SOME COMPARISON THEOREMS FOR KÄHLER MANIFOLDS

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ABSTRACT. In this work, we will verify some comparison results on Kähler manifolds. They are complex Hessian comparison for the distance function from a closed complex submanifold of a Kähler manifold with holomorphic bisectional curvature bounded below by a constant, eigenvalue comparison and volume comparison in terms of scalar curvature. This work is motivated by comparison results of Li and Wang [12].

1. INTRODUCTION

In this work, we will study some comparison theorems on Kähler manifolds. There is a well-known Hessian comparison for the distance function on Riemannian manifolds in terms of lower bound of sectional curvature. It is expected that for Kähler manifolds the lower bound of sectional curvature can be replaced by the lower bound of bisectional curvature to obtain a complex Hessian comparison for the distance function. In fact, in [12], Li and Wang gave a sharp upper estimate for the Laplacian of the distance function from a point. They also gave a sharp upper estimate for the complex Hessian for the distance function in the case that the lower bound of the bisectional curvature $K = 0$, i.e., on Kähler manifolds with nonnegative holomorphic bisectional curvature. This last result was also proved by Cao and Ni [5] using Li-Yau-Hamilton Harnack type inequality for the heat equation. In the note, we shall verify this complex Hessian comparison for general lower bound K and show that the complex Hessian of the distance function is bounded above by that in the complex space forms. See Theorem 2.1 and 2.2 for more details.

Our next result is motivated by the well-known Licherowicz-Obata theorem on Riemannian manifolds, which says that if (M^m, g) is a compact Riemannian manifold with Ricci curvature bounded below by $(m-1)K$, where $K > 0$ is a constant, then the first nonzero eigenvalue λ_1 satisfies $\lambda_1 \geq mK$ and equality holds only if (M, g) is isometric to the standard sphere of radius $1/\sqrt{K}$. It is well-known that if (M^n, g) is a Kähler manifold such that Ricci curvature is such that $R_{\alpha\bar{\beta}} \geq kg_{\alpha\bar{\beta}}$ for some constant $k > 0$, then the first nonzero eigenvalue λ_1 of the complex Laplacian is at least k (see [9]). Our next result is the following:

Let (M^n, g) be as above. Suppose the Kähler form is in the first Chern class. If $\lambda_1 = k$, then M is Kähler-Einstein.

As a corollary, when (M^n, g) is a Kähler manifold with positive holomorphic bisectional curvature such that the Ricci curvature is bounded below by $n+1$, then $\lambda_1 \geq n+1$ and equality holds if and only if (M^n, g) is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric (of constant holomorphic sectional curvature 2). As an application we obtain a partial result on the equality case for the diameter estimate of Li and Wang [12] on compact Kähler manifolds with holomorphic bisectional curvature bounded below by 1. See Corollary 3.2.

In [12], it was proved that if a compact Kähler manifold has bisectional curvature bounded below by 2 (see Definition 2.1), then the volume of the manifold is less than or equal to the volume of $\mathbb{CP}^n$ with the Fubini-Study metric. Our last result is an observation to relax their conditions by replacing the lower bound of the bisectional curvature by the lower bound of the scalar curvature, with the assumption that the bisectional curvature is positive. In fact, we prove:

Let (M^n, g) be a compact manifold with positive holomorphic bisectional curvature. Suppose the scalar curvature $k_2 \geq R \geq k_1 > 0$ for some constants k_1 and k_2. Then the volume $V(M, g)$ of M satisfies $V(\mathbb{CP}^n, h_{k_2}) \leq V(M, g) \leq V(\mathbb{CP}^n, h_{k_1})$. Moreover, if one of the inequalities is an equality, then (M, g) is holomorphically isometric the $\mathbb{CP}^n$ with the Fubini-Study metric.

The result is a consequence of a more general result, which is related to a conjecture of Schoen [17] which states: If (M^n, h) is a closed hyperbolic manifold and g is another metric on M with scalar curvature $R(g) \geq R(h)$, then $V(g) \geq V(h)$. The conjecture was proved for $n = 3$ by Perelmann [15, 16]. We will compare volume of a compact Kähler manifold with the volume of a related Kähler-Einstein metric in terms of upper and lower bounds of the scalar curvature. See Proposition 4.1.

The paper is organized as follows: In §2, we will study comparison of the complex Hessian for the distance function. In §3, we will study eigenvalue comparison and in §4, we will study volume comparison.

The authors would like to thank Xiaowei Wang for some useful discussions.

