Noncontractible periodic orbits in cotangent bundles and Floer homology

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Abstract

For every nontrivial free homotopy class α of loops in any closed connected Riemannian manifold, we prove existence of a noncontractible 1-periodic orbit for every compactly supported time-dependent Hamiltonian on the open unit cotangent bundle whenever it is sufficiently large over the zero section. The proof shows that the Biran-Polterovich-Salamon capacity is finite for every closed connected Riemannian manifold and every free homotopy class of loops. This implies a dense existence theorem for periodic orbits on level hypersurfaces and, consequently, a refined version of the Weinstein conjecture: Existence of closed characteristics (one associated to each nontrivial α) on hypersurfaces in T^*M which are of contact type and contain the zero section.

1 Introduction and main results

Let M be a closed connected smooth manifold. Let $\pi : T^*M \to M$ be the cotangent bundle and $\Omega_{can} = -d\theta$ its canonical symplectic form, where θ is the Liouville form. In addition, we choose a Riemannian metric on M and denote the open unit disc cotangent bundle by DT^*M. Let $\pi_1(M)$ be the set of *free* homotopy classes of loops in M and fix a nontrivial element α. A Hamiltonian $H \in C_0^\infty(S^1 \times DT^*M)$ gives rise to the Hamiltonian vector field X_{H_t} on T^*M, which is defined by $dH_t = \iota(X_{H_t})\Omega_{can}$. Our aim is to detect 1-periodic orbits of X_{H_t} whose projection to M represents α. This set is denoted by

$$\mathcal{P}(H;\alpha) := \{z \in C^\infty(S^1, DT^*M) \mid \dot{z}(t) = X_{H_t}(z(t)), \forall t \in S^1, [\pi(z)] = \alpha\}. \quad (1)$$

Throughout we identify S^1 with $\mathbb{R}/\mathbb{Z}$ and think of H as a smooth, compactly supported function on $\mathbb{R}\times DT^*M$, $(t,z) \mapsto H_t(z)$, satisfying $H_{t+1} = H_t$. The set of lengths of all periodic geodesics representing α is the *marked length spectrum*

$$\Lambda_\alpha := \{length\,(x) \mid x \in C^\infty(S^1, M), \nabla_t\partial_t x \equiv 0, [x] = \alpha\}.$$

The infimum of geodesic lengths in the class α, namely

$$\ell_\alpha := \inf \Lambda_\alpha, \tag{2}$$

is indeed realized by a periodic geodesic according to Lemma 3.3 below.

Theorem A (Existence of noncontractible 1-periodic orbit). *Let M be a closed connected smooth Riemannian manifold and $DT^*M \subset T^*M$ the open unit disc bundle. Then, for every nontrivial free homotopy class α of loops in M, the following is true. Every Hamiltonian system (DT^*M, Ω_{can}, H), where $H \in C^\infty([0,1] \times DT^*M)$ is compactly supported and satisfies*

$$\sup_{[0,1]\times M} H =: -c \leq -\ell_\alpha,$$

admits a 1-periodic orbit z with $[\pi(z)] = \alpha$ and symplectic action $\mathcal{A}_H(z) \geq c$.

The existence problem for periodic orbits has a rich history (see [12], for example). One of the most prominent cornerstones was Floer's solution [5] of the Arnold conjecture by introducing Floer homology theory. Periodic orbits detected in the past were typically contractible. It is only very recently that first steps have been taken in finding *noncontractible* ones. Namely, by Gatien-Lalonde [9] and by Biran-Polterovich-Salamon [2]. Both approaches require rather restrictive assumptions on the manifold, such as flatness. More precisely, Theorem A is proved in [2], if M is either the flat torus or negatively curved. We discard these assumptions *completely*. Generalizations of [9] are given in [14].

Theorem A is sharp in the sense that, firstly, the inequality cannot be improved (to see this perturb and smoothen the Hamiltonian $H(t,x,y) = \ell_\alpha(-1+|y|)$, $|y| \in [0,1]$, appropriately). Secondly, the zero section $\mathcal{O}_M = M$ in the inequality cannot be replaced by an arbitrary smooth section [2, Theorem C].

The proof of Theorem A is based on combining a main tool in [2] – existence of a natural homomorphism T from symplectic homology of DT^*M to relative symplectic homology of (DT^*M, M) which factors through Floer homology of H (see commuting triangle in Figure 1) – and the main result of [17] (see also [23]): Floer homology of the Hamiltonian $T^*M \ni (x,y) \mapsto \frac{1}{2}|y|^2$ is represented by singular homology of the free loop space of M. The filtration provided by the symplectic action functional is another key ingredient.

A major contribution of the present text is to compute symplectic homology $\underleftarrow{SH}$ of DT^*M and relative symplectic homology $\underrightarrow{SH}$ of (DT^*M, M) in terms of the singular homology of the free loop space component $\mathcal{L}_\alpha M$ of M. This is the content of Theorem 3.1 and it is illustrated by the rectangular block in Figure 1. Here ι is the natural inclusion of a sublevel set of $\mathcal{L}_\alpha M$ (see Section 2.2). Note that, in the lower horizontal isomorphism, as the action interval (a,∞) gets bigger homology gets smaller, because the increasing of the chain complex admits more cancellations. Throughout all homologies are with coefficients in $\mathbb{Z}_2$. To compute these symplectic homologies we construct families of functions f_k and h_δ, respectively, as indicated in Figure 1. For appropriate k and δ, they satisfy $f_k \leq H \leq h_\delta$. Together with the composition rule for

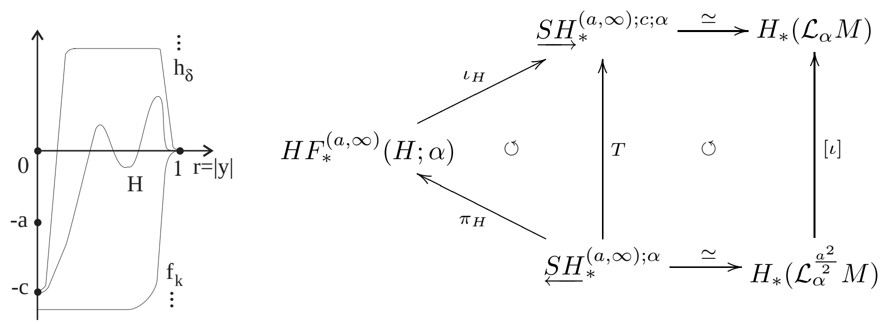

Figure 1: Commutative diagram in case $a \in (0, c]$.

the induced monotone homomorphisms this leads to the commutative triangle in the figure. Now the map $[\iota]$, hence T, and therefore Floer homology of H are nonzero iff $a \in [\ell_\alpha, c]$. If c happens to be a regular value of the symplectic action, Theorem A follows: Set $a = c$ and use nontriviality of Floer homology.

Another consequence of Theorem 3.1 is finiteness of the *Biran-Polterovich-Salamon (BPS) capacity of DT^*M relative to M* (see Theorem 4.3 and (51))

$$
\begin{aligned}
&c_{BPS}(DT^*M, M; \alpha) \\
&:= \inf\{c > 0 \mid \mathcal{P}(H;\alpha) \neq \emptyset \;\; \forall H \in C_0^\infty([0,1] \times DT^*M) : \sup_{S^1 \times M} H \leq -c\} \\
&= \ell_\alpha.
\end{aligned}
$$

In [2] a nontrivial free homotopy class α is called *symplectically essential*, if the relative BPS-capacity is finite. Hence *every* nontrivial α is symplectically essential and this immediately leads to the following two *multiplicity results* (both are proved in [2] under the assumption of symplectically essential α).

Theorem B (Dense existence). *Let M be a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ be a smooth Hamiltonian which is proper and bounded from below. Suppose that the sublevel set $\{H < c\}$ contains M. Then, for every nontrivial free homotopy class α of loops in T^*M, there exists a dense subset $S_\alpha \subset (c, \infty)$ such that the following is true. For every $s \in S_\alpha$, the level set $\{H = s\}$ contains a periodic orbit $z = (x, y)$ of the Hamiltonian system (T^*M, Ω_{can}, H) which represents α and satisfies $\int_0^1 \langle y(t), \dot{x}(t)\rangle dt > 0$.*

Proof. [2, Theorem 3.4.1]. □

The period of the orbit in the previous theorem is not specified. Note that the theorem is not true in case $\alpha = 0$, as the example of the flat torus and Hamiltonian $H(x, y) = |y|^2/2$ shows.

Theorem C (Closed characteristics). *Let M be a closed connected smooth manifold and $W \subset T^*M$ be an open set containing M with compact closure* *and smooth convex boundary $Q = \partial\overline{W}$. Let the characteristic line bundle $\mathcal{L}_Q$ be equipped with its canonical orientation. Then, for every nontrivial free homotopy class α of loops in M, the characteristic foliation of Q has a closed leaf $z \subset Q$ with $j_\#[z] = \alpha$. Here $j_\#$ is the map between free homotopy classes induced by the composition $j : Q \subset T^*M \to M$ of inclusion and projection.*

Proof. [2, Corollary 3.4.2]. □

Here convexity of Q means, by definition, that there exists a smooth Liouville vector field Z (i.e. $\mathcal{L}_Z\Omega_{can} = \Omega_{can}$), defined on a neighbourhood U of Q in $\overline{W}$ and pointing outside $\overline{W}$ along Q. In this case Q is a hypersurface of *contact type*: Z gives rise to the contact form $\lambda_Q := (\iota(Z)\Omega_{can}) \mid_{TQ}$ on Q with $d\lambda_Q = \Omega_{can} \mid_{TQ}$. The characteristic line bundle over Q is given by $\mathcal{L}_Q := \ker(\Omega_{can} \mid_{TQ})$. The corresponding foliation of Q is called *characteristic foliation*. The Reeb vector field R of λ_Q is a nonvanishing section of $\mathcal{L}_Q$ and induces a *canonical orientation*, which orients each leaf of the characteristic foliation.

Theorem C is related to the celebrated *Weinstein conjecture* [25]: Given a symplectic manifold N, then every compact hypersurface $Q \subset N$ of contact type carries a closed characteristic. The conjecture was first proved by Viterbo [20] in case $N = \mathbb{R}^{2n}$. Later Hofer and Viterbo [11] proved it for cotangent bundles under the additional hypothesis $M \subset Q \subset T^*M$. Hence our *multiplicity* result, Theorem C, refines their existence theorem. Viterbo informed us that the techniques of [11] should also provide multiplicities. In [22] he proved the Weinstein conjecture for T^*M, whenever $\pi_1(M)$ is finite. Further references concerning Weinstein conjecture and dense existence are given in the recent survey [8]. Again, multiplicity results are obtained only recently [9, 2, 14] under additional assumptions on M, which we dispose of completely.

This paper is organized as follows. In Section 2 we define action filtered Floer homology for a class of Hamiltonians which are radial and linear outside a compact subset of T^*M. We study Floer continuation and give a geometric criterion to decide if certain variations of the Hamiltonian are action-regular (hence leaving Floer homology invariant). Theorem 2.7 represents Floer homology for radial convex Hamiltonians in terms of the loop space of M. In Section 3 we recall the definition of (relative) symplectic homology and compute them in Theorem 3.1. In Section 4 we compute c_{BPS} and prove Theorem A.

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2 Floer homology

Our aim is to define Floer homology for the class of Hamiltonians which, outside some compact subset of T^*M, are radial and linear of nonnegative slope. In Subsection 2.2 we consider radial convex Hamiltonians and relate its Floer homology to the singular homology of the free loop space of M.

Throughout we use the isomorphism $T_{(x,y)}T^*M \to T_xM \oplus T_x^*M$ which takes the derivative $\dot{z}(t)$ of a curve $\mathbb{R} \to T^*M : t \mapsto z(t) = (x(t), y(t))$, where $y(t) \in T^*_{x(t)}M$, to the pair $(\dot{x}(t), \nabla_t y(t))$. The metric isomorphism $g : TM \to T^*M$ provides an almost complex structure J_g and a Riemannian metric on T^*M, both of which are compatible with Ω_{can} (see [17], for example). For $\alpha \in \pi_1(M)$, define the space of free loops in T^*M representing α by

$$\mathcal{L}_\alpha T^*M := \{z = (x,y) \mid x \in C^\infty(S^1, M),\, [x] = \alpha,\, y(t) \in T^*_{x(t)}M\}.$$

For $H \in C^\infty(S^1 \times T^*M)$, the set $\mathcal{P}(H;\alpha)$ of 1-periodic orbits of H representing α corresponds precisely to the critical points of the symplectic action functional $\mathcal{A}_H$ restricted to $\mathcal{L}_\alpha T^*M$ and given by

$$\mathcal{A}_H(x,y) := \int_0^1 \left(\langle y(t), \dot{x}(t)\rangle - H_t(x(t), y(t))\right) dt.$$

The set of its critical values is called *action spectrum of H in the class α* and it is denoted by $Spec(H;\alpha) := \mathcal{A}_H(\mathcal{P}(H;\alpha))$.

Remark 2.1 (Radial Hamiltonians). Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and such that $f(r) = f(-r)$ for every $r \in \mathbb{R}$. Consider the Hamiltonian $H^f : S^1 \times T^*M \to \mathbb{R}$, $(t,x,y) \mapsto f(|y|)$. For simplicity we denote H^f throughout by f. The set of its 1-periodic orbits can be written as

$$\begin{aligned}
\mathcal{P}(f;\alpha) &= \left\{(x,y) \in C^\infty(S^1, T^*M) \;\middle|\; \dot{x} = \tfrac{f'(|y|)}{|y|} y, \nabla_t y = 0, [x] = \alpha\right\} \\
&= \mathcal{P}^+(f;\alpha) \cup \mathcal{P}^-(f;\alpha) \\
\mathcal{P}^\pm(f;\alpha) :&= \big\{z = (x,y) \in C^\infty(S^1, T^*M) \;\big|\; x \text{ periodic geodesic}, \ell := |\dot{x}|, \\
&\qquad [x] = \alpha,\, y(t) = \pm \tfrac{r_z}{\ell}\dot{x}(t) \text{ where } r_z > 0 \text{ satisfies } f'(r_z) = \pm\ell\big\}.
\end{aligned}$$

Note that $|y(t)| = r_z$ and $\ell \in \Lambda_\alpha$. Therefore we call the elements of Λ_α also *critical slopes*. The symplectic action of $z^\pm \in \mathcal{P}^\pm(f;\alpha)$ with $f'(r_{z^\pm}) = \pm\ell$ is

$$\mathcal{A}_f(z^\pm) = f'(r_{z^\pm}) r_{z^\pm} - f(r_{z^\pm}) = \pm\ell r_{z^\pm} - f(r_{z^\pm}).$$

Given $a \in \mathbb{R}$, there are two methods to determine if the action of $z^+ \in \mathcal{P}^+(f;\alpha)$ is larger than a. *Method 1*: We have $\mathcal{A}_f(z^+) > a$, if and only if $f(r_{z^+})$ is located strictly below the line $r \mapsto -a + r\ell$. *Method 2* (works for $z^\pm \in \mathcal{P}^\pm(f;\alpha)$): Draw the tangent to the graph of f at the point $r_{z^\pm}$ which satisfies $f'(r_{z^\pm}) = \pm\ell$. This tangent $t(r) = \pm\ell r + b$ intersects the vertical coordinate axis at $b = -\mathcal{A}_f(z^\pm)$. Both methods are illustrated in Figure 2.

