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Characterizations of EP and normal Banach algebra elements and Banach space operators

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Abstract

Several characterizations of EP and normal Moore-Penrose invertible Banach algebra elements will be considered. The Banach space operator case will be also studied. The results of the present article will extend well known facts obtained in the frames of matrices and Hilbert space operators.

Keywords: EP element, normal element, Moore-Penrose inverse, group inverse, Banach algebra

AMS classification: 15A09; 47A05

1. Introduction

In [26, 27] EP and normal Moore-Penrose invertible elements were studied in the frame of rings with involution focusing on the pure algebraic structure of the objects under consideration. In addition, these works extended several well known results obtained for matrices, [9, 2], and for Hilbert space operators, [11, 10]. The objective of the present article is to characterize both EP and normal Moore-Penrose invertible elements in arbitrary Banach algebras and EP and normal Moore-Penrose invertible Banach space operators. It is worth noticing that although contexts and arguments are different, above all because of the lack of an involution, results similar to the ones in the above mentioned papers will be presented. Moreover, the proofs of the results of this work give a new insight into the cases where an involution does exist (matrices, Hilbert space operators, C^*-algebras). Furthermore, the results considered also apply to EP and normal matrices defined using an abitrary norm on a finite dimensional vector space, which extends and generalizes the results known for EP and normal matrices defined using the conjugate transpose of a matrix.

From now on, X will denote a Banach space and $L(X)$ the Banach algebra of all bounded and linear maps defined on and with values in X. In addition, if $T \in L(X)$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T respectively. Note also that $I \in L(X)$ will denote the identity operator on X.

Recall that the *descent* and the *ascent* of $T \in L(X)$ are $d(T) = \inf\{n \geq 0 \colon R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \geq 0 \colon N(T^n) = N(T^{n+1})\}$ respectively, where if some of the above sets is empty, its infimum is then defined as ∞, see for example [33]. In particular, note that if $a(T)$ and $d(T)$ are finite, then they coincide, see [33, Theorem 3.6].

On the other hand, A will denote a unital Banach algebra and $e \in A$ will stand for the identity of A. If $a \in A$, then $L_a \colon A \to A$ and $R_a \colon A \to A$ will denote the map defined by left

and right multiplication respectively:

$$L_a(x) = ax, \qquad R_a(x) = xa,$$

where $x \in A$. Note that given $a, b \in A$, $L_{ab} = L_aL_b$ and that $L_a = L_b$ implies that $a = b$. Similarly, $R_{ab} = R_bR_a$ and if $R_a = R_b$, then $a = b$. Moreover, the following notation will be used:

$$N(L_a) = a^{-1}(0), \;\; R(L_a) = aA, \;\; N(R_a) = a_{-1}(0), \;\; R(R_a) = Aa.$$

Recall that an element $a \in A$ is called *regular,* if it has a *generalized inverse,* namely if there exists $b \in A$ such that

$$a = aba.$$

Furthermore, a generalized inverse b of a regular element $a \in A$ will be called *normalized,* if b is regular and a is a generalized inverse of b, equivalently,

$$a = aba, \qquad b = bab.$$

Note that if b is a generalized inverse of a, then $c = bab$ is a normalized generalized inverse of a.

Next follows the key notion in the definition of the Moore-Penrose inverse in context of Banach algebras.

Definition 1.1. *Given a unital Banach algebra A, an element $a \in A$ will be said to be hermitian, if $\| \exp(ita) \| = 1$ for all $t \in \mathbb{R}$.*

As regard equivalent definitions and the main properties of hermitian Banach algebra elements see [34] and [7, pp. 55, 57, 67, 205]. Concerning hermitian Banach space operators, see [13, Chapter 4]. Note that an element of a C^*-algebra is hermitian if and only if it is self-adjoint, see [7, Proposition 20, Chapter I, Section 12].

In [29] V. Rakočević introduced the notion of Moore-Penrose invertible Banach algebra elements. Next the definition of such objects will be recalled.

Definition 1.2. *Let A be a unital Banach algebra and consider $a \in A$. If there exists $x \in A$ such that x is a normalized generalized inverse of a satisfying that xa and ax are hermitian, then the element x will be said to be the Moore-Penrose inverse of a, and it will be denoted by $a^\dagger$.*

In the conditions of Definition 1.2, note that according to [29, Lemma 2.1], there is at most one Moore-Penrose inverse of $a \in A$. Concerning the notion under consideration, see [29, 30, 31, 6]. Note that according to [7, Proposition 20, Chapter I, Section 12], in the frames of matrices with the conjugate transpose, Hilbert space operators and C^*-algebras, Definition 1.2 coincides with the usual definition of the Moore-Penrose inverse. For the original definition of the Moore-Penrose inverse for matrices, see [28]. In the following remark some of the most important properties of the Moore-Penrose inverse in Banach algebras will be recalled.

Remark 1.3. (i) Let A be a unital Banach algebra and consider $a \in A$. If $a^\dagger$ exists, then $a^\dagger$ is Moore-Penrose invertible. In fact, according to Definition 1.2, $(a^\dagger)^\dagger = a$.

(ii) In the conditions of (i), note that according to [6, Theorem 5(ii)], given $a \in A$ a regular element, necessary and sufficient for a to be Moore-Penrose invertible is that $L_a \in L(A)$ has a Moore-Penrose inverse. Moreover, in this case $(L_a)^\dagger = L_{a^\dagger}$.

(iii) Let $A = L(X)$, X a Banach space, and $T \in L(X)$ a Moore-Penrose invertible operator. Then $TT^\dagger \in L(X)$ and $T^\dagger T \in L(X)$ are idempotents such that $R(TT^\dagger) = R(T)$, $N(TT^\dagger) = N(T^\dagger)$, $X = R(T) \oplus N(T^\dagger)$, $R(T^\dagger T) = R(T^\dagger)$, $N(T^\dagger T) = N(T)$ and $X = R(T^\dagger) \oplus N(T)$. In fact, according to Definition 1.2, $T = TT^\dagger T$ and $T^\dagger = T^\dagger T T^\dagger$. Consequently, $T^\dagger T = T^\dagger T T^\dagger T$, $TT^\dagger = TT^\dagger TT^\dagger$, $R(T) \subseteq R(TT^\dagger)$ and $N(TT^\dagger) \subseteq N(T^\dagger)$. Since $R(TT^\dagger) \subseteq R(T)$ and $N(T^\dagger) \subseteq N(TT^\dagger)$, $R(TT^\dagger) = R(T)$ and $N(TT^\dagger) = N(T^\dagger)$. In particular, $X = R(T) \oplus N(T^\dagger)$. The remaining identities can be proved interchanging T with $T^\dagger$.

