

*(iv) For $\varepsilon\in(0,\frac12)$,*

$$\|(g\cdot e_\varepsilon)(A)\|\gtrsim\tfrac{1}{\delta}\cdot|\log(\varepsilon)|^{1-\delta}. \tag{3.11}$$

*Proof.* Let $\beta = \frac{1}{2} - \frac{\delta}{4} \in (\frac{3}{8}, \frac{1}{2})$ and let $\{\phi_n\}_{n\in\mathbb{N}}$ denote the basis from Definition 3.5 and $\{\phi_n^*\}_{n\in\mathbb{N}}$ its dual basis, see Lemma A.3. By Lemma A.3 (i), $\kappa_\phi \lesssim \frac{1}{1-2\beta} = \frac{2}{\delta}$. W.r.t. $\{\phi_n\}_{n\in\mathbb{N}}$, we consider the multiplication operator $A = \mathcal{M}_\lambda$ on $L^2(-\pi,\pi)$, where $\lambda_n = 2^n$. By Theorem 3.1, (i) and (ii) follow.
(iii) follows by (3.8) from Theorem 3.3.

To show (iv) we choose $x(t) = |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|)$ and $y(t) = (\pi - |t|)^{-\beta}\mathbb{1}_{(\frac{\pi}{2},\pi)}(|t|)$. By Lemma A.3 (iii), we have that for $x = \sum_n x_n\phi_n$ and $y = \sum_n y_n\phi_n^*$, the coefficients $x_n$ and $y_n$ are real and that

$$x_{2k} = x_{2k+1} \sim \tfrac{k^{-1+2\beta}}{1-2\beta} \quad \text{and} \quad y_{2k} = y_{2k+1} = (-1)^k 2\pi \cdot x_{2k}. \tag{3.12}$$

Thus, by setting $\mu_{2n} = \mu_{2n+1} = (-1)^n$ for all $n \in \mathbb{N}$, we conclude by using that $\langle \phi_n, \phi_m^*\rangle = \delta_{nm}$,

$$\begin{aligned} |\langle \mathcal{M}_\mu \mathcal{M}_{e^{-\lambda_n\varepsilon}} x, y\rangle| &= 2\pi \sum_{n\in\mathbb{N}} e^{-\lambda_n\varepsilon}|x_n|^2 \\ &\gtrsim \tfrac{1}{(1-2\beta)^2} \sum_{k\in\mathbb{N}} (e^{\lambda_{2k}\varepsilon} + e^{\lambda_{2k+1}\varepsilon})k^{-2+4\beta} \\ &\gtrsim \tfrac{1}{(1-2\beta)^2}|\log(\varepsilon)|^{-1+4\beta}, \end{aligned} \tag{3.13}$$

for $\varepsilon < \frac{1}{2}$, where we have used (3.12) and Lemma A.1. Since $\|x\| \cdot \|y\| \sim \frac{1}{1-2\beta}$, and $2 - 4\beta = \delta$,

$$\|\mathcal{M}_\mu \mathcal{M}_{e^{-\lambda_n\varepsilon}}\| \gtrsim \frac{1}{\delta}|\log(\varepsilon)|^{1-\delta}, \quad \varepsilon \in (0, \tfrac{1}{2}). \tag{3.14}$$

Since $(\lambda_n)$ is an interpolating sequence, we can find $g \in H^\infty(\mathbb{C}_-)$ such that $g(\lambda_n) = \mu_n$ for all $n \in \mathbb{N}$. Thus, $g(A) = \mathcal{M}_\mu$ and (3.11) follows. ☐

The example shows that estimate (2.17) in Theorem 2.10 is sharp in $M(A,\phi)$ and $\varepsilon$ as $\delta \to 0^+$.

**Corollary 3.7.** *Let $X$ be a Banach space, $0 < \omega < \phi < \frac{\pi}{2}$. Then, there exists a $K$ depending only on $\phi$ such that*

$$\sup\left\{\frac{\|(fe_\varepsilon)(A)\|}{M(A,\phi)\|f\|_\infty} : A \in \operatorname{Sect}(\omega) \text{ on } X, \operatorname{dist}(\sigma(A),0) \geq 1, f \in H^\infty(\mathbb{C}_+)\right\} > K|\log\varepsilon|, \tag{3.15}$$

*for all $\varepsilon < \frac{1}{2}$.*

*Remark* 3.8.
(1) As the examples are on Hilbert spaces, the sharpness from Corollary 3.7 even holds on Hilbert spaces. However, we point out that in Theorem 3.6 $M(A,\phi) \to \infty$ as $\delta \to 0^+$. Therefore, for fixed $M(A,\phi)$, the behavior in $\varepsilon \to 0^+$ could be better than $|\log\varepsilon|$. For a similar effect we refer to the question of the sharpness of Spijker's result on the *Kreiss-Matrix-Theorem*, see [36, 37] and the recent contribution by Nikolski [30].
(2) In [41, Theorem 2.1, Remark 2.2], it is shown that estimate (2.14) is indeed sharp in $\varepsilon$ and $\sigma$ on general Banach spaces. Furthermore, Vitse [41, Theorem 2.3 and Remark 2.4] states that for every Hilbert space and every $\delta \in (0,1)$, one can find a sectorial operator $A$ with angle less than $\frac{\pi}{2}$ such that

$$\sup\left\{\|g(A)\| : g \in H^\infty[\varepsilon,\sigma], \|g\|_{\infty,\mathbb{C}_+} \leq 1\right\} \geq a\log\left(\frac{e\sigma}{\varepsilon}\right)^\delta, \tag{3.16}$$

where $a$ depends only on $M(A,\frac{\pi}{2})$. Therefore, item (iii) of Theorem 3.6 and Corollary 3.7 can be seen as a version for $0 \in \rho(A)$ and $\sigma = \infty$. However, Theorem 3.6(iv) shows that the behavior of $\|(fe_\varepsilon)(A)\|$ is indeed better than $|\log(\varepsilon)|$. We remark that Vitse's result, [41, Theorem 2.3] is stated for Banach spaces which *uniformly contain uniformly complemented copies of* $\ell^2$, which is more general than for Hilbert spaces.

## 4. Square function estimates improve the situation

The following notion characterizes bounded $H^\infty$-calculus on Hilbert spaces. It was already used in the early work of McIntosh, [27] and has been investigated intensively since then.

**Definition 4.1.** Let $A \in \mathrm{Sect}(\omega)$ on the Banach space $X$. We say that $A$ satisfies *square function estimates* if there exists $\psi \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$, $\phi > \omega$ and $K_\psi > 0$ such that

$$\int_0^\infty \|\psi(tA)x\|^2 \, \frac{dt}{t} \leq K_\psi^2 \|x\|^2, \quad \forall x \in X. \tag{4.1}$$

The property of having square functions estimates does not rely on the particular function $\psi$. In fact, for $\psi, \eta \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$

$$\exists K > 0 \; \forall h \in H^\infty(\Sigma_\phi): \quad \int_0^\infty \|(\psi_t h)(A)x\|^2 \, \frac{dt}{t} \leq K^2 \|h\|_{\infty,\phi}^2 \int_0^\infty \|\eta_t(A)x\|^2 \, \frac{dt}{t}, \tag{4.2}$$

where $\psi_t(z) = \psi(tz)$ and $\eta_t(z) = \eta(tz)$. We remark that for $\psi = \eta$, $K$ can be chosen only depending on $M(A, \phi)$. The result can be found in [2, Proposition E] for Hilbert spaces, but also holds for general Banach spaces as pointed out in [16, Satz 2.1]. The following result goes back to McIntosh in his early work on $H^\infty$-calculus, [27] and can also be found in [18, Theorem 7.3.1].

