ON ALGEBRAIC AND MORE GENERAL CATEGORIES WHOSE SPLIT EPIMORPHISMS HAVE UNDERLYING PRODUCT PROJECTIONS

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Abstract

We characterize those varieties of universal algebras where every split epimorphism considered as a map of sets is a product projection. In addition we obtain new characterizations of protomodular, unital and subtractive varieties as well as varieties of right Ω-loops and biternary systems.

Introduction

It is well known that in the category of groups if

$$0 \longrightarrow K \xrightarrow{\kappa} A \xrightarrow{\alpha} B \longrightarrow 0$$

is a short exact sequence, then A and $K \times B$ are bijective as sets, moreover when α is split, i.e. for each split extension

$$K \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B\ ,\ \ \alpha\beta = 1_B,\ \ \kappa = \ker(\alpha),$$

this bijection becomes a natural bijection $K \times B \to A$ such that the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\langle 1,0 \rangle} & K \times B & \underset{\langle 0,1 \rangle}{\overset{\pi_2}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle \varphi} & & \| \\ K & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

is a morphism of split extensions in the category **Set**, of sets, that is, $\alpha\varphi = \pi_2$, $\varphi\langle 0, 1\rangle = \beta$, and $\varphi\langle 1, 0\rangle = \kappa$. As shown by E. B. Inyangala, these bijections exists in a more general setting of a variety of right Ω-loops (see

[4, 5]), that is, a pointed variety of universal algebras $\mathcal{V}$ with constant 0 and binary terms $x+y$ and $x-y$ satisfying the identities:

$$x + 0 = x \tag{1}$$

$$x - x = 0 \tag{2}$$

$$(x + y) - y = x \tag{3}$$

$$(x - y) + y = x \tag{4}$$

Moreover, he showed that if a pointed variety $\mathcal{V}$ with constant 0 has binary terms $x+y$ and $x-y$ and there exist bijections (as above) constructed (in the same way as for groups) using those terms, i.e. $\varphi(k,b) = \kappa(k) + \beta(b)$ and $\varphi^{-1}(a) = (\lambda(a), \alpha(a))$, where λ is the unique map such that $\kappa\lambda(a) = a - \beta\alpha(a)$, then $\mathcal{V}$ is a variety of right Ω-loops and in particular the identities (1) - (4) hold for $x+y$ and $x-y$. In this paper we prove that if for a pointed variety $\mathcal{V}$ there exist natural bijections as above, then $\mathcal{V}$ is a variety of right Ω-loops (see Theorem 2.1).

For any category $\mathbb{C}$ let $\mathbf{Pt}(\mathbb{C})$ to be the category of split epimorphisms in $\mathbb{C}$: an object is a quadruple (A, B, α, β) where A and B are objects in $\mathbb{C}$ and $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ with $\alpha\beta = 1_B$; a morphism $(A, B, \alpha, \beta) \to (A', B', \alpha', \beta')$ is a pair of morphisms $(f : A \to A', g : B \to B')$ such that in the diagram

$$\begin{array}{ccc} A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \end{array}$$

$\alpha' f = g\alpha$ and $f\beta = \beta' g$. Throughout this paper for any objects A and B we will denote by π_1 and π_2 the first and second product projections respectively. We will use the same notation for the first and second pullback projections and will write

$$(A \underset{\langle f,g\rangle}{\times} B, \pi_1, \pi_2)$$

for the pullback of $f : A \to C$ and $g : B \to C$ as in the diagram

$$\begin{array}{ccc} A \underset{\langle f,g\rangle}{\times} B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[f]{} & C. \end{array}$$

For any morphisms $u : W \to A$ and $v : W \to B$ with $fu = gv$ we will write

$$\langle u, v\rangle : W \to A \underset{\langle f,g\rangle}{\times} B$$

for the unique morphism with $\pi_1\langle u, v\rangle = u$ and $\pi_2\langle u, v\rangle = v$.

We prove that for a pointed variety $\mathcal{V}$, if for each (A, B, α, β) in $\mathbf{Pt}(\mathcal{V})$ there exists a natural bijection $\varphi : K \times B \to A$, where $\kappa : K \to A$ is the kernel of α, such that the diagram

$$\begin{array}{ccc} K \times B & \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & B \\ {\scriptstyle\varphi}\downarrow & & \| \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

is a morphism in $\mathbf{Pt}(\mathbf{Set})$, then $\mathcal{V}$ is a variety of right Ω-loops (see Corollary 2.2). There is a *natural* generalization of this condition for any variety $\mathcal{V}$, namely asking for each (A, B, α, β) in $\mathbf{Pt}(\mathcal{V})$ and for each morphism $f : E \to B$ that there exists a bijection

$$\varphi : (A \underset{\langle\alpha,f\rangle}{\times} E) \times B \to E \times A$$

natural in both (A, B, α, β) and $f : E \to B$, such that the diagram

$$\begin{array}{ccc} (A \underset{\langle\alpha,f\rangle}{\times} E) \times B & \underset{\langle\beta f,1\rangle\times 1}{\overset{\pi_2\times 1}{\rightleftarrows}} & E \times B \\ {\scriptstyle\varphi}\downarrow & & \| \\ E \times A & \underset{1\times\beta}{\overset{1\times\alpha}{\rightleftarrows}} & E \times B \end{array}$$

is a morphism in $\mathbf{Pt}(\mathbf{Set})$. It is clear that for a pointed variety this condition implies the previous condition, since taking E to be the zero object and f to be the unique morphism from E to B makes

$$\pi_1 : A \underset{\langle\alpha,f\rangle}{\times} E \to A$$

the kernel of α. In Section 4 we prove that this condition is equivalent to the same condition under the restriction that each f as above is an identity morphism (see Theorem 4.6). We also prove that a variety satisfies this condition if and only if it is a *biternary system* [7] that is there exist ternary terms $p(x, y, z)$ and $q(x, y, z)$ satisfying the identities