2. Complex Hessian comparisons

Let M^n be a complex manifold with complex dimension n and let J be the complex structure. Suppose g is a Hermitian metric such that (M, J, g) is Kähler. Suppose $\{e_1, e_2, \cdots, e_n\}$ is a frame on $T^{(1,0)}(M)$, let $g_{\alpha\bar\beta} := g(e_\alpha, \bar e_\beta)$. We also write $g(X, \bar Y)$ as $\langle X, \bar Y \rangle$ and $||X||^2 = g(X, \bar X)$ for $X, Y \in T^{(1,0)}(M)$. Following [12], we have the following definition.

Definition 2.1. Let (M, J, g) be a Kähler manifold. We say that the holomorphic bisectional curvature of M is bounded below by a constant K, denoted by $BK_M \geq K$, if

$$(2.1) \qquad \frac{R(X, \bar X, Y, \bar Y)}{||X||^2 ||Y||^2 + |\langle X, \bar Y \rangle|^2} \geq K$$

for any two nonzero (1,0)-vectors X, Y.

We remark that on a Kähler manifold with constant holomorphic sectional curvature $2K$, (2.1) is an equality for all nonzero $X, Y \in T^{(1,0)}(M)$.

Let S^p be a complex submanifold of M^n with complex dimension $p \geq 0$ and let r be the distance function from S. We want to give a comparison result for the complex Hessian of r. We always assume that S is connected and closed. Let $x_0 \in S$ and let σ be a geodesic from x_0 which is orthogonal to S at x_0 and is parametrized by arc length t with $0 \leq t \leq T$. Assume that σ is within the cut locus of S so that the distance function r is smooth near $\sigma|_{(0,T]}$. Let $e_1 = \frac{1}{\sqrt 2}(\sigma' - \sqrt{-1}J\sigma') = \frac{1}{\sqrt 2}(\nabla r - \sqrt{-1}J\nabla r)$ and let $e_1, e_2, \ldots, e_n$ be unitary frames parallel along γ such that $e_{n-p+1}, \ldots, e_n$ are tangent to S at $t = 0$. In the following $f_{\alpha\bar\beta}$ etc are covariant derivatives of the function f and repeated indices mean summation. The computations in [12] show:

Lemma 2.1. *With the above notations, on σ, we have*

$$(2.2) \qquad \frac{d}{dr}r_{\alpha\bar\beta} + r_{\alpha\bar\gamma}r_{\gamma\bar\beta} + r_{\alpha\gamma}r_{\bar\gamma\bar\beta} = -\frac{1}{2}R_{\alpha\bar\beta 1\bar 1},$$

and

$$(2.3) \qquad \frac{d}{dr}r_{\alpha\beta} + r_{\alpha\gamma}r_{\bar\gamma\beta} + r_{\alpha\bar\gamma}r_{\gamma\beta} = \frac{1}{2}R_{\alpha\bar 1\beta\bar 1}.$$

Proof. Extend e_i to be a unitary frames near a point on σ. Note that on σ

$$
\begin{aligned}
(2.4) \qquad 0 &= (\|\nabla r\|^2)_{\alpha\bar\beta} \\
&= (2r_\gamma r_{\bar\gamma})_{\alpha\bar\beta} \\
&= 2(r_{\gamma\alpha\bar\beta}r_{\bar\gamma} + r_\gamma r_{\bar\gamma\alpha\bar\beta} + r_{\gamma\alpha}r_{\bar\gamma\bar\beta} + r_{\gamma\bar\beta}r_{\bar\gamma\alpha}) \\
&= 2(r_{\alpha\bar\beta\gamma}r_{\bar\gamma} + R_{\alpha\bar\delta\gamma\bar\beta}r_\delta r_{\bar\gamma} + r_\gamma r_{\alpha\bar\beta\bar\gamma} + r_{\gamma\alpha}r_{\bar\gamma\bar\beta} + r_{\gamma\bar\beta}r_{\bar\gamma\alpha}).
\end{aligned}
$$

So,

$$(2.5) \qquad r_{\alpha\bar\beta\gamma}r_{\bar\gamma} + r_{\alpha\bar\beta\bar\gamma}r_\gamma + r_{\alpha\gamma}r_{\bar\gamma\bar\beta} + r_{\gamma\bar\beta}r_{\alpha\bar\gamma} = -R_{\alpha\bar\delta\gamma\bar\beta}r_\delta r_{\bar\gamma}.$$