2.1 Hamiltonians radial and linear outside a compact set

Consider the class of smooth time-1-periodic Hamiltonians on T^*M which are radial and linear of nonnegative slope outside some compact subset of the open disk bundle $D_\rho T^*M$ (of radius $\rho > 0$), namely

$$\begin{aligned}
\mathcal{K}_\rho := \{H \in C^\infty(S^1 \times T^*M) \mid \exists \varepsilon \in (0,\rho)\ \exists \lambda \geq 0\ \exists c \in \mathbb{R} \text{ such that:} \\
H_t(x,y) = -c + \lambda\,|y| \text{ whenever } |y| \geq \rho - \varepsilon\}.
\end{aligned}$$

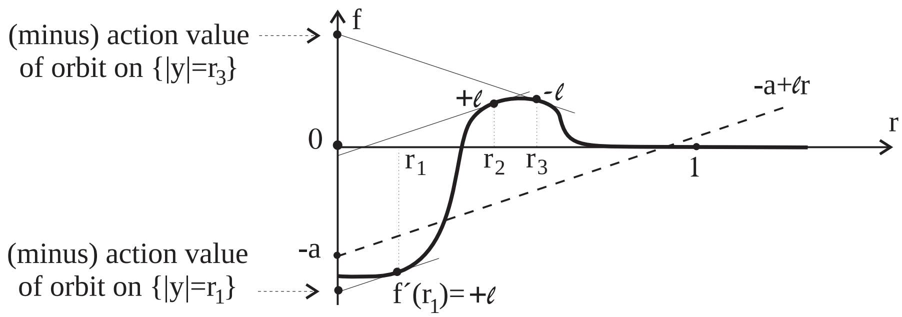


Figure 2: Action of 1-periodic orbits on $\{|y| = r\}$, where $f'(r) = \pm\ell \in \Lambda_\alpha$.

Let $\mathcal{K}_\rho$ be equipped with the C^∞-topology on the closure of $S^1 \times D_\rho T^*M$. Fix $\alpha \in \pi_1(M)$ and real numbers $-\infty \leq a < b \leq \infty$. In order to set up Floer homology for a Hamiltonian H and restricted to the action window (a, b) we shall impose the condition that a and b are not in $Spec(H; \alpha)$. This allows for small perturbations of H. It is also necessary to make sure that the set of relevant periodic orbits is compact. This is not yet satisfied: Consider the case of $H \in \mathcal{K}_\rho$ with $\lambda \in \Lambda_\alpha$ and $c \in [a, b]$, which admits 1-periodic orbits of action c on all sufficiently large hypersurfaces $S_\rho T^*M := \partial D_\rho T^*M$. We simply exclude this situation by definition, namely

$$\mathcal{K}^{a,b}_{\rho;\alpha} := \Big\{ H \in \mathcal{K}_\rho \mid \ \{a, b\} \cap Spec(H; \alpha) = \emptyset \text{ and, if } H \equiv -c + \lambda\,|y| \\ \text{whenever } |y| \geq \rho, \text{ then } \lambda \notin \Lambda_\alpha \text{ or } c \notin [a, b] \Big\}.$$

Next we would like to achieve nondegeneracy of all elements of the set $\mathcal{P}^{(a,b)}(H; \alpha)$ which, by definition, consists of all $z \in \mathcal{P}(H; \alpha)$ with $\mathcal{A}_H(z) \in (a, b)$. Every 1-periodic orbit z corresponds to a fixed point $z_0 := z(0)$ of the time-1-map φ^H of the Hamiltonian flow and vice versa. Recall that z is called *nondegenerate*, if 1 is not in the spectrum of $d\varphi^H(z_0) : T_{z_0}T^*M \circlearrowleft$. It is well known that, in order to achieve nondegeneracy, it suffices to perturb H locally near the images of the elements of $\mathcal{P}^{(a,b)}(H; \alpha)$. A proof in a slightly different setting may be found in [24, Proof of Theorem 3.6]. The proof carries over to the situation at hand and allows for the following conclusion.

Remark 2.2. Fix $H \in \mathcal{K}^{a,b}_{\rho;\alpha}$ and choose an open neighborhood $U \subset S^1 \times D_{\rho-\varepsilon}T^*M$ of the images of the elements of $\mathcal{P}^{(a,b)}(H; \alpha)$. Then there exists a neighborhood $\tilde{\mathcal{U}}$ of 0 in $C^\infty_0(U)$ and a subset $\tilde{\mathcal{U}}_{reg}$ of the second category in the sense of Baire such that the following is true. For every $h \in \tilde{\mathcal{U}}_{reg}$, all elements of $\mathcal{P}^{(a,b)}(H+h; \alpha)$ are nondegenerate and take values in U. Choosing $\tilde{\mathcal{U}}_{reg}$ smaller, if necessary, we may assume without loss of generality $H + h \in \mathcal{K}^{a,b}_{\rho;\alpha}$, $\forall h \in \tilde{\mathcal{U}}_{reg}$.

Whereas, for general $H \in \mathcal{K}^{a,b}_{\rho;\alpha}$, the set $\mathcal{P}(H; \alpha)$ may contain orbits on all sphere bundles S_rT^*M with $r \in [\rho, \infty)$, the set $\mathcal{P}^{(a,b)}(H; \alpha)$ contains only orbits taking values in $D_{\rho-\varepsilon}T^*M$. For all $h \in \tilde{\mathcal{U}}_{reg}$, the latter is also true in case of $\mathcal{P}^{(a,b)}(H+h;\alpha)$, which is therefore a finite set by standard arguments. Now fix $h \in \tilde{\mathcal{U}}_{reg}$. The Conley-Zehnder index $\mu_{CZ}(z)$ is defined naturally (see [24]), for every orbit $z \in \mathcal{P}(H;\alpha)$. Set $\sigma(z) := 0$, if $x^*TM \to S^1$ is trivial, and $\sigma(z) := 1$ otherwise, and define the integer

$$\mu(z) := -\mu_{CZ}(z) + \sigma(z). \tag{3}$$

This index provides the grading for the action filtered Floer chain groups

$$CF_k^{(a,b)}(H+h;\alpha) := \bigoplus_{\substack{z\in\mathcal{P}^{(a,b)}(H+h;\alpha)\\ \mu(z)=k}} \mathbb{Z}_2 z, \tag{4}$$

where $k \in \mathbb{Z}$. We use the convention that $CF_k^{(a,b)}(H+h;\alpha) := 0$, if $\mathcal{P}^{(a,b)}(H+h;\alpha) = \emptyset$. Existence of Floer's boundary operator is based on the following key proposition (needed at this stage in its parameter independent form only).

Proposition 2.3 (C^0-bound). *Let $r, c \geq 0$ and let $f_s : [r,\infty) \to \mathbb{R}$ be a family of smooth functions, whose dependence on the real parameter s is of class C^1, such that $\partial_s f_s' \geq 0$ and $f_s'' \geq -c$ (or $f_s'' \leq c$), for all $s \in \mathbb{R}$. Assume that $H \in C^\infty(\mathbb{R} \times S^1 \times TM)$ satisfies*

$$|y| \geq r \qquad \Rightarrow \qquad H(s,t,x,y) = f_s(|y|), \tag{5}$$

that the pair $(u,v) \in C^\infty(\mathbb{R} \times S^1, TM)$ satisfies

$$\begin{pmatrix} \partial_s u - \nabla_t v \\ \nabla_s v + \partial_t u \end{pmatrix} - \nabla H(s,t,u,v) = 0, \tag{6}$$

and that there exists $T > 0$ such that

$$|s| \geq T \qquad \Rightarrow \qquad |v(s,\cdot)| \leq r. \tag{7}$$

Then $|v| \leq r$ on $\mathbb{R} \times S^1$.

Proof. Assume by contradiction that there exists $(s_*, t_*) \in (-T,T) \times S^1$ such that

$$r_* := |v(s_*,t_*)| = \max_{[-T,T]\times S^1} |v| > r.$$

Let $a \in (r, r_*)$ be a regular value of $|v|$ and define the set

$$\Omega := \{(s,t) \in \mathbb{R} \times S^1 : \ |v(s,t)| \in [a, r_*]\} \subset (-T,T) \times S^1$$

with smooth boundary $\partial\Omega = (|v|^2)^{-1}(a)$. Denote the connected component of Ω containing (s_*, t_*) by Ω_*. Assumptions (5) and (6) show that the pair (u,v) restricted to Ω_* satisfies the equations

$$\partial_s u - \nabla_t v = 0, \qquad \nabla_s v + \partial_t u - \frac{f_s'(|v|)}{|v|} v = 0.$$

Hence, setting $L := \partial_s^2 + \partial_t^2 - f_s''(|v|)\partial_s$, we obtain on Ω_*

$$L\frac{1}{2}|v|^2 = |\partial_s u|^2 + |\nabla_s v|^2 + \big(\partial_s f_s'\big)(|v|)\,|v| \geq 0.$$

Now the proof, but not the statement, of the weak maximum principle [10, Theorem 3.1] carries over to our situation (we have a bound for $|s|$ but not for $|t|$ when viewed as a real variable): Consider first the case $f_s'' \geq -c$ and let $\varepsilon \in (0,1]$. Then on Ω_* we get

$$L(\varepsilon e^{-(c+1)s}) \geq \varepsilon(c+1)e^{-(c+1)s} \geq \varepsilon(c+1)e^{-(c+1)T} > 0.$$

In the first step we used $f_s'' \geq -c$ and the second step follows by $s \leq T$ on Ω_*. It follows that the function $\gamma_\varepsilon : \Omega_* \to \mathbb{R}$, $(s,t) \mapsto |v(s,t)|^2 + \varepsilon e^{-(c+1)s}$, satisfies $L\gamma_\varepsilon > 0$ and therefore cannot have any interior maximum. (Otherwise, if (s',t') is an interior maximum, then $\partial_s\gamma_\varepsilon(s',t') = 0$ and $(\partial_s^2+\partial_t^2)\gamma_\varepsilon(s',t') \leq 0$, which leads to the contradiction $(L\gamma_\varepsilon)(s',t') \leq 0$). In other words $\|\gamma_\varepsilon\|_{L^\infty(\Omega_*)} = \|\gamma_\varepsilon\|_{L^\infty(\partial\Omega_*)}$ and this proves the third of the following equations

$$r_*^2 = \|v\|^2_{L^\infty(\Omega_*)} = \lim_{\varepsilon\to 0}\|\gamma_\varepsilon\|_{L^\infty(\Omega_*)} = \lim_{\varepsilon\to 0}\|\gamma_\varepsilon\|_{L^\infty(\partial\Omega_*)} = \|v\|^2_{L^\infty(\partial\Omega_*)} = a.$$

To obtain the second equation we used $s \geq -T$ in Ω_*, which implies $e^{-(c+1)s} \leq e^{(c+1)T}$, and therefore

$$\sup_{\Omega_*}|v|^2 \leq \sup_{\Omega_*}\left(|v|^2 + \varepsilon e^{-(c+1)s}\right) \leq \left(\sup_{\Omega_*}|v|^2\right) + \varepsilon e^{(c+1)T} \xrightarrow{\varepsilon\to 0} \sup_{\Omega_*}|v|^2.$$

Equation four follows similarly with Ω_* replaced by $\partial\Omega_*$. Now $r_* = a$ contradicts the choice of a and this concludes the proof of the lemma in case $f_s'' \geq -c$. The argument in case $f_s'' \leq c$ is the same up to replacing $\varepsilon e^{-(c+1)s}$ by $\varepsilon e^{(c+1)s}$. ☐

Fix $z^\pm \in \mathcal{P}^{(a,b)}(H+h;\alpha)$ and consider the moduli space $\mathcal{M}(z^-,z^+;H+h,\alpha)$ of so called *connecting trajectories* which consists of smooth solutions $w : \mathbb{R}\times S^1 \to T^*M$ of Floer's elliptic partial differential equation

$$\partial_s w + J_g(w)\partial_t w - \nabla(H+h)(t,w) = 0 \tag{8}$$

subject to the asymptotic boundary conditions

$$\lim_{s\to\pm\infty} w(s,t) = z^\pm(t), \qquad \lim_{s\to\pm\infty}\partial_s w(s,t) = 0, \tag{9}$$

uniformly in $t \in S^1$. Every solution w of (8) and (9) satisfies the energy identity

$$E(w) := \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 |w(s,t)|^2\; dtds = \mathcal{A}_{H+h}(z^-) - \mathcal{A}_{H+h}(z^+).$$

More precisely, for solutions w of (8) which represent α, satisfy (7) with $r = \rho-\varepsilon$ (i.e. its ends lie in $D_{\rho-\varepsilon}T^*M$) and are such that $\mathcal{A}_{H+h}(w(s,\cdot)) \in [a,b]$, for all $s \in \mathbb{R}$, the following are equivalent: Existence of the limits (9) and finite energy of w. Here the key ingredient is nondegeneracy of all elements of $\mathcal{P}^{(a,b)}(H+h;\alpha)$. The following observations enable us to define *Floer's boundary operator*.

(B1) All relevant Floer connecting trajectories stay inside some bounded set $D_{\rho-\varepsilon}T^*M$. More precisely, for all $z^\pm \in \mathcal{P}^{(a,b)}(H+h;\alpha)$ and every $w \in \mathcal{M}(z^-,z^+;H+h,\alpha)$, it holds $w(\mathbb{R}\times S^1) \subset D_{\rho-\varepsilon}T^*M$: All elements of $\mathcal{P}^{(a,b)}(H+h;\alpha)$ lie inside $D_{\rho-\varepsilon}T^*M$, for some $\varepsilon > 0$. Therefore the assumptions of Proposition 2.3 are satisfied for $r = \rho - \varepsilon$ and $H+h$ (which is radial and linear whenever $|y| \geq \rho - \varepsilon$). This proves the claim.

(B2) The moduli spaces $\mathcal{M}(z^-,z^+;H+h,\alpha)$ are compact with respect to C^∞-convergence on compact sets. This is a consequence of the energy identity, exactness of the canonical symplectic form Ω_{can}, and (B1).

(B3) One can achieve surjectivity of the linearized operator for equation (8) by perturbing $H+h$ in any arbitrarily small neighborhood U of the image of w in $S^1 \times D_{\rho-\varepsilon}T^*M$. The perturbation can be chosen to vanish up to second order along the asymptotic orbits $z^\pm$ (see [7, Theorem 5.1 (ii)]). More precisely, there exists a neighborhood $\mathcal{U}$ of zero in $C_0^\infty(U)$ and a subset $\mathcal{U}_{reg}$ of the second category in the sense of Baire such that $\mathcal{U}_{reg} \subset \tilde{\mathcal{U}}_{reg}$ and the linearized operator for equation (8) is surjective, for all $w \in \mathcal{M}(z^-,z^+;H+h,\alpha)$, $z^\pm \in \mathcal{P}^{(a,b)}(H+h;\alpha)$ and $h \in \mathcal{U}_{reg}$. Hamiltonians satisfying this surjectivity condition are called *regular*.