$$p(x, x, y) = y \tag{5}$$

$$p(q(x, y, z), z, y) = x = q(p(x, y, z), z, y). \tag{6}$$

However, there are other generalizations that may be considered. In a variety $\mathcal{V}$ with constants, for each X, let $\theta_X : 1 \to X^n$ be a map (natural in X) such that the composite with each product projection $\pi_i : X^n \to X$

gives a constant. We could then consider the following condition: for each (A, B, α, β) in $\mathbf{Pt}(\mathcal{V})$ there exists a natural split epimorphism (in the category of sets)

$$\varphi : (A^n \underset{\langle \alpha^n, \theta_B \rangle}{\times} 1) \times B \to A$$

with splitting

$$\psi : A \to (A^n \underset{\langle \alpha^n, \theta_B \rangle}{\times} 1) \times B$$

such that in the diagram

$$\begin{array}{ccc} (A^n \underset{\langle \alpha^n, \theta_B \rangle}{\times} 1) \times B & \underset{(\langle \theta_A, 1 \rangle !_B) \times 1}{\overset{\pi_2}{\rightleftarrows}} & B \\ {\scriptstyle \psi} \uparrow \downarrow {\scriptstyle \varphi} & & \| \\ A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

the upward and downward directed sub-diagrams are morphisms in $\mathbf{Pt}(\mathbf{Set})$. We prove in Section 3 that this condition is equivalent to $\mathcal{V}$ being a protomodular variety [2] of *type* n, that is, a variety $\mathcal{V}$ with constants $e_1, \ldots, e_n$, binary terms $s_1(x, y), \ldots, s_n(x, y)$ and an $n + 1$-ary term $p(x_1, \ldots, x_n, z)$ satisfying the identities:

$$s_i(x, x) = e_i \ i \in \{1, \ldots, n\} \tag{7}$$

$$p(s_1(x, z), \ldots, s_n(x, z), z) = x. \tag{8}$$

Note that requiring φ to be a bijection gives the addition conditions

$$s_i(p(x_1, ..., x_n, y), y)) = x_i \text{ for all } i \in 1, ..., n. \tag{9}$$

In order to study these conditions simultaneously we make a further generalization described in Section 1.

1 The general setting

In this section we replace a forgetful functor from a variety into the category of sets (or pointed sets) with an abstract functor (satisfying certain conditions) and consider a generalization allowing us to study simultaneously both generalizations discussed in the introduction.

For a set $\mathbf{n}$, a category $\mathbb{D}$ with finite products and products indexed over $\mathbf{n}$, and for functors $F, G, H : \mathbb{C} \to \mathbb{D}$ we denote by $F^{\mathbf{n}}$ the $\mathbf{n}$ indexed product of F with itself and by $G \times H$ the product of G and H in the functor category $\mathbb{D}^{\mathbb{C}}$.

Throughout this section we will assume that:

1. $\mathbb{A}$ is a category with finite products;
2. $\mathbf{m}$ and $\mathbf{n}$ are sets;
3. $\mathbb{X}$ is a category with finite limits and products indexed by the sets $\mathbf{m}$ and $\mathbf{n}$;
4. $U : \mathbb{A} \to \mathbb{X}$ is a functor preserving finite products;
5. $\theta : U^{\mathbf{m}} \to U^{\mathbf{n}}$ is a natural transformation.

Let $\Delta : \mathbb{A} \to \mathbf{Pt}(\mathbb{A})$ be the functor sending X in $\mathbb{A}$ to $(X \times X, X, \pi_2, \langle 1, 1\rangle)$ and let $D_{\mathbb{A}}$ be the functor $\mathbf{Pt}(\mathbb{A}) \to \mathbb{A}$ taking (A, B, α, β) to B. Let $V : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$ and $W : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$ be the functors sending (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ to

$$((U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B\rangle}{\times} U(B)^{\mathbf{m}}) \times U(B), U(B)^{\mathbf{m}} \times U(B), \pi_2 \times 1, \langle U(\beta)^{\mathbf{n}}\theta_B, 1\rangle \times 1)$$

and

$$(U(B)^{\mathbf{m}} \times U(A), U(B)^{\mathbf{m}} \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively.

From the beginning of the next section we will consider the case where $\mathbb{A}$ is a variety, $\mathbb{X}$ is the category of sets, U is the usual forgetful functor from the variety to the category of sets, $\mathbf{m} = \{1, \ldots, m\}$, $\mathbf{n} = \{1, \ldots, n\}$, and θ is constructed from n m-ary terms of $\mathbb{A}$. In particular when $\mathbb{A}$ is pointed with constant 0, $\mathbf{n} = \{1\}$, $\mathbf{m} = \emptyset$, and $\theta : U^{\mathbf{m}} \to U^{\mathbf{n}}$ is the natural transformation with component at X $\theta_X(1) = 0$ (where 1 is the unique element in $U^{\mathbf{m}}(X)$), it can be seen that

$$\pi_1 : U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B\rangle}{\times} U(B)^{\mathbf{m}} \to U(A)$$

is up to isomorphism the image under U of the kernel of α and the bijections mentioned at the start of the introduction become components of a natural transformation $V \to W$.

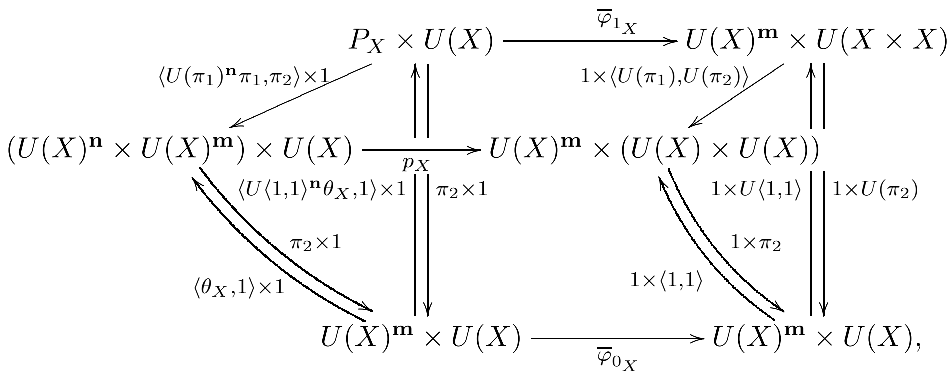
Lemma 1.1. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\tau : V \to W$;