In the following definition the notion of EP Banach algebra element will be recalled, see [6].

Definition 1.4. *Given a unital Banach algebra A, an element $a \in A$ will be said to be EP, if there exists $a^\dagger$, and $aa^\dagger = a^\dagger a$.*

As regard the properties of EP Hilbert space operators see [8, 16], of EP C^*-algebra elements see [20, 24, 3, 12], and of EP Banach space operators and EP Banach algebra elements see [6]. In the following remark some of the main results concerning EP Banach algebra elements will be considered.

Remark 1.5. (i) Let A be a unital Banach algebra and consider $a \in A$. Note that $a \in A$ is EP if and only if $a^\dagger$ is EP.

(ii) In the conditions of (i), according to [6, Remark 12], necessary and sufficient for $a \in A$ to be EP is the fact that $L_a \in L(A)$ is EP.

(iii) Let $A = L(X)$, X a Banach space, and consider $T \in L(X)$. Then, according to [6, Theorem 16], T is EP if and only if $R(T) = R(T^\dagger)$ or $N(T) = N(T^\dagger)$.

The group inverse is a notion closely related to the one of EP Banach algebra element. Since in the following section this notion will be intensively used, it will be recalled.

Definition 1.6. *Given a unital Banach algebra A and $a \in A$, an element $b \in A$ will be said to be the group inverse of a, if the following set of equations is satisfied:*

$$a = aba, \qquad b = bab, \qquad ab = ba.$$

In the conditions of Definition 1.6, note that according to [19, Theorem 9], if the group inverse of $a \in A$ exists, then it is unique. In this case, the group inverse of $a \in A$ will be denoted by $a^\sharp$. In the following remark some of the most relevant properties of the group inverse will be given.

Remark 1.7. (i) Let A be a unital Banach algebra and consider $a \in A$. Suppose that $b \in A$ is a normalized generalized inverse of a. Then, necessary and sufficient for b to be the group inverse of a is that $L_b \in L(A)$ (respectively $R_b \in L(A)$) is the group inverse of $L_a \in L(A)$ (respectively $R_a \in L(A)$). In fact, since L_b is a normalized generalized inverse of L_a, according to Definition 1.6 the statement under consideration is equivalent to saying that a and b commute if and only if L_a and L_b commute, which is clear. A similar argument

proves the statement for the right multiplication operator on $L(A)$. In addition, note that in this case, according to [19, Theorem 9], $(L_a)^\sharp = L_{a^\sharp}$ (respectively $(R_a)^\sharp = R_{a^\sharp}$).

(ii) In the conditions of (i), note that if $a \in A$ is group invertible, then necessary and sufficient for a to be EP is that $aa^\sharp = a^\sharp a$ is a hermitian element. In fact, if $a \in A$ is EP, then according to [19, Theorem 9], $a^\sharp$ exists, actually $a^\sharp = a^\dagger$. In particular, $aa^\sharp = a^\sharp a = a^\dagger a$ is hermitian. On the other hand, if $a^\sharp$ exists and $aa^\sharp = a^\sharp a$ is hermitian, then according to Definition 1.2, $a^\dagger$ exists. What is more, according to [29, Lemma 2.1], $a^\dagger = a^\sharp$. Since $aa^\dagger = aa^\sharp = a^\sharp a = a^\dagger a$, a is EP.

(iii) Let $A = L(X)$, X a Banach space, and consider $T \in L(X)$. Then, according to [22, Lemma 1 and Theorem 4], the following statements are equivalent:

$$(1)\quad T^\sharp \text{ exists}, \quad (2)\quad X = N(T) \oplus R(T), \quad (3)\quad a(T) \leq 1 \text{ and } d(T) \leq 1.$$

(iv) In the conditions of (iii), note that if $T \in L(X)$ is group invertible, then an argument similar to the one in Remark 1.3(iii) proves that $N(T) = N(T^\sharp T) = N(TT^\sharp) = N(T^\sharp)$ and $R(T) = R(TT^\sharp) = R(T^\sharp T) = R(T^\sharp)$.

On the other hand, to prove the characterizations of Section 3, normal Banach algebra elements need to be considered.

Let A be a unital Banach algebra and denote by $\mathcal{H}(A)$ the set of all hermitian elements of A. Set $\mathcal{V}(A) = \mathcal{H}(A) + i\mathcal{H}(A)$. Recall that according to [34, Hilfssatz 2(c)], for each $a \in \mathcal{V}(A)$ there exist necessary unique hermitian elements u, $v \in \mathcal{H}(A)$ such that $a = u + iv$. As a result, the operation $a^* = u - iv$ is well defined. Note that $^*\colon \mathcal{V}(A) \to \mathcal{V}(A)$ is not an involution, in particular $(ab)^*$ does not in general coincide with b^*a^*, where a, $b \in \mathcal{V}(A)$. However, if $A = \mathcal{V}(A)$ and for every $h \in \mathcal{H}(A)$, $h^2 = u + iv$, where u, $v \in \mathcal{H}(A)$ and $uv = vu$, then A is a C^*-algebra whose involution is the just considered operation, see [4, 18, 34]. Next follows the definition of normal Banach algebra element.

Definition 1.8. *Given a unital Banach algebra A, $a = u + iv \in \mathcal{V}(A)$ will be said to be normal if $uv = vu$.*

In the follwing remark several properties of normal Banach algebra elements will be recalled.

Remark 1.9. (i) Let A be a unital Banach algebra and consider $a \in \mathcal{V}(A)$. Note that necessary and sufficient for a to be normal is the fact that $aa^* = a^*a$.