Changing notation, if necessary, we may assume without loss of generality that H is regular. In this case all moduli spaces $\mathcal{M}(z^-,z^+;H,\alpha)$ are smooth manifolds of dimension $\mu_{CZ}(z^+) - \mu_{CZ}(z^-) = \mu(z^-) - \mu(z^+)$ (see [18] and [16]) and they admit a free $\mathbb{R}$-action, given by shifting the s-variable: $w(s,t) \mapsto w(s+\sigma,t)$. For $z^- \in \mathcal{P}^{(a,b)}(H;\alpha)$ with $\mu(z^-) = k$, define

$$\partial_k = \partial_k(H;\alpha) : CF_k^{(a,b)}(H;\alpha) \to CF_{k-1}^{(a,b)}(H;\alpha)$$

by

$$\partial_k z^- := \sum_{\substack{z^+ \in \mathcal{P}^{(a,b)}(H;\alpha) \\ \mu(z^+)=k-1}} \#_2 \left(\mathcal{M}(z^-,z^+;H,\alpha)/\mathbb{R}\right) \ z^+.$$

Here $\#_2$ denotes the number of elements modulo two. Define $\partial_k z^- := 0$, if the sum is over the empty set. Since the index difference is one, the coefficients in the sum are finite. This is a consequence of the compactness property (B2) of the 1-dimensional moduli space components. The identity $\partial^2 = 0$ follows as in Floer's original work [5] using the compactness property (B2) of the 2-dimensional moduli space components. Action filtered Floer homology of the Hamiltonian system (T^*M, Ω_{can}, H) and with respect to the free homotopy class α is defined by

$$HF_k^{(a,b)}(H;\alpha) := \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}, \qquad k \in \mathbb{Z}.$$

We suppress the almost complex structure J_g in the notation, since it is fixed once and for all.

Continuation

We define filtered Floer homology $HF_*^{(a,b)}(H;\alpha)$ for *every* $H \in \mathcal{K}_{\rho;\alpha}^{a,b}$ and show that on connected components of this set Floer homology is independent of H.

First observe that the subset $\mathcal{K}_{\rho;\alpha}^{a,b} \subset \mathcal{K}_\rho$ is open: The second condition in the definition of $\mathcal{K}_{\rho;\alpha}^{a,b}$, namely $\lambda \notin \Lambda_\alpha \vee c \notin [a,b]$ is clearly open, given Lemma 3.3 below. The first condition $a, b \notin Spec(H;\alpha)$ is open, because its negation "$a \in Spec(H;\alpha)$ or $b \in Spec(H;\alpha)$" is closed: Let H_ν be a sequence in $\mathcal{K}_\rho$ such that $a \in Spec(H_\nu;\alpha)$ and which converges to $H \in \mathcal{K}_\rho$ in C^∞. Hence there is a sequence of orbits $z_\nu \in \mathcal{P}(H_\nu;\alpha)$ of action a. This sequence admits a uniform C^2-bound, which implies existence of a C^1-convergent subsequence. It follows that the limit z is a 1-periodic orbit of H of action a and with $[\pi(z)] = \alpha$.

Now fix $H \in \mathcal{K}_{\rho;\alpha}^{a,b}$ and choose a convex neighbourhood $\mathcal{W}$ of H *in* $\mathcal{K}_{\rho;\alpha}^{a,b}$ and regular Hamiltonians $H^0, H^1 \in \mathcal{W}$. Outside $D_{\rho-\varepsilon}T^*M$, for some $\varepsilon > 0$, they are of the form $H_t^k(x,y) = -c^k + \lambda^k |y|$, for $k = 0,1$. Renaming H^0 and H^1, if necessary, we may assume that $\lambda^0 \leq \lambda^1$. Let $\beta : \mathbb{R} \to [0,1]$ be a smooth nondecreasing cutoff function which is zero on $(-\infty,-1]$ and one on $[1,\infty)$. Consider the smooth homotopy

$$\mathbb{R} \to \mathcal{W}, \qquad s \mapsto H_s := H^0 + \beta(s)(H^1 - H^0). \tag{10}$$

Observe that outside $D_{\rho-\varepsilon}T^*M$ it is of the form

$$f_s(|y|) = -c^0 + \lambda^0 |y| + \beta(s)\left(c^0 - c^1 + |y|(\lambda^1 - \lambda^0)\right)$$

and satisfies $\partial_s f_s'(|y|) = \dot\beta(s)(\lambda^1 - \lambda^0) \geq 0$. For $z_k \in \mathcal{P}^{(a,b)}(H^k;\alpha)$, define the moduli space $\mathcal{M}(z_0, z_1; H_s, \alpha)$ of so called *continuation trajectories* to be the set of smooth solutions $w : \mathbb{R} \times S^1 \to T^*M$ of Floer's parameter-dependent equation

$$\partial_s w + J_g(w)\partial_t w - \nabla H_{s,t}(w) = 0 \tag{11}$$

which satisfy (uniformly in $t \in S^1$) the asymptotic boundary conditions

$$\lim_{s\to-\infty} w(s,t) = z_0(t), \quad \lim_{s\to+\infty} w(s,t) = z_1(t), \quad \lim_{s\to\pm\infty} \partial_s w(s,t) = 0. \tag{12}$$

Moreover, every solution w of (11) and (12) satisfies the energy identity

$$E(w) = \mathcal{A}_{H^0}(z_0) - \mathcal{A}_{H^1}(z_1) - \int_{-\infty}^{\infty}\int_0^1 \left(\partial_s H_{s,t}\right)(w(s,t))\, dtds. \tag{13}$$

The following observations lead to the definition of Floer's chain map.

(C1) All continuation trajectories $w \in \mathcal{M}(z_0, z_1; H_s, \alpha)$ stay in $D_{\rho-\varepsilon}T^*M$, for all $z_k \in \mathcal{P}^{(a,b)}(H^k;\alpha)$ and $k = 0,1$: As observed earlier in (B1), this is true for the elements of $\mathcal{P}^{(a,b)}(H^k;\alpha)$, for $k = 0,1$. Since $\partial_s f_s' \geq 0$, it follows that H_s satisfies the assumptions of Proposition 2.3 and this proves the claim.

(C2) The moduli spaces $\mathcal{M}(z_0, z_1; H_s, \alpha)$ are compact with respect to C^∞-convergence on compact sets. Again this is a consequence of the energy identity (13), exactness of the canonical symplectic form Ω_{can} and (C1).

(C3) A perturbation argument similar to (B3) shows that there is a subset (of the second category in the sense of Baire) of *regular homotopies* among all homotopies $\mathbb{R} \to \mathcal{W}$. By definition, this means that the linearized operator for equation (11) is surjective, for all elements w of all moduli spaces $\mathcal{M}(z_0, z_1; H_s, \alpha)$.

By (C3) we may assume without loss of generality that H_s is a regular homotopy. In this case all moduli spaces $\mathcal{M}(z_0, z_1; H_s, \alpha)$ are smooth manifolds of dimension $\mu(z_0) - \mu(z_1)$. By the argument in [2, Section 4.4], we can define *Floer's continuation map* $\Phi^{10} : CF_k^{(a,b)}(H^0; \alpha) \to CF_k^{(a,b)}(H^1; \alpha)$, whenever H^0 and H^1 are *sufficiently close* to $H \in \mathcal{K}^{a,b}_{\rho;\alpha}$. For $z_0 \in \mathcal{P}^{(a,b)}(H^0; \alpha)$ with $\mu(z_0) = k$, set

$$\Phi^{10} z_0 := \sum_{\substack{z_1 \in \mathcal{P}^{(a,b)}(H_1;\alpha) \\ \mu(z_1)=k}} \#_2 \left(\mathcal{M}(z_0, z_1; H_s, \alpha)\right) \; z_1. \tag{14}$$

That the action window is preserved is a consequence of the energy identity (13). The fact that Φ^{10} is a chain map, i.e. commutes with the two boundary operators, is standard and follows by investigating the boundary of the 1-dimensional components of the moduli spaces $\mathcal{M}(z_0^-, z_1^+; H_s, \alpha)$ (see [16, Section 3.4]). Consequently Φ^{10} descends to filtered Floer homology. Moreover, again by standard arguments, it is independent of the choice of homotopy $\mathbb{R} \to \mathcal{W}$.

In order to prove that Φ^{10} induces an isomorphism on homology, it is common to consider the reverse homotopy H_{-s} from H^1 to H^0 and show that it induces the inverse map on homology. Unfortunately, this approach fails here, because H_{-s} leads to $\partial_s f'_{-s} \leq 0$ and so we can't justify $(C1)$ anymore. We overcome this problem by introducing an appropriate intermediate Hamiltonian H^2 (see Figure 3) and replacing the homotopy $H^{01} := H_s$ from H^0 to H^1 by $H_s^{02} := H^0 + \beta(s)(H^2 - H^0)$ followed by $H_s^{21} := H^2 + \beta(s)(H^1 - H^2)$. The corresponding continuation maps Φ^{10} and $\Phi^{12} \circ \Phi^{20}$ are chain homotopy equivalent by standard Floer theory (see [16, Section 3.4]) and therefore the induced maps on homology are equal. In other words, the following diagram commutes

$$\begin{array}{ccccc} & & HF_*^{(a,b)}(H^2;\alpha) & & \\ & \overset{[\Phi^{20}]}{\nearrow} & \circlearrowleft & \overset{[\Phi^{12}]}{\searrow} & \\ HF_*^{(a,b)}(H^0;\alpha) & & \xrightarrow{\quad [\Phi^{10}] \quad} & & HF_*^{(a,b)}(H^1;\alpha). \end{array}$$

We will prove that $\Phi^{20} = \mathrm{id}$ and Φ^{12} induces an isomorphism on homology. Therefore $[\Phi^{10}]$ is an isomorphism.

The Hamiltonian H^2 is defined as follows. On $D_{\rho-\varepsilon}T^*M$ it coincides with H^0 and on the complement it is radial: Start at $r = |y| = \rho - \varepsilon$. Increase r

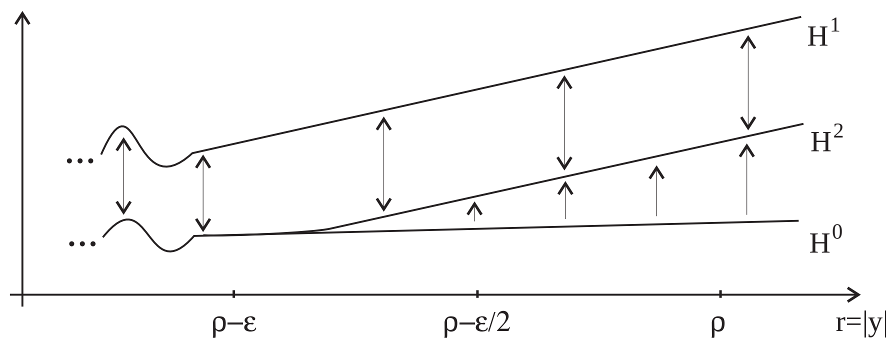


Figure 3: The Hamiltonians H^0, H^1, H^2 and homotopies.

and make a smooth left turn and continue, say outside $D_{\rho-\varepsilon/2}T^*M$, with the constant slope λ^1 of H^1 (see Figure 3). Proof of $\Phi^{20} = \mathrm{id}$: If H^0 and H^1 are sufficiently close, then the chain groups associated to H^0 and H^2 (and H^1) are identical. This follows, since $\mathbb{R} \setminus \Lambda_\alpha$ is open. Hence all relevant periodic orbits lie inside $D_{\rho-\varepsilon}T^*M$. By Proposition 2.3, this is also true for all relevant continuation trajectories (11). But in $D_{\rho-\varepsilon}T^*M$ the homotopy from H^0 to H^2 is constant and so they are all independent of s, which means that the continuation map Φ^{20} is the identity. We show that Φ^{12} induces an isomorphism on homology: The key fact is that the homotopy H^{21} is outside $D_{\rho-\varepsilon/2}T^*M$ of the form $f_s(|y|)$ with $\partial_s f_s' \equiv 0 \equiv \partial_s f_{-s}'$. Hence, by Proposition 2.3, all continuation trajectories corresponding to H^{21} and also those corresponding to the reverse homotopy H^{21}_{-s} remain in the bounded domain $D_{\rho-\varepsilon/2}T^*M$. Therefore the standard argument of reversing the homotopy applies and shows that $\Phi(H^{21}_{-s})$ induces on homology the inverse of $\Phi(H^{21}_s)$.

Definition 2.4. (1) For $H \in \mathcal{K}^{a,b}_{\rho;\alpha}$, define $HF_*^{(a,b)}(H;\alpha) := HF_*^{(a,b)}(H^0;\alpha)$, where H^0 is any regular Hamiltonian sufficiently close to H.
(2) A homotopy $\{H_s\}_{s\in[0,1]}$ taking values in a connected component of $\mathcal{K}^{a,b}_{\rho;\alpha}$ is called *action-regular*.

The observations above allow to draw the following conclusions. Firstly, the definition in (1) does not depend on the choice of H^0. Secondly, given any action-regular homotopy $\{H_s\}_{s\in[0,1]}$, define $s_k := k/N$ and choose $N \in \mathbb{N}_0$ sufficiently large to obtain the composition of isomorphisms

$$[\Phi^{s_N s_{N-1}}] \circ \cdots \circ [\Phi^{s_1 s_0}] \;:\; HF_*^{(a,b)}(H_0;\alpha) \to HF_*^{(a,b)}(H_1;\alpha). \tag{15}$$

Note that it is an open question, if the resulting isomorphism depends on the choice of homotopy between H_0 and H_1 (see [2, Remark 4.4.3]). In the case of a monotone homotopy this isomorphism can be directly defined in terms of the solutions of (11) and it is independent of the choice of the monotone homotopy. This will be discussed in the next section.

Monotone homotopies

Given $H^0, H^1 \in \mathcal{K}^{a,b}_{\rho;\alpha}$ satisfying $H^0 \leq H^1$ pointwise, there exists a homotopy $s \mapsto H_s$ from H^0 to H^1 such that $\partial_s H_s \geq 0$. Such a homotopy is called *monotone*. Observe that a monotone homotopy is outside $S^1 \times D_{\rho-\varepsilon} T^*M$, for some $\varepsilon > 0$, of the form $f_s(|y|)$ and satisfies necessarily $\partial_s f_s' \geq 0$. By the perturbation argument (B3) we may assume that H^0 and H^1 are regular. The energy identity (13) shows that the map defined by (14) in terms of the solutions of (11) is indeed a chain map from $CF_*^{(a,b)}(H^0;\alpha)$ to $CF_*^{(a,b)}(H^1;\alpha)$ (here the point is preservation of the action window). This shows that a monotone homotopy induces a homomorphism, called *monotone homomorphism*, namely

$$\sigma^{10} : HF_*^{(a,b)}(H^0;\alpha) \to HF_*^{(a,b)}(H^1;\alpha). \tag{16}$$

Lemma 2.5 ([6, 4]). *The monotone homomorphism does not depend on the choice of the monotone homotopy used to define it and*

$$\sigma^{21} \circ \sigma^{10} = \sigma^{20}, \qquad \sigma^{00} = id, \tag{17}$$

whenever $H^0, H^1, H^2 \in \mathcal{K}^{a,b}_{\rho;\alpha}$ *satisfy* $H^0 \leq H^1 \leq H^2$.