**Theorem 4.2** (McIntosh '86)**.** *Let $X$ be a Hilbert space, $A \in \mathrm{Sect}(\omega)$, densely defined and with dense range. Then, the following assertions are equivalent.*

*(1) The $H^\infty(\Sigma_\mu)$-calculus for $A$ is bounded for some (all) $\mu \in (\omega, \pi)$.*
*(2) $A$ and $A^*$ satisfy square function estimates.*

Note that on a Hilbert space, $\overline{D(A)} = X$ follows from sectorality, see [18, Proposition 2.1.1].
Le Merdy showed in [25, Theorem 5.2] that having square function estimates for only $A$ or $A^*$ is not sufficient to get a bounded calculus. However, we will show that the validity of single square function estimates always yields an improved growth of $\|(fe_\varepsilon)(A)\|$ near zero. Roughly speaking, having 'half of the assumptions' in McIntosh's result indeed interpolates the general logarithmic behavior of $\|(fe_\varepsilon)(A)\|$.

**Theorem 4.3.** *Let $\omega < \phi < \frac{\pi}{2}$ and $A \in \mathrm{Sect}(\omega)$ be densely defined on the Banach space $X$. Assume that*

- $0 \in \rho(A)$ *and that*
- *$A$ satisfies square function estimates.*

*Then for every $\kappa \in (0,1)$ there exists $C = C(\kappa, M(A,\phi), \cos(\phi)) > 0$ such that for all $\varepsilon > 0$ and for $f \in H^\infty(\Sigma_\phi)$,*

$$\|(fe_\varepsilon)(A)\| \leq C K_\psi \cdot \left[\mathrm{Ei}\left(\frac{\kappa\varepsilon\cos\phi}{\|A^{-1}\|}\right)\right]^{\frac{1}{2}} \cdot \|f\|_{\infty,\phi}, \tag{4.3}$$

*where $K_\psi$ denotes the constant in (4.1) for $\psi(z) = z^{\frac{1}{2}} e^{-z}$.*

*Proof.* Let $\psi(z) = ze^{-z}$. Since $\sqrt{\psi} \in H_0^\infty(\Sigma_\phi)$, we have by (4.2) that

$$\int_0^\infty \|(fe_\varepsilon\sqrt{\psi_t})(A)x\|^2 \,\frac{dt}{t} \leq K^2 \,\|fe_\varepsilon\|_{\infty,\phi}^2 \cdot \int_0^\infty \|(\sqrt{\psi_t})(A)x\|^2 \,\frac{dt}{t}, \tag{4.4}$$

where $K > 0$ only depends on $M(A,\phi)$. The integral on the right-hand side is finite because $A$ satisfies square function estimates (for $\sqrt{\psi}$). It is easy to see that $\int_0^\infty \psi_t(z)\,\frac{dt}{t} = 1$ for $z \in \Sigma_\phi$, and applying the Convergence Lemma, [18, Proposition 5.1.4], yields $y = \int_0^\infty \psi_t(A)y\frac{dt}{t}$ for $y \in X$. Thus,

$$\begin{aligned}
\|(fe_\varepsilon)(A)x\| &= \left\|\int_0^\infty (fe_\varepsilon\psi_t)(A)x \,\frac{dt}{t}\right\| \\
&\leq \int_0^\infty \left\|\left(e_{\frac{\varepsilon}{2}}\sqrt{\psi_t}\right)(A)\left(fe_{\frac{\varepsilon}{2}}\sqrt{\psi_t}\right)(A)x\right\| \frac{dt}{t} \\
&\leq \left(\int_0^\infty \|\left(e_{\frac{\varepsilon}{2}}\sqrt{\psi_t}\right)(A)\|^2\frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^\infty \|\left(fe_{\frac{\varepsilon}{2}}\sqrt{\psi_t}\right)(A)x\|^2\frac{dt}{t}\right)^{\frac{1}{2}}. \qquad (4.5)
\end{aligned}$$

In the last step we used that $t \mapsto (e_{\frac{\varepsilon}{2}}\sqrt{\psi_t})(A)$ is continuous in the operator norm which makes the first integral exist. In fact, $e^{-\frac{\varepsilon z}{2}}\sqrt{\psi_t(z)} = (zt)^{\frac{1}{2}}e^{-z\frac{t+\varepsilon}{2}} \in H_0^\infty(\Sigma_\phi)$, and hence by the functional calculus for sectorial operators,

$$\left[e^{-\frac{\varepsilon z}{2}}\sqrt{\psi_t(z)}\right](A) = t^{\frac{1}{2}}A^{\frac{1}{2}}T\left(\frac{t+\varepsilon}{2}\right). \tag{4.6}$$

For $s > 0$ we have that $A^{\frac{1}{2}}T(s) = A^{-\frac{1}{2}}AT(s) = A^{-\frac{1}{2}}\frac{\partial}{\partial s}T(s)$. Since $s \mapsto T(s)$ is $C^\infty(\mathbb{R}_+, \mathcal{B}(X))$ for analytic semigroups and $A^{-\frac{1}{2}} \in \mathcal{B}(X)$ as $0 \in \rho(A)$, we get indeed that $t \mapsto (e_{\frac{\varepsilon}{2}}\sqrt{\psi_t})(A)$ is continuous in the operator norm.
By (4.4) we can estimate the second integral in (4.5) and find

$$\|(fe_\varepsilon)(A)x\| \leq \left(\int_0^\infty \|\left(e_{\frac{\varepsilon}{2}}\sqrt{\psi_t}\right)(A)\|^2\frac{dt}{t}\right)^{\frac{1}{2}} \cdot K \cdot \|f\|_{\infty,\phi} \cdot K_{\sqrt{\psi}}\|x\|. \tag{4.7}$$

Hence, it remains to study the first term in (4.7). By (4.6) and Lemma A.5

$$\begin{aligned}
\int_0^\infty \|\left(e^{-\frac{\varepsilon}{2}\cdot}\sqrt{\psi_t}\right)(A)\|^2\frac{dt}{t} &= \int_{\frac{\varepsilon}{2}}^\infty \|A^{\frac{1}{2}}T(t)\|^2 \,dt \\
&\leq \tilde{C}^2 \int_{\frac{\varepsilon}{2}}^\infty t^{-1}e^{-2tR\kappa\cos\omega} \,dt \\
&= \tilde{C}^2 \cdot \operatorname{Ei}(\kappa\varepsilon R\cos\phi), \qquad (4.8)
\end{aligned}$$

for $\kappa \in (0,1)$, $R = \frac{1}{\|A^{-1}\|}$ and $\tilde{C} = C_{\frac{1}{2},\kappa}M(A,\phi)(\cos\phi)^{-\frac{1}{2}} > 0$, see Lemma A.5. □