Since $e_1 = \frac{1}{\sqrt{2}}(\nabla r - J\nabla r)$, and $e_1, e_2, \cdots, e_n$ are parallel along σ, we have

$$(2.6) \qquad r_1 = \frac{1}{\sqrt{2}} \text{ and } r_\alpha = 0 \text{ for any } \alpha > 1$$

and

$$(2.7) \qquad r_{\alpha\bar\beta\gamma}r_{\bar\gamma} + r_{\alpha\bar\beta\bar\gamma}r_\gamma = \frac{1}{\sqrt{2}}(r_{\alpha\bar\beta 1} + r_{\alpha\bar\beta\bar 1}) = \frac{d}{dr}r_{\alpha\bar\beta}.$$

By (2.5)–(2.7), we conclude that (2.2) is true.

The proof of (2.3) is similar. $\qquad\square$

The following fact may be well-known. We include the proof for the sake of completeness.

Lemma 2.2. *Let M^{n+m} be a complete Riemannian manifold and S^m be a closed submanifold of M. Let r be the distance function to S. Let $\gamma(s)$ be a normal geodesic orthogonal to S with $\gamma(0) \in S$. Let $e_1, e_2, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}$ are parallelled orthonormal frames along γ such that $e_{n+1}(0), e_{n+2}(0), \cdots, e_{n+m}(0)$ is tangent to S and $e_1 = \gamma'$. Then*
(2.8)
$$
\lim_{s\to 0}\left(\left[\nabla^2 r(\gamma(s))(e_i, e_j)\right]_{1\le i,j\le n+m} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{s}I_{n-1} & 0 \\ 0 & 0 & [h_{ij}]_{n+1\le i,j\le n+m} \end{pmatrix} \right) = 0.
$$

where $h_{ij} = \langle h(e_i, e_j), \gamma'\rangle$, and $h(X, Y) = -(\nabla_X Y)^\perp$ for $X, Y \in T(S)$ is the second fundamental form of S.

Proof. Let $p = \gamma(0)$ and let $\{\nu_1, \ldots, \nu_n\}$ be a unit normal frame of the normal bundle $T^\perp S$ near p such that $D\nu_i = 0$ at p for $i = 1, \cdots, n$, where D is the normal connection given by $D_X\nu = (\nabla_X\nu)^\perp$ for any normal vector field ν and any X tangent to S. Moreover, we may choose ν_i such that $\nu_i(p) = e_i(0)$ for $i = 1, 2, \cdots, n$.

Now let $x_1, \ldots, x_m$ be a local coordinates at p in S, then one can parametrize M by $F(x, y) = \exp_x(\sum_{j=1}^n y_j \nu_j)$, where $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$. Let $X_i = \frac{\partial}{\partial x^i}$, $Y_j = \frac{\partial}{\partial y_j}$. Then $r^2(x, y) = \sum_{j=1}^n y_j^2$. We can moreover assume that $X_i(p) = e_{i+n}(0)$ for $i = 1, 2, \cdots, m$. Similar to normal coordinates, we have

$$(2.9) \qquad\qquad \nabla_{Y_i} Y_j = 0$$

on S. Extend e_i's to be smooth vector fields near p. For any i,

$$e_i = \sum_{k=1}^m a_i^k X_k + \sum_{l=1}^n b_i^l Y_l.$$

Note that $a_i^k = 0$ and $b_i^l = \delta_i^l$ at $s = 0$ for $1 \leq i \leq n$, and $a_i^k = \delta_{i-n}^k$ and $b_i^l = 0$ at $s = 0$ for $n + 1 \leq i \leq m + n$.

Since e_i is parallel along γ, by (2.9) and the initial conditions of a_i^k and b_i^l, we conclude that if $1 \leq i \leq n$, then

$$(2.10) \qquad\qquad a_i^k = o(s),\, b_i^l - \delta_i^l = o(s)$$

as $s \to 0$. Using (2.9), the fact that $D_{X_k} Y_1 = 0$ at $s = 0$ and the fact that X_k are tangential at $s = 0$, if $n + 1 \leq i \leq m + n$, then

$$(2.11) \qquad\qquad b_i^l = o(s)$$

as $s \to 0$. It is easy to see that on γ, $||\nabla^2 r|| = O(\frac{1}{s})$. Hence by (2.10) for $1 \leq i, j \leq n$, we have on γ

$$
\begin{aligned}
0 &= \lim_{s \to 0} \left[\nabla^2 r(\gamma(s))(e_i, e_j) - \nabla^2 r(\gamma(s))(Y_i, Y_j) \right] \\
(2.12) \\
&= \lim_{s \to 0} \left(\nabla^2 r(\gamma(s))(e_i, e_j) - \frac{\delta_{ij} - \delta_{i1}\delta_{1j}}{s} \right)
\end{aligned}
$$

where we have used (2.9) and the facts that on γ, $y_1 = s > 0$, $y_j = 0$ for $2 \leq j \leq n$, and $\nabla r = \gamma'$.