Lemma 2.6 ([21]). *The monotone homomorphism associated to an action-regular monotone homotopy is an isomorphism.*

Proof. ([2, Proof of Proposition 4.5.1]) The local isomorphisms (15) are induced by monotone homotopies; now apply (17). ☐

Given $a \leq b \leq c$ and Hamiltonians $H^0 \leq H^1 \in \mathcal{K}^{a,b}_{\rho;\alpha} \cap \mathcal{K}^{a,c}_{\rho;\alpha}$, we obtain the following commuting diagram (we temporarily suppress α in the notation), whose rows are the short exact sequences for H^0 and for H^1 (where ι^F and π^F denote the natural inclusion and projection, respectively, and α is fixed)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CF_*^{(a,b)}(H^0) & \xrightarrow{\iota^F} & CF_*^{(a,c)}(H^0) & \xrightarrow{\pi^F} & CF_*^{(b,c)}(H^0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi^{10}} & \circlearrowleft & \downarrow{\scriptstyle \Phi^{10}} & \circlearrowleft & \downarrow{\scriptstyle \Phi^{10}} & & \\ 0 & \longrightarrow & CF_*^{(a,b)}(H^1) & \xrightarrow{\iota^F} & CF_*^{(a,c)}(H^1) & \xrightarrow{\pi^F} & CF_*^{(b,c)}(H^1) & \longrightarrow & 0 \end{array} .$$

The associated long exact sequences fit into the commutative diagram

$$\begin{array}{ccccccccc} \cdot\cdot & \longrightarrow & HF_*^{(a,b)}(H^0) & \xrightarrow{[\iota^F]} & HF_*^{(a,c)}(H^0) & \xrightarrow{[\pi^F]} & HF_*^{(b,c)}(H^0) & \longrightarrow & \cdot\cdot \\ & & \downarrow{\scriptstyle \sigma^{10}} & \circlearrowleft & \downarrow{\scriptstyle \sigma^{10}} & \circlearrowleft & \downarrow{\scriptstyle \sigma^{10}} & & \\ \cdot\cdot & \longrightarrow & HF_*^{(a,b)}(H^1) & \xrightarrow{[\iota^F]} & HF_*^{(a,c)}(H^1) & \xrightarrow{[\pi^F]} & HF_*^{(b,c)}(H^1) & \longrightarrow & \cdot\cdot \end{array} . \tag{18}$$

Given $a \leq b_1 \leq c$ and Hamiltonians $H^0 \in \mathcal{K}^{a,c}_{\rho;\alpha}$ and $H^1 \in \mathcal{K}^{a,c}_{\rho;\alpha} \cap \mathcal{K}^{a,b_1}_{\rho;\alpha}$ with $H^0 \leq H^1$, let H_s be a monotone homotopy between them such that the induced continuation map satisfies $\Phi^{10}(\mathcal{P}^{(a,c)}(H^0;\alpha)) \subset \mathcal{P}^{(-\infty,b_1)}(H^1;\alpha)$. Then H_s induces a homomorphism

$$\check{\sigma}^{10} : HF_*^{(a,c)}(H^0;\alpha) \to HF_*^{(a,b_1)}(H^1;\alpha).$$

If, in addition, $b_2 \in [b_1, c]$, $\Phi^{10}(\mathcal{P}^{(a,b_2)}(H^0;\alpha)) \subset \mathcal{P}^{(-\infty,b_1)}(H^1;\alpha)$ and $H^0 \in \mathcal{K}^{a,b_2}_{\rho;\alpha}$, then the following diagram commutes (even on the chain level)

$$\begin{array}{ccc} HF_*^{(a,c)}(H^0;\alpha) & \xrightarrow{\sigma^{10}} & HF_*^{(a,c)}(H^1;\alpha) \\ {\scriptstyle [\iota^F]}\uparrow & \searrow^{\check{\sigma}^{10}} & \uparrow{\scriptstyle [\iota^F]} \\ HF_*^{(a,b_2)}(H^0;\alpha) & \xrightarrow{\check{\sigma}^{10}} & HF_*^{(a,b_1)}(H^1;\alpha) \end{array} \qquad . \tag{19}$$

2.2 Computation for radial convex Hamiltonians

Let $\mathcal{L}M = C^\infty(S^1, M)$ denote the free loop space of M and $\mathcal{L}_\alpha M$ the subspace whose elements represent a given free homotopy class α of loops in M. The *classical action functional* $\mathcal{S}_V : \mathcal{L}M \to \mathbb{R}$, for $V \in C^\infty(S^1 \times M)$, is given by

$$\mathcal{S}_V(x) := \int_0^1 \frac{1}{2}\left|\dot{x}(t)\right|^2 - V_t(x(t))\, dt. \tag{20}$$

For $a \in \mathbb{R}$, we define the sublevel set $\mathcal{L}^a_\alpha M := \{x \in \mathcal{L}_\alpha M \mid \mathcal{S}_0(x) \le a\}$ and denote its singular homology with coefficients in $\mathbb{Z}_2$ by $H_*(\mathcal{L}^a_\alpha M)$. For $b \ge a$, let $\iota_{ba} : \mathcal{L}^a_\alpha M \hookrightarrow \mathcal{L}^b_\alpha M$ denote the natural inclusion. Set $\iota_a := \iota_{\infty a}$.

Theorem 2.7 (Floer homology of radial convex Hamiltonians). *Fix a free homotopy class α. For every smooth symmetric function f with $f'' \ge 0$ the following is true. If λ is a positive noncritical slope of f, i.e. $\lambda \in \mathbb{R}^+ \setminus \Lambda_\alpha$ and $f'(\rho) = \lambda$ for some $\rho > 0$, then there is a natural isomorphism*

$$\Psi^f_\lambda : HF_*^{(-\infty,c_{f,\lambda})}(f;\alpha) \to H_*(\mathcal{L}^{\lambda^2/2}_\alpha M), \qquad c_{f,\lambda} := \rho f'(\rho) - f(\rho).$$

If g is another such function, then the following diagram commutes

$$\begin{array}{ccc} HF_*^{(-\infty,c_{f,\lambda})}(f;\alpha) & \xrightarrow[\simeq]{\Phi^{gf}_\lambda} & HF_*^{(-\infty,c_{g,\lambda})}(g;\alpha) \\ & {\scriptstyle \Psi^f_\lambda}\searrow^{\simeq} \quad {}^{\simeq}\swarrow{\scriptstyle \Psi^g_\lambda} & \\ & H_*(\mathcal{L}^{\lambda^2/2}_\alpha M) & \end{array} \qquad . \tag{21}$$

If $\mu \in (0, \lambda] \setminus \Lambda_\alpha$ is another slope of f, then the following diagram commutes

$$\begin{array}{ccc} HF_*^{(-\infty,c_{f,\mu})}(f;\alpha) & \xrightarrow{[\iota^F]} & HF_*^{(-\infty,c_{f,\lambda})}(f;\alpha) \\ {\scriptstyle \Psi^f_\mu}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \Psi^f_\lambda} \\ H_*(\mathcal{L}^{\mu^2/2}_\alpha M) & \xrightarrow{[\iota_{\frac{\lambda^2}{2}\frac{\mu^2}{2}}]} & H_*(\mathcal{L}^{\lambda^2/2}_\alpha M) \end{array} \qquad . \tag{22}$$

The proof of the theorem is based on the main result in [17] and the apriori bound provided by Proposition 2.8 below.

Proof. Let $f^{(\lambda)}$ be given by following the graph of f until slope λ turns up for the first time, say at a point r_λ, then continue linearly with slope λ. This function is only C^1. After smoothing it out nearby r_λ, we may assume that $f^{(\lambda)}$ is C^∞. Let $f_0(|y|) := |y|^2/2$. Define Ψ^f_λ to be the composition of the vertical maps on the right hand side of the following diagram

$$
\begin{array}{ccc}
HF_*^{(-\infty,c_{f,\mu})}(f+h;\alpha) & \xrightarrow{[\iota^F]} & HF_*^{(-\infty,c_{f,\lambda})}(f+h;\alpha) \\
\Big\downarrow \equiv & & \equiv \Big\downarrow [\mathrm{id}] \\
HF_*^{(-\infty,\infty)}(f^{(\mu)}+\tilde{h};\alpha) & \xrightarrow[{[\Phi]}]{[\iota^F]} & HF_*^{(-\infty,\infty)}(f^{(\lambda)}+h;\alpha) \\
\Big\downarrow \simeq & & \simeq \Big\downarrow [\Phi] \\
HF_*^{(-\infty,\infty)}(f_0^{(\mu)}+\tilde{h_0};\alpha) & \xrightarrow[{[\iota^F]}]{[\Phi]} & HF_*^{(-\infty,\infty)}(f_0^{(\lambda)}+h_0;\alpha) \\
\Big\downarrow \equiv & & \equiv \Big\downarrow [\mathrm{id}] \\
HF_*^{(-\infty,\mu^2/2)}(f_0+h_0;\alpha) & \xrightarrow{[\iota^F]} & HF_*^{(-\infty,\lambda^2/2)}(f_0+h_0;\alpha) \\
\Big\downarrow \simeq & & \simeq \Big\downarrow [\Phi] \\
HF_*^{(-\infty,\mu^2/2)}(f_0+V;\alpha) & \xrightarrow{[\iota^F]} & HF_*^{(-\infty,\lambda^2/2)}(f_0+V;\alpha) \\
\Big\downarrow \simeq & & \simeq \Big\downarrow [17,\text{ Main Theorem}] \\
H_*(\mathcal{L}_\alpha^{\mu^2/2}M) & \xrightarrow{[\iota_{\frac{\lambda^2}{2}\frac{\mu^2}{2}}]} & H_*(\mathcal{L}_\alpha^{\lambda^2/2}M)
\end{array}
\quad . \tag{23}
$$

Definition of the vertical maps on the right hand side of (23): By definition,

$$HF_*^{(-\infty,c_{f,\lambda})}(f;\alpha) := HF_*^{(-\infty,c_{f,\lambda})}(f+h;\alpha),$$

where $h \in \mathcal{U}_{reg}(f)$ as in (B3) is a small perturbation of compact support in $S^1 \times D_{r_\lambda-\varepsilon}T^*M$, for some small $\varepsilon > 0$. As observed repeatedly, all 1-periodic orbits of $f+h$ of energy less than $c_{f,\lambda}$ and all connecting trajectories (use Proposition 2.3) lie entirely in $D_{r_\lambda-\varepsilon}T^*M$. The same is true for the Hamiltonian $f^{(\lambda)}+h$ and the action window $(-\infty,\infty)$. But in $D_{r_\lambda-\varepsilon}T^*M$ both Hamiltonians coincide, hence both chain complexes are identical.

The second vertical map in (23) is Floer's continuation map $[\Phi]$ which is an isomorphism by the standard *reverse homotopy argument* (see Section *Continuation*). Given f_0, let ${f_0}^{(\lambda)}$ and $r_{0,\lambda}$ be defined as above. Pick a small perturbation $h_0 \in \mathcal{U}_{reg}(f_0)$. Since the action window is the whole real line, we are not restricted to *monotone* homotopies. Let the homotopy H_s be a convex combination of $H_0 = f^{(\lambda)} + h$ and $H_1 = {f_0}^{(\lambda)} + h_0$ as in (10). Then outside of D_RT^*M with $R := \max\{r_\lambda, r_{0,\lambda}\}$ it is of the form $f_s(|y|)$ and satisfies $\partial_s f_s' = 0$.

Hence $\Phi(H_{-s})$ is defined and induces a map on homology inverse to $[\Phi(H_s)]$. Moreover, any two convex combinations such as H_s are homotopic through such convex combinations and therefore $[\Phi(H_s)]$ is actually independent of the choice of H_s.

The third vertical map in (23) is the identity by the same argument as the first map.

The fourth vertical map is again a version Floer's continuation map. Its definition and the proof that it is an isomorphism require new arguments: Floer homology as in line five of diagram (23) is defined in [17]. The difference is the perturbation $V = V(t,x)$ which is a sufficiently small element of the set of *regular potentials*. This is a subset of the second category in the sense of Baire of $C^\infty(S^1 \times M)$ (see [24, Theorem 1.1]). For every $(x,y) \in \mathcal{P}^{(-\infty,\lambda^2/2)}(f_0 + V;\alpha)$, it follows as in [24, App. A] that

$$\begin{aligned} \|y\|_\infty^2 = \|\dot{x}\|_\infty^2 &\leq \|\dot{x}\|_2^2 + \|\nabla V\|_\infty \left(1 + \|\dot{x}\|_2^2\right) \\ &\leq \lambda^2 + 2\|V\|_\infty + \|\nabla V\|_\infty \left(1 + \lambda^2 + 2\|V\|_\infty\right). \end{aligned}$$

Hence we may assume without loss of generality, by choosing λ slightly larger in the same connected component of $\mathbb{R}\setminus\Lambda_\alpha$ if necessary, that all relevant 1-periodic orbits are nondegenerate and take values in $D_\lambda T^*M$. The same is true for the elements of $\mathcal{P}^{(-\infty,\lambda^2/2)}(f_0 + h_0;\alpha)$. Let Φ be the continuation map (14) with respect to the homotopy

$$H_s = H(s,t,x,y) := \tfrac{1}{2}|y|^2 + h_0(t,x,y) + \beta(s)\left(V(t,x) - h_0(t,x,y)\right), \qquad (24)$$

where β is a cutoff function as in (10). Even though this homotopy is not monotone, the map Φ respects the action window, whenever the perturbations V and h_0 are sufficiently C^0-small (see [2, Section 4.4]). Among conditions (C1-C3) only the apriori bound in (C1) is nonstandard. It is provided by Proposition 2.8 below, which says that there is a uniform bound $R > \lambda$ for the fibre components of all relevant continuation trajectories. Again the reverse homotopy argument applies and shows that $[\Phi]$ is an isomorphism (see [16, Theorem 3.6]).

Existence of the fifth and last vertical map in (23) is the main result in [17]. This proves existence of the homomorphism Φ_λ^f.

