

Negative Binomial and Multinomial States: probability distributions and coherent states

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Abstract

Following the relationship between probability distribution and coherent states, for example the well known Poisson distribution and the ordinary coherent states and relatively less known one of the binomial distribution and the $su(2)$ coherent states, we propose *interpretation* of $su(1,1)$ and $su(r,1)$ coherent states *in terms of probability theory*. They will be called the *negative binomial (multinomial) states* which correspond to the *negative binomial (multinomial) distribution*, the non-compact counterpart of the well known binomial (multinomial) distribution. Explicit forms of the negative binomial (multinomial) states are given in terms of various boson representations which are naturally related to the probability theory interpretation. Here we show fruitful interplay of probability theory, group theory and quantum theory.

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1 Introduction

It is well known that the photon number distribution of the ordinary coherent states [1, 2, 3, 4] is the Poisson distribution, one of the most fundamental probability distributions, which governs random events (such as radioactive decays) occurring in a time (space) interval. As we will show in this paper the relationship between the coherent states in quantum optics and the probability distributions are neither coincidental nor superficial but essential. The main purpose of the present paper is to give unified probabilistic interpretation of the various coherent states.

For the elementary binomial distribution of the probability theory, corresponding to the binomial expansion $(1 + x)^M = \sum_0^M \binom{M}{n} x^n$, we have $su(2)$ coherent states (the ‘binomial states’ (BS) [5]) based on the spin $M/2$ representation. For the multinomial distributions corresponding to the multinomial expansion

$$(1 + x_1 + \cdots + x_r)^M = \sum_{n_0+n_1+\cdots+n_r=M} \frac{M!}{n_0!n_1!\cdots n_r!} x_1^{n_1} \cdots x_r^{n_r}, \quad (1.1)$$

we have certain types of $su(r + 1)$ coherent states. These coherent states are known for some time [6] but the probabilistic interpretation seems new. Let us call them *multinomial states* (MS). They are based on the *symmetric* representations corresponding to the Young diagram

$$\square\square\square \cdots \square\square \quad M \text{ boxes.} \quad (1.2)$$

In probability theory the non-compact version of the binomial distribution is well known and called *negative binomial distribution*. In this paper the *negative binomial states* (NBS) of quantised radiation field will be introduced in a parallel way as the binomial states. It will be shown that they are the well known coherent states of $su(1, 1)$ algebra [3, 6, 7, 8], the non-compact counterpart of the compact $su(2)$ algebra. They belong to the discrete series of irreducible representations. Similarly the *negative multinomial states* (NMS), the coherent states of $su(r, 1)$ algebra belonging to discrete symmetric representations, will be introduced in terms of the *negative multinomial distributions*. It is easy to see that in certain limits these coherent states reduce to the ordinary coherent states and their tensor products, since the (negative) binomial and (negative) multinomial distributions tend to the Poisson and multiple Poisson distributions.

This paper is organised as follows: In section 2 the *negative binomial states* are introduced directly as a *square root* of the negative binomial distribution. In other words they

are constructed in such a way that their photon number distribution is the negative binomial distribution. Then these coherent states are shown to have the displacement operator forms. Namely, they are created by the action of the unitary operators in $SU(1, 1)$ acting on certain highest (lowest) weight states (“vacuum”). In section 3 we relate the inhomogeneous representation of $su(1, 1)$ suggested by the negative binomial states to the symmetric two boson realisation. The two boson formulation provides natural interpretation and more explicit formulas than those of the formal representation theory of $su(1, 1)$. At the same time this section uncovers some Lie algebraic structures hidden in the probability distribution. The physical and statistical properties of the NBS as well as their dynamical generation are discussed in some detail in our recent publication [9]. Section 4 deals with the generalisation to $su(r, 1)$, the negative multinomial states. One formulation of the negative multinomial states is closely related with the Holstein-Primakoff (H-P) [10] type realisation of $su(r, 1)$ in terms of r ($=$ rank of $su(r, 1)$) bosons. Whereas the comparison with the $r + 1$ boson realisation gives natural interpretation of various quantities and concepts. By explicit Lie algebraic calculation which goes quite parallel with probability theory, it is shown that the negative multinomial states are $su(r, 1)$ coherent states belonging to discrete symmetric representations. Section 5 is for summary and comments. Appendix A serves to give general background of the paper, relating probability theory, coherent states and Lie algebra theory by taking elementary examples such as the ordinary coherent states and the binomial states. Appendix B also provides some basic elements like quantum mechanical generation of coherent states. A collection of two level atoms is discussed. It gives a good physical example of the binomial states and at the same time it provides simple interpretation of the H-P realisations as well as the relationship with the ordinary coherent states. Appendix C gives the higher rank generalisation of the results of the previous two Appendices. Here we advocate a seemingly ill-recognised fact that the multinomial states are coherent states of $su(r + 1)$ belonging to the symmetric representations. We stress, here as in the main text, the interplay of probability theory, Lie algebra theory and quantum mechanics exemplified in various coherent states. Appendix D gives a short explanation of the negative binomial distribution as a distribution of “waiting time”. We adopt such notation as to reveal the essential features underlying this subject which sometimes results in deviating from the conventional notation.

2 Negative Binomial State

Let us start with the *negative binomial distribution* (For an elementary introduction of the negative binomial distribution from probability theory see Appendix D. For more details,

see for example, Chap.VI of [11])

$$B_n^-(\eta; M) = \binom{M+n-1}{n} \eta^{2n} (1-\eta^2)^M, \quad n = 0, 1, \dots, \quad (2.1)$$

in which $0 < \eta^2 < 1$ and M is a positive integer. This can be rewritten as

$$(1-\eta^2)^{-M} B_n^-(\eta; M) = \binom{-M}{n} (-\eta^2)^n, \quad n = 0, 1, \dots, \quad (2.2)$$

and it is easy to see that the right hand side corresponds to the power series expansion of $(1-\eta^2)^{-M}$, the *negative binomial expansion*. Thus the normalisation

$$\sum_{n=0}^{\infty} B_n^-(\eta; M) = 1 \quad (2.3)$$

is obvious. From this it is also easy to see that the negative binomial distribution (and later the negative binomial states) can be defined for any positive number M . In this case we have to interpret

$$\binom{M+n-1}{n} = \frac{\Gamma(M+n)}{\Gamma(M) n!}. \quad (2.4)$$

Let us introduce the ‘*negative binomial state*’ (NBS) by taking a ‘square root’ of the negative binomial distribution (2.1). To be more precise, we follow the analogy *Poisson distribution* \Leftrightarrow *coherent state* (for details see Appendix A):

$$P_n(\alpha) = e^{-\alpha^2} \frac{\alpha^{2n}}{n!} \Leftrightarrow |\alpha e^{i\theta}\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{i\theta})^n}{\sqrt{n!}} |n\rangle, \quad (2.5)$$

in which $\alpha > 0$. Namely we define NBS

$$|\eta e^{i\theta}; M\rangle^- = (1-\eta^2)^{M/2} \sum_{n=0}^{\infty} \sqrt{\binom{M+n-1}{n}} (\eta e^{i\theta})^n \|n\rangle, \quad (2.6)$$

in which $\{\|n\rangle \mid n = 0, 1, \dots\}$ are the number states of an oscillator:

$$[b, b^\dagger] = 1, \quad b\|0\rangle = 0, \quad \|n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}} \|0\rangle. \quad (2.7)$$

(The reason for using a slightly unconventional notation $\|n\rangle$ will become clear in the next section.) Then the number distribution in the NBS is the negative binomial distribution (2.1):

$$|\langle n \| \eta e^{i\theta}; M \rangle^-|^2 = (1-\eta^2)^M \binom{M+n-1}{n} \eta^{2n} = B_n^-(\eta; M). \quad (2.8)$$

The condition $0 < \eta^2 < 1$ is necessary for the NBS to be normalisable. In the next section we will have a geometrical interpretation of the same condition as characterising the parameter space (the Poincaré disk) of the $su(1,1)$ coherent states.