*Remark* 4.4. In [12], Galé, Miana and Yakubovich draw a remarkable connection between the $H^\infty$-calculus for sectorial operators and the theory of functional models for Hilbert space operators. They prove (however, without using this connection) a, as they call it, *logarithmic gap* between the Hilbert space $H$ and $H_A$. $H_A$ is the space of elements of $H$ such that

$$\|x\|_A^2 = \int_0^\infty \|\psi(tA)\|^2\frac{dt}{t} < \infty,$$

for some $\psi \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$. Loosely speaking $H_A$ is the space on which one has square function estimates. From Theorem 4.2 it is clear that the $H^\infty(\Sigma_\phi)$-calculus is bounded if and only if the norm $\|\cdot\|_A$ is equivalent to the norm of the space $H$. The *logarithmic gap* refers to the result that for all $r > \frac{1}{2}$ there exist $c_r > 0$ such that

$$c_r^{-1}\|\Lambda_1(A)^{-r}x\| \leq \|x\|_A \leq c_r\|\Lambda_1(A)^r x\|,$$

for all $x \in \Lambda_1(A)^{-r}H$, where $\Lambda_1(z) = Log(z) + 2\pi i$ (here, $Log$ denotes the principle branch of the logarithm) and where $\Lambda_1^{-r}(A)H$ has to be interpreted as a subspace of $H$, see Theorem 2.1 in [12]. We learned from D. Yakubovich that it seems that this result can be used to derive estimates of $\|(fe_\varepsilon)(A)\|$ of the form in (1.2), which are slightly weaker than our results presented here. However, it is an interesting question how the techniques can be related. This is subject to future work.

The following theorem proves that the result in Theorem 4.3 is essentially sharp.

**Theorem 4.5.** *There exists a Hilbert space $X$ and $g \in H^\infty(\mathbb{C}_+)$ such that for any $\delta \in (0, \frac{1}{2})$ there exists a $A \in \mathrm{Sect}(0)$ on $X$ with*

*(i)* $0 \in \rho(A)$,

*(ii)* $A^*$ *satisfies square function estimates,*

*(iii)* *for some* $\tilde{C} > 0$,

$$\|(ge_\varepsilon\cdot)(A)\| \geq \tilde{C} \cdot |\log(\varepsilon)|^{\frac{1}{2}-\delta}, \qquad \varepsilon \in (0, \tfrac{1}{2}). \tag{4.9}$$

*(iv)* *For all* $\varepsilon > 0$ *and* $f \in H^\infty(\mathbb{C}_+)$

$$\|(f \cdot e_\varepsilon)(A)\| \leq c_\delta \cdot \mathrm{Ei}(\varepsilon)^{\frac{1}{2}-\frac{\delta}{6}} \cdot \|f\|_\infty, \tag{4.10}$$

*where* $c_\delta$ *depends only on* $\delta$.

*Proof.* The example is a multiplication operator w.r.t. to a Schauder basis. It is well-known and easy to see that if the basis is Besselian, the multiplication operator $\mathcal{M}_{\{2^n\}}$ satisfies square function estimates, see e.g. [25, Proof of Theorem 5.2]. We consider a basis $\{\psi_n\}_{n\in\mathbb{N}}$ such that the dual basis $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, i.e.

$$\forall y = \sum_{n\in\mathbb{N}} y_n\psi_n^* \in X \;\Rightarrow\; (y_n) \in \ell^2(\mathbb{N}). \tag{4.11}$$

Hence, $A^* = \mathcal{M}_{\{2^n\}}$ w.r.t. to $\{\psi_n^*\}_{n\in\mathbb{N}}$ satisfies square function estimates.
In fact, let $X = L^2(-\pi, \pi)$, $\beta \in (\frac{1}{3}, \frac{1}{2})$, and define $\psi_n$ by

$$\psi_{2k}(t) = |t|^\beta e^{ikt}, \quad \psi_{2k+1}(t) = |t|^\beta e^{-ikt}, \; k \in \mathbb{N}_0,$$

see Lemma A.4. The dual basis $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, see [35, Example 11.2]. Further note that $\psi_n(t) = \phi_n(t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with $\phi_n$ from Definition 3.5. Let $x(t) = |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|) \in X$. Since $x(t) = 0$ for $t \in (\frac{\pi}{2}, \pi)$ we get the same coefficients $x_n$ w.r.t to $\{\psi_n\}$ as for the basis $\{\phi_n\}$. Thus, by Lemma A.3 (iii), the coefficients are positive and $x_{2n} = x_{2n+1} \sim \frac{n^{-1+2\beta}}{1-2\beta}$. Furthermore, let

$$y(t) = |t|^{-\beta}(\pi - |t|)^{-\beta}\mathbb{1}_{(\frac{\pi}{2},\pi)},$$

which lies in $L^2(-\pi, \pi)$ and has, w.r.t. $\left\{\psi_{n,\beta}^*\right\}$, the coefficients $y_n$,

$$y_{2k} = \langle y, \psi_{2k,\beta}\rangle = \frac{1}{2\pi}\int_{\frac{\pi}{2}<|t|<\pi} (\pi - |t|)^{-\beta}e^{ikt}dt = \frac{(-1)^n}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^{-\beta}e^{-ikt}dt,$$

and $y_{2k+1} = y_{2k}$ which can be seen easily. By Lemma A.2, we conclude that $|y_{2k}| = |y_{2k+1}| \sim (1-\beta)^{-1}k^{-1+\beta} \sim k^{-1+\beta}$. Thus, because $(2^n)$ is interpolating,we find $g \in H^\infty(\mathbb{C}_+)$ such that $g(2^n) = \operatorname{sgn}(y_n)$ for all $n \in \mathbb{N}$, and we get

$$\begin{aligned} \langle (g \cdot e^{-\varepsilon\cdot})(A)x, y\rangle &= \sum_{n\in\mathbb{N}} g(2^n) e^{-2^n\varepsilon} x_n y_n \\ &= \sum_{n\in\mathbb{N}} e^{-2^n\varepsilon} |x_n y_n| \qquad (4.12)\\ &\gtrsim \tfrac{1}{1-2\beta} \sum_{n\in\mathbb{N}} (e^{-2^n\varepsilon} + e^{-2^{n+1}\varepsilon}) n^{-2+3\beta} \gtrsim \tfrac{1}{1-2\beta} |\log(\varepsilon)|^{-1+3\beta}, \qquad (4.13)\end{aligned}$$

where in the last step we used Lemma A.1. Since $\|x\|_{L^2} \cdot \|y\|_{L^2} \sim \frac{1}{1-2\beta}$ and by defining $\beta = \frac{1}{2} - \frac{\delta}{3}$ the assertion follows.
To show (4.10) let $x, y \in H$ and $x = \sum x_n \psi_n$, $y = \sum y_n \psi_n^*$. For $f \in H^\infty(\mathbb{C}_+)$,

$$\begin{aligned} \langle (f e_\varepsilon)(A)x, y\rangle &= \langle \sum_{n\in\mathbb{N}} f(2^n) e^{-2^n\varepsilon} x_n \psi_n, \sum_{n\in\mathbb{N}} y_n \psi_n^*\rangle \\ &= \sum_{n\in\mathbb{N}} f(2^n) e^{-2^n\varepsilon} x_n y_n, \qquad (4.14)\end{aligned}$$

where we used that $\langle \psi_n, \psi_m^*\rangle = \delta_{nm}$. By the Cauchy-Schwarz inequality

$$|\langle (f e_\varepsilon)(A)x, y\rangle| \le \|f\|_\infty \cdot \|(e^{-2^n\varepsilon/2} x_n)\|_2 \cdot \|(e^{-2^n\varepsilon/2} y_n)\|_2.$$