Commutativity of diagram (21) follows by replacing the second vertical isomorphism on the right hand side of (23) according to the following commuting diagram (invariance of Floer homology; see [16, Theorem 3.7])

$$\begin{array}{ccc} HF_*^{(-\infty,\infty)}(f^{(\lambda)} + h;\alpha) & \xrightarrow{\Phi^{gf}} & HF_*^{(-\infty,\infty)}(g^{(\lambda)} + h_g;\alpha) \\ {\scriptstyle \Phi^{f_0 f}}\downarrow & \swarrow {\scriptstyle \Phi^{f_0 g}} & \\ HF_*^{(-\infty,\infty)}(f_0^{(\lambda)} + h_0;\alpha) & & \end{array}.$$

To prove the final claim (22) we show that diagram (23) commutes. The vertical maps on the left hand side of (23) are the analogues of the ones on the right hand side. The perturbations on the left (marked by ˜) are restrictions of the ones on the right hand side. The first five horizontal maps can be viewed as being induced by inclusion of subcomplexes. Here convexity of the unperturbed Hamiltonians enters. Now blocks one, three and four in diagram (23) commute already on the chain level. Horizontal maps two and three can alternatively be interpreted as being induced by continuation maps associated to homotopies. Note that, due to the fact that orbits could enter from infinity during the homotopy, the induced maps on homology are in general not isomorphisms. Continuation maps satisfy the obvious composition rule for catenations of homotopies (see [16, Lemmata 3.11 and 3.12]) and this implies commutativity of block two. Block five commutes by [17, Main Theorem]. This concludes the proof of Theorem 2.7. □

Proposition 2.8 (Apriori bound). *Fix constants $c_0, \lambda > 0$ and functions $h \in C_0^\infty(S^1 \times D_\lambda T^*M)$ and $V \in C^\infty(S^1 \times M)$. Let $H_{s,t}$ be given by (24). Then there is a constant $C = C(c_0, h, V) > 0$ such that the following holds. If $w = (u, v) : \mathbb{R} \times S^1 \to T^*M$ is a solution of (11) such that*

$$E(u,v) = \|\partial_s u\|_2^2 + \|\nabla_s v\|_2^2 \le c_0, \qquad \sup_{s\in\mathbb{R}} \mathcal{A}_{H_{s,t}}(u(s,\cdot), v(s,\cdot)) \le c_0,$$

then $\|v\|_\infty \le C$.

Remark 2.9. An apriori bound for s-independent Hamiltonians was obtained by Cieliebak [3, Theorem 5.4]. Our proof uses the techniques developed by Salamon and the present author in [17]. There is yet another approach towards C^0-bounds by Oancea [15]. It requires $\partial_s H_s \ge 0$, which is not necessarily satisfied by (24). However, Proposition 2.3 fits into the framework [15], which yields some uniform C^0-bound c. In contrast, our proof of Proposition 2.3 results in the specific value $c = r$, which is important for our purposes.

Proof. The proof of the apriori estimate in [17, Chapter 4, case $\varepsilon = 1$] carries over almost literally. Just three estimates have to be verified in the situation at hand. Identify T^*M and TM via the Riemannian metric. For $i, j \in \{1, 2\}$, let $\nabla_i h_t(x, \xi) \in T_xM$ and $\nabla_{ij} h_t(x, \xi) \in End(T_xM)$ be determined by the identities

$$\begin{aligned} \frac{d}{d\tau} h_t(x,\xi) &=: \langle \nabla_1 h_t(x,\xi), \partial_s x\rangle + \langle \nabla_2 h_t(x,\xi), \nabla_s \xi\rangle, \\ \nabla_\tau \nabla_i h_t(x,\xi) &=: \nabla_{i1} h_t(x,\xi)\partial_s x + \nabla_{i2} h_t(x,\xi)\nabla_s \xi, \end{aligned}$$

for every path $\mathbb{R} \to TM : \tau \mapsto (x(\tau), \xi(\tau))$. Then equation (11) is equivalent to

$$\begin{aligned} 0 &= \partial_s u - \nabla_t v - (1-\sigma)\nabla V_t(u) - \sigma \nabla_1 h_t(u,v), \\ 0 &= \nabla_s v + \partial_t u - v - \sigma \nabla_2 h_t(u,v). \end{aligned} \tag{25}$$

Use these equations to derive estimate I:

$$
\begin{aligned}
(\partial_s^2+\partial_t^2)\frac{|v|^2}{2} &= |\nabla_s v|^2+|\nabla_t v|^2+\langle\nabla_s\left(v-\partial_t u+\sigma\nabla_2 h_t(u,v)\right),v\rangle \\
&\quad+\langle\nabla_t\left(\partial_s u-(1-\sigma)\nabla V_t(u)-\sigma\nabla_1 h_t(u,v)\right),v\rangle \\
&= |\nabla_s v|^2+|\nabla_t v|^2+\langle\nabla_s v,v\rangle \\
&\quad+\langle\sigma'\nabla_2 h_t(u,v)+\sigma\nabla_{21}h_t(u,v)\partial_s u+\sigma\nabla_{22}h_t(u,v)\nabla_s v,v\rangle \\
&\quad-(1-\sigma)\langle\nabla_{\partial_t u}\nabla V_t(u)+\nabla(\partial_t V_t)(u),v\rangle \\
&\quad-\sigma\langle\nabla_{11}h_t(u,v)\partial_t u+\sigma\nabla_{12}h_t(u,v)\nabla_t v+\nabla_1(\partial_t h_t)(u,v),v\rangle \\
&\geq \frac{6}{8}|\nabla_s v|^2+\frac{7}{8}|\nabla_t v|^2-2|v|^2\left(\|\nabla_{22}h\|_\infty^2+\|\nabla_{12}h\|_\infty^2\right) \\
&\quad-|v|\Big(2\|\nabla_2 h\|_\infty+\|\nabla(\partial_t V_t)\|_\infty+\|\nabla_1(\partial_t h_t)\|_\infty \\
&\quad+|\partial_s u|\,\|\nabla_{21}h\|_\infty+|\partial_t u|\,\|\nabla\nabla V\|_\infty+|\partial_t u|\,\|\nabla_{11}h\|_\infty\Big) \\
&\geq \frac{5}{8}|\nabla_s v|^2+\frac{6}{8}|\nabla_t v|^2-\mu_1|v|-\mu_2|v|^2 \\
&\geq \frac{5}{8}|\nabla_s v|^2+\frac{6}{8}|\nabla_t v|^2-\frac{1}{2}-({\mu_1}^2+2\mu_2)\frac{|v|^2}{2}
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1 &= \|\nabla(\partial_t V_t)\|_\infty+\|\nabla_1(\partial_t V_t)\|_\infty+\|\nabla_{21}h\|_\infty\left(\|\nabla V\|_\infty+\|\nabla_1 h\|_\infty\right) \\
&\quad+\|\nabla_2 h\|_\infty\left(2+\|\nabla\nabla V\|_\infty+\|\nabla_{11}h\|_\infty\right) \\
\mu_2 &= 2\|\nabla_{22}h\|_\infty^2+2\|\nabla_{12}h\|_\infty^2+2\|\nabla_{21}h\|_\infty^2 \\
&\quad+2\left(\|\nabla\nabla V\|_\infty+\|\nabla_{11}h\|_\infty\right)^2+\|\nabla\nabla V\|_\infty+\|\nabla_{11}h\|_\infty .
\end{aligned}
$$

All L^∞ norms are finite by compact support of V and h. Here is estimate II: By assumption, definition of $\mathcal{A}_{H_{s,t}}$ and replacing $\partial_t u$ according to (25), we obtain for every $s \in \mathbb{R}$

$$
\begin{aligned}
c_0 &\geq \mathcal{A}_{H_{s,t}}\left(u(s,\cdot),v(s,\cdot)\right) \\
&= \int_0^1\left(\langle v,v-\nabla_s v+\sigma\nabla_2 h_t(u,v)\rangle-\frac{1}{2}|v|^2-(1-\sigma)V_t(u)-\sigma h_t(u,v)\right)dt \\
&\geq \int_0^1\left(\frac{1}{4}|v|^2-2|\nabla_s v|^2\right)dt-2\|\nabla_2 h\|_\infty^2-\|V\|_\infty-\|h\|_\infty .
\end{aligned}
$$

Estimate III: There is a constant $a=a(V,h)>0$, such that

$$
(\partial_s^2+\partial_t^2-\partial_s)\frac{|v|^2}{2} \geq -a\left(|v|^2+1\right).
$$

This is derived similarly to estimate I, and a depends on the C^2-norms of V and h similarly to μ_1 and μ_2. Given the three estimates, the proof of Proposition 2.8 proceeds precisely as in [17, Chapter 4, case $\varepsilon=1$]. □

3 (Relative) symplectic homology of DT^*M

3.1 Definition

To define symplectic homology we need to introduce direct and inverse limits. Refering to [2, Section 4.6] for the general case, we shall discuss these notions only in the particular situation at hand. The set $\mathcal{H} := C_0^\infty(S^1 \times DT^*M)$ is partially ordered via

$$H_0 \preceq H_1 \qquad :\Leftrightarrow \qquad H_0(t,z) \le H_1(t,z) \quad \forall (t,z) \in S^1 \times DT^*M.$$

Fix $\alpha \in \pi_1(M)$, two reals $-\infty \le a < b \le \infty$ and denote by $\mathcal{H}_\alpha^{a,b}$ the set of $H \in \mathcal{H}$ such that $a, b \notin Spec(H;\alpha)$. The monotone homomorphisms $\sigma^{H_1H_0}$ of Section 2 give rise to the *partially ordered system* (HF, σ) *of* $\mathbb{Z}_2$*-vector spaces over* $\mathcal{H}_\alpha^{a,b}$. By definition, this means that HF assigns to each $H \in \mathcal{H}_\alpha^{a,b}$ the $\mathbb{Z}_2$-vector space ${HF_*}^{(a,b)}(H;\alpha)$, and σ assigns to all elements $H_0 \preceq H_1$ of $\mathcal{H}_\alpha^{a,b}$ the monotone homomorphism $\sigma^{H_1H_0}$ subject to composition rule (17).

The partially ordered set $(\mathcal{H}_\alpha^{a,b}, \preceq)$ is *downward directed*: For all $H_1, H_2 \in \mathcal{H}_\alpha^{a,b}$ there exists $H_0 \in \mathcal{H}_\alpha^{a,b}$ such that $H_0 \preceq H_1$ and $H_0 \preceq H_2$. The functor (HF, σ) is called an *inverse system of* $\mathbb{Z}_2$*-vector spaces over* $\mathcal{H}_\alpha^{a,b}$. Its *inverse limit*, called *symplectic homology of* DT^*M *with respect to* α *and the action window* (a,b), is defined by

$$\underleftarrow{SH}_*^{(a,b)}(DT^*M;\alpha) := \varprojlim_{H \in \mathcal{H}_\alpha^{a,b}} HF_*^{(a,b)}(H;\alpha)$$

$$:= \left\{ \{a_H\}_{H \in \mathcal{H}_\alpha^{a,b}} \in \prod_{H \in \mathcal{H}_\alpha^{a,b}} HF_*^{(a,b)}(H;\alpha) \;\middle|\; H_0 \preceq H_1 \Rightarrow \sigma^{H_1H_0}(a_{H_0}) = a_{H_1} \right\}.$$

For $H \in \mathcal{H}_\alpha^{a,b}$, let

$$\pi_H : \underleftarrow{SH}_*^{(a,b)}(DT^*M;\alpha) \to HF_*^{(a,b)}(H;\alpha) \tag{26}$$

be the projection to the component corresponding to H. It holds $\pi_{H_1} = \sigma^{H_1H_0} \circ \pi_{H_0}$, whenever $H_0 \preceq H_1$.

To define *relative* symplectic homology, fix $c > 0$ and consider the subset

$$\mathcal{H}_{\alpha;c}^{a,b} = \mathcal{H}_{\alpha;c}^{a,b}(DT^*M, M) := \left\{ H \in \mathcal{H}_\alpha^{a,b} \mid \sup_{S^1 \times M} H \le -c \right\}.$$

This set is upward directed: For all $H_0, H_1 \in \mathcal{H}_{\alpha;c}^{a,b}$ there exists $H_2 \in \mathcal{H}_{\alpha;c}^{a,b}$ such that $H_0 \preceq H_2$ and $H_1 \preceq H_2$. The functor (HF, σ) is called a *direct system of* $\mathbb{Z}_2$*-vector spaces over* $\mathcal{H}_{\alpha;c}^{a,b}$. Its *direct limit*, called *relative symplectic homology of the pair* (DT^*M, M) *with respect to* α*, the action window* (a,b) and the bound c, is defined to be the quotient

$$\underrightarrow{SH}_*^{(a,b);c}(DT^*M, M;\alpha) := \varinjlim_{H \in \mathcal{H}_{\alpha;c}^{a,b}} HF_*^{(a,b)}(H;\alpha)$$

$$:= \left\{ (H, a_H) \;\middle|\; H \in \mathcal{H}_{\alpha;c}^{a,b},\ a_H \in HF_*^{(a,b)}(H;\alpha) \right\} \Big/ \sim .$$

Here $(H_0,a_0) \sim (H_1,a_1)$ iff there exists $H_2 \in \mathcal{H}^{a,b}_{\alpha;c}$ such that $H_0 \preceq H_2$, $H_1 \preceq H_2$ and $\sigma^{H_2H_0}(a_0) = \sigma^{H_2H_1}(a_1)$. This is an equivalence relation, since $\mathcal{H}^{a,b}_{\alpha;c}$ is upward directed. The direct limit is a $\mathbb{Z}_2$-vector space with the operations

$$k[H_0,a_0] := [H_0,ka_0], \quad [H_0,a_0]+[H_1,a_1] := [H_2, \sigma_1 H_2 H_0(a_0) + \sigma_{H_2H_1}(a_1)],$$

for all $k \in \mathbb{Z}_2$ and $H_2 \in \mathcal{H}^{a,b}_{\alpha;c}$ such that $H_0 \preceq H_2$ and $H_1 \preceq H_2$. For $H \in \mathcal{H}^{a,b}_{\alpha;c}$ define the homomorphism

$$\iota_H : HF^{(a,b)}_*(H;\alpha) \to \underrightarrow{SH}^{(a,b);c}_*(DT^*M,M;\alpha), \qquad a_H \mapsto [H,a_H]. \tag{27}$$

It satisfies $\iota_{H_0} = \iota_{H_1} \circ \sigma_{H_1H_0}$, whenever $H_0 \preceq H_1$.

The main feature of symplectic and relative symplectic homology for our purposes is the existence of a unique homomorphism between them which factors through Floer homology (see [2, Proposition 4.8.2]), i.e. such that the diagram

$$\begin{array}{ccc} \underleftarrow{SH}^{(a,b)}_*(DT^*M;\alpha) & \xrightarrow{T^{(a,b)}_{\alpha;c}} & \underrightarrow{SH}^{(a,b);c}_*(DT^*M,M;\alpha) \\ & {\scriptstyle \pi_{H'}} \searrow \quad \nearrow {\scriptstyle \iota_{H'}} & \\ & HF^{(a,b)}_*(H';\alpha) & \end{array} \tag{28}$$

commutes for every $H' \in \mathcal{H}^{a,b}_{\alpha;c}$. This homomorphism is given by

$$T^{(a,b)}_{\alpha;c} : \{a_H\}_{H\in\mathcal{H}^{a,b}_{\alpha}} \mapsto [H'', a_{H''}], \tag{29}$$

where H'' is any element of $\mathcal{H}^{a,b}_{\alpha;c}$ (and $T^{(a,b)}_{\alpha;c}$ is independent of this choice).