Next let us rewrite (2.6) ($\eta_C \equiv \eta e^{i\theta}$)

$$|\eta_C; M\rangle^- = (1 - |\eta_C|^2)^{\frac{M}{2}} \sum_{n=0}^{\infty} \frac{\sqrt{M(M+1)\cdots(M+n-1)}}{n!} (\eta_C)^n (b^\dagger)^n \|0\rangle. \quad (2.9)$$

This can be reexpressed in the exponential form

$$|\eta_C; M\rangle^- = (1 - |\eta_C|^2)^{\frac{M}{2}} \exp[\eta_C \mathcal{K}_+] \|0\rangle, \quad (2.10)$$

in which

$$\mathcal{K}_+ = b^\dagger \sqrt{M+N} \equiv \sqrt{M+N-1} b^\dagger. \quad (2.11)$$

Here use is made of the following identity [12]

$$(b^\dagger g(N))^n \|0\rangle = (b^\dagger)^n g(0)g(1)\cdots g(n-1)\|0\rangle, \quad \text{with } g(N) \equiv \sqrt{M+N}, \quad N = b^\dagger b. \quad (2.12)$$

Eq.(2.10) and (2.11) reveal the $su(1, 1)$ structure of NBS since \mathcal{K}_+ and its hermitian conjugate

$$\mathcal{K}_- = \sqrt{M+N} b \equiv b \sqrt{M+N-1} \quad (2.13)$$

generate the $su(1, 1)$ algebra via H-P [10] realisation of the discrete irreducible representation¹ with the Bargman index $M/2$:

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0, \quad [\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \quad \mathcal{K}_0 = N + \frac{M}{2}, \quad (2.14)$$

and the “vacuum” $\|0\rangle$ is the lowest weight state:

$$\mathcal{K}_- \|0\rangle = 0, \quad \mathcal{K}_0 \|0\rangle = \frac{M}{2} \|0\rangle. \quad (2.15)$$

It is easy to see that (2.10) is expressed in the displacement operator form by using the disentangling theorem for $su(1, 1)$:

$$|\eta e^{i\theta}; M\rangle^- = \exp[\zeta_C \mathcal{K}_+ - \zeta_C^* \mathcal{K}_-] \|0\rangle, \quad \zeta_C = e^{i\theta} \operatorname{arctanh} \eta. \quad (2.16)$$

In other words the negative binomial states are $su(1, 1)$ coherent states in the definition of [4, 3, 6], although the $su(1, 1)$ structure is not obvious in the original definition of the binomial state (2.6). It should be remarked that in contrast to the binomial states which cover all the coherent states of $su(2)$ the negative binomial states give only part of the $su(1, 1)$ coherent states. (There are other types of $su(1, 1)$ coherent states: for example those which are eigenstates of K_- , [13].)

¹The generalised case with the real non-integer M gives the continuous irreducible representation of the universal covering group of $SU(1, 1)$.

It should be remarked that the generating function of the negative binomial state

$$\begin{aligned} G^-(\eta; M; t) &= \sum_{n=0}^{\infty} t^n B_n^-(\eta; M), \quad |t| \leq 1 \\ &= \frac{(1 - \eta^2)^M}{(1 - \eta^2 t)^M} \end{aligned} \quad (2.17)$$

has a succinct “quantum” definition

$$G^-(\eta; M; t) = \langle \eta e^{i\theta}; M \| t^N \| \eta e^{i\theta}; M \rangle. \quad (2.18)$$

As is well known in probability theory [11] the generating function is quite useful for calculating various statistical quantities of the negative binomial states [9].

3 Two Boson Formulation of NBS

As with the binomial states discussed in Appendix A and B, the simplest way to understand the negative binomial states algebraically is to introduce two bosonic oscillators to express the $su(1, 1)$ generators as bilinear forms rather than the inhomogeneous forms as given (2.11), (2.13) and (2.14). (We choose the formalism that the oscillators define the ordinary positive definite Hilbert space but the generators of the algebra reflect the non-compactness.) Let us introduce two bosonic oscillators

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad j, k = 0, 1, \quad (3.1)$$

and the Fock space

$$a_j |0, 0\rangle = 0, \quad j = 0, 1, \quad |n_0, n_1\rangle = \frac{a_0^{\dagger n_0} a_1^{\dagger n_1}}{\sqrt{n_0! n_1!}} |0, 0\rangle. \quad (3.2)$$

Define

$$K_+ = a_0^\dagger a_1^\dagger, \quad K_- = a_0 a_1, \quad K_0 = \frac{1}{2}(N_0 + N_1 + 1), \quad N_j = a_j^\dagger a_j, \quad (3.3)$$

which satisfy $su(1, 1)$ algebra

$$[K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\pm. \quad (3.4)$$

These operators either increase (K_+) or decrease (K_-) n_0 and n_1 simultaneously by 1 or keep them unchanged (K_0). In other words the above Fock space gives a reducible representation of $su(1, 1)$ since the subspaces with different $\Delta \equiv n_0 - n_1$ are always separated. So we can restrict it as in the case of the binomial states

$$\Delta \equiv n_0 - n_1 = M - 1 \geq 0, \quad (3.5)$$

in which M is a *positive integer*^{2 3}. Thus we arrive at the discrete representation of $su(1, 1)$ with Bargman index $M/2$

$$|M + n - 1, n\rangle, \quad n = 0, 1, \dots, \quad (3.6)$$

with the lowest weight state

$$K_- |M - 1, 0\rangle = 0, \quad K_0 |M - 1, 0\rangle = \frac{M}{2} |M - 1, 0\rangle. \quad (3.7)$$

Obviously this representation is irreducible. Since these states are uniquely specified by $n \equiv n_1$, we can identify them with the number states defined in the previous section (2.7) together with the “reduced” oscillator b and b^\dagger :

$$\|n\rangle = |M + n - 1, n\rangle, \quad n = 0, 1, \dots \quad (3.8)$$

Thus we obtain the H-P representation of K_\pm and K_0 :

$$K_+ \rightarrow \mathcal{K}_+ = b^\dagger \sqrt{M + N}, \quad K_- \rightarrow \mathcal{K}_- = \sqrt{M + N} b, \quad K_0 \rightarrow \mathcal{K}_0 = N + \frac{M}{2}. \quad (3.9)$$

One advantage of the H-P type realisation as above is that it admits the generalisation to the continuous representation for non-integer M .

The other group theoretical aspects of the negative binomial states are about the same as those in the binomial states. The physical and statistical properties of the NBS as well as their dynamical generation are discussed in some detail in our recent publication [9]. The content of this section, though known in Lie algebra theory, can be considered to provide some Lie algebraic backgrounds for the probability distribution, which are new to the best of our knowledge.

4 Negative Multinomial States

The negative multinomial distribution is

$$M_{\mathbf{n}'}^-(\boldsymbol{\eta}; M) = (1 - \boldsymbol{\eta}^2)^M \frac{(M + n_1 + \dots + n_r - 1)!}{\mathbf{n}'!(M - 1)!} \eta_1^{2n_1} \dots \eta_r^{2n_r}, \quad (4.1)$$

in which M is a positive integer and

$$\begin{aligned} \mathbf{n} &= (n_0, n_1, \dots, n_r), \quad \mathbf{n}' = (n_1, \dots, n_r) \quad \boldsymbol{\eta} = (\eta_1, \dots, \eta_r) \in \mathbf{R}^r, \\ 0 &< \boldsymbol{\eta}^2 = \eta_1^2 + \dots + \eta_r^2 < 1, \quad \mathbf{n}'! = n_1! \dots n_r!. \end{aligned} \quad (4.2)$$

²For the other sign of r.h.s. we can change the role of 0 and 1.

³In terms of the “waiting time” interpretation of the negative binomial distribution (see Appendix D) n_0 is the total number of trials but the last (which is always a *success*), n_1 is the number of *failures* and M is the preset number of *successes* to be achieved.

In particular, the negative *trinomial distribution* reads

$$M_{n_1, n_2}^-(\eta_1, \eta_2; M) = (1 - \boldsymbol{\eta}^2)^M \frac{(M + n_1 + n_2 - 1)!}{n_1! n_2! (M - 1)!} \eta_1^{2n_1} \eta_2^{2n_2}. \quad (4.3)$$

This can be easily obtained from the negative binomial distribution

$$B_n^-(\eta; M) = \binom{M + n - 1}{n} \eta^{2n} (1 - \eta^2)^M, \quad n = 0, 1, \dots,$$

by a binomial expansion

$$\eta^2 = \eta_1^2 + \eta_2^2, \quad \eta^{2n} = \sum_{n_1 + n_2 = n} \frac{n!}{n_1! n_2!} \eta_1^{2n_1} \eta_2^{2n_2},$$

and collecting appropriate terms. By repeating the same thing or by applying a multinomial expansion we arrive at the general form of the negative multinomial distribution (4.1). As we will see later this procedure also explains the generation of *negative multinomial states*.