(ii) In the conditions of (i), if $a \in A$, then according to the proof of [6, Theorem 5], necessary and sufficient for $a \in A$ to belong to $\mathcal{H}(A)$ is that $L_a \in \mathcal{H}(L(A))$. Therefore, $a \in \mathcal{V}(A)$ is normal if and only if $L_a \in \mathcal{V}(L(A))$ is normal. A similar statement can be proved if $R_a \in L(A)$ instead of $L_a \in L(A)$ is considered. Note also that if $a \in \mathcal{V}(A)$, then L_a, $R_a \in \mathcal{V}(L(A))$, and $(L_a)^* = L_{a^*}$ and $(R_a)^* = R_{a^*}$.

(iii) When $A = L(X)$, X a Banach space, if $T \in L(X)$ is a normal operator, then according to [14, Lemma 3], $N(T^*) = N(T)$. In addition, if $R(T)$ is closed, then according to [17, Corollary 4], $T^\sharp$ exists. Since $TT^* = T^*T$, according to Remark 1.7(iii), $R(T^*) \subseteq R(T)$.

(iv) In the conditions of (i), when $a \in A$ is a normal element, according to what has been recalled, it is not difficult to prove that $a^{-1}(0) = (a^*)^{-1}(0)$, $a_{-1}(0) = (a^*)_{-1}(0)$, and if aA (respectively Aa) is closed, then $a^*A \subseteq aA$ (respectively $Aa^* \subseteq Aa$).

Next a characterization of normal invertible elements will be presented. The following Theorem presents a new proof of the main result in [32].

Theorem 1.10. *Let A be a unital Banach algebra and consider $a \in \mathcal{V}(A)$ a normal element. Then, if a is one side invertible, a is invertible.*

Proof. Suppose that a is right invertible but not invertible. Then, there exists $b \in A$ such that $ab = e$ but $0 \in \sigma(a)$. Next consider $L_a \in L(A)$. Then, since according to [7, Proposition 4(ii), Chapter I, Section 5] $\sigma(a) = \sigma(L_a)$, $0 \in \sigma(L_a)$. Moreover, according to Remark 1.9(ii), $L_a \in L(A)$ is normal. Furthermore, since $ab = e$, $aA = A$ is closed. Therefore, according to Remark 1.9(iii), $(L_a)^\sharp$ exists. However, according to Remark 1.7(iii), $A = a^{-1}(0) \oplus aA$. In particular, $a^{-1}(0) = 0$ and then $L_a \in L(A)$ is an invertible map, which is impossible for $0 \in \sigma(L_a)$. Therefore, a is invertible.

A similar argument proves that it is impossible for a to be left invertible but not invertible. □

2. EP Banach algebra elements

In this section characterizations of EP Banach algebra elements will be presented. However, to this end some preliminary results must be considered.

Lemma 2.1. *Let X be a Banach space and consider $T \in L(X)$ such that $T^\dagger$ and $T^\sharp$ exist. Then the following statements are equivalent.*

(i) T *is EP* , (ii) $R(T) \subseteq R(T^\dagger)$, (iii) $R(T^\dagger) \subseteq R(T)$,
(iv) $N(T) \subseteq N(T^\dagger)$, (v) $N(T^\dagger) \subseteq N(T)$.

Proof. According to Remark 1.5(iii), statement (i) implies statements (ii)-(v).

Suppose that statement (ii) holds. Then, according to Remark 1.3(iii) and Remark 1.7(iii),

$$X = R(T^\dagger) \oplus N(T) = R(T) \oplus N(T).$$

Thus $N(T)$ is a common complement of $R(T^\dagger)$ and $R(T)$ and so $R(T) \subseteq R(T^\dagger)$ implies that $R(T^\dagger) = R(T)$ (see [15, p.142] where it is mentioned without proof that if M, N are subspaces of X and N is a proper subspace of M, then M and N cannot have a common complement). To prove the above mentioned implication, let $m \in R(T^\dagger)$. Next consider $n \in N(T)$ and $l \in R(T)$ such that $m = n + l$. Since $R(T) \subseteq R(T^\dagger)$, $l - m = -n \in R(T^\dagger) \cap N(T) = 0$. In particular, $n = 0$ and $m = l \in R(T)$. Therefore, $R(T^\dagger) = R(T)$, and according to Remark 1.5(iii), T is EP.

The equivalences among statement (i) and statements (iii)-(v) follow in a similar manner. □

Next results of [11, Theorem 4.1] will be extended from Hilbert space operators to Banach algebra elements, see also [2, Theorems 1, 3 and 4], where matrices were considered. Compare with [27, Theorem 2.1], where EP elements in rings with involution were studied.

Theorem 2.2. *Let A be a unital Banach algebra and consider $a \in A$ such that $a^\dagger$ exists. Then a is EP if and only if $a^\sharp$ exists and one of the following statements holds.*

$$
\begin{aligned}
&(i)\ aa^\dagger a^\sharp = a^\dagger a^\sharp a, && (ii)\ aa^\dagger a^\sharp = a^\sharp aa^\dagger,\\
&(iii)\ aa^\sharp a^\dagger = a^\dagger aa^\sharp, && (iv)\ aa^\sharp a^\dagger = a^\sharp a^\dagger a,\\
&(v)\ a^\dagger aa^\sharp = a^\sharp a^\dagger a, && (vi)\ (a^\dagger)^2 a^\sharp = a^\dagger a^\sharp a^\dagger,\\
&(vii)\ aa^\sharp a^\dagger = a^\sharp, && (viii)\ a^\dagger a^\sharp = a^\sharp a^\dagger,\\
&(ix)\ a^\dagger a^\sharp a^\dagger = a^\sharp (a^\dagger)^2, && (x)\ a^\dagger (a^\sharp)^2 = a^\sharp a^\dagger a^\sharp,\\
&(xi)\ a^\dagger (a^\sharp)^2 = (a^\sharp)^2 a^\dagger, && (xii)\ (a^\sharp)^2 a^\dagger = a^\sharp a^\dagger a^\sharp,\\
&(xiii)\ aa^\sharp = a^\dagger a, && (xiv)\ a^\dagger a^\dagger = a^\dagger a^\sharp,\\
&(xv)\ a^\dagger a^\dagger = a^\sharp a^\dagger, && (xvi)\ (a^\dagger)^2 = (a^\sharp)^2,\\
&(xvii)\ a^\dagger a^\sharp = (a^\sharp)^2, && (xviii)\ a(a^\dagger)^2 = a^\sharp,\\
&(xix)\ a^\sharp a^\dagger a = a^\dagger, && (xx)\ a^\dagger aa^\sharp = a^\dagger.
\end{aligned}
$$

Proof. It is clear that if a is EP, then all the statement hold.