Since $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, (4.11) and the uniform boundedness principle imply that there exists a constant $C_\beta$ such that $\|(y_n)\|_{\ell^2} \le C_\beta \|y\|$ for all $y \in X$. Therefore,

$$|\langle (f e_\varepsilon)(A)x, y\rangle| \le C_\beta \cdot \|f\|_\infty \cdot \|(e^{-2^{n-1}\varepsilon} x_n)\|_2 \cdot \|y\|_{L^2}. \qquad (4.15)$$

By (A.13) in Lemma A.4,

$$|\langle (f e_\varepsilon)(A)x, y\rangle| \lesssim C_\beta K_\beta \cdot \|f\|_\infty \cdot \operatorname{Ei}(\varepsilon)^{\frac{1+2\beta}{4}} \cdot \|x\|_{L^2} \cdot \|y\|_{L^2}.$$

Substituting $\beta = \frac{1}{2} - \frac{\delta}{3}$ and $c_\delta := C_\beta K_\beta$ yields (4.10). □

## 5. Discussion and Outlook

5.1. **Comparison with a result of Haase & Rozendaal.** In [21] Haase and Rozendaal derived a result of the type of Theorem 2.3 for Hilbert spaces, but for general bounded, not necessarily analytic, $C_0$-semigroups. We devote this subsection to compare the results, in particular the dependence on the semigroup bound and the sectorality constant, respectively. We define the right half-plane $\mathrm{R}_\delta = \{z \in \mathbb{C} : \Re z > \delta\}$. Using transference principles developed by Haase in [20] the following result was proved in [21].

**Theorem 5.1** (Haase, Rozendaal, Corollary 3.10 in [21])**.** *Let $H$ be a Hilbert space and $-A$ generate a bounded semigroup $T$ on $H$ and define $B = \sup_{t>0} \|T(t)\|$. Then, there exists an absolute constant $c > 0$ such that for all $\varepsilon, \delta > 0$ the following holds.*
*For $f \in H^\infty(\mathrm{R}_\delta)$, the operator $(f e_\delta)(A) = f(A)T(\delta)$ is bounded and*

$$\|(f e_\varepsilon)(A)\| \le B^2 \cdot \eta(\delta, \varepsilon) \cdot \|f\|_{\infty, \mathrm{R}_\delta}, \qquad (5.1)$$

*where*

$$\eta(\delta, \varepsilon) = \begin{cases} c|\log(\varepsilon\delta)|, & \delta\varepsilon \le \frac{1}{2}, \\ 2c, & \delta\varepsilon > \frac{1}{2}. \end{cases}$$

We can now compare Theorems 2.3 and 5.1 by setting $r_0 = \delta$. Then $\Omega_{\phi,\delta} \subset \mathrm{R}_\delta$ for all $\phi \in (0, \frac{\pi}{2}]$ and thus, for functions $f \in H^\infty(\mathrm{R}_\delta)$, we have $\|f\|_{\infty,\Omega_{\phi,\delta}} \leq \|f\|_{\infty,\mathrm{R}_\delta}$. Hence, Theorem 2.3 yields

$$\|(fe_\varepsilon)(A)\| \leq M(A,\phi) \cdot b(\varepsilon,\delta,\phi) \cdot \|f\|_{\infty,\mathrm{R}_\delta}, \tag{5.2}$$

for all $\phi \in (\omega_A, \frac{\pi}{2})$ and $f \in H^\infty(\mathrm{R}_\delta)$, where

$$b(\varepsilon,\delta,\phi) \sim \begin{cases} |\log(\varepsilon\delta\cos\phi)|, & \varepsilon\delta < \frac{1}{2}, \\ |\log\frac{\cos\phi}{2}|, & \varepsilon\delta \geq \frac{1}{2}. \end{cases}$$

Let us collect the key observations when comparing (5.1) and (5.2).

(1) We see that the square of the semigroup bound $B$ gets replaced by the sectorality constant $M(A,\phi)$ in our result.
(2) Our estimate depends on another parameter $\phi$ that accounts for the fact that the spectrum is truly lying in a sector rather than the half-plane. Taking the infimum over all $\phi \in (\omega_A, \frac{\pi}{2})$ in (5.2) yields an optimized estimate. However, then the constant dependence on $M(A,\phi)$ becomes unclear. See also Theorem 2.6.
(3) The dependence on $\phi$ also explains how the estimate explodes when considering $A$'s with sectorality angle $\omega_A$ tending to $\frac{\pi}{2}$. However, one can cover this behavior in terms of the constant $M = M(A, \frac{\pi}{2})$: Taking $\phi = \arccos\frac{1}{2M}$, we get by (2.12) that $M(A,\phi) \leq 2M$ and thus (5.2) becomes

$$\|(fe_\varepsilon)(A)\| \leq M \cdot b(\varepsilon,\delta,\arccos\tfrac{1}{2M}) \cdot \|f\|_{\infty,\mathrm{R}_\delta}. \tag{5.3}$$

Therefore, we get an $M$-dependence of the form $\mathcal{O}(M(\log(M)+1))$.
(4) By Theorem 2.8, the semigroup bound of $e_t(A)$ is also of order $\mathcal{O}(M(\log(M)+1))$. Whether $B \sim M(\log(M)+1))$ in general is still an open problem, see also [41, Remark 1.3]. However, it is easy to see that, in general, $M(A,\pi) \leq B$. Therefore,

$$M(A,\pi) \leq B \lesssim M(\log(M)+1). \tag{5.4}$$

5.2. **Besov calculus.** We briefly introduce the following homogenous Besov space and refer to [41, Section 1.7] and the references therein for details, see also [20]. The notation follows [41]. The space $B^0_{\infty,1}$ can be defined as the space of holomorphic functions $f$ on $\mathbb{C}_+$ such that

$$\|f\|_B := \|f\|_\infty + \int_0^\infty \|f'(t+i\cdot)\|_\infty dt < \infty.$$

Clearly, $B^0_{\infty,1}$, equipped with the above norm, is continuously embedded in $H^\infty(\mathbb{C}_+)$. Moreover, $\cup_{0<\varepsilon<\sigma} H^\infty[\varepsilon,\sigma]$, see Section 2.2, lies dense in $B^0_{\infty,1}$ and the following norm is equivalent to $\|\cdot\|_B$, see [41, Theorem A.1],

$$\|f\|_{*B} = |f(\infty)| + \sum_{k\in\mathbb{Z}} \|f * \hat{\phi_k}\|_\infty,$$

where $\phi_k$ is the continuous, triangular-shaped function that is linear on the intervals $[2^{k-1},2^k]$ and $[2^k,2^{k+1}]$, vanishes outside $[2^{k-1},2^{k+1}]$, and such that $\phi_k(2^k) = 1$. Thus, $\{\phi_k\}_{k\in\mathbb{N}}$ is a partition of unity with $\sum_{k\in\mathbb{Z}} \phi_k \equiv 1$ locally finite on $(0,\infty)$, see [20, 41]. Obviously, the (inverse) Fourier-Laplace transform of $f * \hat{\phi_k}$ has support in $[2^{k-1},2^{k+1}]$, hence, $f * \hat{\phi_k} \in H^\infty[2^{k-1},2^{k+1}]$. Therefore, it follows directly from Theorem 2.8 that for $f \in B^0_{\infty,1}$

$$\|(f * \hat{\phi_k})(A)\| \leq cM(\log(M)+1) \cdot 4 \cdot \|f * \hat{\phi_k}\|_\infty, \tag{5.5}$$

where $c$ is an absolute consant and $M = M(A, \frac{\pi}{2})$. The following Theorem is a slight improvement of Theorem 1.7 in [41], see also [20, Corollary 5.5].