The *negative multinomial state* (NMS) is defined by taking a “square root” of the negative multinomial distribution (4.1):

$$|\boldsymbol{\eta}_C; M\rangle^- = (1 - |\boldsymbol{\eta}_C|^2)^{M/2} \sum_{\mathbf{n}'} \sqrt{\frac{(M + n_1 + \dots + n_r - 1)!}{\mathbf{n}'! (M - 1)!}} (\eta_{1C})^{n_1} \dots (\eta_{rC})^{n_r} \|\mathbf{n}'\rangle, \quad (4.4)$$

in which the “reduced” states $\{\|\mathbf{n}'\rangle = \|n_1, \dots, n_r\rangle \mid n = 0, 1, \dots\}$ are the number states of r bosonic oscillators:

$$\begin{aligned} [b_j, b_k^\dagger] &= \delta_{jk}, \quad b_j \|\mathbf{0}\rangle = 0, \quad j = 1, \dots, r, \quad \|\mathbf{0}\rangle = |0, \dots, 0\rangle, \\ \|\mathbf{n}'\rangle &= \frac{(\mathbf{b}^\dagger)^{\mathbf{n}'}}{\sqrt{\mathbf{n}'!}} \|\mathbf{0}\rangle, \quad (\mathbf{b}^\dagger)^{\mathbf{n}'} = b_1^{\dagger n_1} \dots b_r^{\dagger n_r}. \end{aligned} \quad (4.5)$$

It should be remarked that both negative multinomial distribution (4.1) and state (4.4) are also well defined for M positive real number.

In order to show that the negative multinomial states are the coherent states of $su(r, 1)$, we need to realise the algebra. Let us first construct $su(r, 1)$ generators on the Fock space generated by $r + 1$ bosonic oscillators:

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \quad a_j \|\mathbf{0}\rangle = 0, \quad j = 0, 1, \dots, r, \quad \|\mathbf{0}\rangle = |0, 0, \dots, 0\rangle, \\ |\mathbf{n}\rangle &= \frac{(\mathbf{a}^\dagger)^{\mathbf{n}}}{\sqrt{\mathbf{n}!}} \|\mathbf{0}\rangle, \quad (\mathbf{a}^\dagger)^{\mathbf{n}} = a_0^{\dagger n_0} a_1^{\dagger n_1} \dots a_r^{\dagger n_r}. \end{aligned} \quad (4.6)$$

Let us define the $u(r, 1)$ generators as bilinears in a_j and a_k^\dagger :

$$\begin{aligned} K_{+j} &= a_0^\dagger a_j^\dagger, \quad K_{-k} = a_0 a_k, \quad 1 \leq j, k \leq r, \\ K_{jk} &= a_j^\dagger a_k \quad (j \neq k \neq 0), \quad N_j = a_j^\dagger a_j. \end{aligned} \quad (4.7)$$

It is easy to see that they leave the combination

$$\Delta \equiv N_0 - (N_1 + \cdots + N_r)$$

invariant and the above Fock space (4.6) is a disjoint sum of subspaces characterised by the value of Δ . As before, let us impose a constraint

$$\Delta \equiv N_0 - (N_1 + \cdots + N_r) = M - 1 \geq 0, \quad (4.8)$$

in which M is a positive integer. Then for fixed M the restricted space provides an irreducible representation of $u(r, 1)$. It has a lowest weight vector

$$K_{-j}|M - 1, 0, \dots, 0\rangle = 0, \quad j = 1, \dots, r \quad (4.9)$$

which is invariant under $su(r)$:

$$K_{jk}|M - 1, 0, \dots, 0\rangle = N_j|M - 1, 0, \dots, 0\rangle = 0, \quad 1 \leq j, k \leq r. \quad (4.10)$$

Let us connect the states $\|\mathbf{n}'\rangle$ and $|\mathbf{n}\rangle$. Each state in the above representation is uniquely specified by \mathbf{n}' only and we identify

$$\|\mathbf{n}'\rangle \equiv |\mathbf{n}\rangle = |n_0, n_1, \dots, n_r\rangle, \quad n_0 = M + n_1 + \cdots + n_r - 1$$

or

$$\|n_1, \dots, n_r\rangle \equiv |M + n_1 + \cdots + n_r - 1, n_1, \dots, n_r\rangle, \quad \text{and} \quad \|\mathbf{0}\rangle = |M - 1, 0, \dots, 0\rangle. \quad (4.11)$$

On these states the $su(r, 1)$ generators are expressed inhomogeneously:

$$\begin{aligned} K_{+j} \rightarrow \mathcal{K}_{+j} &= b_j^\dagger \sqrt{M + N_1 + \cdots + N_r - 1}, & K_{+j} \rightarrow \mathcal{K}_{+j} &= \sqrt{M + N_1 + \cdots + N_r - 1} b_j, \\ K_{jk} \rightarrow \mathcal{K}_{jk} &= b_j^\dagger b_k, & \mathcal{N}_j &= N_j. \end{aligned} \quad (4.12)$$

Note that the invariant subalgebra $u(r)$ is expressed bilinearly.

It is not difficult to generate the negative multinomial states explicitly by applying the $SU(r, 1)$ operator on the lowest weight state $\|\mathbf{0}\rangle$. For simplicity and concreteness let us show this for the $su(2, 1)$ case:

$$|\eta_1 e^{i\theta_1}, \eta_2 e^{i\theta_2}; M\rangle^- = (1 - \boldsymbol{\eta}^2)^{\frac{M}{2}} \sum_{n_1, n_2=0}^{\infty} \sqrt{\frac{(M + n_1 + n_2 - 1)!}{n_1! n_2! (M - 1)!}} (e^{i\theta_1} \eta_1)^{n_1} (e^{i\theta_2} \eta_2)^{n_2} \|n_1, n_2\rangle. \quad (4.13)$$

For the ‘*negative trinomial state*’ we only have to use the $su(1, 1)$ and $su(2)$ disentangling theorems for the two subalgebras spanned by $(0, 1)$ and $(1, 2)$ oscillators, respectively:

$$e^{r(e^{i\theta_1}\mathcal{K}_{10}-e^{-i\theta_1}\mathcal{K}_{01})} \quad (4.14)$$

$$= \exp[e^{i\theta_1} \tanh r \mathcal{K}_{10}] \exp[\log(1 - \tanh^2 r)(N_1 + M/2)] \exp[-e^{-i\theta_1} \tanh r \mathcal{K}_{01}],$$

$$e^{r'(e^{i\theta_2}\mathcal{K}_{21}-e^{-i\theta_2}\mathcal{K}_{12})} \quad (4.15)$$

$$= \exp[e^{i\theta_2} \tan r' \mathcal{K}_{21}] \exp[-\frac{1}{2} \log(1 + \tan^2 r')(N_1 - N_2)] \exp[-e^{-i\theta_2} \tan r' \mathcal{K}_{12}].$$

First let us choose r such that

$$\tanh^2 r = \eta_1^2 + \eta_2^2,$$

to obtain

$$e^{r(e^{i\theta_1}\mathcal{K}_{10}-e^{-i\theta_1}\mathcal{K}_{01})} \|0, 0\rangle = (1 - \eta_1^2 - \eta_2^2)^{\frac{M}{2}} \sum_{n=0}^{\infty} \sqrt{\frac{(M+n-1)!}{(M-1)!n!}} \left(e^{i\theta_1} \sqrt{\eta_1^2 + \eta_2^2} \right)^n \|n, 0\rangle, \quad (4.16)$$

which is a negative binomial state in the $(0, 1)$ subspace. Next let us choose r' such that

$$\tan^2 r' = \frac{\eta_2^2}{\eta_1^2},$$

to obtain

$$e^{r'(e^{i\theta_2}\mathcal{K}_{21}-e^{-i\theta_2}\mathcal{K}_{12})} \times e^{r(e^{i\theta_1}\mathcal{K}_{10}-e^{-i\theta_1}\mathcal{K}_{01})} \|0, 0\rangle = |\eta_1 e^{i\theta_1}, \eta_2 e^{i\theta_2}; M\rangle^-. \quad (4.17)$$

Thus we have shown that the negative trinomial states are the coherent states of $su(2, 1)$ belonging to discrete symmetric representations. Note the parallelism with the negative trinomial distribution at the beginning of this section. The generalisation to higher rank cases is straightforward. One has to apply first $su(1, 1)$ disentangling theorem and $su(2)$ disentangling theorems in the following sequence of $su(2)$ algebras spanned by $(1, 2)$, $(2, 3)$, \dots , $(r-1, r)$ oscillators.