By (2.10) and (2.11), for $n + 1 \leq i \leq m + n$ and $1 \leq j \leq n$, we have on γ

$$
\begin{aligned}
(2.13) \qquad \lim_{s \to 0} \nabla^2 r(e_i, e_j) &= \lim_{s \to 0} \sum_{k=1}^m a_i^k \nabla^2 r(X_k, Y_j) \\
&= 0
\end{aligned}
$$

where we have used the fact r does not depend on x and the fact that $DY_j = 0$ at $p = \gamma(0)$.

By (2.11), for $n + 1 \leq i, j \leq n + m$, on γ we have

$$
(2.14) \qquad \lim_{s \to 0} \nabla^2 r(e_i, e_j) = \lim_{s \to 0} \sum_{k,l=1}^{m} a_i^k a_j^l \nabla^2 r(X_k, X_l)
$$

$$
= \langle h(e_i, e_j), \gamma'(0) \rangle
$$

where $h(X_i, X_j) = -(\nabla_{X_i} X_j)^\perp$ is the second fundamental form of S. The completes the proof of the lemma.

$\square$

We also need the following result in [8]:

Lemma 2.3. *Let $A(t)$ and $B(t)$ be two smooth curves in the space of $n \times n$ complex Hermitian matrices on the interval $(0, T)$, satisfying the equality*

$$
(2.15) \qquad A'(t) + A^2(t) \leq B'(t) + B^2(t)
$$

for any $t \in (0, T)$. Suppose

$$
(2.16) \qquad \lim_{t \to 0^+} (B(t) - A(t)) = 0.
$$

Then

$$
(2.17) \qquad A(t) \leq B(t)
$$

for $t \in (0, T)$. Here $X \leq Y$ means that $Y - X$ is positive semi definite.

Theorem 2.1. *Let (M^n, g) be a complete Kähler manifold with holomorphic bisectional curvature bounded below by K. Let S^p be a connected closed complex submanifold of M with complex dimension p. Then within the cut-locus of S,*

$$
(2.18) \qquad r_{\alpha\bar{\beta}} \leq F_K \left(g_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}}^S \right) + G_K r_\alpha r_{\bar{\beta}} + H_K g_{\alpha\bar{\beta}}^S
$$

where

$$
(2.19) \qquad F_K = \begin{cases} \sqrt{\frac{K}{2}} \cot(\sqrt{\frac{K}{2}} r), & \text{if } K > 0; \\ \frac{1}{r}, & \text{if } K = 0; \\ \sqrt{-\frac{K}{2}} \coth(\sqrt{-\frac{K}{2}} r), & \text{if } K < 0, \end{cases}
$$

$$
(2.20)
$$

$$
G_K = \begin{cases} \sqrt{2K} \left(\cot(\sqrt{2K} r) - \cot(\sqrt{\frac{K}{2}} r) \right) & \text{if } K > 0; \\ -\frac{1}{r} & \text{if } K = 0; \\ \sqrt{-2K} \left(\coth(\sqrt{-2K} r) - \coth(\sqrt{-\frac{K}{2}} r) \right) & \text{if } K < 0, \end{cases}
$$

and

$$(2.21) \qquad H_K = \begin{cases} -\sqrt{K/2}\tan\left(\sqrt{K/2}\,r\right) & \text{if } K > 0; \\ 0 & \text{if } K = 0; \\ -\sqrt{-K/2}\tanh\left(\sqrt{-K/2}\,r\right) & \text{if } K < 0, \end{cases}$$

where g^S is the metric of S parallel transported along geodesics emanating from S that is orthogonal to S.

Proof. We only prove the case $K > 0$ and the proofs of the other two cases are similar.