(b) a natural transformation $\overline{\tau} : V\Delta \to W\Delta$;

(c) natural transformations $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$ and $\zeta : U^{\mathbf{m}} \times U \to U^{\mathbf{m}}$;

Proof. For each (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ and X in $\mathbb{A}$, let $({\varphi_1}_{(A,B,\alpha,\beta)}, {\varphi_0}_{(A,B,\alpha,\beta)}) = \tau_{(A,B,\alpha,\beta)}$ and $(\overline{\varphi}_{1_X}, \overline{\varphi}_{0_X}) = \overline{\tau}_X$. The diagram



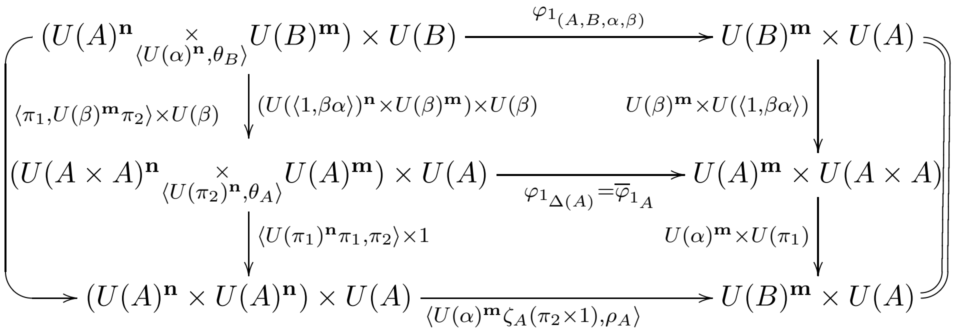
in which

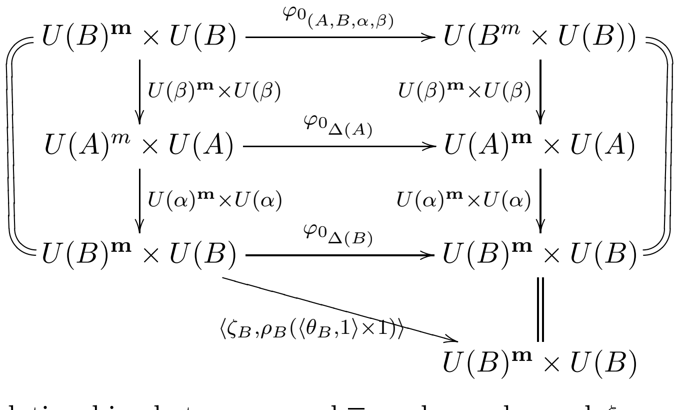
$$P_X = U(X \times X)^{\mathbf{n}} \underset{\langle U(\pi_2)^n, \theta_X \rangle}{\times} U(X)^{\mathbf{m}}$$

and

$$p_X = \langle \zeta_X(\pi_2 \times 1), \langle \rho_X, \rho_X(\langle \theta_X \pi_2, \pi_2 \rangle \times 1)\rangle\rangle,$$

is a commutative diagram of morphisms in $\mathbf{Pt}(\mathbb{X})$, and shows the relationship between $\overline{\tau}$ and ρ and ζ. The commutative diagrams





show the relationships between τ and $\overline{\tau}$, and τ and ρ and ζ. $\square$

Lemma 1.2. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\gamma : W \to V$*;*

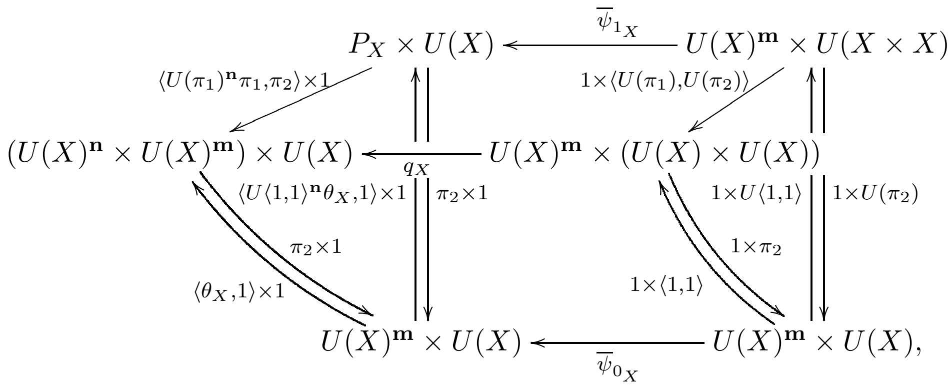
(b) a natural transformation $\overline{\gamma} : W\Delta \to V\Delta$*;*

(c) natural transformations $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$, $\eta : U^{\mathbf{m}} \times U \to U^{\mathbf{m}}$ *and* $\epsilon : U^{\mathbf{m}} \times U \to U$ *with components at each* X *in* $\mathbb{A}$ *making the diagram*

$$\begin{array}{ccc} U(X)^{\mathbf{m}} \times U(X) & \xrightarrow{1\times\langle 1,1\rangle} & U(X)^{\mathbf{m}} \times (U(X) \times U(X)) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \sigma_X} \\ U(X)^{\mathbf{m}} & \xrightarrow[\theta_X]{} & U(X)^{\mathbf{n}} \end{array} \tag{10}$$

commute.