3.2 Computation

Theorem 3.1 ((Relative) symplectic homology of DT^*M). *Let M be a closed smooth Riemannian manifold and let α be a free homotopy class of loops in M. Then the following are true.*

(i) For every $a \in \mathbb{R} \setminus \Lambda_\alpha$ (with $a > 0$, if $\alpha = 0$), there is a natural isomorphism

$$\underleftarrow{SH}^{(a,\infty)}_*(DT^*M;\alpha) \simeq H_*(\mathcal{L}^{a^2/2}_\alpha M).$$

(ii) For all real numbers $a, c > 0$, there is a natural isomorphism

$$\underrightarrow{SH}^{(a,\infty);c}_*(DT^*M,M;\alpha) \simeq \begin{cases} H_*(\mathcal{L}_\alpha M), & \textit{if } a \in (0,c], \\ 0, & \textit{if } a > c. \end{cases}$$

(iii) For every $a \in (0,c] \setminus \Lambda_\alpha$, the following diagram commutes

$$\begin{array}{ccc} \underrightarrow{SH}^{(a,\infty);c}_*(DT^*M,M;\alpha) & \xrightarrow[\simeq]{(ii)} & H_*(\mathcal{L}_\alpha M) \\ {\scriptstyle T^{(a,\infty)}_{\alpha;c}} \uparrow & & \uparrow {\scriptstyle [\iota_{a^2/2}]} \\ \underleftarrow{SH}^{(a,\infty)}_*(DT^*M;\alpha) & \xrightarrow[\simeq]{(i)} & H_*(\mathcal{L}^{a^2/2}_\alpha M) \end{array} .$$

Remark 3.2. 1) In part (i) of Theorem 3.1, assumption $a > 0$ in case $\alpha = 0$ avoids taking into account the trivial 1-periodic orbits near the boundary of DT^*M. Their existence is caused by the compact support condition for the Hamiltonians and their symplectic action is zero.
2) Assumption $a > 0$ in part (ii) is to avoid the elements of $\mathcal{P}^-(f;\alpha)$ (see Remark 2.1) whose symplectic action is negative.
3) The homomorphism T in (iii) is nonzero, if and only if $a \in [\ell_\alpha, c]$.

The proof of Theorem 3.1 uses the following two lemmata.

Lemma 3.3. *Let $\alpha \in \pi_1(M)$. The marked length spectrum $\Lambda_\alpha \subset \mathbb{R}$ is a closed and nowhere dense subset. Equivalently, its complement is open and dense.*

Proof. The set Λ_α is closed: Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \Lambda_\alpha$ be such that $\lambda := \lim_{j\to\infty} \lambda_j$ exists. For each length λ_j choose a periodic geodesic x_j in the class α of length λ_j. Since $|\dot{x}_j(t)| = \lambda_j$, $\forall t \in S^1$, there exists a compact subset $K \subset TM$ such that $(q_j, v_j) := (x_j(0), \dot{x}_j(0)) \in K$, for every $j \in \mathbb{N}$. Passing to a subsequence, if necessary, we may assume without loss of generality that the limit

$$(q, v) := \lim_{j\to\infty} (q_j, v_j)$$

exists. Let $x : [0, 1] \to M$ be the solution of the initial value problem $\nabla_t \dot{x} \equiv 0$ and $x(0) = q$, $\dot{x}(0) = v$. It follows that x is periodic:

$$(x(0), \dot{x}(0)) = \lim_{j\to\infty} (x_j(0), \dot{x}_j(0)) = \lim_{j\to\infty} (x_j(1), \dot{x}_j(1)) = (x(1), \dot{x}(1)).$$

The last step uses the fact that the solution to a second order ordinary differential equation depends continuously on the initial values. It remains to show $[x] = \alpha$: Since x_j converges to x uniformly in $t \in S^1$ and the injectivity radius of M is positive, it follows that x_j is homotopic to x, for all sufficiently large j.

The set Λ_α does not contain intervals: It suffices to prove this property for the set Σ_α of critical values of the classical action $\mathcal{S}_0 : \mathcal{L}_\alpha M \to \mathbb{R}$, since $\Lambda_\alpha = \{\sqrt{2\sigma} \mid \sigma \in \Sigma_\alpha\}$. Fix $a \in \mathbb{R}^+$ and consider the space $L^a_\alpha M$ of piecewise smooth maps $x : S^1 \to M$ such that $[x] = \alpha$ and $\mathcal{S}_0(x) \le a$. For some sufficiently large $N = N(a) \in \mathbb{N}$, there is a smooth finite dimensional manifold $P^N_\alpha M$ (the set of geodesic polygons; see [1]) and the functional $\mathcal{S}_0$ restricted to $P^N_\alpha M$ is smooth. The critical points of $\mathcal{S}_0$ restricted to the three domains

$$P^N_\alpha M \subset L^a_\alpha M \supset \mathcal{L}^a_\alpha M$$

all coincide. Hence $\Sigma_\alpha \cap \{\mathcal{S}_0(x) \le a\}$ equals {critical values of $\mathcal{S}_0 : P^N_\alpha M \to \mathbb{R}$}. The latter set is of measure zero by the Theorem of Sard [19].

The set Λ_α is nowhere dense in $\mathbb{R}$: By definition of *nowhere dense* we have to show that every nonempty open set $U \subset \mathbb{R}$ contains a nonempty open set V such that $V \cap \Lambda_\alpha = \emptyset$. Choose any such U and an open interval $I \subset \mathbb{R}$ contained in U. Since Λ_α does not contain intervals, we can find an element $s \in I$ such that $s \notin \Lambda_\alpha$. Because Λ_α is closed, it is possible to choose an open neighbourhood V of s in I which is disjoint from Λ_α. □

Lemma 3.4. *For every $\alpha \in \pi_1(M)$, there is a natural isomorphism*

$$\varinjlim_{a\in\mathbb{R}_0^+} H_*(\mathcal{L}_\alpha^a M) \to H_*(\mathcal{L}_\alpha M), \qquad [a, x_a] \mapsto [\iota_a](x_a).$$

Proof. The direct limit is defined by

$$\varinjlim_{a\in\mathbb{R}_0^+} H_*(\mathcal{L}_\alpha^a M) := \{(a, x_a) \mid a \geq 0,\ x_a \in H_*(\mathcal{L}_\alpha^a M)\}/\sim,$$

where $(a, x_a) \sim (b, x_b)$ iff there exists $c \geq \max\{a, b\}$ with $[\iota_{ca}](x_a) = [\iota_{cb}](x_b)$.

1) well defined: Assume $(a, x_a) \sim (b, x_b)$, i.e. $[\iota_{ca}](x_a) = [\iota_{cb}](x_b)$ for some $c \geq \max\{a, b\}$. Apply $[\iota_c]$ to both sides to obtain $[\iota_a](x_a) = [\iota_b](x_b)$.

2) surjective: Represent $x \in H_*(\mathcal{L}_\alpha M)$ by a cycle $f(C)$. Here C is the total space of a closed simplicial complex and $f : C \to \mathcal{L}_\alpha M$ is a continuous map. Let $\mathcal{L}_\alpha M$ be equipped with the $W^{1,2}$-metric, then $\mathcal{S}_0$ given by (20) is continuous (see [13, section 5.4]). Since $f(C)$ is compact and $\mathcal{S}_0$ is continuous, we obtain the real number $a - 1 := \max_{f(C)} \mathcal{S}_0$. Now $f(C) \subset \mathcal{L}_\alpha^a M$ shows $x_a := [f(C)] \in H_*(\mathcal{L}_\alpha^a M)$.

3) injective: Assuming $[\iota_a](x_a) = 0 \in H_*(\mathcal{L}_\alpha M)$, we need to show that $[a, x_a]$ is the zero element of the direct limit, i.e. that there exists $b \geq a$ such that $[\iota_{ba}](x_a) = 0 \in H_*(\mathcal{L}_\alpha^b M)$. Choose a cycle $f_a : C \to \mathcal{L}_\alpha^a M$ representing x_a. By assumption there is a cycle $F : W \to \mathcal{L}_\alpha M$, where W is the total space of a simplicial complex with $\partial W = C$ and F is a continuous map which coincides with f_a on $\partial W = C$. Defining $b - 1 := \max_{F(W)} \mathcal{S}_0 \geq a$ and $F_b := F : W \to \mathcal{L}_\alpha^b M$, we have $F_b(p) = F(p) = f_a(p) = (\iota_{ba} \circ f_a)(p)$, for every $p \in \partial W = C$. Hence $[\iota_{ba}](x_a) = [(\iota_{ba} \circ f_a)(C)] = [F_b(C)] = 0$. □

Proof of Theorem 3.1. Fix a free homotopy class α of loops in M. We prove Theorem 3.1 in five steps. Part (i) is equivalent to

STEP 1. (a) *For every $a \in \mathbb{R}^+ \setminus \Lambda_\alpha$, there is a natural isomorphism*

$$\underleftarrow{SH}_*^{(a,\infty)}(DT^*M; \alpha) \simeq H_*(\mathcal{L}_\alpha^{a^2/2} M).$$

(b) *If $\alpha \neq 0$ and $a \leq 0$, then $\underleftarrow{SH}_*^{(a,\infty)}(DT^*M; \alpha) = 0$.*

Proof. (a) Consider the infimum of all geodesic lengths strictly greater than a

$$\ell_+ := \inf\left(\Lambda_\alpha \cap (a, \infty)\right). \tag{30}$$

Indeed, by Lemma 3.3, we have $a < \ell_+ \in \Lambda_\alpha$. We use the convention that $\inf \emptyset = \infty$. Consider the sequence of linear functions T_k (of increasing slopes) through the points $(r_{k1}, -a)$ and $(r_{k2}, 0)$, where

$$r_{k1} = r_{k,1;a,\ell_+} := 1 - \frac{1}{k} + \frac{3}{16k}\left(1 - \frac{a}{\ell_+}\right),$$
$$r_{k2} = r_{k,2;a,\ell_+} := 1 - \frac{3}{16k}\left(1 - \frac{a}{\ell_+}\right),$$

and

$$T_k(r) = T_{k;a,\ell_+}(r) := ak \frac{r - 1 + \frac{3}{16k}\left(1 - \frac{a}{\ell_+}\right)}{1 - \frac{3}{8}\left(1 - \frac{a}{\ell_+}\right)}, \qquad k \in \mathbb{N}.$$

Note that $0 < r_{k1} < r_{k2} < 1$, for all $k \in \mathbb{N}$, and that both numbers converge to $+1$ as $k \to \infty$. Moreover, the slope m_1 of T_1 satisfies

$$a < m_1 = \frac{a}{1 - \frac{3}{8}\left(1 - \frac{a}{\ell_+}\right)} < \ell_+. \tag{31}$$

Define the sequence of piecewise linear functions (see Figure 4)

$$\hat{f}_k = \hat{f}_{k;a,\ell_+}(r) := \begin{cases} -A_k := T_k(r_{k1}/2), & \text{if } r \in [0, r_{k1}/2], \\ T_k(r), & \text{if } r \in [r_{k1}/2, r_{k2}], \\ 0, & \text{if } r \geq r_{k2}. \end{cases} \tag{32}$$

We obtain a sequence of smooth functions $f_k = f_{k;a,\ell_+}$ by smoothing out $\hat{f}_k$

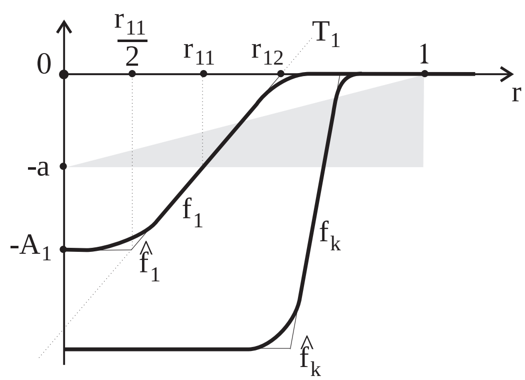


Figure 4: The functions f_k.

nearby $r_{k1}/2$ and r_{k2} subject to the following conditions. The new functions f_k coincide with the old functions $\hat{f}_k$ away from small neighbourhoods of $r_{k1}/2$ and r_{k2}. In particular, these neighbourhoods are chosen sufficiently small such that graph $f_k =$ graph $\hat{f}_k$ in the region which lies below the line $r \mapsto -a + ar$ and above the line $r \mapsto -a$ (grey shaded in Figure 4). Moreover, the derivatives are required to satisfy $f_k' \geq 0$ everywhere, $f_k'' \geq 0$ near $r_{k1}/2$, $f_k'' \leq 0$ near r_{k2} and $f_k'' = 0$ elsewhere (see Figure 4).

The significance of the functions f_k lies in the fact that they give rise to natural isomorphisms

$$\psi_*^{(k)} : HF_*^{(a,\infty)}(f_k; \alpha) \to H_*(\mathcal{L}_\alpha^{a^2/2} M), \qquad \forall k \in \mathbb{N}. \tag{33}$$

Let us first check that, $\forall k \in \mathbb{N}$, indeed $a \notin Spec(f_k; \alpha)$: The graph of f_k admits a tangent through the point $(0, -a)$, but the slope of this tangent lies in the interval (a, ℓ_+) (see Figure 5) and is therefore noncritical.

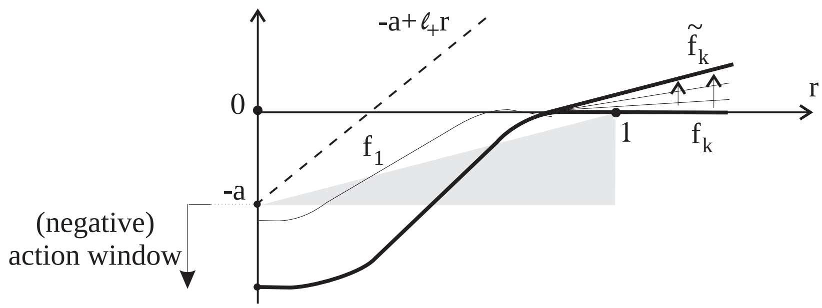


Figure 5: An action-regular monotone homotopy between f_k and $\tilde{f}_k$.

Let $\tilde{f}_k$ be the function obtained by following the graph of f_k until it takes on slope a for the second time (near r_{k2}, say at a point p), then continue linearly with slope a. Yet the resulting function is only of class C^1 at p. Local smoothing near p yields a smooth function $\tilde{f}_k$ (see Figure 5). Figure 5 also indicates a monotone action-regular homotopy which induces the monotone isomorphism

$$\sigma_{\tilde{f}_k f_k} : HF_*^{(a,\infty)}(f_k;\alpha) \to HF_*^{(a,\infty)}(\tilde{f}_k;\alpha). \tag{34}$$

The homotopy is action-regular, because all points which do not remain constant during the homotopy intersect the vertical coordinate axis strictly above the point $-a$. Therefore all 1-periodic orbits arising (if any) are of action strictly less than a (see Remark 2.1; method 2).