Before concluding this section let us remark that the generalisation of the discussion (4.13)–(4.17) to the coherent states of $su(r, s)$ is rather straightforward.

5 Summary and Comments

Stimulated by the well known fact that the photon number distribution of the ordinary coherent states is Poissonian, we have constructed quantum mechanical states which have the other well known probability distributions such as the binomial, multinomial, negative binomial and negative multinomial distributions as their particle number distributions. They

turn out to be the coherent states of the well known Lie algebras of $su(2)$, $su(r+1)$, $su(1,1)$ and $su(r,1)$, respectively, belonging to certain symmetric representations. Interpretation of these coherent states in terms of probability theory is obtained and it is quite useful. At the same time Lie algebraic structure of these most fundamental probability distributions is revealed.

The results of the present paper provoke many questions, to most of which we do not have answers yet. For example: What about the coherent states of $su(r+1)$ ($su(r,1)$) belonging to the other representations? ⁴ Are they also characterised by some probability distributions? The same question for the other Lie algebras, in particular the exceptional ones. Or the affine Lie algebras and other infinite dimensional algebras like the Virasoro and the w -algebras...?

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Appendix A Binomial States

In the Appendix A-C, we reformulate the mathematical theory of coherent states for $su(2)$ and $su(r+1)$ algebras. Most of the results are known in one way or another but we believe that the elementary exposition and the resulting explicit and concrete formulas and the emphasis on the connection with probability distributions are helpful and useful for most readers. It is also hoped that the comparison and the contrast with the compact cases will provide deeper understanding of the non-compact cases treated in the main sections.

We follow the schematic path

$$\text{Probability Distribution} \iff \text{Coherent States} \tag{A.1}$$

by imitating the well known example of the Poisson distribution

$$P_n(\alpha) = e^{-\alpha^2} \frac{\alpha^{2n}}{n!}, \quad n = 0, 1, 2, \dots \tag{A.2}$$

⁴Some coherent states can be constructed in terms of fermion oscillators [2, 4].

Let us introduce the ‘‘probability amplitude’’ by taking its ‘square root’⁵

$$|\alpha e^{i\theta}\rangle = e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{i\theta})^n}{\sqrt{n!}} |n\rangle, \quad (\text{A.3})$$

in which $\{|n\rangle | n = 0, 1, \dots\}$ are the number states of the ordinary oscillator

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle. \quad (\text{A.4})$$

The origin of the additional phase factor $e^{i\theta}$ is obvious, $\alpha^2 = \alpha e^{i\theta} \alpha e^{-i\theta}$. By using the last formula of (A.4) we can rewrite (A.3) as ($\alpha_C \equiv \alpha e^{i\theta}$)

$$|\alpha_C\rangle = e^{-|\alpha_C|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha_C a^\dagger)^n}{n!} |0\rangle = e^{-|\alpha_C|^2/2} e^{\alpha_C a^\dagger} |0\rangle = \exp[\alpha_C a^\dagger - \alpha_C^* a] |0\rangle. \quad (\text{A.5})$$

At the last step use is made of the Baker-Campbell-Hausdorff formula. Eq.(A.5) tells that the parameter space is the ordinary complex plane \mathbf{C} , which is a coset space

$$\mathbf{C} = \text{Heisenberg-Weyl Group}/U(1), \quad (\text{A.6})$$

in which the Heisenberg-Weyl Group is generated by a, a^\dagger and the identity operator. The stability subgroup $U(1)$ is just the group of complex numbers of unit modulus, $U(1) = \{e^{i\theta} | \theta : \text{real}\}$.

The binomial distribution

$$B_n(\eta; M) = \binom{M}{n} \eta^{2n} (1 - \eta^2)^{M-n}, \quad n = 0, 1, \dots, M, \quad (\text{A.7})$$

is a well known elementary probability distribution related with binomial expansion⁶

$$1 = 1^M = (1 - x + x)^M = \sum_{n=0}^M \binom{M}{n} x^n (1 - x)^{M-n}. \quad (\text{A.8})$$

This gives the probability of n ‘successes’ among M times repeated Bernoulli’s trials with the *success* probability $0 < \eta^2 < 1$. The associated ‘probability amplitude’ is

$$|\eta e^{i\theta}; M\rangle = \sum_{n=0}^M \sqrt{\binom{M}{n}} (\eta e^{i\theta})^n (1 - \eta^2)^{\frac{M-n}{2}} |n\rangle, \quad (\text{A.9})$$

in which $\{|n\rangle | n = 0, 1, \dots\}$ are the number states of the oscillator

$$[b, b^\dagger] = 1, \quad b|0\rangle = 0, \quad |n\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}} |0\rangle. \quad (\text{A.10})$$

⁵Apart from the well known example Klein-Gordon eq. \Rightarrow Dirac eq., let us mention that the creation and annihilation operators a^\dagger, a are also ‘square roots’ of the oscillator Hamiltonian. In all these cases, including the probability amplitudes, the ‘square roots’ are complex in spite of the reality (hermiticity) of the original objects.

⁶Note that $e^{\alpha^2} P_n(\alpha) = \frac{\alpha^{2n}}{n!}$ is obtained by a power series expansion of e^{α^2} .

(The reason for using a different oscillator b, b^\dagger from the above coherent state one a, a^\dagger and the slightly unconventional notation $\|n\rangle$ will become clear in Appendix B.) Let us call the state (A.9) the ‘*binomial state*’ (BS) [5, 14, 15]. At first glance one might be tempted to give a phase to the second factor

$$(\sqrt{1 - \eta^2} e^{i\theta_2})^{M-n}. \quad (\text{A.11})$$

But this is unnecessary since it is decomposed to an overall phase $e^{iM\theta_2}$ (which is immaterial) and $e^{-in\theta_2}$ which can be absorbed by the redefinition of θ , $\theta \rightarrow \theta - \theta_2$.

Next let us rewrite (A.9) ($\eta_C \equiv \eta e^{i\theta}$)

$$|\eta_C; M\rangle = (1 - |\eta_C|^2)^{\frac{M}{2}} \sum_{n=0}^M \frac{\sqrt{M(M-1)\cdots(M-n+1)}}{n!} \left(\frac{\eta_C}{\sqrt{1 - |\eta_C|^2}} \right)^n (b^\dagger)^n \|0\rangle. \quad (\text{A.12})$$

Then, by making use of the following identity [12]

$$(b^\dagger g(N))^n \|0\rangle = (b^\dagger)^n g(0)g(1)\cdots g(n-1)\|0\rangle, \quad \text{with } g(N) \equiv \sqrt{M-N}, \quad N = b^\dagger b, \quad (\text{A.13})$$

we can write (A.12) in the exponential form

$$|\eta_C; M\rangle = (1 - |\eta_C|^2)^{\frac{M}{2}} \exp \left[\frac{\eta_C}{\sqrt{1 - |\eta_C|^2}} \mathcal{J}_+ \right] \|0\rangle. \quad (\text{A.14})$$

Here \mathcal{J}_+

$$\mathcal{J}_+ = b^\dagger \sqrt{M-N} \equiv \sqrt{M-N+1} b^\dagger \quad (\text{A.15})$$

together with its hermitian conjugate

$$\mathcal{J}_- = \sqrt{M-N} b \equiv b \sqrt{M-N+1} \quad (\text{A.16})$$

generate the $su(2)$ algebra via H-P [10] realisation in the spin $M/2$ representation:

$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_0, \quad [\mathcal{J}_0, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \quad \mathcal{J}_0 = N - \frac{M}{2}, \quad (\text{A.17})$$

and the ‘vacuum’ $\|0\rangle$ is the lowest weight state

$$\mathcal{J}_- \|0\rangle = 0, \quad \mathcal{J}_0 \|0\rangle = -\frac{M}{2} \|0\rangle. \quad (\text{A.18})$$

By using the disentangling theorem for $su(2)$ we can rewrite (A.14) as

$$|\eta e^{i\theta}; M\rangle = \exp [\zeta_C \mathcal{J}_+ - \zeta_C^* \mathcal{J}_-] \|0\rangle, \quad \zeta_C = e^{i\theta} \arctan \left(\frac{\eta}{\sqrt{1 - \eta^2}} \right). \quad (\text{A.19})$$

In other words the binomial states are $su(2)$ coherent states in the definition of [3, 4, 6], although the $su(2)$ structure is not obvious in the original definition of the binomial state

(A.9). Since all the irreducible representations of $su(2)$ are exhausted by the representations (A.10)-(A.18) for all non-negative integer values of M , the binomial states give all the $su(2)$ coherent states.