On the other hand, to prove the converse implications, first of all consider $A = L(X)$, X a Banach space, and let $T \in L(X)$ be a bounded operator defined on X such that $T^\dagger$ and $T^\sharp$ exist. The statements of the present Theorem will be divided in several cases.

First case: $R(T) = R(T^\dagger)$.

Suppose that statement (i) holds, that is $TT^\dagger T^\sharp = T^\dagger T^\sharp T$. In particular, according to Remark 1.7(iv),

$$R(TT^\dagger T^\sharp) = TT^\dagger(R(T^\sharp)) = TT^\dagger(R(T)) = R(TT^\dagger T) = R(T).$$

On the other hand, according to Remark 1.7(iv) and Remark 1.3(iii),

$$R(T^\dagger T^\sharp T) = T^\dagger(R(T^\sharp T)) = T^\dagger(R(T)) = R(T^\dagger T) = R(T^\dagger).$$

Therefore, $R(T) = R(T^\dagger)$, and according to Remark 1.5(iii), T is an EP operator.

Next suppose that statement (viii) is satisfied, that is $T^\dagger T^\sharp = T^\sharp T^\dagger$. Consequently, according to Remark 1.7(iv) and Remark 1.3(iii),

$$R(T^\dagger T^\sharp) = T^\dagger(R(T^\sharp)) = T^\dagger(R(T)) = R(T^\dagger T) = R(T^\dagger).$$

As regard the other range space, according to Remark 1.7(iii)-(iv) and Remark 1.3(iii),

$$R(T^\sharp T^\dagger) = T^\sharp(R(T^\dagger) \oplus N(T)) = R(T^\sharp) = R(T).$$

As before, T is EP. Moreover, the equivalence between statement (xiii) and the condition of being EP can be proved in a similar way.

Second case: $R(T^\dagger) \subseteq R(T)$.

Suppose that statement (v) is true, that is $T^\dagger TT^\sharp = T^\sharp T^\dagger T$. Then, according to Remark 1.7(iv) and Remark 1.3(iii),

$$R(T^\dagger TT^\sharp) = T^\dagger(R(TT^\sharp)) = T^\dagger(R(T)) = R(T^\dagger),$$

while $R(T^\sharp T^\dagger T) \subseteq R(T^\sharp) = R(T)$ (Remark 1.7(iv)). Therefore, $R(T^\dagger) \subseteq R(T)$. However, according to Lemma 2.1(iii), T is EP. The equivalences among statements (ix)-(xi) and the condition of being EP can be proved in a similar way.

Third case: $R(T) \subseteq R(T^\dagger)$.

Suppose that statement (xv) holds, that is $T^\dagger T^\dagger = T^\sharp T^\dagger$. Thus, according to Remark 1.7(iv) and Remark 1.3(iii),

$$R(T^\sharp T^\dagger) = T^\sharp(R(T^\dagger)) = T^\sharp(R(T^\dagger) \oplus N(T)) = R(T^\sharp) = R(T).$$

Therefore, $R(T) \subseteq R((T^\dagger)^2) \subseteq R(T^\dagger)$. Now well, according to Lemma 2.1(ii), T is EP. The equivalences among statements (xvi)-(xvii) and the condition of being EP can be proved in a similar way.

Fourth case: $N(T) \subseteq N(T^\dagger)$.

Suppose that statement (ii) holds, that is $TT^\dagger T^\sharp = T^\sharp TT^\dagger$. Consider $x \in N(T)$. Then, $TT^\dagger(x) \in N(T^\sharp) = N(T)$ (Remark 1.7(iv)). Consequently, $TT^\dagger(x) \in N(T) \cap R(T) = 0$ (Remark 1.7(iii)). In particular, $x \in N(TT^\dagger) = N(T^\dagger)$ (Remark 1.3(iii)). Therefore, $N(T) \subseteq N(T^\dagger)$. However, according to Lemma 2.1(iv), T is EP. The equivalences among statements (vi), (xii) and (xix)-(xx) and the condition of being EP can be proved in a similar way.

The fifth and last case: $N(T^\dagger) \subseteq N(T)$.

Suppose that statement (iii) holds, that is $TT^\sharp T^\dagger = T^\dagger TT^\sharp$. Let x belong to $N(T^\dagger)$. Then, $TT^\sharp(x) \in N(T^\dagger) \cap R(T) = 0$ (Remark 1.3(iii)). Thus, $x \in N(TT^\sharp) = N(T)$ (Remark 1.7(iv)). Now well, since $N(T^\dagger) \subseteq N(T)$, according to Lemma 2.1(v), T is EP. The equivalences among statements (iv), (vii), (xiv) and (xviii) and the condition of being EP can be proved in a similar way.

Next consider an arbitrary Banach algebra A. According to Remark 1.3(ii) and Remark 1.7(i), it is possible to consider L_a, $L_{a^\dagger}$, $L_{a^\sharp} \in L(A)$. What is more, $(L_a)^\dagger = L_{a^\dagger}$ and $(L_a)^\sharp = L_{a^\sharp}$. Then, if one of the statements holds, the same statement holds for L_a and, according to what has been proved, L_a is EP, which, according to Remark 1.5(ii), implies that a is EP. □

Next some results of [10, Theorem 5.1], see also [9, Theorem 2.3], will be proved in the context of Banach algebras. Compare with [27, Theorem 2.1], where EP elements in rings with involution were studied. However, recall first that given a Banach algebra A and $a \in A$, the element a is said to be *quasinilpotent*, if $\sigma(a) = \{0\}$, where $\sigma(a)$ denotes the spectrum of a. In particular, if $a \in A$ is quasinilpotent, then $r(a) = 0$, where $r(a)$ stands for the *spectral radius* of a, i. e., $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

Theorem 2.3. *Let A be a unital Banach algebra and consider $a \in A$ such that $a^\dagger$ exists. Then, necessary and suffcient for $a \in A$ to be EP is that one of the following statements holds.*

$$(i)\ a^2a^\dagger + a^\dagger a^2 = 2a,\ \ (ii)\ (a^\dagger)^2a + a(a^\dagger)^2 = 2a^\dagger,\ \ (iii)\ a^\sharp \text{ exists and } a^\dagger a^\sharp a + aa^\sharp a^\dagger = 2a^\dagger.$$

Proof. Clearly, if $a \in A$ is EP, then all the statements hold.