Next consider the function ${f_k}^{(a)}$ obtained by following the graph of $\tilde{f}_k$ (likewise f_k) until it takes on slope a for the first time (say at a point q), then continue linearly with slope a (see Figure 6). Again smoothing near q yields ${f_k}^{(a)}$. Figure 6 shows a monotone homotopy between ${f_k}^{(a)}$ and $\tilde{f}_k$ which is also

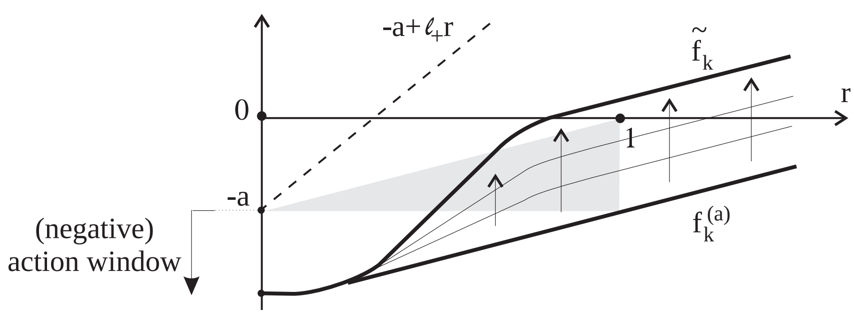


Figure 6: An action-regular monotone homotopy between ${f_k}^{(a)}$ and $\tilde{f}_k$.

action-regular: All tangents to members of the homotopy which run through $(0,-a)$ lie above the ray $\mathbb{R}^+ \ni r \mapsto -a + ar$ and strictly below $-a + \ell_+ r$. Hence their slopes lie in the interval $[a, \ell_+)$ and so are noncritical. Consequently the homotopy induces a monotone isomorphism with inverse

$$\sigma_{\tilde{f}_k f_k^{(a)}}{}^{-1} : HF_*^{(a,\infty)}(\tilde{f}_k;\alpha) \to HF_*^{(a,\infty)}({f_k}^{(a)};\alpha). \tag{35}$$

Since ${f_k}^{(a)}$ does not admit 1-periodic orbits of action less than a, the map $[\pi^F]$ in (18) in case $(a,b,c)=(-\infty,a,\infty)$ is an isomorphism with inverse

$$[\pi^F]^{-1} : HF_*^{(a,\infty)}({f_k}^{(a)};\alpha) \to HF_*^{(-\infty,\infty)}({f_k}^{(a)};\alpha). \tag{36}$$

Similarly, there is the isomorphism

$$[\iota^F]^{-1} : HF_*^{(-\infty,\infty)}({f_k}^{(a)};\alpha) \to HF_*^{(-\infty,\tilde{c})}({f_k}^{(a)};\alpha), \tag{37}$$

where the constant $\tilde{c}=\tilde{c}({f_k}^{(a)},a)$ is defined in Theorem 2.7. Here we need the assumption that a is not a length. Theorem 2.7 provides the isomorphism

$$\Psi^a : HF_*^{(-\infty,\tilde{c})}({f_k}^{(a)};\alpha) \to H_*(\mathcal{L}_\alpha^{a^2/2}M). \tag{38}$$

Composing (34-38) gives the desired isomorphisms (33).

To conclude the proof of part (a) of Step 1 we show that $\{f_k\}_{k\in\mathbb{N}}$ is a downward exhausting sequence (see [2, Section 4.7]) for the inverse limit defining symplectic homology. There are two properties to be checked. Firstly, given any $H\in\mathcal{H}_\alpha^{a,\infty}$, there exists f_k such that $H(t,x,y)\geq f_k(|y|)$, for every $(t,x,y)\in S^1\times DT^*M$. This can be achieved by choosing $k\in\mathbb{N}$ sufficiently large. Secondly, for every $k\in\mathbb{N}$, the map

$$\sigma_{f_k f_{k+1}} : HF_*^{(a,\infty)}(f_{k+1};\alpha) \to HF_*^{(a,\infty)}(f_k;\alpha)$$

induced by a monotone homotopy between f_{k+1} and f_k is an isomorphism. To see this think of k as a continuous parameter in $[1,\infty)$ instead of a positive integer. Consider the family $\{f_r\}_{r\in[1,\infty)}$ defined analogous to $\{f_k\}_{k\in\mathbb{N}}$. Then the homotopy $\{f_{k-s}\}_{s\in[-1,0]}$ from f_{k+1} to f_k is clearly monotone and in addition action-regular: Similar arguments as above show that all tangents to members of the homotopy which run through $(0,-a)$ are located between the lines $-a+ar$ and $-a+\ell_+ r$. Hence their slope is in the interval $[a,\ell_+)$ which does not contain critical slopes. Consequently the boundary $+a$ of the action window remains disjoint from the action spectrum throughout the homotopy. This proves property two and therefore that the sequence is exhausting. Hence the homomorphism (26), given in the situation at hand by

$$\pi_{f_k} : \underleftarrow{SH}_*^{(a,\infty)}(DT^*M;\alpha) \to HF_*^{(a,\infty)}(f_k;\alpha), \tag{39}$$

is an isomorphism by [2, Lemma 4.7.1 (ii)]. Then (33) proves Step 1 (a).

(*b*) We follow the line of argument in [2, Pf. of Thm. 5.1.2 Step 1]. Fix $a\leq 0$ and $\alpha\neq 0$. Then the smallest length bigger than a is given by $\ell_\alpha>0\geq a$, where ℓ_α is defined by (2). Consider the *new* family $f_k=f_{k;-a,\ell_\alpha}$ as defined subsequent to (32). It is easy to see that all points of slope $\lambda\in\Lambda_\alpha$ on the graph of f_k (which is nonpositive on $\mathbb{R}_0^+$), are located strictly below the line $r \mapsto -a + \ell_\alpha r$ (which is strictly positive on $\mathbb{R}^+$ due to $-a \geq 0$ and $\ell_\alpha > 0$). Therefore they are located strictly below the line $r \mapsto -a + \lambda r$. Hence, by method 1 of Remark 2.1, all 1-periodic orbits (representing $\alpha \neq 0$) are of action strictly bigger than a. This shows that the action window (a, ∞) is admissible and $Spec(f_k; \alpha) \subset (a, \infty)$. Consider the upper horizontal row in diagram (18) in case $(a, b, c) = (-\infty, a, \infty)$ and $H^0 = f_k$. It follows $[\iota^F] = 0$ and

$$[\pi^F] : HF_*^{(a,\infty)}(f_k; \alpha) \to HF_*^{(-\infty,\infty)}(f_k; \alpha)$$

is an isomorphism. The target space is independent of the Hamiltonian, by Floer continuation, and therefore zero: Every sufficiently C^2-small compactly supported Hamiltonian admits only *contractible* 1-periodic orbits. The same exhausting sequence argument as in part (a) proves (b) and therefore Step 1. □

Fix $a, c > 0$. Step 2 and Step 3 of the proof rely on a family of functions $h_\delta \in \mathcal{H}_{\alpha;c}^{a;\infty}(DT^*M, M)$ defined as follows. Choose a real number δ such that

$$\delta \in \left(0, \frac{a}{c}\right), \qquad \mu_\delta := \frac{a}{\delta} - (c - a) \notin \Lambda_\alpha. \tag{40}$$

It follows $\mu_\delta > 0$. Using the conventions $\sup \emptyset = 0$ and $\inf \emptyset = \infty$, we define

$$\ell_- := \sup\left((0, \mu_\delta) \cap \Lambda_\alpha\right), \quad \ell_+ := \inf\left((\mu_\delta, \infty) \cap \Lambda_\alpha\right), \quad \ell_-' := \frac{\mu_\delta + \ell_-}{2}. \tag{41}$$

Since $\mu_\delta \notin \Lambda_\alpha$, Lemma 3.3 shows

$$0 \leq \ell_- < \ell_-' < \mu_\delta < \ell_+ \leq +\infty, \qquad (\ell_-, \ell_+) \cap \Lambda_\alpha = \emptyset.$$

Assumption $\delta < a/c$ ensures that the lines $-c + (a/\delta)r$ and $-a + \mu_\delta r$ intersect strictly above the r-axis (see Figure 7). As a first approximation for h_δ consider the piecewise linear curve in $\mathbb{R}^2$ obtained by starting at the point $(0, -c)$ with slope a/δ. Upon meeting the horizontal line through $(0, \max\{0, \ell_-' - a\})$, follow this horizontal line to the right until it intersects the vertical line through $(1, 0)$. Now go straight down to the point $(1, 0)$ and follow the horizontal coordinate axis to $+\infty$. Smooth out this piecewise linear curve near its corners such that the result h_δ is the graph of a smooth function, as indicated in Figure 7, of slope zero at $r = 0$ and at $r = 1$.

Step 2. *If $a > c > 0$, then $\underrightarrow{SH}_*^{(a,\infty);c}(DT^*M, M; \alpha) = 0$.*

Proof. Let δ be as in (40) and h_δ the associated function (see Figure 7), then

$$HF_*^{(a,\infty)}(h_\delta; \alpha) = 0. \tag{42}$$

The reason is that the action of all 1-periodic orbits is strictly less than a. This is because all tangents to the graph of h_δ intersect the vertical coordinate axis strictly above $-a$ (see Remark 2.1 method 2). Now choose a strictly decreasing

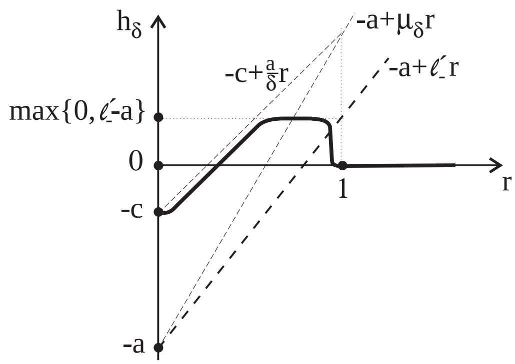


Figure 7: The function h_δ in case $c < a$.

real sequence $\delta_k \to 0$ such that each δ_k satisfies (40). Then $\{h_k := h_{\delta_k}\}_{k\in\mathbb{N}}$ is an upward exhausting sequence for the direct limit. We have to check two properties (see [2, Section 4.7]): Firstly, given any $H \in \mathcal{H}^{a;\infty}_{\alpha;c}(DT^*M, M)$, there exists $h_k \geq H$. This is clearly true. Secondly, the monotone homomorphism

$$\sigma_{h_{k+1},h_k} : HF_*^{(a,\infty)}(h_k;\alpha) \to HF_*^{(a,\infty)}(h_{k+1};\alpha),$$

is an isomorphism, $\forall k \in \mathbb{N}$. This holds trivially, since both Floer homologies are zero by (42). By Lemma 4.7.1 (i) in [2], the homomorphism in (27)

$$\iota_{h_k} : HF_*^{(a,\infty)}(h_k;\alpha) \to \underrightarrow{SH}_*^{(a,\infty);c}(DT^*M, M;\alpha), \qquad x_{h_k} \mapsto [h_k, x_{h_k}],$$

is an isomorphism, $\forall k \in \mathbb{N}$. By (42), the proof of Step 2 is complete. ☐

Step 3. *Given $0 < a \leq c$ and δ satisfying (40), there is a natural isomorphism*

$$\phi_*^{(\delta)} : HF_*^{(a,\infty)}(h_\delta;\alpha) \to H_*(\mathcal{L}_\alpha^{\mu_\delta^2/2}M), \qquad \mu_\delta := \frac{a}{\delta} - (c-a). \tag{43}$$

Proof. The idea is to deform h_δ (see Figure 8) by action-regular monotone homotopies to a convex function, so that Theorem 2.7 applies. Let $0 \leq \ell_- < \ell'_- < \mu_\delta < \ell_+ \leq +\infty$ be given by (41). First we check that the action window (a,∞) is indeed admissible for h_δ: We need to check only positive critical slopes. They appear in two clusters, one of which is located near the point $(0,-c)$. Their tangents hit the vertical coordinate axis below $-c < -a$. So they correspond to action values strictly bigger than a. The other cluster is located in the region between the lines $-a+\ell_- r$ and $-a+\ell_+ r$. The slope of any such tangent running through $(0,-a)$ lies in the interval (ℓ_-,ℓ_+) and is therefore noncritical. Here we use the assumption $\mu_\delta \notin \Lambda_\alpha$, which implies $\ell_- < \mu_\delta < \ell_+$ and $(\ell_-,\ell_+)\cap\Lambda_\alpha = \emptyset$.

Next we define the graph of a new function $\tilde{h}_\delta$ by initially following the graph of h_δ. When h_δ makes its first right turn, keep going with constant

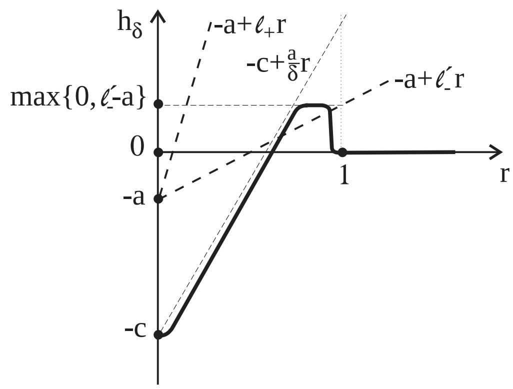


Figure 8: The function h_δ in case $c \geq a$.

slope a/δ until you are about to hit the line $-a + \mu_\delta r$. Perform a smooth right turn and follow that line closely and linearly as indicated in Figure 9. The figure also shows an action-regular monotone homotopy between h_δ and

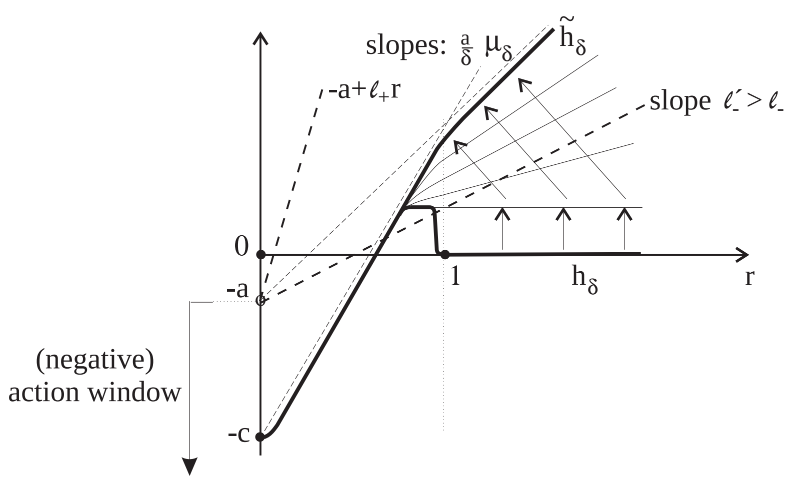


Figure 9: An action-regular monotone homotopy between h_δ and $\tilde{h}_\delta$.

$\tilde{h}_\delta$: We need to make sure that no 'critical slope tangents' to members of the homotopy move over the point $(0, -a)$, because this corresponds to periodic orbits entering or leaving the action window (a, ∞). The former is true, since all points on members of the homotopy whose tangents run through $(0, -a)$ lie strictly between the lines $-a + \ell_- r$ and $-a + \ell_+ r$ (see Figure 9). Hence these slopes are noncritical and we obtain the monotone isomorphism

$$\sigma_{\tilde{h}_\delta h_\delta} : HF_*^{(a,\infty)}(h_\delta; \alpha) \to HF_*^{(a,\infty)}(\tilde{h}_\delta; \alpha). \tag{44}$$

Define the function $h_\delta^{(\mu_\delta)}$ by following $\tilde h_\delta$ until slope μ_δ shows up for the first time. Then make a smooth turn and continue linearly with slope μ_δ as indicated in Figure 10. The monotone homotopy shown is also action-regular:

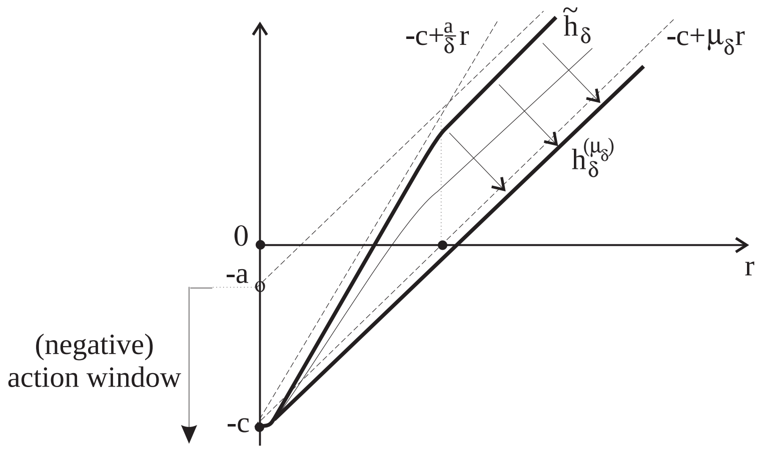


Figure 10: An action-regular monotone homotopy between $h_\delta^{(\mu_\delta)}$ and $\tilde h_\delta$.