**Theorem 5.2.** *Let $A \in \mathrm{Sect}(\omega)$ on the Banach space $X$ with $\omega < \frac{\pi}{2}$. Let $M = M(A, \frac{\pi}{2})$. Then,*

$$\|f(A)\| \leq cM(\log(M) + 1)\|f\|_{*B},$$

*for all $f \in B^0_{\infty,1}$, where $c > 0$ is an absolute constant. Thus, the $B^0_{\infty,1}$-calculus is bounded.*

*Proof.* It is easy to see that for $g \in H^\infty[\varepsilon, \sigma]$ with $0 < \varepsilon < \sigma < \infty$,

$$g(z) = \sum_{k\in\mathbb{Z}} (\hat{\phi_k} * g)(z), \quad z \in \mathbb{C}_+ \tag{5.6}$$

because the inverse Fourier transform of $g$ has compact support. Let $f \in B^0_{\infty,1}$. Since $\cup_{0<\varepsilon<\sigma} H^\infty[\varepsilon, \sigma]$ is dense in $B^0_{\infty,1}$, see [41], we find a sequence $g_n \in H^\infty[\frac{1}{n}, n]$ such that $g_n \to (f - f(\infty))$ in $B^0_{\infty,1}$ as $n \to \infty$. Thus, $g_n \to f - f(\infty)$ in $\|\cdot\|_\infty$ and $\|\cdot\|_{*B}$. Therefore, by (5.6) and the fact that $\hat{\phi_k} * (f - f(\infty)) = \hat{\phi_k} * f$ we have that

$$f(z) = f(\infty) + \sum_{k\in\mathbb{Z}} (\hat{\phi_k} * f)(z), \quad z \in \mathbb{C}_+. \tag{5.7}$$

Since $\| \sum_{|k|\leq N} (\hat{\phi_k} * f)\|_\infty \leq \|f\|_{*B}$ for any $N \in \mathbb{N}$, the Convergence Lemma, [18, Proposition 5.1.1], implies

$$f(A) = f(\infty) + \sum_{k\in\mathbb{Z}} (\hat{\phi_k} * f)(A)$$

and the assertion follows from (5.5). □

*Remark* 5.3.

1. In [41, Theorem 1.7] Vitse already showed that the $B^0_{\infty,1}$-calculus is bounded where the bound of the calculus was estimated by $31M^3$. Like in our proof, she derived the result from an $H^\infty$-calculus estimate for $H^\infty[\varepsilon, \sigma]$.
2. In [20] Haase showed that for (polynomially) bounded semigroups on Hilbert spaces, one can consider more general homogenous Besov spaces $B^s_{\infty,1}$, $s \geq 0$. $B^s_{\infty,1}$ consists of functions $f$, holomorphic on $\mathbb{C}_+$, and such that $\lim_{z\to\infty} f(z)$ exists and

$$\|f\|_{*B^s} := |f(\infty)| + \sum_{k<0} \|\hat{\phi_k} * f\|_\infty + \sum_{k\geq 0} 2^{ks}\|\hat{\phi_k} * f\|_\infty < \infty.$$

   It is easy to see that Theorem 5.2 holds for $B^s_{\infty,1}$ with the analogous proof as for $B^0_{\infty,1}$.

5.3. **Final remarks and outlook.** Let us conclude by mentioning the well-known relation between analytic semigroup generators and Tadmor-Ritt operators, see e.g. [18, 40, 41]. A bounded operator $T$ is called *Tadmor-Ritt* if its spectrum lies in the closed unit disc and its resolvent satisfies that

$$C(T) := \sup_{|z|>1} \|(z - 1)R(z, T)\| < \infty,$$

see [32, 38]. Such operators are of interest in studying stability of numerical schemes. Moreover, they can be seen as the discrete counterpart of sectorial operators. In [39, 40] Vitse discussed $H^\infty$- and Besov space functional calculi for Tadmor-Ritt operators with similar ideas as in the continuous case, [41]. It seems natural to use discrete versions of the techniques used in this paper to improve these results. This is subject to on-going work, [34].

We point out that in Theorems 2.3 and 2.10 the operator $A$ need not be densely defined. Thus, in the view of analytic semigroups, $e_t(A)$ need not be strongly continuous at 0, see [18, Chapter 3.3].
Looking back to Propositions 1.1 and 2.2 which served as a starting point to study $\|(fe_\varepsilon)\|$ to quantify the (un)boundedness, we can ask ourselves which other functions $g_\varepsilon$ with $g_\varepsilon \mapsto 1$ as $\varepsilon \to 0$ can be studied in order to characterize a bounded calculus. Moreover, one could consider $g_\varepsilon(z) = z^\varepsilon e^{-\varepsilon z}$ which yields that $fg_\varepsilon \in H_0^\infty(\Sigma_\delta)$ for $f \in H^\infty(\Sigma_\delta)$.
An interesting question is how Theorem 4.3 generalizes to general Banach spaces. As Theorem 4.2 is not true on general Banach spaces, one has to use generalized square function estimates to characterize bounded $H^\infty$-calculus then, see e.g. [7, 16, 24]. This is subject to future work.

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## Appendix A.

**Lemma A.1** (Growth Lemma). *Let $b > 1$ and $\gamma < 0$.*

*(1) For $0 < \varepsilon < \frac{1}{b}$,*

$$e^{-1}F_\gamma(\varepsilon, b) \le \sum_{n=1}^{\infty} n^\gamma e^{-b^n\varepsilon} \le F_\gamma(\varepsilon, b) + 1 + \tfrac{\mathrm{Ei}(1)}{\log(b)}, \tag{A.1}$$

*where*

$$F_\gamma(\varepsilon, b) = \begin{cases} \frac{\log(1/\varepsilon)^{1+\gamma} - \log(b)^{1+\gamma}}{\log(b)^{1+\gamma}(1+\gamma)}, & \gamma \neq -1, \\ \log\log(1/\varepsilon) - \log\log(b), & \gamma = -1. \end{cases}$$

*Moreover, for all $\gamma_0 \in (-1, 0)$ there exists $C_{\gamma_0,b} > 0$ such that*

$$\forall \gamma \in (\gamma_0, 0): \quad F_\gamma(\varepsilon, b) \ge C_{\gamma_0,b} \log\left(\tfrac{1}{\varepsilon}\right)^{1+\gamma}. \tag{A.2}$$

*(2) For all $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} e^{-b^n\varepsilon} \le \frac{\mathrm{Ei}(\varepsilon)}{\log(b)}.$$

*Proof.* We estimate $\int_1^\infty x^\gamma e^{-b^x\varepsilon} dx$. Substitute $y = b^x\varepsilon$, thus, $x = \frac{\log(y/\varepsilon)}{\log(b)}$,

$$\begin{aligned} \int_1^\infty x^\gamma e^{-b^x\varepsilon}\, dx &= \frac{1}{\log(b)^{1+\gamma}} \int_{\varepsilon b}^\infty \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy \\ &= \frac{1}{\log(b)^{1+\gamma}} \Big( \int_{\varepsilon b}^1 \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy + \underbrace{\int_1^\infty \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy}_{\le \log(\frac{1}{\varepsilon})^\gamma \mathrm{Ei}(1) < \log(b)^\gamma \mathrm{Ei}(1)} \Big). \end{aligned}$$