Note that statement (i) is equivalent to $a(aa^\dagger - a^\dagger a) = (aa^\dagger - a^\dagger a)a$. Now well, according to [23], $aa^\dagger - a^\dagger a$ is quasinilpotent. However, according to [7, Theorem 17, Chapter I, Section 10], the spectral radius of $aa^\dagger - a^\dagger a$ coincides with $\| aa^\dagger - a^\dagger a \|$. Therefore, $aa^\dagger = a^\dagger a$.

To prove that statement (ii) implies that a is EP, apply (i) interchanging a with $a^\dagger$.

Suppose that statement (iii) holds. Multiplying by a it is not difficult to obtain

$$aa^\sharp + aa^\dagger = 2aa^\dagger.$$

Thus, $aa^\sharp = aa^\dagger$. However, multiplying by $a^\dagger$, $a^\dagger aa^\sharp = a^\dagger$. Consequently, according to Theorem 2.2(xx), a is EP. □

In what follows, the ascent and the descent of a Banach space operators will be used to characterize EP bounded and linear maps and EP Banach algebra elements.

Theorem 2.4. *Let X be a Banach space, and consider $T \in L(X)$ such that $T^\dagger$ exists. Then T is EP if and only if one of the following statements holds.*

(i)	$a(T) < \infty$ *and* $T(T^\dagger)^2 = T^\dagger$,	(ii)	$a(T^\dagger) < \infty$ *and* $T^\dagger T^2 = T$,
(iii)	$d(T^\dagger) < \infty$ *and* $T^2T^\dagger = T$,	(iv)	$d(T) < \infty$ *and* $(T^\dagger)^2T = T^\dagger$,
(v)	$a(T) < \infty$ *and* $T^2T^\dagger = T$,	(vi)	$a(T^\dagger) < \infty$ *and* $(T^\dagger)^2T = T^\dagger$,
(vii)	$d(T) < \infty$ *and* $T^\dagger T^2 = T$,	$(viii)$	$d(T^\dagger) < \infty$ *and* $T(T^\dagger)^2 = T^\dagger$.

Furthermore, necessary and sufficient for T to be EP is that $T^\sharp$ exists and one of the following identities holds.

$$(ix)\ T = T^\dagger T^2,\ (x)\ T = T^2T^\dagger,\ (xi)\ T^\dagger = T(T^\dagger)^2,\ (xii)\ T^\dagger = (T^\dagger)^2T.$$

Proof. It is clear that if T is an EP operator, then all the statements hold.

On the other hand, if statement (i) holds, then $T^2(T^\dagger)^2 = TT^\dagger$. In particular, $R(T) = R(TT^\dagger) \subseteq R(T^2)$ (Remark 1.3(iii)). Therefore, $d(T) \leq 1$, and since $a(T)$ and $d(T)$ are finite, according to [33, Theorem 3.6], $a(T) \leq 1$ and $d(T) \leq 1$. Consequently, according to Remark 1.7(iii), $T^\sharp$ exists. However, since $R(T^\dagger) \subseteq R(T)$, according to Lemma 2.1(iii), T is EP.

Suppose that statement (iii) holds and let $x \in N((T^\dagger)^2)$. Then $0 = T^2(T^\dagger)^2(x) = TT^\dagger(x)$. Consequently, $x \in N(TT^\dagger) = N(T^\dagger)$ (Remark 1.3(iii)). Thus $a(T^\dagger) \leq 1$. However, since $d(T^\dagger)$ is finite and $a(T^\dagger) \leq 1$, according to [33, Theorem 3.6], $(T^\dagger)^\sharp$ exists. Now well, since $N(T^\dagger) \subseteq N(T)$, according to Lemma 2.1(iv) and Remark 1.5(i), T is EP.

If statement (v) holds, then $d(T) \leq 1$. Consequently, using an argument similar to the one in the previous paragraphs, $T^\sharp$ exists. However, since $N(T^\dagger) \subseteq N(T)$, according to Lemma 2.1(v), T is EP.

If statement (vii) holds, then $a(T) \leq 1$, and then as before, $T^\sharp$ exists. However, since $R(T) \subseteq R(T^\dagger)$, according to Lemma 2.1(ii), T is EP.

To prove that statements (ii), (iv), (vi) and (viii) imply the condition of being EP, use what has been proved and interchage T with $T^\dagger$.

Concerning the second characterization, if T is EP, it is clear that $T^\sharp$ exists and statements (ix)-(xii) hold. On the other hand, if T has a group inverse and one of the statements (ix) - (xii) holds, then, according to Remark 1.7(iii) and what has been proved, T is EP. □

Next Theorem 2.4 will be applied to prove new characterizations of EP Banach algebra elements.

Theorem 2.5. *Let A be a unital Banach algebra and consider $a \in A$ such that $a^\dagger$ exists. Then necessary and sufficient for $a \in A$ to be EP is that $a^\sharp$ exists and one of the following statements holds.*

$$(i)\ \ a = a^\dagger a^2,\ (ii)\ \ a = a^2 a^\dagger,\ (iii)\ \ a^\dagger = a(a^\dagger)^2,\ (iv)\ \ a^\dagger = (a^\dagger)^2 a.$$

Proof. According to Remark 1.3(ii) and Remark 1.7(i), $(L_a)^\dagger = L_{a^\dagger}$ and $(L_a)^\sharp = L_{a^\sharp}$ respectively. Moreover, according to Remark 1.5(ii), a is EP if and only if $L_{a^\dagger}$ is EP. To prove the Theorem it is then enough to apply Theorem 2.4 to $L_a \in L(A)$. □

3. Normal Banach algebra elements

In the present section Moore-Penrose invertible normal elements in arbitrary Banach algebras will be studied. In first place, a well known property will be considered.