Before closing this Appendix, let us recall the fact that the binomial distribution tends to the Poisson distribution in a certain limit. Let $M \rightarrow \infty$, $\eta \rightarrow 0$ in such a way that the average value $\langle n \rangle$ is fixed: $\langle n \rangle = \eta^2 M = \alpha^2$. Then for finite n

$$B_n(\eta; M) \rightarrow \frac{\alpha^{2n}}{n!} e^{-\alpha^2} = P_n(\alpha). \quad (\text{A.20})$$

(Of course there are many other ways of showing this, e.g. in terms of the generating functions of these distributions.) We have the corresponding limit at the level of the ‘‘probability amplitude’’ (A.9)

$$|\eta e^{i\theta}; M\rangle \rightarrow e^{-\alpha^2/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{i\theta})^n}{\sqrt{n!}} \|n\rangle, \quad (\text{A.21})$$

namely the binomial state tends to the ordinary coherent state. This limit can also be visualised as a contraction of $su(2)$ (A.15),(A.16) into the Heisenberg-Weyl algebra :

$$\eta \mathcal{J}_+ \rightarrow \alpha b^\dagger, \quad \eta \mathcal{J}_- \rightarrow \alpha b. \quad (\text{A.22})$$

Thus (A.19) tends to

$$|\eta e^{i\theta}; M\rangle \rightarrow \exp [\alpha e^{i\theta} b^\dagger - \alpha e^{-i\theta} b] \|0\rangle. \quad (\text{A.23})$$

In Appendix B we will discuss the physical problem of dynamical generation of BS starting from certain Hamiltonian. This, in turn, will provide a mathematical framework in which (i) $su(2)$ structure is more visible and, (ii) generalisation to the coherent states of $su(r+1)$ algebra, the ‘*multinomial states*’, is straightforward.

Appendix B Binomial States: Two Boson Formulation

In order to discuss the generation of the binomial states, let us recapitulate the process of the physical generation of the ordinary coherent states, for comparison. This is an oversimplified model retaining only the most essential features of the coherent states. We focus on one particular mode of the photon since the system is decomposed into a sum of such subsystems:

$$H = H_0 + H_1, \quad H_0 = \omega a^\dagger a, \quad H_1 = j(t) a^\dagger + j(t)^* a, \quad (\text{B.1})$$

in which a^\dagger, a are the creation and annihilation operators of the photon and $j(t)$ is the classical current (with complex phase). The state vector in the interaction picture $|\psi(t)\rangle_I$

obeys the equation of motion

$$\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle_I &= \mathcal{H}_I(t)|\psi(t)\rangle_I, \\ \mathcal{H}_1(t) &= e^{iH_0t}H_1e^{-iH_0t} = j(t)e^{i\omega t}a^\dagger + j(t)^*e^{-i\omega t}a. \end{aligned} \quad (\text{B.2})$$

Let us suppose that the system is in the ‘vacuum’ $|0\rangle$ at $t = 0$. Then we obtain

$$\begin{aligned} |\psi(t)\rangle_I &= T e^{-i\int_0^t \mathcal{H}_1(t')dt'}|0\rangle, \\ &= e^{i\Omega(t)}e^{(\alpha(t)a^\dagger - \alpha(t)^*a)}|0\rangle, \\ \alpha(t) &= -i\int_0^t j(t')e^{i\omega t'}dt', \end{aligned} \quad (\text{B.3})$$

in which T is the time-ordering operator and $\Omega(t)$ is a calculable function giving the immaterial overall phase.

For the *binomial states* let us consider a slightly different model consisting of a number of identical *two level atoms* (bosons). Let us also assume that the space extension of the system is not big compared with the wavelength of the photon corresponding to the energy gap and that the interactions between different atoms are negligible. The system can be described by the ‘spin’ operators

$$\begin{aligned} H_B &= H_{B0} + H_{B1}, \\ H_{B0} &= \epsilon \sum_{j=1}^M \sigma_0^{(j)}, \\ H_{B1} &= \sum_{j=1}^M \left(\lambda(t)\sigma_+^{(j)} + \lambda(t)^*\sigma_-^{(j)} \right), \end{aligned} \quad (\text{B.4})$$

in which M is the number of the two level atoms. As is well known [16] a collection of identical particles can also be described by oscillators corresponding to each energy eigenstate. Let us denote the lower (upper) state and the corresponding oscillator by 0 (1) ⁷:

$$[a_j, a_k^\dagger] = \delta_{jk}, \quad j, k = 0, 1. \quad (\text{B.5})$$

Then the above Hamiltonian is equivalent to

$$\begin{aligned} H'_B &= H'_{B0} + H'_{B1}, \\ H'_{B0} &= \omega_1 N_1 + \omega_0 N_0, \quad N_j = a_j^\dagger a_j, \quad j = 0, 1, \quad \omega_1 - \omega_0 = \epsilon \\ H'_{B1} &= \mu(t)a_1^\dagger a_0 + \mu(t)^*a_0^\dagger a_1, \end{aligned} \quad (\text{B.6})$$

⁷ In quantum optics situations one may call 0 ‘wiggler’ photon and 1 ‘laser’ photon [14].

and its Fock space is

$$|n_0, n_1\rangle, \quad n_0 + n_1 = M. \quad (\text{B.7})$$

Now the $su(2)$ structure is obvious, since

$$J_+ = a_1^\dagger a_0, \quad J_- = a_0^\dagger a_1, \quad J_0 = \frac{1}{2}(N_1 - N_0) \quad (\text{B.8})$$

generate an $su(2)$ algebra and the Fock space (B.7) gives the $M + 1$ dimensional (spin $M/2$) irreducible representation corresponding to the Young diagram

$$\square\square\square \cdots \square\square \quad M \text{ boxes} \quad (\text{B.9})$$

with the lowest weight state

$$J_- |M, 0\rangle = 0, \quad J_0 |M, 0\rangle = -\frac{M}{2} |M, 0\rangle. \quad (\text{B.10})$$

The state vector in the interaction picture $|\psi(t)\rangle_I$ obeys the equation of motion

$$\begin{aligned} i \frac{d}{dt} |\psi(t)\rangle_I &= \mathcal{H}'_{B1}(t) |\psi(t)\rangle_I, \\ \mathcal{H}'_{B1}(t) &= e^{iH'_{B0}t} H'_{B1} e^{-iH'_{B0}t} = \mu(t) e^{i\epsilon t} J_+ + \mu(t)^* e^{-i\epsilon t} J_-. \end{aligned} \quad (\text{B.11})$$

Let us suppose that the system is in the lowest weight state $|M, 0\rangle$ at $t = 0$. Then we obtain

$$\begin{aligned} |\psi(t)\rangle_I &= T \exp \left[-i \int_0^t \mathcal{H}'_{B1}(t') dt' \right] |M, 0\rangle, \\ &= T \exp \left[-i \int_0^t \mu(t') e^{i\epsilon t'} dt' J_+ - i \int_0^t \mu(t')^* e^{-i\epsilon t'} dt' J_- \right] |M, 0\rangle. \end{aligned} \quad (\text{B.12})$$

Since $g = T e^{-i \int_0^t \mathcal{H}'_{B1}(t') dt'} \in SU(2)$ it can always be decomposed into $g = \exp(\zeta J_+ - \zeta^* J_-) \exp(i\nu J_0)$ ($\zeta \in \mathbf{C}, \nu \in \mathbf{R}$) and the obtained state is the binomial state. For illustration purpose let us choose a special form of $\mu(t)$:

$$\mu(t) = i e^{-i\epsilon t + i\theta} \eta, \quad \eta \in \mathbf{R}. \quad (\text{B.13})$$

Then we obtain [14, 17]

$$\begin{aligned} |\psi(t)\rangle_I &= \exp \left[e^{i\theta} \eta t J_+ - e^{i\theta} \eta t J_- \right] |M, 0\rangle, \\ &\propto \exp \left[e^{i\theta} \tan(\eta t) J_+ \right] |M, 0\rangle \propto | \tan(\eta t) e^{i\theta}; M \rangle. \end{aligned} \quad (\text{B.14})$$