Proof. For each (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ and X in $\mathbb{A}$, let $(\psi_{1_{(A,B,\alpha,\beta)}}, \psi_{0_{(A,B,\alpha,\beta)}}) = \gamma_{(A,B,\alpha,\beta)}$ and $(\overline{\psi}_{1_X}, \overline{\psi}_{0_X}) = \overline{\gamma}_X$. The diagram



in which

$$P_X = U(X \times X)^{\mathbf{n}} \underset{\langle U(\pi_2)^n, \theta_X\rangle}{\times} U(X)^{\mathbf{m}}$$

and

$$q_X = \langle\langle \sigma_X, \eta_X(1 \times \pi_2)\rangle, \epsilon_X(1 \times \pi_2)\rangle,$$

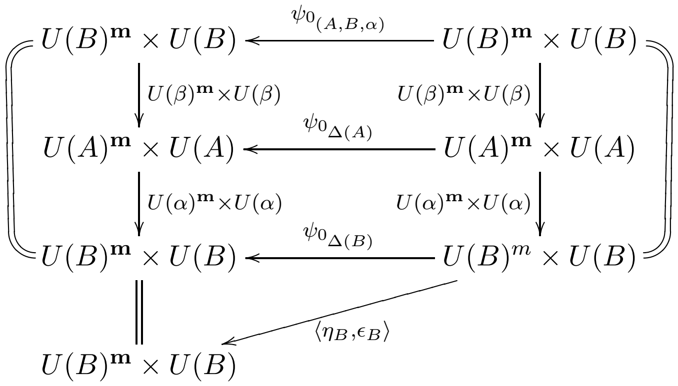
is a commutative diagram of morphisms in $\mathbf{Pt}(\mathbb{X})$, and shows the relationship between $\overline{\gamma}$ and σ, η and ϵ. The equations

$$\gamma_{\Delta(X)} = \overline{\gamma}_X$$

and

$$\psi_{1_{(A,B,\alpha,\beta)}} = \langle\langle \sigma_A(U(\beta)^{\mathbf{m}} \times U(\langle 1, \beta\alpha\rangle)), \eta_B(1 \times U(\alpha))\rangle \epsilon_B(1 \times U(\alpha))\rangle,$$

and the commutative diagram



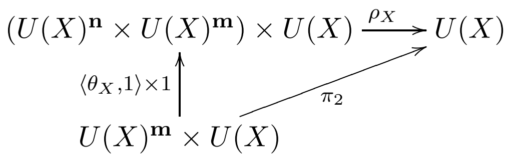
show the relationships between γ and $\overline{\gamma}$, and γ and σ, η and ϵ. □

From the two lemmas above we easily prove the following corollaries.

Corollary 1.3. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\tau : V \to W$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D^{\mathbf{m}}_{\mathbb{A}} \times D_{\mathbb{A}}}$*;*

(b) a natural transformation $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$ *with component at each* X *in* $\mathbb{C}$ *making the diagram*



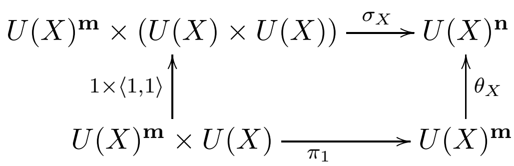
(11)

commute.

Corollary 1.4. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\gamma : W \to V$ *with* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D^{\mathbf{m}}_{\mathbb{A}} \times D_{\mathbb{A}}}$*;*

(b) a natural transformation $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$ *with component at each* X *in* $\mathbb{C}$ *making the diagram*



(12)

commute.

Corollary 1.5. *Each of the following types of data uniquely determine each other:*

(a) natural transformations $\tau : V \to W$ *and* $\gamma : W \to V$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and such that* $\tau\gamma = 1_W$*;*

(b) natural transformations $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$ *and* $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$ *with components at each* X *in* $\mathbb{C}$ *making the diagrams (11), (12) and*

$$\begin{array}{ccc} U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & & \\ \big\downarrow{\scriptstyle \langle\langle \sigma, \pi_1\rangle, \pi_2\pi_2\rangle} & \overset{\pi_1\pi_2}{\searrow} & \\ (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & \xrightarrow[\rho_X]{} & U(X) \end{array} \tag{13}$$

commute.

Corollary 1.6. *Each of the following types of data uniquely determine each other:*

(a) natural transformations $\tau : V \to W$ *and* $\gamma : W \to V$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and such that* $\gamma\tau = 1_V$*;*

(b) natural transformations $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$ *and* $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$ *with components at each* X *in* $\mathbb{C}$ *making the diagrams (11), (12) and*

$$\begin{array}{ccc} (U(X)^{\mathbf{n}} \times U(X)^{\mathbf{m}}) \times U(X) & & \\ \big\downarrow{\scriptstyle \langle \pi_2\pi_1, \langle \rho_X, \pi_2\rangle\rangle} & \overset{\pi_1\pi_1}{\searrow} & \\ U(X)^{\mathbf{m}} \times (U(X) \times U(X)) & \xrightarrow[\sigma_X]{} & U(X)^{\mathbf{n}} \end{array} \tag{14}$$

commute.

Corollary 1.7. *Each of the following types of data uniquely determine each other:*

(a) natural transformations $\tau : V \to W$ *and* $\sigma : W \to V$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}^{\mathbf{m}} \times D_{\mathbb{A}}}$ *and inverse to each other;*

(b) natural transformations $\rho : (U^{\mathbf{n}} \times U^{\mathbf{m}}) \times U \to U$ *and* $\sigma : U^{\mathbf{m}} \times (U \times U) \to U^{\mathbf{n}}$ *with components at each* X *in* $\mathbb{C}$ *making the diagrams (11), (12), (13) and (14) commute.*

We now consider the case where $\mathbf{m} = \emptyset$ and $\mathbf{n} = \{1\}$, the results proved here will be used in Section 2.

When $\mathbf{m} = \emptyset$ and $\mathbf{n} = \{1\}$, the functors V and W are up to isomorphism the functors $\tilde{V}, \tilde{W} : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$ sending (A, B, α, β) to

$$((U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1) \times U(B), U(B), \pi_2, \langle\langle\theta_A, 1\rangle !_{U(B)}, 1\rangle)$$

and

$$(U(A), U(B), U(\alpha), U(\beta))$$

respectively.