Recall that necessary and sufficient for a $m \times m$ complex matrix M to be normal is that M is EP and $M^*M^\dagger = M^\dagger M^*$, where M^* denotes the conjugate transpose of M, see [9, Lemma 1.1(d)] and the references mentioned there. In the contex of C^*-algebras, a similar result holds for normal Moore-Penrose invertible elements of the algebra. For sake of completeness, this fact will be proved.

Theorem 3.1. *Let A be C^*-algebra and consider $a \in A$ such that $a^\dagger$ exists. Then, necessary and sufficient for a to be normal is that a is EP and $a^*a^\dagger = a^\dagger a^*$.*

Proof. Let $a \in A$ be a normal Moore-Penrose invertible element. Then, according to [19, Theorem 5], a is EP and a^* and $a^\dagger$ commute.

Now suppose that $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$, and consider $v = aa^* - a^*a$. A straightforward calculation, using in particular that $aa^\dagger$ and $a^\dagger a$ are self-adjoint, proves that $va^\dagger = 0$. Consequently, $a^\dagger A \subseteq v^{-1}(0)$. In addition, since according to [24, Theorem 3.1(iv)] $a^{-1}(0) = (a^*)^{-1}(0)$, $a^{-1}(0) \subseteq v^{-1}(0)$. However, according to Remark 1.3(ii), $L_a \in L(A)$ is Moore-Penrose invertible, what is more, according to Remark 1.3(ii)-(iii), $A = a^\dagger A \oplus a^{-1}(0)$. Therefore, $v^{-1}(0) = A$, equivalently, a is normal. □

To characterize normal Moore-Penrose invertible Banach algebra elements, some preparation is needed.

Theorem 3.2. *Let X be a Banach space and consider $T \in L(X)$ such that $T^\dagger$ exists and $T \in \mathcal{V}(L(X))$. Then, the following statements hold.*
(i) $R(T^*) \subseteq R(T)$ if and only if $T = TTT^\dagger$.
(ii) $N(T) \subseteq N(T^*)$ if and only if $T = T^\dagger TT$.

In addition, necessary and sufficient for T to be EP is that the conditions of statements (i) and (ii) hold.

Proof. Since $R(T) = R(TT^\dagger) = N(I - TT^\dagger)$, $R(T^*) \subseteq R(T)$ is equivalent to $T^* = TT^\dagger T^*$.

Next consider U, $V \in \mathcal{H}(L(X))$ such that $T = U + iV$ and $T^* = U - iV$. Recall also that $T = TT^\dagger T$. Then, adding and substracting T and T^*, $U = TT^\dagger U$ and $V = TT^\dagger V$. In addition, according to [5, Theorem 2.13], $UTT^\dagger = TT^\dagger U$ and $VTT^\dagger = TT^\dagger V$. Therefore $T = U + iV = UTT^\dagger + iVTT^\dagger = TTT^\dagger$.

On the other hand, if $T = TTT^\dagger$, then

$$U + iV = (U + iV)TT^\dagger = TT^\dagger(U + iV).$$

In particular $(UTT^\dagger - TT^\dagger U) + i(VTT^\dagger - TT^\dagger V) = 0$. However, since V, $TT^\dagger \in \mathcal{H}(L(X))$, according to [34, Hilfssatz 2(b)], $i(VTT^\dagger - TT^\dagger V) \in \mathcal{H}(L(X))$. In addition, according to [34, Hilfssatz 2(a)], $(UTT^\dagger - TT^\dagger U) = -i(VTT^\dagger - TT^\dagger V) \in \mathcal{H}(L(X))$. Moreover, multiplying by $-i$ the identity in the third line of the present paragraph, $(VTT^\dagger - TT^\dagger V) + i(TT^\dagger U - UTT^\dagger) = 0$. Then, an argument similar to the previous one proves that $(VTT^\dagger - TT^\dagger V) \in \mathcal{H}(L(X))$. However, according to [34, Hilfssatz 2(c)], $UTT^\dagger = TT^\dagger U$ and $VTT^\dagger = TT^\dagger V$. Consequently, since $TT^\dagger$ is an idempotent, $R(T) = R(TT^\dagger)$ and $N(T^\dagger) = N(TT^\dagger)$ are closed invariant subspaces both for U and V.

Consider $U' = U\ |_{N(T^\dagger)}^{N(T^\dagger)} \in L(N(T^\dagger))$ and $V' = V\ |_{N(T^\dagger)}^{N(T^\dagger)} \in L(N(T^\dagger))$. According to [13, Proposition 4.12], U', $V' \in \mathcal{H}(L(N(T^\dagger)))$. If T' is the restrictions of T to $N(T^\dagger)$, then it is clear that $T' = U' + iV'$. However, since $T = TTT^\dagger$, $N(T^\dagger) \subseteq N(T)$, which, according to [34, Hilfssatz 2(c)], implies that $U' = V' = 0$. In particular, $T^*(N(T^\dagger)) = 0$. In addition, according to what has been proved at the end of the previous paragraph, it is clear that $T^*(R(T)) \subseteq R(T)$. Therefore, accoding to Remark 1.3(iii), $R(T^*) \subseteq R(T)$.

Next suppose that $N(T) \subseteq N(T^*)$. Since $N(T) = N(T^\dagger T) = R(I - T^\dagger T)$, $N(T) \subseteq N(T^*)$ is equivalent to $T^* = T^* T^\dagger T$. Now well, as in the proof of statement (i), if $T = U + iV$ and $T^* = U - iV$, with U, $V \in \mathcal{H}(L(X))$, adding and substracting T and T^* and using that $T = TT^\dagger T$, it is then clear that $U = UT^\dagger T$ and $V = VT^\dagger T$. In particular, $UT^\dagger T$, $VT^\dagger T \in \mathcal{H}(L(X))$. However, according again to [5, Theorem 2.13], $UT^\dagger T = T^\dagger TU$ and $VT^\dagger T = T^\dagger TV$. Consequently $T = (U + iV)T^\dagger T = T^\dagger TT$.