Because $e^{-1} \leq e^{-y} \leq 1$ for $y \in (\varepsilon b, 1)$ and since the primitive of $\frac{\log(y/\varepsilon)^\gamma}{y}$ is

$$\begin{cases} \frac{(\log(y/\varepsilon))^{1+\gamma}}{1+\gamma}, & \gamma \neq -1, \\ \log\log(y/\varepsilon), & \gamma = -1, \end{cases}$$

we get

$$e^{-1} F_\gamma(\varepsilon, b) \leq \int_1^\infty x^\gamma e^{-b^x \varepsilon}\, dx \leq F_\gamma(\varepsilon, b) + \tfrac{\mathrm{Ei}(1)}{\log(b)}.$$

Next we use the fact that for the decreasing, integrable function $f : [1,\infty) \to (0,\infty)$, $x \mapsto x^\gamma e^{-b^x\varepsilon}$ there holds

$$\int_1^\infty f(x)\, dx \leq \sum_{n=1}^\infty f(n) \leq f(1) + \int_1^\infty f(x)\, dx, \tag{A.3}$$

and so we conclude (A.1). (A.2) can be easily seen by the definition of $F_\gamma(\varepsilon, b)$. Finally, 2. follows by

$$\sum_{n=1}^\infty e^{-b^n\varepsilon} \leq \int_0^\infty e^{-b^x\varepsilon}\, dx = \frac{1}{\log(b)} \int_\varepsilon^\infty \frac{e^{-y}}{y}\, dy = \frac{\mathrm{Ei}(\varepsilon)}{\log(b)}.$$

□

**Lemma A.2.** *Let $\alpha \in (-1,1)$. Then, for all $n \in \mathbb{N}$,*

$$c_{n,\alpha} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^\alpha e^{int}\, dt = C_{1,\alpha} n^{-1-\alpha} + B_n, \tag{A.4}$$

*where $C_{1,\alpha} = -2\sin\left(\alpha\frac{\pi}{2}\right)\Gamma(\alpha+1)$, $|B_n| \leq C_2 n^{-1}$ and $C_2$ is an absolute constant. Moreover, $c_{n,\alpha} \in \mathbb{R}$ and for $\alpha \in (-1, 0]$,*

$$d_{3,\alpha} n^{-1-\alpha} \leq c_{n,\alpha} \leq d_{1,\alpha} n^{-1-\alpha}, \quad n \in \mathbb{N}, \tag{A.5}$$

*where $d_{k,\alpha} = 2\int_0^{\frac{k\pi}{2}} t^\alpha \cos t\, dt \sim \frac{1}{1+\alpha}$, for $k \in \{1,3\}$. If $\alpha \in (-1, -\frac{1}{2}]$, then $d_{3,\alpha} > 0$, hence $c_{n,\alpha} > 0$ for all $n \in \mathbb{N}$.*

*Proof.* By

$$c_{n,\alpha} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^\alpha e^{int} dt = 2\Re \int_0^{\frac{\pi}{2}} t^\alpha e^{int} dt,$$

it is clear that $c_{n,\alpha}$ is a real number and we can consider

$$\int_0^{\frac{\pi}{2}} t^\alpha e^{int} dt = n^{-1-\alpha} \int_0^{n\frac{\pi}{2}} t^\alpha e^{it} dt. \tag{A.6}$$

Consider the contour consisting of the lines segments $[\varepsilon, n\frac{\pi}{2}]$ and $i[\varepsilon, n\frac{\pi}{2}]$ connected via quarter circles with radii $n\frac{\pi}{2}$ and $\varepsilon$ respectively, orientated counterclockwise. Then, since $h(z) = z^\alpha e^{iz}$ is holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$,

(A.7)

$$\int_\varepsilon^{n\frac{\pi}{2}} h(t)dt = \int_\varepsilon^{n\frac{\pi}{2}} h(it)idt - i\int_0^{\frac{\pi}{2}} (n\frac{\pi}{2}e^{i\theta})^{\alpha+1} e^{in\frac{\pi}{2}e^{i\theta}} d\theta + i\int_0^{\frac{\pi}{2}} (\varepsilon e^{i\theta})^{\alpha+1} e^{i\varepsilon e^{i\theta}} d\theta.$$

The last two integrals can both be estimated using the fact that $|e^{ire^{i\theta}}| = e^{-r\sin\theta} \leq e^{-r\frac{2\theta}{\pi}}$ for $\theta \in [0, \frac{\pi}{2}]$, $r > 0$. This yields

$$\left| \int_0^{\frac{\pi}{2}} (re^{i\theta})^{\alpha+1} e^{ire^{i\theta}} d\theta \right| \leq \frac{\pi}{2} r^\alpha (1 - e^{-r}).$$

Therefore, the integral for $r=\varepsilon$ goes to zero as $\varepsilon\to 0^+$ because $\alpha>-1$. The integral for $r=n\frac{\pi}{2}$ can be estimated by $\left(\frac{\pi}{2}\right)^{\alpha+1}n^{\alpha}$. It remains to consider

$$\begin{aligned}\lim_{\varepsilon\to 0^+} i\int_{\varepsilon}^{n\frac{\pi}{2}} h(it)dt = i\int_0^{n\frac{\pi}{2}} h(it)dt &= e^{i(\alpha+1)\frac{\pi}{2}}\int_0^{n\frac{\pi}{2}} t^{\alpha}e^{-t}dt\\ &= e^{i(\alpha+1)\frac{\pi}{2}}\left[\Gamma(\alpha+1)-\int_{n\frac{\pi}{2}}^{\infty} t^{\alpha}e^{-t}dt\right].\end{aligned}$$

It is easily seen that there exists a constant $C$ such that $\int_n^{\infty} t^{\alpha}e^{-t}dt\le Cn^{\alpha}e^{-n}$ for all $\alpha\in(-1,1)$. Altogether we get by (A.6) and the estimates for the terms in (A.7) that

$$\int_0^{\frac{\pi}{2}} t^{\alpha}e^{int}dt = e^{i(\alpha+1)\frac{\pi}{2}}\Gamma(\alpha+1)n^{-1-\alpha}+B_{n,\alpha},$$

with $|B_{n,\alpha}|\le \frac{1}{n}\left[\left(\frac{\pi}{2}\right)^{\alpha+1}+Ce^{-n}\right]$. This yields (A.4).

To show (A.5) for $\alpha\in(-1,0)$, note that with (A.6)

$$c_{n,\alpha}=n^{-1-\alpha}2\int_0^{\frac{n\pi}{2}} t^{\alpha}\cos(t)dt.$$

We define $d_{n,\alpha}=2\int_0^{\frac{n\pi}{2}} t^{\alpha}\cos t dt$. It remains to show that $d_{3,\alpha}\le d_{n,\alpha}\le d_{1,\alpha}$ for all $n\in\mathbb{N}$. Since $t\mapsto t^{\alpha}$ is positive and decreasing on $(0,\infty)$, it follows by the periodicity of the cosine that for all for all $m\in\mathbb{N}_0$

(1) since $\cos(\frac{t\pi}{2})<0$ on $((4m+1),(4m+3))$, $d_{4m+1,\alpha}>d_{4m+2,\alpha}>d_{4m+3,\alpha}$,
(2) since $\cos(\frac{t\pi}{2})>0$ on $((4m+3),(4m+5))$, $d_{4m+3,\alpha}<d_{4m+4,\alpha}<d_{4m+5,\alpha}$,
(3) since $t\mapsto t^{\alpha}$ is decreasing, $d_{4m+5,\alpha}<d_{4m+1,\alpha}$ and $d_{4m+3,\alpha}<d_{4(m+1)+3,\alpha}$.