All tangents to all members of the homotopy hit the vertical coordinate axis strictly below the point $-a$. Hence the corresponding monotone homomorphism is an isomorphism with inverse

$$\sigma_{\tilde h_\delta h_\delta^{(\mu_\delta)}}{}^{-1} : HF_*^{(a,\infty)}(\tilde h_\delta;\alpha) \to HF_*^{(a,\infty)}(h_\delta^{(\mu_\delta)};\alpha). \tag{45}$$

Since $h_\delta^{(\mu_\delta)}$ does not admit 1-periodic orbits of action less than a, the map $[\pi^F]$ in (18) with $(a,b,c,)=(-\infty,a,\infty)$ is an isomorphism with inverse

$$(\pi^F)^{-1} : HF_*^{(a,\infty)}(h_\delta^{(\mu_\delta)};\alpha) \to HF_*^{(-\infty,\infty)}(h_\delta^{(\mu_\delta)};\alpha). \tag{46}$$

Similarly, there is the isomorphism

$$[\pi^F]^{-1} : HF_*^{(-\infty,\infty)}(h_\delta^{(\mu_\delta)};\alpha) \to HF_*^{(-\infty,\hat c)}(h_\delta^{(\mu_\delta)};\alpha), \tag{47}$$

where the constant $\hat c = \hat c(h_\delta^{(\mu_\delta)},\mu_\delta)$ is defined in Theorem 2.7. Using again $\mu_\delta \notin \Lambda_\alpha$, we may apply the theorem to obtain the isomorphism

$$\Psi^{\mu_\delta} : HF_*^{(-\infty,\hat c)}(h_\delta^{(\mu_\delta)};\alpha) \to H_*(\mathcal{L}_\alpha^{{\mu_\delta}^2/2}M). \tag{48}$$

The proof of Step 3 concludes by composing (44-48). □

Step 4. *If $0 < a \le c$, then $\underrightarrow{SH}_*^{(a,\infty);c}(DT^*M,M;\alpha) \simeq H_*(\mathcal{L}_\alpha M)$ naturally.*

Proof. Consider the set $\Delta := \{\delta \in (0,a/c) \mid (40) \text{ holds}\}$ with partial order defined by $\delta_0 \preccurlyeq \delta_1$ iff $\delta_0 \ge \delta_1$. Observe that $(\Delta,\preccurlyeq)$ is upward directed and that

$\delta_0 \geq \delta_1$ is equivalent to $\mu_{\delta_0} \leq \mu_{\delta_1}$ and to $h_{\delta_0} \leq h_{\delta_1}$. Setting $\mu_\Delta := \{\mu_\delta \mid \delta \in \Delta\}$ and $\mathcal{H}_\Delta := \{h_\delta \mid \delta \in \Delta\}$, we can therefore identify the upward directed sets $(\Delta, \preccurlyeq)$, $(\mu_\Delta, \leq)$ and $(\mathcal{H}_\Delta, \leq)$. Taking the direct limit of both sides of (43) with respect to $(\Delta, \preccurlyeq)$ leads to

$$\varinjlim_{h_\delta \in \mathcal{H}_\Delta} HF_*^{(a,\infty)}(h_\delta; \alpha) \simeq \varinjlim_{\mu \in \mu_\Delta} H_*(\mathcal{L}_\alpha^{\mu^2/2} M).$$

By Lemma 3.4, the right hand side is naturally isomorphic to $H_*(\mathcal{L}_\alpha M)$. Concerning the left hand side, there is the natural inclusion homomorphism

$$\varinjlim_{h_\delta \in \mathcal{H}_\Delta} HF_*^{(a,\infty)}(h_\delta; \alpha) \to \varinjlim_{H \in \mathcal{H}^{a;\infty}_{\alpha;c}} HF_*^{(a,\infty)}(H; \alpha), \qquad [h_\delta, x_{h_\delta}] \mapsto [h_\delta, x_{h_\delta}].$$

Using (17) and the fact that $(\mathcal{H}_\Delta, \leq)$ is upward directed, it is easy to see that this map is well defined, injective and surjective. $\square$

Steps 2 and 4 prove part (ii) of Theorem 3.1.

STEP 5. *We prove part* (iii) *of Theorem 3.1.*

Proof. Throughout we use the notation introduced in Steps 1-4. Fix $0 < a \leq c$ and choose $\delta \in \Delta$ sufficiently small such that $a)$ $\mu_\delta > a$, and $k \in \mathbb{N}$ sufficiently large such that $b)$ $f_k \in \mathcal{H}^{a;\infty}_{\alpha;c}$ and $c)$ $f_k \leq h_\delta$. Let $\psi^{(k)}$ and $\phi^{(\delta)}$ be given by (33) and (43), respectively, and assume that the following diagram commutes

$$\begin{array}{ccc} HF_*^{(a,\infty)}(f_k;\alpha) & \xrightarrow{\sigma_{h_\delta f_k}} & HF_*^{(a,\infty)}(h_\delta;\alpha) \\ {\scriptstyle \psi^{(k)}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \phi^{(\delta)}} \\ H_*(\mathcal{L}_\alpha^{a^2/2}M) & \xrightarrow{[\iota_{\frac{\mu_\delta^2}{2}\frac{a^2}{2}}]} & H_*(\mathcal{L}_\alpha^{\mu_\delta^2/2}M) \end{array}, \qquad (49)$$

Then the subsequent diagram, whose existence uses $b)$, commutes too (thereby proving Step 5)

$$\begin{array}{ccc} \underleftarrow{SH}_*^{(a,\infty)}(DT^*M;\alpha) & & \\ {\scriptstyle \pi_{f_k}}\downarrow\simeq & \searrow^{T^{(a,\infty)}_{\alpha;c}} & \\ HF_*^{(a,\infty)}(f_k;\alpha) & \xrightarrow{\iota_{f_k}} & \underrightarrow{SH}_*^{(a,\infty);c}(DT^*M,M;\alpha) \\ {\scriptstyle \psi^{(k)}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \phi} \\ H_*(\mathcal{L}_\alpha^{a^2/2}M) & \xrightarrow{I_{a^2/2}} & \varinjlim_{\mu\in\mu_\Delta} H_*(\mathcal{L}_\alpha^{\mu^2/2}M) \simeq H_*(\mathcal{L}_\alpha M) \end{array}. \qquad (50)$$

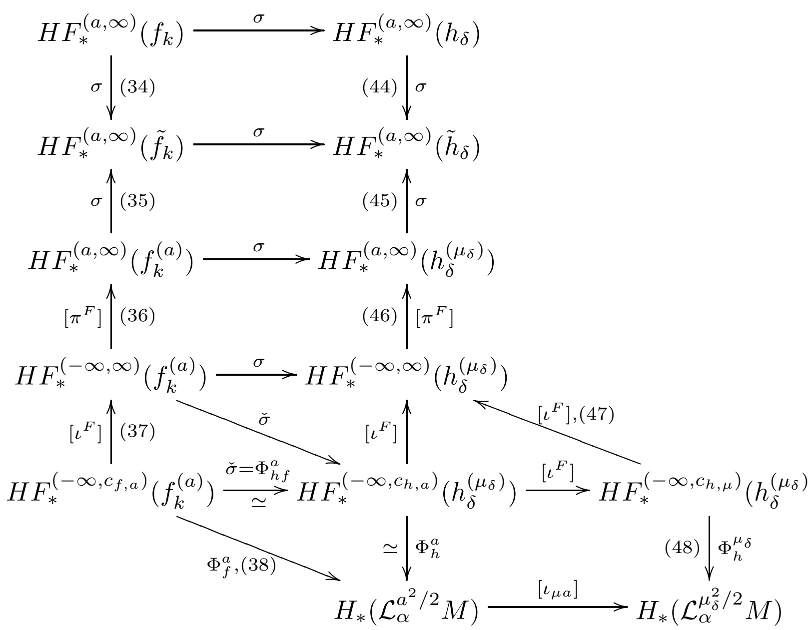
To see this observe that the upper triangle commutes by (28). Definition of ϕ and $I_{a^2/2}$ is obvious. Commutativity of the lower rectangular block follows by applying (49) to the identities

$$I_{a^2/2}(\psi^{(k)}x_{f_k}) = \big[\mu_\delta^2/2, [\iota_{\frac{\mu_\delta^2}{2}\frac{a^2}{2}}](\psi^{(k)}x_{f_k})\big],$$

which uses a), and

$$\phi \circ \iota_{f_k}(x_{f_k}) = \phi([f_k, x_{f_k}]) = \phi([h_\delta, \sigma_{h_\delta f_k} x_{f_k}]) = [\mu_\delta^2/2, \phi^{(\delta)}(\sigma_{h_\delta f_k} x_{f_k})],$$

which uses c). It remains to prove that diagram (49) commutes, which we rewrite as follows (simplifying notation slightly)



The first two rectangular blocks of the diagram commute due to property (17) of the monotone homomorphisms. The third block commutes by (18). The fourth block, which consists of two triangles, commutes by (19) with $(a, b_1, b_2, c) = (-\infty, c_{h,a}, c_{f,a}, \infty)$. The triangle to its right commutes already on the chain level. The final rectangular block commutes by Theorem 2.7 (22) and the triangle to its left by (21). This concludes the proof of Step 5. □

The proof of Theorem 3.1 is complete. □

4 Relative BPS-capacity

We compute the BPS-capacity of DT^*M relative to M and show that every nontrivial free homotopy class of loops in M is symplectically essential. For $c > 0$, set $\mathcal{H}_c := \{H \in C_0^\infty(S^1 \times DT^*M) \mid \sup_{S^1\times M} H \le -c\}$. In the following definition we use the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Definition 4.1 ([2, Sections 3.2 and 4.9]). For $\alpha \in \pi_1(M)$ and $a \in \mathbb{R}$, the *relative Biran-Polterovich-Salamon capacity* is defined by

$$\begin{aligned} c_{BPS}(DT^*M, M; \alpha, a) := \inf\{c > 0 \mid \forall H \in \mathcal{H}_c \ \exists z \in \mathcal{P}(H;\alpha) \\ \text{such that } \mathcal{A}_H(z) \geq a\} \\ c_{BPS}(DT^*M, M; \alpha) := \inf\{c > 0 \mid \ \mathcal{P}(H;\alpha) \neq \emptyset \ \forall H \in \mathcal{H}_c\}. \end{aligned}$$

For $c > 0$, define the set

$$A_c(DT^*M, M; \alpha) := \{a \in \mathbb{R} \ (a > 0 \text{ if } \alpha = 0) \mid T^{(a,\infty)}_{\alpha;c} \neq 0\}.$$

Define the *homological relative BPS-capacity* by

$$\hat{c}_{BPS}(DT^*M, M; \alpha, a) := \inf\{c > 0 \mid \sup A_c(DT^*M, M; \alpha) > a\}.$$

The significance of the homological relative capacity lies in the fact that it is accessible to computation and bounds c_{BPS} from above [2, Proposition 4.9.1].

Proposition 4.2. $\hat{c}_{BPS}(DT^*M, M; \alpha, a) = \max\{\ell_\alpha, a\}$, $\forall \alpha \in \pi_1(M)$, $\forall a \in \mathbb{R}$.

Proof. Theorem 3.1 shows that

$$\sup A_c(DT^*M, M; \alpha) = \sup[\ell_\alpha, c] = \begin{cases} c, & \text{if } c \geq \ell_\alpha, \\ -\infty, & \text{else,} \end{cases}$$

and this implies

$$\hat{c}_{BPS}(DT^*M, M; \alpha, a) = \inf\{c > 0 \mid c > a \text{ and } c \geq \ell_\alpha\} = \max\{a, \ell_\alpha\}.$$

□

Theorem 4.3 (BPS-capacity of DT^*M relative M). *Let M be a closed connected smooth Riemannian manifold. Then, for every free homotopy class α of loops in M and every $a \in \mathbb{R}$, the relative BPS-capacity is finite and given by*

$$c_{BPS}(DT^*M, M; \alpha, a) = \max\{\ell_\alpha, a\}.$$

Proof. The proof of [2, Proof of Theorem 3.2.1] carries over almost literally. In case $\alpha = 0$ every $H \in \mathcal{H}$ admits constant 1-periodic orbits of action zero, because it is zero near the boundary of DT^*M. Therefore $c_{BPS}(DT^*M, M; 0, a) = 0$ whenever $a \leq 0$. The remaining part of proof proceeds like [2, Proof of Theorem 3.2.1; case $\alpha \neq 0$] with Theorem 5.1.1 and Theorem 5.1.2 replaced by Proposition 4.2, ℓ by ℓ_α, and with

$$m := \begin{cases} \max\{\ell_\alpha, a\}, & \text{in case } \alpha \neq 0 \text{ and } a \in \mathbb{R}, \\ a, & \text{in case } \alpha = 0 \text{ and } a > 0. \end{cases}$$

□

Definition 4.4 ([2, Section 3.4]). Let M be a closed connected smooth manifold. A nontrivial free homotopy class α of loops in M is called *symplectically essential* if there exists a domain $W \subset T^*M$ containing M and such that $c_{BPS}(W, M; \iota_\# \alpha)$ is finite. Here $\iota_\#$ is the map between free homotopy classes induced by the inclusion $\iota : M \hookrightarrow W$.

Corollary 4.5. *Every nontrivial free homotopy class α of loops in a closed connected smooth Riemannian manifold M is symplectically essential.*

Proof. Let $W := DT^*M$, then $\iota_\#$ is an isomorphism and in abuse of notation we write $\iota_\# \alpha = \alpha$. Theorem 4.3 shows

$$c_{BPS}(DT^*M, M; \alpha) = c_{BPS}(DT^*M, M; \alpha, -\infty) = \ell_\alpha. \tag{51}$$

□

Proof of Theorem A. By [2, Proof of Theorem B], we may assume without loss of generality that H is periodic, i.e. $H \in C_0^\infty(S^1 \times DT^*M)$. Theorem 4.3 shows $c_{BPS}(DT^*M, M; \alpha, c) = \max\{\ell_\alpha, c\} = c$. By [2, Proposition 3.3.4], the set which appears in the definition of $c_{BPS}(DT^*M, M; \alpha, c)$ has a minimum and this proves the theorem. □

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