However this is not exactly the same as the binomial state (A.14) given in the previous Appendix. In order to relate these two forms let us note that the state in the Fock space (B.7) is uniquely specified by $n_1 \equiv n$ only:

$$|n\rangle \equiv |M - n, n\rangle. \quad (\text{B.15})$$

Let us understand that the states $\{|n\rangle, n = 0, 1, \dots, M\}$ are generated by the “reduced” single boson operators b^\dagger and b as in (A.10). Then the $su(2)$ operators are expressed in terms of b^\dagger and b as

$$\begin{aligned} J_+ |n\rangle &= J_+ |M-n, n\rangle = a_1^\dagger a_0 |M-n, n\rangle, \\ &= \sqrt{n+1} \sqrt{M-n} |M-n-1, n+1\rangle = \sqrt{n+1} \sqrt{M-n} |n+1\rangle, \\ &= b^\dagger \sqrt{M-N} |n\rangle, \quad N = b^\dagger b, \end{aligned} \quad (\text{B.16})$$

namely

$$\mathcal{J}_+ = b^\dagger \sqrt{M-N}, \quad \mathcal{J}_- = \sqrt{M-N} b, \quad \mathcal{J}_0 = N - \frac{M}{2}. \quad (\text{B.17})$$

Thus we have naturally “derived” the H-P realisation of $su(2)$ used in the previous Appendix. The lowest weight state in this notation is

$$|0\rangle \equiv |M, 0\rangle, \quad \mathcal{J}_- |0\rangle = 0, \quad \mathcal{J}_0 |0\rangle = -\frac{M}{2} |0\rangle.$$

At the end of the previous Appendix we have shown that the binomial state tends to the ordinary coherent state in a certain limit. Here we will show a result in an opposite direction. That is, the binomial states can be obtained from the ordinary coherent states with two degrees of freedom by appropriate ‘slicing’ or restriction. This reveals some features of the binomial states quite naturally. As before let us start with the corresponding result in the probability theory, which is rather elementary. A double Poisson distribution is given by

$$P_{n_0, n_1}(\alpha_0, \alpha_1) \equiv P_{\mathbf{n}}(\boldsymbol{\alpha}) = e^{-\alpha_0^2 - \alpha_1^2} \frac{\alpha_0^{2n_0} \alpha_1^{2n_1}}{n_0! n_1!}, \quad \mathbf{n} = (n_0, n_1), \quad \boldsymbol{\alpha} = (\alpha_0, \alpha_1). \quad (\text{B.18})$$

If we restrict it to a line

$$n_0 + n_1 = M,$$

we obtain the binomial distribution up to normalisation:

$$\begin{aligned} P_{M-n, n}(\alpha_0, \alpha_1) &= e^{-\alpha_0^2 - \alpha_1^2} \frac{\alpha_0^{2(M-n)} \alpha_1^{2n}}{(M-n)! n!}, \quad \eta \equiv \frac{\alpha_1}{\sqrt{\alpha_0^2 + \alpha_1^2}}. \\ &\propto \binom{M}{n} \eta^{2n} (1 - \eta^2)^{M-n} = B_n(\eta; M). \end{aligned} \quad (\text{B.19})$$

The same proposition at the level of the “probability amplitude” including the normalisation can be easily obtained by considering the projection operator onto the representation space (B.7) [18]:

$$\mathbf{P}_M = \sum_{n_0+n_1=M} |n_0, n_1\rangle \langle n_0, n_1|. \quad (\text{B.20})$$

With the aid of the resolution of unity (over-completeness relation)

$$\int \frac{d^2\alpha_{0C}d^2\alpha_{1C}}{\pi^2} |\alpha_C\rangle\langle\alpha_C| = 1, \quad \alpha_C = (\alpha_{0C}, \alpha_{1C}), \quad d^2\alpha_{jC} = d(\alpha_{jC})_R d(\alpha_{jC})_I, \quad j = 0, 1, \quad (\text{B.21})$$

for the double coherent state

$$|\alpha_C\rangle = e^{-|\alpha_C|^2/2} \sum_{n_0, n_1} \frac{(\alpha_{0C})^{n_0} (\alpha_{1C})^{n_1}}{\sqrt{n_0! n_1!}} |n_0, n_1\rangle$$

we have

$$\mathbf{P}_M = \int \frac{d^2\alpha_{0C}d^2\alpha_{1C}}{\pi^2} |\alpha_C\rangle\langle\alpha_C| \mathbf{P}_M. \quad (\text{B.22})$$

By a change of variables

$$\begin{pmatrix} \alpha_{0C} \\ \alpha_{1C} \end{pmatrix} = \frac{\zeta_C}{\sqrt{1 + |\xi_C|^2}} \begin{pmatrix} 1 \\ \xi_C \end{pmatrix} \quad (\text{B.23})$$

we have

$$\begin{aligned} \text{r.h.s. of (B.22)} &= \int \frac{d^2\xi_C}{\pi(1 + |\xi_C|^2)^2} \int \frac{|\zeta_C|^2 d^2\zeta_C}{\pi} e^{-|\zeta_C|^2} \\ &\times \sum_{m_0, m_1=0}^{\infty} \frac{1}{\sqrt{m_0! m_1!}} \left(\frac{\zeta_C}{\sqrt{1 + |\xi_C|^2}} \right)^{m_0+m_1} \xi_C^{m_1} |m_0, m_1\rangle \\ &\times \sum_{n_0+n_1=M} \langle n_0, n_1 | \frac{1}{\sqrt{n_0! n_1!}} \left(\frac{\zeta_C^*}{\sqrt{1 + |\xi_C|^2}} \right)^M \xi_C^{*n_1} \\ &= \int \frac{d^2\xi_C}{\pi} \frac{(M+1)!}{(1 + |\xi_C|^2)^{M+2}} \sum_{m_0+m_1=M} \frac{1}{\sqrt{m_0! m_1!}} \xi_C^{m_1} |m_0, m_1\rangle \\ &\times \sum_{n_0+n_1=M} \langle n_0, n_1 | \frac{1}{\sqrt{n_0! n_1!}} \xi_C^{*n_1} \\ &= \int d\mu(\xi_C, \xi_C^*) |\xi_C\rangle\langle\xi_C|, \end{aligned} \quad (\text{B.24})$$

in which

$$\begin{aligned} |\xi_C\rangle &= \frac{1}{(1 + |\xi_C|^2)^{M/2}} \sum_{n=0}^M \sqrt{\binom{M}{n}} \xi_C^n |M-n, n\rangle \\ &= \frac{1}{(1 + |\xi_C|^2)^{M/2}} \sum_{n=0}^M \sqrt{\binom{M}{n}} \xi_C^n |n\rangle \end{aligned} \quad (\text{B.25})$$

and the measure is

$$d\mu(\xi_C, \xi_C^*) = \frac{(M+1)!}{M!} \frac{d^2\xi_C}{\pi(1 + |\xi_C|^2)^2}. \quad (\text{B.26})$$

By introducing a parameter $\eta_C \equiv \xi_C/\sqrt{1 + |\xi_C|^2}$, we can identify $|\xi_C\rangle$ as the binomial state $|\eta_C; M\rangle$ (A.9). This process shows elementarily that the parameter space of the binomial states is $SU(2)/U(1) = \mathbf{CP}^1$ ($\xi_C = \alpha_{1C}/\alpha_{0C}$) obtained from \mathbf{C}^2 ($\alpha_C = (\alpha_{0C}, \alpha_{1C})$) by integrating out the overall factor ζ_C .