Corollary 1.8. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\tau : \tilde{V} \to \tilde{W}$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$ *and with component at each* (A, B, α, β) *in* $\mathbf{Pt}(\mathbb{A})$ *such that the diagram*

$$\begin{array}{ccc} U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1 & \xrightarrow{\langle 1,\theta_B !\rangle} & (U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1) \times U(B) \\ \| & & \downarrow {\scriptstyle \varphi 1_{(A,B,\alpha,\beta)}} \\ U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1 & \xrightarrow[\pi_1]{} & U(A) \end{array} \tag{15}$$

commutes;

(b) a natural transformation $\rho : U \times U \to U$ *with component at each* X *in* $\mathbb{A}$ *making the diagram*

$$\begin{array}{ccc} U(X) & & \\ {\scriptstyle \langle 1,\theta_X !\rangle} \downarrow & \searrow^{1_{U(X)}} & \\ U(X) \times U(X) & \xrightarrow{\rho_X} & U(X) \\ {\scriptstyle \langle \theta_X !,1\rangle} \uparrow & \nearrow_{1_{U(X)}} & \\ U(X) & & \end{array} \tag{16}$$

commute.

Corollary 1.9. *Each of the following types of data uniquely determine each other:*

(a) a natural transformation $\gamma : \tilde{W} \to \tilde{V}$ *with* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$ *and with component at each* (A, B, α, β) *in* $\mathbf{Pt}(\mathbb{A})$ *such that the diagram*

$$\begin{array}{ccc} U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1 & \xrightarrow{\langle 1,\theta_B !\rangle} & (U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1) \times U(B) \\ \| & & \uparrow {\scriptstyle \psi 1_{(A,B,\alpha,\beta)}} \\ U(A) \underset{\langle U(\alpha),\theta_B\rangle}{\times} 1 & \xrightarrow[\pi_1]{} & U(A) \end{array} \tag{17}$$

commutes;

(b) *a natural transformation* $\sigma : U \times U \to U$ *with component at each* X *in* $\mathbb{A}$ *making the diagram*

$$\begin{array}{ccc} U(X) & & \\ {\scriptstyle \langle 1,\theta_X ! \rangle}\downarrow & \searrow^{1_{U(X)}} & \\ U(X) \times U(X) & \xrightarrow[\sigma_X]{} & U(X) \\ {\scriptstyle \langle 1,1 \rangle}\uparrow & & \uparrow{\scriptstyle \theta_X} \\ U(X) & \xrightarrow[!_{U(X)}]{} & 1 \end{array} \tag{18}$$

commute.

Corollary 1.10. *Each of the following types of data uniquely determine each other:*

(a) *natural transformations* $\tau : \tilde{V} \to \tilde{W}$ *and* $\gamma : \tilde{W} \to \tilde{V}$ *with* $1_{D_{\mathbb{X}}} \circ \tau = 1_{D_{\mathbb{A}}}$ *and* $1_{D_{\mathbb{X}}} \circ \gamma = 1_{D_{\mathbb{A}}}$ *inverse to each other and with components at each* (A, B, α, β) *in* $\mathbf{Pt}(\mathbb{A})$ *making the diagrams (15) and (17) commute;*

(b) *natural transformations* $\rho : U \times U \to U$ *and* $\sigma : U \times U \to U$ *with component at each* X *in* $\mathbb{A}$ *making the diagrams (16), (18),*

$$\begin{array}{ccc} U(X) \times U(X) & & \\ {\scriptstyle \langle \sigma_X ,\pi_2 \rangle}\downarrow & \searrow^{\pi_1} & \\ U(X) \times U(X) & \xrightarrow[\rho_X]{} & U(X) \end{array} \tag{19}$$

and

$$\begin{array}{ccc} U(X) \times U(X) & & \\ {\scriptstyle \langle \rho_X ,\pi_2 \rangle}\downarrow & \searrow^{\pi_1} & \\ U(X) \times U(X) & \xrightarrow[\sigma_X]{} & U(X) \end{array} \tag{20}$$

commute.

In the sections that follows we use the fact that the set of natural transformation $U^{\mathbf{n}} \to U$ (where $\mathbf{n} = \{1, \ldots, n\}$ and U is the forgetful functor from a variety to sets) is in bijection with the set of n-ary terms of the variety. Since this is no longer true for arbitrary internal varieties (every term determines a natural transformation but not conversely) the results in the sections that follow hold only partially in arbitrary internal varieties, i.e. the existence of certain terms determine natural transformations between appropriate V and W but not conversely.

2 Pointed varieties

In this section we apply the results from Section 1 to the special case where $\mathbb{A} = \mathcal{V}$ is a pointed variety, $\mathbb{X} = \mathbf{Set}_*$ is the category of pointed sets, U is the usual forgetful functor, $\mathbf{m} = \emptyset$, $\mathbf{n} = \{1\}$, and θ is constructed using the constant of $\mathcal{V}$.

For any category $\mathbb{C}$ we define $\mathbf{SplExt}(\mathbb{C})$ to be the category of split extensions: an object is a sextuple $(K, A, B, \kappa, \alpha, \beta)$ where K, A and B are objects in $\mathbb{C}$ and $\kappa : K \to B$, $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ with (K, κ) the kernel of α and $\alpha\beta = 1_B$; a morphism $(K, A, B, \kappa, \alpha, \beta) \to (K', A', B', \kappa', \alpha', \beta')$ is a triple (u, v, w) of morphisms $u : K \to K'$, $v : A \to A'$ and $w : B \to B'$ such that in the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ K' & \xrightarrow[\kappa']{} & A' & \underset{\beta'}{\overset{\alpha'}{\rightleftarrows}} & B' \end{array}$$

$v\kappa = \kappa' u$, $\alpha' v = w\alpha$ and $v\beta = \beta' w$.