Inductively, this shows that $\max_n d_{n,\alpha}=d_{1,\alpha}$ and $\min_n d_{n,\alpha}=d_{3,\alpha}$. Finally we check that $d_{3,\alpha}>0$ if $\alpha\in(-1,-\frac{1}{2}]$,

$$\begin{aligned}d_{3,\alpha}&=\int_0^{\frac{3\pi}{2}} t^{\alpha}\cos(t)dt\ge\cos(1)\int_0^1 t^{\alpha}dt+\left(\frac{\pi}{2}\right)^{\alpha}\int_1^{\frac{\pi}{2}}\cos(t)dt+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} t^{\alpha}\cos(t)dt\\ &\ge 2\cos(1)+\frac{2}{\pi}(1-\sin(1))+\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\cos(t)}{\sqrt{t}}dt>0,\end{aligned}$$

where the last integral can be computed via a Fresnel integral and is approximately 0.0314. □

**Lemma A.3.** *Let $\beta\in(\frac{1}{4},\frac{1}{2})$, $X=L^2(-\pi,\pi)$ and $\{\phi_n\}_{n\in\mathbb{N}}\subset X$ as in Definition 3.5. The following assertions hold. (See (3.3) for the definitions of $m_\phi$ and $\kappa_\phi$.)*

*(i) $\{\phi_n\}_{n\in\mathbb{N}}$ forms a bounded Schauder basis of $X$ with $\kappa_\phi\sim\frac{1}{1-2\beta}$.*
*(ii) The family $\{\phi_n^*\}_{n\in\mathbb{N}}\subset X$ given by*

$$\phi_{2k}^*(t)=\tfrac{1}{2\pi w_\beta(t)}e^{ikt},\quad \phi_{2k+1}^*(t)=\tfrac{1}{2\pi w_\beta(t)}e^{-ikt},$$

*satisfies $\langle\phi_n^*,\phi_m\rangle_{L^2}=\delta_{nm}$ and forms a Schauder basis of $X$ with $\kappa_{\phi^*}\sim\frac{1}{1-2\beta}$. Here, $w_\beta$ is defined as in Definition 3.5.*
*(iii) The coefficients of $x(t)=|t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|)$, $x=\sum_n x_n\phi_n$ are positive and satisfy*

$$x_{2k}=x_{2k+1}\sim\frac{k^{-1+2\beta}}{1-2\beta},\qquad k\in\mathbb{N}. \tag{A.8}$$

*For the coefficients of* $y(t) = (\pi - |t|)^{-\beta}\mathbb{1}_{(\frac{\pi}{2},\pi)}(|t|)$, $y = \sum_n y_n \phi_n^*$, *we have that*

$$y_{2k} = (-1)^k 2\pi \cdot x_{2k}, \quad y_{2k+1} = (-1)^k 2\pi \cdot x_{2k+1}, \qquad k \in \mathbb{N}. \tag{A.9}$$

*Proof.* Lemma A.3 (i)-A.3 (ii) follow from [35, Example 11.2].
To see A.3 (iii) we point out that for all $x = \sum_k x_k \phi_{k,\beta} \in X$ there holds

$$x_n = \langle x, \phi_n^* \rangle_{L^2}, \qquad n \in \mathbb{N}. \tag{A.10}$$

Thus, for $x = (t \mapsto |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|))$, $k \in \mathbb{N}$,

$$x_{2k} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |t|^{-2\beta} e^{-ikt} dt = \frac{c_{k,-2\beta}}{2\pi}, \quad x_{2k+1} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |t|^{-2\beta} e^{ikt} dt = x_{2k}, \tag{A.11}$$

where $c_{k,-2\beta}$ are the coefficients from Lemma A.2. Since $-2\beta \in (-1, -\frac{1}{2})$, they are even positive and (A.8) follows. The assertion for $y$ follows similarly. □

**Lemma A.4.** *Let* $X = L^2 = L^2(-\pi,\pi)$, $\beta \in (\frac{1}{3}, \frac{1}{2})$. *Then* $\{\psi_n\}_{n\in\mathbb{N}}$ *defined by*

$$\psi_{2k}(t) = |t|^\beta e^{ikt}, \quad \psi_{2k+1}(t) = |t|^\beta e^{-ikt}, \ k \in \mathbb{N}_0,$$

*is a bounded Schauder basis. For* $x = \sum_{n\in\mathbb{N}} x_n \psi_n \in X$, *we have that* $\{x_n\} \in \ell^r$ *for* $r > \frac{2}{1-2\beta}$ *and*

$$\|(x_n)\|_r \lesssim \|x\|_{L^2} \cdot \|n^{-1+\beta}\|_q, \qquad \tfrac{1}{q} = \tfrac{1}{2} + \tfrac{1}{r}. \tag{A.12}$$

*Furthermore,*

$$\|(e^{-2^n\varepsilon} x_n)\|_{\ell^2} \lesssim K_\beta \cdot \mathrm{Ei}(\varepsilon)^{\frac{1+2\beta}{4}} \|x\|_{L^2}, \tag{A.13}$$

*with* $K_\beta = \|(n^{\beta-1})\|_{\frac{3-2\beta}{4}}$.

*Proof.* That $\{\psi_n\}$ is a Schauder basis can e.g. be found in [35, Example 11.2]. Let $w_\beta(t) = |t|^\beta$ on $(-\pi,\pi)$. Since $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of $L^2$ it follows that for $x = \sum_{n\in\mathbb{N}} x_n \psi_n \in X$,

$$x_{2k} = \frac{1}{2\pi}\langle x w_\beta^{-1}, e^{ik\cdot}\rangle_{L^2} = \mathcal{F}(x w_\beta^{-1})[k],$$

where $\mathcal{F}$ denotes the discrete Fourier transform. Thus,

$$x_{2k} = \left(\mathcal{F}(x) * \mathcal{F}(w_\beta^{-1})\right)[k]. \tag{A.14}$$

By $x \in L^2$, $\{\mathcal{F}(x)[n]\} \in \ell^2$. From [17, Proof of Theorem 2.4, p.861] (see also Lemma A.2) we have that

$$\int_{-\pi}^{\pi} |t|^{\gamma-1} e^{-int}\, dt = 2n^{-\gamma}\cos(\gamma\tfrac{\pi}{2})\Gamma(\gamma) + \mathcal{O}(\tfrac{1}{n}),$$

for $\gamma > 0$. Thus, with $\gamma = 1-\beta \in (\frac{1}{2}, \frac{2}{3})$, $\mathcal{F}(w_\beta^{-1})[n] \in \ell^q$ with $q > q_0 := \frac{1}{1-\beta}$ and

$$\|\mathcal{F}(w_\beta^{-1})[n]\|_{\ell^q} \lesssim \|(n^{-1+\beta})\|_{\ell^q}.$$

We use Young's inequality with $\frac{1}{2} + \frac{1}{q} = 1 + \frac{1}{r}$ and $q \in (q_0, 2)$ to estimate the right-hand-side of (A.14). Hence, $\{x_{2k}\} \in \ell^r$ for $r > \frac{2}{1-2\beta}$. Analogously, $\{x_{2k+1}\} \in \ell^r$. Eq. (A.12) then follows since the discrete Fourier transform is isometric from $L^2$ to $\ell^2$.