On the other hand, if $T = T^\dagger TT$, applying an argument similar to the one used in the proof of statement (i) but considering $T^\dagger T$ instead of $TT^\dagger$, it is then not difficult to prove that $UT^\dagger T = T^\dagger TU$ and $VT^\dagger T = T^\dagger TV$. As a result, since $T^\dagger T$ is an idempotent such that $N(T^\dagger T) = N(T)$, $N(T)$ is a closed invariant subspaces both for U and V.

Consider $\tilde{U} = U\ |_{N(T)}^{N(T)}$, $\tilde{V} = V\ |_{N(T)}^{N(T)} \in L(N(T))$. According again to [13, Proposition 4.12], $\tilde{U}$, $\tilde{V} \in \mathcal{H}(L(N(T)))$. However, since $T(N(T)) = 0$, according to [34, Hilfssatz 2(c)], $\tilde{U} = \tilde{V} = 0$. Consequently, $T^*(N(T)) = 0$, equivalently, $N(T) \subseteq N(T^*)$.

The last statement is a consequence of [6, Theorem 18(xii)]. □

Next normal Moore-Penrose invertible Banach algebra elements will be characterized. Compare with [21, Proposition 27] and [26, Lemma 1.2] where normal Moore-Penrose invertible elements of rings with involution were considered.

Theorem 3.3. *Let A be a unital Banach algebra and consider $a \in \mathcal{V}(A)$ such that $a^\dagger$ exists. Then, necessary and sufficient for a to be normal is the fact that a is EP and $a^\dagger a^* = a^* a^\dagger$.*

Proof. First of all suppose that a is normal. According to Remark 1.3(ii), $L_a \in L(A)$ has a Moore-Penrose inverse, what is more $(L_a)^\dagger = L_{a^\dagger}$, and according to Remark 1.9(ii), $L_a \in \mathcal{V}(L(A))$ and L_a is normal. However, according to Remark 1.9(iii) and Theorem 3.2, L_a is an EP operator, which, according to Remark 1.5(ii), is equivalent to the fact that a is EP. Furthermore, according to [1, Theorem], $a^\dagger a^* = a^* a^\dagger$.

On the other hand, suppose that a is EP and $a^\dagger a^* = a^* a^\dagger$. Since, according to Remark 1.9(ii) and Remark 1.3(ii), $(L_a)^* = L_{a^*}$ and $(L_a)^\dagger = L_{a^\dagger}$ respectively, according to Remark 1.5(ii), it is clear that $L_a \in L(A)$ is EP and $(L_a)^\dagger (L_a)^* = (L_a)^* (L_a)^\dagger$. It will be proved that $L_a (L_a)^* = (L_a)^* L_a$, which, according to Remark 1.9(i)-(ii), is equivalent to the fact that a is normal.

Since L_a is EP, according to [6, Theorem18(xii)] and Theorem 3.2(i), $R((L_a)^*) \subseteq R(L_a)$. In particular, $R(L_a(L_a)^* - (L_a)^* L_a) \subseteq R(L_a)$. According again to [6, Theorem18(xii)] and Theorem 3.2(ii), $(L_a(L_a)^* - (L_a)^* L_a)(N(L_a)) = 0$. Moreover, since $(L_a)^\dagger (L_a(L_a)^* - (L_a)^* L_a)(L_a)^\dagger = 0$, $(L_a(L_a)^* - (L_a)^* L_a)(R(L_a)^\dagger) \subseteq N((L_a)^\dagger)$. Thus, according to Remark 1.3(iii), $R(L_a(L_a)^* - (L_a)^* L_a) \subseteq N((L_a)^\dagger)$. Therefore, since according to Remark 1.3(iii) $R(L_a) \cap N((L_a)^\dagger) = 0$, $L_a(L_a)^* - (L_a)^* L_a = 0$. □

In the following theorem results of [11, Theorem 3.2] will be extended to normal Moore-Penrose invertible Banach algebra elements, see also [2, Theorems 2 and 6] and [26, Theorem 2.2].

Theorem 3.4. *Let A be a unital Banach algebra and consider $a \in \mathcal{V}(A)$ such that $a^\dagger$ exists. Then, necessary and sufficient for a to be normal is the fact that $a^\sharp$ exists and one of the following conditions holds.*

(i)	$aa^*a^\sharp = a^*a^\sharp a$ *and* $a = a^\dagger aa$,	(ii)	$aa^*a^\sharp = a^\sharp aa^*$ *and* $a = aaa^\dagger$,
(iii)	$aa^\sharp a^* = a^\sharp a^* a$ *and* $a = aaa^\dagger$,	(iv)	$a^*aa^\sharp = a^\sharp a^* a$ *and* $a = a^\dagger aa$,
(v)	$a^\dagger a^* a^\sharp = a^\dagger a^\sharp a^*$ *and* $a = aaa^\dagger$,	(vi)	$a^*a^\sharp a^\dagger = a^\sharp a^* a^\dagger$ *and* $a = a^\dagger aa$,
(vii)	$a^* = aa^*a^\sharp$,	$(viii)$	$a^* = a^\sharp a^* a$,
(ix)	$aa^*a = a^*aa$ *and* $a = a^\dagger aa$,	(x)	$aaa^* = aa^*a$ *and* $a = aaa^\dagger$.

Proof. According to Remark 1.9(ii), if $a \in \mathcal{V}(A)$ is normal, then $L_a \in L(A)$ is normal. In addition, since $a^\dagger$ exists, according to Remark 1.3(ii), $(L_a)^\dagger$ exists, in particular $R(L_a)$ is closed. Consequently, according to Remark 1.9(iii), $(L_a)^\sharp$ exists, which, according to Remark 1.7(i), implies that $a^\sharp$ exists. Moreover, since a is a normal Moore-Penrose invertible element, according to Remark 1.7(ii), Theorem 3.3 and [6, Theorem 18(xii)], all the statements hold.