Theorem 2.1. *Let $\mathcal{V}$ be a pointed variety and let $P, Q : \mathbf{SplExt}(\mathcal{V}) \to \mathbf{SplExt}(\mathbf{Set}_*)$ be the functors taking $(K, A, B, \kappa, \alpha, \beta)$ to $(U(K), U(K) \times U(B), \langle 1, 0\rangle, \pi_2, \langle 0, 1\rangle)$ and $(U(K), U(A), U(B), U(\kappa), U(\alpha), U(\beta))$ respectively.*

(a) $\mathcal{V}$ is a unital variety [1] if and only if there exists a natural transformation $P \to Q$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form

$$\begin{array}{ccccc} U(K) & \longrightarrow & U(K) \times U(B) & \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & U(B) \\ \| & & \downarrow & & \| \\ U(K) & \longrightarrow & U(A) & \underset{U(\beta)}{\overset{U(\alpha)}{\rightleftarrows}} & U(B); \end{array}$$

(b) $\mathcal{V}$ is a subtractive variety [6] if and only if there exists a natural transformation $Q \to P$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form

$$\begin{array}{ccccc} U(K) & \longrightarrow & U(A) & \underset{U(\beta)}{\overset{U(\alpha)}{\rightleftarrows}} & U(B) \\ \| & & \downarrow & & \| \\ U(K) & \longrightarrow & U(K) \times U(B) & \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & U(B); \end{array}$$

(c) $\mathcal{V}$ is a variety of right Ω-loops if and only if there exists a natural isomorphism $P \to Q$ with component at $(K, A, B, \kappa, \alpha, \beta)$ of the form

$$\begin{array}{ccccc}
U(K) & \longrightarrow & U(K) \times U(B) & \underset{\langle 0,1\rangle}{\overset{\pi_2}{\rightleftarrows}} & U(B) \\
\| & & \downarrow & & \| \\
U(K) & \longrightarrow & U(A) & \underset{U(\beta)}{\overset{U(\alpha)}{\rightleftarrows}} & U(B).
\end{array}$$

Proof. It is easy to see that to give a natural transformation $P \to Q$ as in (a) above is the same as to give a natural transformation $\tilde{V} \to \tilde{W}$ as in (a) of Corollary 1.8 which, by Corollary 1.8, is uniquely determined by a natural transformation $\rho : U \times U \to U$ with components making the diagram (16) commute. And, such a natural transformation determines and is determined by a binary term $+$ such that for each x, y in X, an algebra, $x + y = \rho_X(x, y)$. The commutativity of (16) then implies that $x + 0 = x = 0 + x$. The statements (b) and (c) follow from Corollaries 1.9, and 1.10 in a similar way. $\square$

Corollary 2.2. *Let $\tilde{P}, \tilde{Q} : \mathbf{Pt}(\mathbb{A}) \to \mathbf{Pt}(\mathbb{X})$ be the functors sending (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ to $(U(K \times B), U(B), U(\pi_2), U(\langle 0, 1\rangle))$ (where $K = \mathrm{Ker}(\alpha)$) and $(U(A), U(B), U(\alpha), U(\beta))$ respectively. $\mathcal{V}$ is a variety of right Ω-loops if and only if there exists a natural bijection $\tilde{P} \to \tilde{Q}$ with component (A, B, α, β) of the form*

$$\begin{array}{ccc}
U(K \times B) & \underset{U(\langle 0,1\rangle)}{\overset{U(\pi_2)}{\rightleftarrows}} & U(B) \\
\downarrow & & \| \\
U(A) & \underset{U(\beta)}{\overset{U(\alpha)}{\rightleftarrows}} & U(B).
\end{array}$$

Proof. It follows from Corollary 1.7 that a natural bijection $\tilde{P} \to \tilde{Q}$ as above is completely determined by and determines binary terms $\rho(x, y)$ and $\sigma(x, y)$ satisfying the identities $\sigma(x, x) = 0$, $\rho(\sigma(x, y), y) = x$ and $\sigma(\rho(x, y), y) = x$. Setting $x + y = \rho(\sigma(x, 0), y)$ and $x - y = \rho(\sigma(x, y), 0)$ determines terms that satisfy the right loop identities. $\square$

Remark 2.3. *In fact it can be shown that $\mathcal{V}$ is a variety of right Ω-loops if and only if there exists a natural isomorphism $\tilde{P} \to \tilde{Q}$.*

3 Protomodular varieties

In this section we give a new classification of protomodular varieties by applying the results from Section 1 to the case where $\mathbb{A} = \mathcal{V}$ is an arbitrary variety with constants, $\mathbb{X} = \mathbf{Set}$ is the category of sets, and U is the usual forgetful functor.

Theorem 3.1. *$\mathcal{V}$ is a protomodular variety if and only if for some $\mathbf{m} = \{1, \ldots, m\}$, $\mathbf{n} = \{1, \ldots, n\}$ and θ there exist natural transformations $\tau : V \to W$ and $\gamma : W \to V$ with $\tau\gamma = 1_W$ and with components at each (A, B, α, β) in $\mathbf{Pt}(\mathbb{C})$ of the form*

$$\begin{array}{ccc} (U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}}, \theta_B \rangle}{\times} U(B)^{\mathbf{m}}) \times U(B) & \underset{\langle U(\beta)^{\mathbf{m}}\theta_B, 1\rangle \times 1}{\overset{\pi_2 \times 1}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B) \\ \Big\updownarrow & & \Big\| \\ U(B)^{m} \times U(A) & \underset{1 \times U(\beta)}{\overset{1 \times U(\alpha)}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B). \end{array}$$

Proof. It follows from Corollary 1.5 that natural transformations $\tau : V \to W$ and $\gamma : W \to V$ as above determine terms

$$\rho(x_1, \ldots, x_n, y_1, \ldots, y_m, z) \text{ and } \sigma_i(y_1, \ldots, y_m, x, z) \ i \in \mathbf{n}$$

satisfying the identities

$$\sigma_i(y_1, \ldots, y_m, x, x) = \theta_i(y_1, \ldots, y_m) \ i \in \mathbf{n}$$

$$\rho(\sigma_1(y_1, \ldots, y_m, x, z), \ldots, \sigma_n(y_1, \ldots, y_m, x, z), y_1, \ldots, y_m, z) = x.$$

For any constant e we may form new terms $e_i = \theta_i(e, \ldots, e)$ $i \in \mathbf{n}$, $s_i(x, z) = \sigma_i(e, \ldots, e, x, z)$ $i \in \mathbf{n}$, and $p(x_1, \ldots, x_n, z) = \rho(x_1, \ldots, x_n, e, \ldots, e, z)$. It easy to check that these terms make $\mathcal{V}$ a protomodular variety. The converse follows from Corollary 1.5 with $\mathbf{m} = \emptyset$. □