Let $x_0 \in S$ and let γ be a geodesic emanating from x_0 orthogonal to S. Let $e_1, e_2, \cdots, e_n$ be parallel unitary frame along γ with $e_1 = \frac{1}{\sqrt{2}}(\gamma' - \sqrt{-1}J\gamma')$ and $e_{n-p+1}(0), \cdots, e_n(0)$ be tangent to S. Then, by Lemma 2.2 and the fact that $h(u, \bar{v}) = 0$ for $u, v \in T^{1,0}S$,

$$(2.22) \qquad \lim_{r \to 0}\left((r_{\alpha\bar{\beta}})_{1\leq\alpha,\beta\leq n} - \begin{pmatrix} \frac{1}{2r} & 0 & 0 \\ 0 & \frac{1}{r}I_{n-p-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = 0.$$

Since $e_1 = \frac{1}{\sqrt{2}}(\nabla r - \sqrt{-1}J\nabla r)$,

$$(2.23) \qquad\qquad r_1 = \frac{1}{\sqrt{2}} \text{ and } r_\alpha = 0 \text{ for any } \alpha > 1.$$

Moreover, since $(\|\nabla r\|^2)_\alpha = 0$, we have

$$(2.24) \qquad\qquad\qquad r_{\alpha 1} = -r_{\alpha\bar{1}}.$$

Therefore, by Lemma 2.2, we have

$$(2.25) \qquad \frac{d}{dr}r_{\alpha\bar{\beta}} + r_{\alpha\bar{\gamma}}r_{\gamma\bar{\beta}} + r_{\alpha\bar{1}}r_{1\bar{\beta}} + \sum_{\gamma=2}^{n} r_{\alpha\gamma}r_{\bar{\gamma}\bar{\beta}} = -\frac{1}{2}R_{\alpha\bar{\beta}1\bar{1}}.$$

Let $B = (r_{\alpha\bar{\beta}})$. Since the holomorphic bisectional curvature is bounded below by K, we conclude that

$$(2.26) \qquad \frac{d}{dr}B + BB^* + B_1 B_1^* \leq \begin{pmatrix} -K & 0 \\ 0 & -\frac{K}{2}I_{n-1} \end{pmatrix}.$$

where B_1 is the first column of B. This implies that

$$(2.27) \qquad\qquad \frac{d}{dr}\tilde{B} + \tilde{B}^2 \leq \begin{pmatrix} -2K & 0 \\ 0 & -\frac{K}{2}I_{n-1} \end{pmatrix}$$

where $\tilde{B} = DBD$ and $D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & I_{n-1} \end{pmatrix}$. By (2.22),

(2.28) $\qquad \lim_{r \to 0} \left(\tilde{B} - \begin{pmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r}I_{n-p-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = 0.$

Note that
(2.29)
$$X = \begin{pmatrix} \sqrt{2K}\cot(\sqrt{2K}r) & 0 & 0 \\ 0 & \sqrt{\frac{K}{2}}\cot(\sqrt{\frac{K}{2}}r)I_{n-p-1} & 0 \\ 0 & 0 & -\sqrt{\frac{K}{2}}\tan(\sqrt{\frac{K}{2}}r)I_p \end{pmatrix}$$

is a solution of

(2.30) $\qquad \frac{d}{dr}X + X^2 = \begin{pmatrix} -2K & 0 \\ 0 & -\frac{K}{2}I_{n-1} \end{pmatrix}$

with initial condition (2.28). By Lemma 2.3, $\tilde{B} \le X$. By the definition of $\tilde{B}$, we conclude that the theorem is true for the case $K > 0$. The other cases are similar. $\qquad\qquad\square$

In case S is a point $x_0 \in M$, then it is understood that g^S is zero. So within the cut locus of x_0, we have

$$r_{\alpha\bar\beta} \le F_K g_{\alpha\bar\beta} + G_K r_\alpha r_{\bar\beta}.$$

Next, we want to discuss the equality case.

Theorem 2.2. *With the same assumptions in the Theorem 2.1.*
  (i) *Suppose equality holds in* (2.18) *in a neighborhood of S, then S is totally geodesic.*
  (ii) *If $S = x_0$ is a point and equality holds in* (2.18) *at all points in the geodesic ball $B_{x_0}(r)$ which are within the cut locus of x_0, then $B_{x_0}(r)$ is holomorphically isometric to the geodesic ball of radius r in the simply connected Kähler manifold with constant holomorphic sectional curvature $2K$. Here if $K > 0$, it is assumed that $r < \frac{\pi}{\sqrt{2K}}$.*

Proof. The proof is similar to the Riemannian case (see [11, 6]).
  (i) When equality of (2.18) holds near S, we know that

(2.31) $\qquad\qquad\qquad\qquad r_{\alpha\beta} = 0$

for all $\alpha, \beta \ge 2$ near S. By Lemma 2.2 and the fact that $h(u, \bar v) = 0$ for $u, v \in T^{1,0}S$, we know that the second fundamental form h of S is zero, hence S is totally geodesic.