To prove the converse implications, first of all the case $A = L(X)$, X a Banach space, and $T \in L(X)$ such that $T^\dagger$ and $T^\sharp$ exist will be considered.

Suppose that $TT^*T^\sharp = T^*T^\sharp T$ and $T = T^\dagger TT$. Then, according to Remark 1.7(iv), $(TT^* - T^*T)(R(T)) = 0$, and according to Theorem 3.2(ii), $(TT^* - T^*T)(N(T)) = 0$. However, according to Remark 1.7(iii), T is normal.

If statement (ii) holds, then $R(T^*T^\sharp - T^\sharp T^*) \subseteq N(T)$. In addition, according to Theorem 3.2(i), $R(T^*) \subseteq R(T)$. Then, $R(T^*T^\sharp - T^\sharp T^*) \subseteq R(T)$. Consequently, according to Remark

1.7(iii), $T^*T^\sharp = T^\sharp T^*$. However, according to Theorem 2.4(x), T is EP. Then, according to Remark 1.7(ii) and Theorem 3.3, T is normal.

If $TT^\sharp T^* = T^\sharp T^* T$ and $T = TTT^\dagger$, then, according to Remark 1.7(iv), $R(TT^* - T^*T) \subseteq N(T^\sharp) = N(T)$. In addition, according to Theorem 3.2(i), $R(T^*) \subseteq R(T)$. Then $R(TT^* - T^*T) \subseteq R(T)$. However, according to Remark 1.7(iii), T is normal.

Next suppose that statement (iv) holds. According to Remark 1.7(iii)-(iv), $R(T^*T) \subseteq R(T)$. In addition, note that since $TT^\sharp(T^*T) = (TT^*)TT^\sharp$,

$$TT^\sharp(T^*T)TT^\sharp = TT^\sharp(TT^*)TT^\sharp.$$

However, according to Theorem 3.2(ii), $N(T) \subseteq N(T^*)$. Therefore, according to Remark 1.7(iii), $TT^* = T^*T$.

Concerning statement (v), it can be proved that T is normal as in the case of statement (ii) using that $N(T^\dagger) \cap R(T) = 0$ (Remark 1.3(iii)).

Suppose that statement (vi) holds. Then $R(T^\dagger) \subseteq N(T^*T^\sharp - T^\sharp T^*)$. Moreover, according to Remark 1.7(iv) and Theorem 3.2(ii), $N(T) \subseteq N(T^*T^\sharp - T^\sharp T^*)$. However, according to Remark 1.3(iii), $T^*T^\sharp = T^\sharp T^*$. Now proceed as in statement (ii).

If statement (vii) holds, then according to Remark 1.7(iv), $N(T) \subseteq N(T^*)$. Clearly, $R(T^*) \subseteq R(T)$. Therefore, according to Theorem 3.2, T is EP. In particular, according to Remark 1.7(ii), $T^\sharp = T^\dagger$. To prove that T is normal, according to Theorem 3.3, it is enough to show that $T^*T^\dagger = T^\dagger T^*$.

Note that according to Remark 1.5(iii), $R(T^*T^\dagger - T^\dagger T^*) \subseteq R(T)$. In addition, according to the proof of Theorem 3.2(i), since $R(T^*) \subseteq R(T)$, $T^* = TT^\dagger T^*$. Now well,

$$T(T^*T^\dagger - T^\dagger T^*) = TT^*T^\dagger - TT^\dagger T^* = T^* - TT^\dagger T^* = 0.$$

Therefore, $R(T^*T^\dagger - T^\dagger T^*) \subseteq N(T)$. According to Remark 1.7(iii), $T^*T^\dagger = T^\dagger T^*$.

Suppose that statement (viii) holds. Then, it can be proved that T is normal as in the case of statement (vii) using $N(T^\dagger) \cap R(T) = 0$ (Remark 1.3(iii)).

If statement (ix) holds, then it can be proved that T is normal as in the case of statement (i) using T instead of $T^\sharp$.

Concerning statement (x), it can be proved that T is normal as in the case of statement (iii) using T instead of $T^\sharp$.

Next consider an arbitrary Banach algebra A and $a \in A$ satisfying the hypothesis of the Theorem. According to Remark 1.3(ii) and Remark 1.7(i), $L_a \in L(A)$ satisfies the hypotesis of the Theorem. What is more, according to Remark 1.7(i), Remark 1.3(ii) and Remark 1.9(ii), if a satisfies one of the statements of the Theorem, then L_a satisfies the same statement. Consequently, according to what has been proved, $L_a \in L(A)$ is normal. Then, according to Remark 1.9(ii), a is normal. □

Remark 3.5. Given A a C^*-algebra, it is well known that if a and $x \in A$, then $a^*ax = 0$ implies that $ax = 0$. Similarly, $xaa^* = 0$ implies $xa = 0$. More generally, if $\mathcal{R}$ is a ring with involution, an element $a \in \mathcal{R}$ that satisfies this property is said to be $*$-*cancellable*, see for example [25, Definition 5.2] and [26, Definition 1.1]. In particular, if $a \in \mathcal{R}$ is Moore-Penrose invertible, see [25, 26], then according to [25, Theorem 5.3], a is $*$-cancellable. However, in the context of the present article, not only $^*\colon \mathcal{V}(A) \to \mathcal{V}(A)$ is not in general an involution,

but also it is not clear if the cancellation property holds for Moore-Penrose invertible elements of $\mathcal{V}(A)$, see the proof of [25, Theorem 5.3]. The conjecture is that an $a \in \mathcal{V}(A)$ such that $a^\dagger$ exists is not in general $*$-cancellable.

Anyway, concerning equivalent statements characterizing the condition of being normal, in the frame of a C^*-algebras or more generally in a ring with involution $\mathcal{R}$, the cancellation property and the identity $(ab)^* = b^*a^*$, a and $b \in \mathcal{R}$, are two important properties, see the proof of [26, Theorem 2.2]. Since in the conditions of Theorem 3.4 these two properties fail to hold, in most statements an additional condition needs to be considered. Note that in these cases, according to Theorem 2.5, an element a that satisfies the conditions of Theorem 3.4 is EP. Compare with Theorem 3.3.

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