Remark 3.2. *The results in this section can easily be extended to $\mathcal{V}$ an infinitary variety, with $\mathbf{m}$ and $\mathbf{n}$ possibly infinite sets, giving, by Theorem 2.1 of [3], a new classification of infinitary protomodular varieties.*

Remark 3.3. *It could also be interesting to study when $\gamma\tau = 1_V$ (without $\tau\gamma = 1_W$) which can be seen to be equivalent to the existence of ρ and σ as above, satisfying the identities:*

$$\sigma_i(y_1, \ldots, y_m, x, x) = \theta_i(y_1, \ldots, y_m) \ i \in \mathbf{n}$$
$$\rho(\theta_1(y_1, \ldots, y_m), \ldots, \theta_n(y_1, \ldots, y_m), y_1, \ldots, y_m, x) = x$$
$$\sigma_i(y_1, \ldots, y_m, \rho(x_1, .., x_n, y_1, .., y_m, z), z) = x_i \ i \in \mathbf{n}$$

instead.

4 General varieties

In this section we consider the case where $\mathbb{A} = \mathcal{V}$ is a variety, $\mathbb{X} = \mathbf{Set}$ is the category sets, and U is the usual forgetful functor.

For a variety $\mathcal{V}$ consider the condition:

Condition 4.1. *There exist ternary terms p and q satisfying the identities: $p(x,x,y) = y$ and $p(q(x,y,z),z,y) = x = q(p(x,y,z),z,y)$.*

It is easy to see that $q(x,x,y) = y$ follows from the conditions above, as remarked in [7], where such a variety was called a *biternary system*.

Remark 4.2. *It is easy to see that if a variety $\mathcal{V}$ satisfies Condition 4.1 then every regular epimorphism $f : E \to B$ is up to bijection a product projection $\pi_2 : X \times B \to B$ for some X (since for each b and b' choosing e and e' in $f^{-1}(\{b\})$ and $f^{-1}(\{b'\})$ respectively gives a bijection $p(-,e,e') : f^{-1}(\{b\}) \to f^{-1}(\{b\})$).*

Proposition 4.3. *For a variety $\mathcal{V}$ the following conditions are equivalent:*

1. *$\mathcal{V}$ satisfies Condition 4.1;*
2. *There exist ternary terms $\tilde{p}$ and $\tilde{p}$ satisfying the identities: $\tilde{p}(x,x,y) = y = \tilde{q}(x,x,y)$, $\tilde{p}(x,y,y) = x = \tilde{q}(x,y,y)$ and $\tilde{p}(\tilde{q}(x,y,z),z,y) = x = \tilde{q}(\tilde{p}(x,y,z),z,y)$;*
3. *There exists a quaternary term u satisfying the identities: $u(a,b,b,a) = b$ and $u(u(a,b,c,d),b,d,c) = a$;*
4. *There exists a quaternary term $\tilde{u}$ satisfying the identities: $\tilde{u}(a,b,b,a) = b = \tilde{u}(a,a,b,a)$ and $\tilde{u}(a,b,c,c) = a = \tilde{u}(\tilde{u}(a,b,c,d),b,d,c)$;*

If in addition $\mathcal{V}$ has at least one constant, those conditions are further equivalent to:

5. *For each constant e there exist binary terms $x+y$ and $x-y$ satisfying the right loop identities (for that constant e).*

Proof. The implications $2 \Rightarrow 1$ and $4 \Rightarrow 3$ are trivial.
$1 \Rightarrow 2$: Given p and q define $\tilde{p}(x,y,z) = p(q(x,y,y),y,z)$ and $\tilde{q}(x,y,z) = p(q(x,y,z),z,z)$.
$2 \Rightarrow 4$: Given $\tilde{p}$ and $\tilde{q}$ define $\tilde{u}(a,b,c,d) = \tilde{p}(\tilde{q}(a,b,c),d,b)$.
$3 \Rightarrow 1$: Given u define $p(x,y,z) = u(x,z,z,y)$ and $q(x,y,z) = u(x,y,z,y)$.
If in addition $\mathcal{V}$ has at least one constant.
$2 \Rightarrow 5$: Given $\tilde{p}$ and $\tilde{q}$ for each constant e define $x+y = \tilde{p}(x,e,y)$ and $x-y = \tilde{q}(x,y,e)$.
$5 \Rightarrow 1$: Given $x+y$ and $x-y$ for some constant e define $p(x,y,z) = q(x,y,z) = (x-y)+z$. ☐

Remark 4.4. *It follows that a variety satisfying Condition 4.1 is a Mal'tsev variety.*

Theorem 4.5. *(a) If $\mathcal{V}$ satisfies Condition 4.1, then for $\mathbf{n} = \{1\}$, $\mathbf{m} = \{1\}$ and $\theta = 1_U$ there exists a natural isomorphism $\tau : V \to W$ with component at each (A, B, α, β) in $\mathbf{Pt}(\mathbb{C})$ of the form*

$$\begin{array}{ccc} (U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}},\theta_B\rangle}{\times} U(B)^{\mathbf{m}}) \times U(B) & \underset{\langle U(\beta)^{\mathbf{m}}\theta_B,1\rangle\times 1}{\overset{\pi_2\times 1}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow & & \| \\ U(B)^{\mathbf{m}} \times U(A) & \underset{1\times U(\beta)}{\overset{1\times U(\alpha)}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B); \end{array}$$

(b) If for some $\mathbf{n} = \{1, \ldots, n\}$, $\mathbf{m} = \{1, \ldots, m\}$ and θ there exists a natural isomorphism $\tau : V \to W$ with component at each (A, B, α, β) in $\mathbf{Pt}(\mathbb{C})$ of the form

$$\begin{array}{ccc} (U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}},\theta_B\rangle}{\times} U(B)^{\mathbf{m}}) \times U(B) & \underset{\langle U(\beta)^{\mathbf{m}}\theta_B,1\rangle\times 1}{\overset{\pi_2\times 1}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow & & \| \\ U(B)^{\mathbf{m}} \times U(A) & \underset{1\times U(\beta)}{\overset{1\times U(\alpha)}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B), \end{array}$$

then $\mathcal{V}$ satisfies Condition 4.1.