Appendix C Multinomial States

The multinomial distribution is

$$M_{\mathbf{n}}(\boldsymbol{\eta}; M) = \frac{M!}{\mathbf{n}!} \eta_1^{2n_1} \cdots \eta_r^{2n_r} (1 - \boldsymbol{\eta}^2)^{n_0}, \quad (\text{C.1})$$

in which

$$\begin{aligned} \mathbf{n} &= (n_0, n_1, \dots, n_r), \quad n_0 + n_1 + \cdots + n_r = M, \quad \boldsymbol{\eta} = (\eta_1, \dots, \eta_r) \in \mathbf{R}^r, \\ 0 &< \boldsymbol{\eta}^2 = \eta_1^2 + \cdots + \eta_r^2 < 1, \quad \mathbf{n}! = n_0! n_1! \cdots n_r!. \end{aligned} \quad (\text{C.2})$$

Let us first define the ‘*multinomial state*’ in the linear representation form

$$|\boldsymbol{\eta}_C; M\rangle = \sum_{\mathbf{n}} \sqrt{\frac{M!}{\mathbf{n}!}} (\eta_{1C})^{n_1} \cdots (\eta_{rC})^{n_r} (1 - |\boldsymbol{\eta}_C|^2)^{n_0/2} |\mathbf{n}\rangle, \quad (\text{C.3})$$

in which the Fock states

$$|\mathbf{n}\rangle = |n_0, n_1, \dots, n_r\rangle, \quad n_0 + n_1 + \cdots + n_r = M, \quad (\text{C.4})$$

are generated by $r + 1$ bosonic oscillators

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \quad a_j |\mathbf{0}\rangle = 0, \quad j = 0, 1, \dots, r, \quad |\mathbf{0}\rangle = |0, 0, \dots, 0\rangle, \\ |\mathbf{n}\rangle &= \frac{(\mathbf{a}^\dagger)^{\mathbf{n}}}{\sqrt{\mathbf{n}!}} |\mathbf{0}\rangle, \quad (\mathbf{a}^\dagger)^{\mathbf{n}} = a_0^{\dagger n_0} a_1^{\dagger n_1} \cdots a_r^{\dagger n_r}. \end{aligned} \quad (\text{C.5})$$

Obviously the above Fock space (C.4) provides an irreducible representation of $su(r + 1)$ with generators

$$J_{jk} = a_j^\dagger a_k, \quad j \neq k, \quad N_j = a_j^\dagger a_j, \quad (\text{C.6})$$

in which J_{jk} ($j > k$) are considered as shift-up operators. It is a symmetric representation corresponding to the same Young diagram as before:

$$\square\square\square \cdots \square\square \quad M \text{ boxes}, \quad (\text{C.7})$$

and the lowest weight state is

$$|\mathbf{0}'\rangle \equiv |M, 0, \dots, 0\rangle, \quad J_{0k} |\mathbf{0}'\rangle = 0, \quad J_{jk} |\mathbf{0}'\rangle = 0, \quad j, k > 0. \quad (\text{C.8})$$

The last equation shows that the lowest weight state $|\mathbf{0}'\rangle$ is invariant under $u(r)$. The dimension of the above irreducible representation is

$$\binom{M+r}{M} = \binom{M+r}{r} \quad (\text{C.9})$$

which is the same as the number of terms in the multinomial expansion, the number of the partitions of M into $r + 1$ non-negative integers and the number of M -th order partial derivatives of analytic functions of $r + 1$ variables. It should be remarked that there are other types of coherent states of $su(r + 1)$ ($r \geq 2$) algebra belonging to the Young diagrams other than those given above (C.7). They cannot be constructed by bosons only.

It is not difficult to generate the multinomial states explicitly by applying the $SU(r+1)$ operator on the lowest weight (energy) state $|\mathbf{0}'\rangle$. For simplicity and concreteness let us show this for the $su(3)$ case:

$$|\eta_1 e^{i\theta_1}, \eta_2 e^{i\theta_2}; M\rangle = \sum_{n_1, n_2=0}^M \sqrt{\frac{M!}{n_0! n_1! n_2!}} (\eta_1 e^{i\theta_1})^{n_1} (\eta_2 e^{i\theta_2})^{n_2} (1 - \eta_1^2 - \eta_2^2)^{n_0/2} |n_0, n_1, n_2\rangle. \quad (\text{C.10})$$

This process is essentially the same as the generation of negative trinomial state given in section 4. For the ‘*trinomial state*’ we only have to use the $su(2)$ disentangling theorems twice for two $su(2)$ subalgebras spanned by $(0, 1)$ and $(1, 2)$ oscillators:

$$e^{r(e^{i\theta_1} J_{10} - e^{-i\theta_1} J_{01})} \quad (\text{C.11})$$

$$= \exp[e^{i\theta_1} \tan r J_{10}] \exp[\log(1 + \tan^2 r)(N_1 + N_2/2 - M/2)] \exp[-e^{-i\theta_1} \tan r J_{01}],$$

$$e^{r'(e^{i\theta_2} J_{21} - e^{-i\theta_2} J_{12})} \quad (\text{C.12})$$

$$= \exp[e^{i\theta_2} \tan r' J_{21}] \exp[-\frac{1}{2} \log(1 + \tan^2 r')(N_1 - N_2)] \exp[-e^{-i\theta_2} \tan r' J_{12}].$$

We choose r and r' such that

$$\tan^2 r = \frac{\eta_1^2 + \eta_2^2}{1 - \eta_1^2 - \eta_2^2}, \quad \tan^2 r' = \frac{\eta_2^2}{\eta_1^2}.$$

The generalisation to higher rank cases is straightforward. One has to apply $su(2)$ disentangling theorems in the following sequence of $su(2)$ algebras spanned by $(0, 1)$, $(1, 2)$, $(2, 3)$, \dots , $(r-1, r)$ oscillators.

To obtain the $su(r+1)$ multinomial states from the $R+1$ -fold coherent states is also straightforward. One only needs to develop clever notation to express the essential features succinctly.

A multiple Poisson distribution is given by

$$P_{\mathbf{n}}(\boldsymbol{\alpha}) = e^{-\alpha_0^2 - \dots - \alpha_r^2} \frac{\alpha_0^{2n_0} \alpha_1^{2n_1} \dots \alpha_r^{2n_r}}{n_0! \dots n_r!}, \quad \mathbf{n} = (n_0, n_1, \dots, n_r), \quad \boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_r). \quad (\text{C.13})$$

If we restrict it to a hyperplane

$$n_0 + n_1 + \dots + n_r = M, \quad \text{or} \quad n_0 = M - n_1 - \dots - n_r,$$

we obtain the multinomial distribution up to normalisation :

$$\eta_j \equiv \frac{\alpha_j}{\sqrt{\boldsymbol{\alpha}^2}}, \quad \boldsymbol{\alpha}^2 = \sum_{j=0}^r \alpha_j^2,$$

$$P_{\mathbf{n}}(\boldsymbol{\alpha}) \propto \frac{M!}{\mathbf{n}!} \eta_1^{2n_1} \dots \eta_r^{2n_r} (1 - \boldsymbol{\eta}^2)^{n_0} = M_{\mathbf{n}}(\boldsymbol{\eta}; M). \quad (\text{C.14})$$

The same proposition at the level of the “probability amplitude” including the normalization can be easily obtained by considering the projection operator onto the representation space (C.4) [18]:

$$\mathbf{P}_M = \sum_{n_0+n_1+\dots+n_r=M} |n_0, n_1, \dots, n_r\rangle \langle n_0, n_1, \dots, n_r|. \quad (\text{C.15})$$

With the aid of the resolution of unity (over-completeness relation)

$$\int \frac{\prod_{j=0}^r d^2\alpha_{jC}}{\pi^{r+1}} |\boldsymbol{\alpha}_C\rangle \langle \boldsymbol{\alpha}_C| = 1, \quad \boldsymbol{\alpha}_C = (\alpha_{0C}, \alpha_{1C}, \dots, \alpha_{rC}),$$

$$d^2\alpha_{jC} = d(\alpha_{jC})_{Rd}(\alpha_{jC})_I, \quad j = 0, \dots, r \quad (\text{C.16})$$

for the multiple coherent states

$$|\boldsymbol{\alpha}_C\rangle = e^{-|\boldsymbol{\alpha}_C|^2/2} \sum_{n_0, \dots, n_r} \frac{(\alpha_{0C})^{n_0} \dots (\alpha_{rC})^{n_r}}{\sqrt{\mathbf{n}!}} |n_0, n_1, \dots, n_r\rangle$$

we have

$$\mathbf{P}_M = \int \frac{\prod_{j=0}^r d^2\alpha_{jC}}{\pi^{r+1}} |\boldsymbol{\alpha}_C\rangle \langle \boldsymbol{\alpha}_C| \mathbf{P}_M. \quad (\text{C.17})$$