To show (A.13), we use Hölder's inequality and (A.12),

$$
\begin{aligned}
\|(e^{-2^{n-1}\varepsilon}x_n)\|_2^2 &= \|(e^{-2^n\varepsilon}|x_n|^2)\|_1 \\
&\leq \|(e^{-2^n\varepsilon})\|_{r_0'} \cdot \|(x_n)\|_{2r_0}^2 \\
&\lesssim \|(e^{-2^n\varepsilon})\|_{r_0'} \cdot \|(n^{-1+\beta})\|_q^2 \cdot \|x\|_{L^2}^2 \qquad \text{(A.15)}
\end{aligned}
$$

for $r_0' = (1-\frac{1}{r_0})^{-1} = \frac{2}{1+2\beta}$ and $\frac{1}{q} = \frac{1}{2} + \frac{1}{2r_0} = \frac{3-2\beta}{4}$. By Lemma A.1,

$$
\|(e^{-2^n\varepsilon})\|_{r_0'=\frac{2}{1+2\beta}} \lesssim \operatorname{Ei}(r_0'\varepsilon)^{\frac{1+2\beta}{2}} \overset{(1.6)}{\leq} \operatorname{Ei}(\varepsilon)^{\frac{1+2\beta}{2}}, \qquad \text{(A.16)}
$$

where we used that $r_0' > 1$. Thus, (A.15) shows (A.13). □

A version of the following Lemma can be found in [31, Theorem 6.13], however, the constant dependence is unclear there.

**Lemma A.5.** *Let $A \in \operatorname{Sect}(\omega)$ with $\omega < \phi < \frac{\pi}{2}$ and $\alpha \in (0,1]$. Set $R = 0$ if $0 \notin \rho(A)$, and $R = \frac{1}{\|A^{-1}\|}$ otherwise. Then, for every $\kappa \in [0,1)$*

$$
\|A^\alpha T(t)\| \leq Ct^{-\alpha}e^{-t\kappa R\cos\phi} \qquad \forall t > 0, \qquad \text{(A.17)}
$$

*with $C = C_{\alpha,\kappa}M(A,\phi)(\cos\phi)^{-\alpha}$.*
*Note that by the assumptions, the growth bound $\omega_0$ of $T$ satisfies $\omega_0 \leq -R\cos\phi$.*

*Proof.* Let $r = \eta R$ for $\eta \in [0,1)$. Thus, if $R = 0$, then $r = 0$. We define the path $\Gamma^r = \Gamma_{1,r} \cup \Gamma_{2,r} \cup \Gamma_{3,r}$ with

$$
\Gamma_1^r = \left\{\tilde{r}e^{i\phi}, \tilde{r} \geq r\right\}, \Gamma_2^r = \left\{re^{i\phi} - it, t \in (0, 2\Im(re^{i\phi})\right\}, \Gamma_3^r = \left\{-\tilde{r}e^{-i\phi}, \tilde{r} \leq -r\right\},
$$

orientated counter-clockwise. Note that $\Gamma^0 = \partial\Sigma_\phi$. Since $z \mapsto z^\alpha e^{-tz} \in H_0^\infty(\Sigma_\phi)$, and $\Gamma \subset \rho(A)$ if $0 \in \rho(A)$, we get (see (1.4))

$$
A^\alpha T(t) = \frac{1}{2\pi i}\int_{\Gamma^r} z^\alpha e^{-tz}R(z,A)\,dz. \qquad \text{(A.18)}
$$

Splitting up the integral and taking norms, we derive for $\Gamma_1^r$,

$$
\begin{aligned}
\int_{\Gamma_1^r} |z|^\alpha e^{-t\Re z}\|R(z,A)\|\,|dz| &\leq M(A,\phi)\int_r^\infty s^{\alpha-1}e^{-ts\cos\phi}\,ds \\
&= M(A,\phi)(t\cos\phi)^{-\alpha}\int_{rt\cos\phi}^\infty s^{\alpha-1}e^{-s}\,ds.
\end{aligned}
$$

By the definition of the Gamma function it follows that $\int_a^\infty s^{\alpha-1}e^{-s}ds \leq \Gamma(\alpha)e^{-a}$ for $a \geq 0$ and $\alpha \in (0,1]$. Thus,

$$
\left\|\int_{\Gamma_1^r} z^\alpha e^{-tz}R(z,A)dz\right\| \leq M(A,\phi)(t\cos\phi)^{-\alpha}\Gamma(\alpha)e^{-tR\eta\cos\phi}. \qquad \text{(A.19)}
$$

The estimate $\Gamma_3^r$ can be derived analogously. Since $\Gamma_2^0 = \{\}$, it remains to consider $\Gamma_2^r$ for $r > 0$. Thus, $R, \eta > 0$ which means that $0 \in \rho(A)$. By the resolvent identity follows

$$
\|R(z,A)\| \leq \frac{\|A^{-1}\|}{1-\eta} \quad \text{for} \quad |z| \leq r = \eta R,
$$

see also the proof of Theorem 2.10. Therefore,

$$\begin{aligned}\int_{\Gamma_2^r} |z|^\alpha e^{-t\Re z}\|R(z,A)\|dz &\leq e^{-tr\cos\phi}\frac{\|A^{-1}\|}{1-\eta}r^\alpha \int_{\Gamma_{2,r}}|dz| \\ &\leq e^{-tr\cos\phi}\frac{\|A^{-1}\|}{1-\eta}r^{\alpha+1}(2\sin\phi) \\ &= r^\alpha e^{-tr\cos\phi}\frac{2\eta}{1-\eta}.\end{aligned}$$

Since $r \mapsto f(r) = r^\alpha e^{-rb}$ attains its maximum $(\frac{\alpha}{b})^\alpha e^{-\alpha}$ at $r = \frac{\alpha}{b}$, we conclude that

$$r^\alpha e^{-rb} = r^\alpha e^{-rb(1-\eta)}e^{-rb\eta} \leq \left(\tfrac{\alpha}{b(1-\eta)}\right)^\alpha e^{-\alpha}e^{-rb\eta},$$

and thus

$$\left\|\int_{\Gamma_2^r} z^\alpha e^{-tz}R(z,A)dz\right\| \leq M(A,\phi)\frac{2\eta}{1-\eta}\left(\tfrac{e^{-1}\alpha}{1-\eta}\right)^\alpha (t\cos\phi)^{-\alpha}e^{-tR\eta^2\cos\phi}. \tag{A.20}$$

where we use that $M(A,\phi) \geq 1$, see [18, Prop. 2.1.1]. Since $e^{-tR\eta^2\cos\phi} \geq e^{-tR\eta\cos\phi}$, combining (A.18), (A.19) and (A.20) yields the assertion by setting $\eta = \sqrt{\kappa}$. □

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Felix L. Schwenninger, Department of Applied Mathematics, P.O. Box 217, 7500 AE Enschede, The Netherlands, Phone: +31534893457.
*E-mail address*: f.l.schwenninger@utwente.nl