Proof. (a) Let $\mathbf{n} = \mathbf{m} = \{1\}$ and $\theta = 1_U$. Given ternary terms p and q as in Condition 4.1, it is easy to check that $\rho = p$ and $\sigma(x, y, z) = q(y, z, x)$ define natural transformations making the diagrams (11), (12), (13) and (14) commute. Therefore by Corollary 1.7 determine a natural isomorphism $V \to W$, as required.

(b) If for some $\mathbf{n} = \{1, \ldots, n\}$, $\mathbf{m} = \{1, \ldots, m\}$ and θ there exists an isomorphism $V \to W$ then by Corollary 1.7 there exist terms $\rho(x_1, \ldots, x_n, y_1, \ldots, y_m, z)$ and $\sigma_i(y_1, \ldots, y_m, x, z)$ $i \in \mathbf{n}$ satisfying the identities:

$$\sigma_i(y_1, \ldots, y_m, x, x) = \theta_i(y_1, \ldots, y_m)$$
$$\rho(\sigma_1(y_1, \ldots, y_m, x, z), \ldots, \sigma_n(y_1, \ldots, y_m, x, z), y_1, \ldots, y_m, z) = x$$
$$\sigma_i(y_1, \ldots, y_m, \rho(x_1, .., x_n, y_1, .., y_m, z), z) = x_i.$$

Let p and q be the terms defined by

$$p(x, y, z) = \rho(\sigma_1(y, \ldots, y, x, y), \ldots, \sigma_n(y, \ldots, y, x, y), y, \ldots, y, z)$$
$$q(x, y, z) = \rho(\sigma_1(z, \ldots, z, x, y), \ldots, \sigma_n(z, \ldots, z, x, y), z, \ldots, z, z).$$

It is easy to check that p and q satisfy the desired identities as in Condition 4.1.

□

Recall that for any category $\mathbb{C}$ the functor $D_{\mathbb{C}}$ is the functor $\mathbf{Pt}(\mathbb{C}) \to \mathbb{C}$ taking (A, B, α, β) to B.

Theorem 4.6. *Let V and W be the functors defined in Section 1 with $\mathbb{A} = \mathcal{V}$, $\mathbb{X} = \mathbf{Set}$, U the usual forgetful functor, $\mathbf{n} = \mathbf{m} = \{1\}$ and $\theta = 1_U$. Let $P, Q : (\mathbb{A} \downarrow D_{\mathbb{A}}) \to \mathbf{Pt}(\mathbb{X})$ be the functors sending $(E, (A, B, \alpha, \beta), f)$ to*

$$(U(A \underset{\langle\alpha,f\rangle}{\times} E) \times U(B), U(E) \times U(B), U(\pi_2) \times 1, U(\langle\beta f, 1\rangle) \times 1)$$

and

$$(U(E) \times U(A), U(E) \times U(B), 1 \times U(\alpha), 1 \times U(\beta))$$

respectively. The following are equivalent:

1. *There exists an isomorphism $\tau : V \to W$ with component at each (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ of the form*

$$\begin{array}{ccc} (U(A)^{\mathbf{n}} \underset{\langle U(\alpha)^{\mathbf{n}},\theta_B\rangle}{\times} U(B)^{\mathbf{m}}) \times U(B) & \underset{\langle U(\beta)^{\mathbf{m}}\theta_B,1\rangle\times 1}{\overset{\pi_2\times 1}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B) \\ \downarrow & & \| \\ U(B)^{\mathbf{m}} \times U(A) & \underset{1\times U(\beta)}{\overset{1\times U(\alpha)}{\rightleftarrows}} & U(B)^{\mathbf{m}} \times U(B); \end{array}$$

2. *There exists an isomorphism $\chi : P \to Q$ with component at each $(E, (A, B, \alpha, \beta), f)$ in $(\mathbb{A} \downarrow D_{\mathbb{A}})$ of the form*

$$\begin{array}{ccc} (U(A \underset{\langle\alpha,f\rangle}{\times} E) \times U(B) & \underset{U(\langle\beta f,1\rangle)\times 1}{\overset{U(\pi_2)\times 1}{\rightleftarrows}} & U(E) \times U(B) \\ \downarrow & & \| \\ U(E) \times U(A) & \underset{1\times U(\beta)}{\overset{1\times U(\alpha)}{\rightleftarrows}} & U(E) \times U(B); \end{array}$$

3. *$\mathcal{V}$ satisfies Condition 4.1.*

Proof. The equivalence of 1 and 3 follows from Theorem 4.5. It is easy to show that $2 \Rightarrow 1$ since P and Q composed with the functor sending (A, B, α, β) in $\mathbf{Pt}(\mathbb{A})$ to $(B, (A, B, \alpha, \beta), 1_B)$ in $(\mathbb{A} \downarrow D_{\mathbb{A}})$ are up to natural isomorphism the functors V and W respectively. We will show that $3 \Rightarrow 2$.

Let p and q be ternary terms as in Condition 4.1. It is easy to check that χ with component at each $(E, (A, B, \alpha, \beta), f)$ defined by $\chi_{(E,(A,B,\alpha,\beta),f)} = (\varphi_{(E,(A,B,\alpha,\beta),f)}, 1_{U(B)})$ where $\varphi_{(E,(A,B,\alpha,\beta),f)}((a, e), b) = (e, p(a, \beta f(e), \beta(b)))$ is an isomorphism with inverse $\chi^{-1}_{(E,(A,B,\alpha,\beta),f)} = (\psi_{(E,(A,B,\alpha,\beta),f)}, 1_U(B))$ where $\psi_{(E,(A,B,\alpha,\beta),f)}(e, a) = ((q(a, \beta\alpha(a), \beta f(e)), e), \alpha(a))$. ☐

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