By a change of variables

$$\begin{pmatrix} \alpha_{0C} \\ \alpha_{1C} \\ \vdots \\ \alpha_{rC} \end{pmatrix} = \frac{\zeta_C}{\sqrt{1 + |\boldsymbol{\xi}_C|^2}} \begin{pmatrix} 1 \\ \xi_{1C} \\ \vdots \\ \xi_{rC} \end{pmatrix} \quad (\text{C.18})$$

we have

$$\begin{aligned} \mathbf{P}_M &= \int \frac{\prod_{j=1}^r d^2\xi_{jC}}{\pi^r (1 + |\boldsymbol{\xi}_C|^2)^2} \int \frac{|\zeta_C|^{2r} d^2\zeta_C}{\pi} e^{-|\zeta_C|^2} \\ &\times \sum_{m_0, m_1, \dots, m_r=0}^{\infty} \frac{1}{\sqrt{\mathbf{m}!}} \left(\frac{\zeta_C}{\sqrt{1 + |\boldsymbol{\xi}_C|^2}} \right)^{m_0+m_1+\dots+m_r} \xi_{1C}^{m_1} \xi_{2C}^{m_2} \dots \xi_{rC}^{m_r} |m_0, m_1, \dots, m_r\rangle \\ &\times \sum_{n_0+n_1+\dots+n_r=M} \langle n_0, n_1, \dots, n_r | \frac{1}{\sqrt{\mathbf{n}!}} \left(\frac{\zeta_C^*}{\sqrt{1 + |\boldsymbol{\xi}_C|^2}} \right)^M \xi_{1C}^{*n_1} \xi_{2C}^{*n_2} \dots \xi_{rC}^{*n_r} \\ &= \int \frac{\prod_{j=1}^r d^2\xi_{jC}}{\pi^r} \frac{(M+r)!}{(1 + |\boldsymbol{\xi}_C|^2)^{M+r+1}} \sum_{m_0+m_1+\dots+m_r=M} \frac{1}{\sqrt{\mathbf{m}!}} \xi_{1C}^{m_1} \xi_{2C}^{m_2} \dots \xi_{rC}^{m_r} |m_0, m_1, \dots, m_r\rangle \\ &\times \sum_{n_0+n_1+\dots+n_r=M} \langle n_0, n_1, \dots, n_r | \frac{1}{\sqrt{\mathbf{n}!}} \xi_{1C}^{*n_1} \xi_{2C}^{*n_2} \dots \xi_{rC}^{*n_r} \\ &= \int d\mu(\boldsymbol{\xi}_C, \boldsymbol{\xi}_C^*) |\boldsymbol{\xi}_C\rangle \langle \boldsymbol{\xi}_C|, \end{aligned} \quad (\text{C.19})$$

in which

$$\begin{aligned} |\boldsymbol{\xi}_C\rangle &= \frac{1}{(1 + |\boldsymbol{\xi}_C|^2)^{M/2}} \sum_{\mathbf{n}} \sqrt{\frac{M!}{\mathbf{n}!}} \xi_{1C}^{n_1} \xi_{2C}^{n_2} \dots \xi_{rC}^{n_r} |M - \sum n'_j, \mathbf{n}'\rangle \\ &= \frac{1}{(1 + |\boldsymbol{\xi}_C|^2)^{M/2}} \sum_{\mathbf{n}} \sqrt{\frac{M!}{\mathbf{n}!}} \xi_{2C}^{n_2} \dots \xi_{rC}^{n_r} |\mathbf{n}'\rangle \end{aligned} \quad (\text{C.20})$$

and the measure is

$$d\mu(\boldsymbol{\xi}_C, \boldsymbol{\xi}_C^*) = \frac{(M+r)!}{M!} \frac{\prod_{j=1}^r d^2 \xi_{jC}}{\pi^r (1 + |\boldsymbol{\xi}_C|^2)^{r+1}}. \quad (\text{C.21})$$

By introducing parameters $\eta_{jC} \equiv \xi_{jC}/\sqrt{1 + |\boldsymbol{\xi}_C|^2}$, we can identify $|\boldsymbol{\xi}_C\rangle$ as the multinomial state $|\boldsymbol{\eta}_C; M\rangle$ (C.3). This process shows elementarily that the parameter space of the multinomial states is $SU(r+1)/U(1) \times SU(r) = \mathbf{CP}^r$ ($\xi_{jC} = \alpha_{jC}/\alpha_{0C}$) obtained from \mathbf{C}^{r+1} ($\boldsymbol{\alpha}_C$) by integrating out the overall factor ζ_C .

A few words about the multiple coherent states limit of the multinomial states. For the multinomial state (C.20) we let $M \rightarrow \infty$ and $\xi_{jC} \rightarrow 0$ while keeping the ‘average’ fixed, $\xi_{jC}^2 M = \alpha_{jC}^2$ to obtain for fixed \mathbf{n}'

$$|\boldsymbol{\xi}_C\rangle \rightarrow e^{-|\boldsymbol{\alpha}_C|^2/2} \sum_{\mathbf{n}'} \frac{(\alpha_C)^{\mathbf{n}'}}{\mathbf{n}'!} \|\mathbf{n}'\rangle. \quad (\text{C.22})$$

Like in the case of the binomial states one can express the states and the $su(r+1)$ generators in the ‘reduced’ notation using only r boson oscillators. This gives rise to the generalisation of the Holstein-Primakoff realisation. But as remarked above it is applicable only to the *symmetric representations*. Because of the constraint

$$n_0 + n_1 + \dots + n_r = M,$$

the state $|\mathbf{n}\rangle$ is uniquely specified by

$$\mathbf{n}' = (n_1, n_2, \dots, n_r)$$

only. So we identify

$$\|\mathbf{n}'\rangle = \|n_1, n_2, \dots, n_r\rangle \equiv |M - \sum_j n_j, n_1, \dots, n_r\rangle = |\mathbf{n}\rangle, \quad (\text{C.23})$$

and introduce r independent boson oscillators

$$[b_j, b_k^\dagger] = \delta_{jk}, \quad b_j |\mathbf{0}\rangle = 0, \quad j = 1, \dots, r,$$

which create the ‘reduced’ states

$$\|\mathbf{n}'\rangle = \frac{(\mathbf{b}^\dagger)^{\mathbf{n}'}}{\mathbf{n}'!} \|\mathbf{0}\rangle, \quad (\mathbf{b}^\dagger)^{\mathbf{n}'} = b_1^{\dagger n_1} \dots b_r^{\dagger n_r}, \quad \mathbf{n}'! = n_0! \dots n_r!. \quad (\text{C.24})$$

Then we have

$$\mathcal{J}_{j0} = b_j^\dagger \sqrt{M - N_1 - \dots - N_r}, \quad \mathcal{J}_{0j} = \sqrt{M - N_1 - \dots - N_r} b_j, \quad \mathcal{J}_{jk} = b_j^\dagger b_k. \quad (\text{C.25})$$

Note that the ‘vacuum’ $\|\mathbf{0}\rangle$ is the lowest weight state and it is invariant under $su(r)$ which is expressed *linearly*:

$$\mathcal{J}_{j0} \|\mathbf{0}\rangle = 0, \quad \mathcal{J}_{jk} \|\mathbf{0}\rangle = 0. \quad (\text{C.26})$$

Before closing this Appendix, let us remark on the dynamical generation of the multinomial states. This is essentially the same as that of the binomial states. Let us consider a collection (total number M) of identical $r + 1$ -level atoms (bosons). It is assumed that the interactions among different atoms are negligibly small compared with the interactions within the same atoms among different energy levels. As before the system is described in terms of $r + 1$ bosonic oscillators and the Hamiltonian at the zero-th order approximation is quadratic in the oscillators keeping the total number of atoms fixed. In other words the Hamiltonian is a hermitian linear combination of the $u(r + 1)$ generators given in (C.6). If we assume that the system is in the lowest energy (weight) state $|M, 0, \dots, 0\rangle$ at $t = 0$, then at time t it is

$$e^{-iHt}|M, 0, \dots, 0\rangle,$$

which is a multinomial state since the time evolution operator e^{-iHt} is an element of $U(r + 1)$ and the $U(1)$ part and the $SU(r)$ is immaterial when they act on $|M, 0, \dots, 0\rangle$.

Appendix D Waiting Time : Negative Binomial Distribution

For those who are not familiar with probability theory, we give here a simple example in which the *negative binomial distribution* occurs. We follow Feller's textbook [11]. Let us consider a succession of Bernoulli's trials each of which has the probability of *failure* $0 < \eta^2 < 1$. We ask a question: How long it will take for the M -th success to turn up? Here M is a positive integer. Since M -th success comes not earlier than M -th try, we denote by $B_n^-(\eta; M)$ the probability that the M -th success occurs at the trial number $M + n$, $n \geq 0$. This occurs, if and only if, among the $M + n - 1$ trials there are exactly n failures and the $M + n$ -th trial results in success: so that

$$B_n^-(\eta; M) = \binom{M + n - 1}{n} (1 - \eta^2)^M \eta^{2n}.$$

For the very unlucky the waiting time (n) can be infinite. This corresponds to the fact that the irreducible unitary representations of non-compact algebras are infinite dimensional.

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