(ii) We only prove the case $K > 0$, the other two cases are similar. Under the assumptions of the theorem, at all points within the cut locus of x_0, we have

$$(2.32) \qquad (R_{\alpha\bar\beta1\bar1}) = \begin{pmatrix} 2K & 0 \\ 0 & KI_{n-1} \end{pmatrix}$$

and

$$(2.33) \qquad (r_{\alpha\beta}) = \begin{pmatrix} -\frac{\sqrt{2K}}{2}\cot(\sqrt{2K}r) & 0 \\ 0 & 0_{n-1} \end{pmatrix}.$$

By Lemma 2.1, we have

$$(2.34) \qquad \frac{d}{dr}r_{\alpha\beta} + r_{\gamma\alpha}r_{\bar\gamma\beta} + r_{\gamma\beta}r_{\bar\gamma\alpha} = R_{\alpha\bar\delta\beta\bar\gamma}r_\delta r_\gamma = \frac{1}{2}R_{\alpha\bar1\beta\bar1}.$$

$$(2.35) \qquad R_{\alpha\bar1\beta\bar1} = 0 \text{ if } \alpha \neq 1 \text{ or } \beta \neq 1$$

by substituting the equality in (2.18) and (2.33) into (2.34). Let $J = x_\alpha e_\alpha + \bar x_\beta \bar e_\beta$ be a Jacobi field along a geodesic emanating from p. Then
(2.36)
$$x''_\alpha = \langle R(\frac{e_1 + \bar e_1}{\sqrt{2}}, x_\beta e_\beta + \bar x_\gamma \bar e_\gamma)\frac{e_1 + \bar e_1}{\sqrt{2}}, \bar e_\alpha\rangle = \frac{1}{2}(R_{1\bar\gamma1\bar\alpha}\bar x_\gamma - R_{\beta\bar\alpha1\bar1}x_\beta)$$

This is the same as the Jocobi field equation on the space form with constant holomorphic sectional curvature $2K$ because of (2.32) and (2.35). Let $o \in \mathbb{CP}^n$ with Kähler metric with constant holomorphic sectional curvature $2K$. Let I be an isometry from $T_{x_0}(M) \to T_o(\mathbb{CP}^n)$ such that I is holomorphic. Then suppose $r_0 <$ injectivity radius of x_0 and $r_0 < r$, then the map $\exp_o \circ I \circ \exp_{x_0}^{-1} : B_{x_0}(r_0) \to B_o(r_0)$ is an isometry by the proof of Catan-Ambrose-Hicks Theorem [11]. Since the injectivity radius of o is $\frac{\pi}{\sqrt{2K}}$, it is easy to see that $\phi = \exp_o \circ I \circ \exp_{x_0}^{-1}$ can be extended to be an isometry from $B_{x_0}(r)$ to $B_o(r)$. We may arrange so that $\phi^* J_0 = J_M$ at x_0 where J_0 is the complex structure of $\mathbb{CP}^n$. Since ϕ is an isometry, $\nabla(J_M - \phi^* J_0) = 0$. Hence ϕ is holomorphic.

$\square$

Remark 2.1. By the theorem and the proofs in [12], we may obtain results on the equality case for the volume comparison and first Dirichlet eigenvalue comparison for geodesic balls in [12].

3. Eigenvalue comparison

Consider the first nonzero eigenvalue of compact Kähler manifold (M^n, g). It is well known that if the Ricci curvature of M satisfies $\mathrm{Ric} \geq kg$ for some $k > 0$, then the first nonzero eigenvalue of the

complex Laplacian is bounded below by k (see [9, Theorem 2.4.5]). Here the complex Laplacian of a function u is given by

$$\Delta u = g^{\alpha\bar{\beta}} u_{\alpha\bar{\beta}}.$$

Let $\Delta^{\mathbb{R}}$ be the Laplacian of the underlining Riemannian metric, then $\Delta = \frac{1}{2}\Delta^{\mathbb{R}}$. We want to discuss the equality case.

Theorem 3.1. *Let (M^n, g) be a compact Kähler manifolds with $Ric \geq kg$, $k > 0$. Suppose the first nonzero eigenvalue λ for the complex Laplacian is k and that the Ricci form ρ and Kähler form ω satisfy $[\rho] = c[\omega]$ for some c. Then M is Kähler-Einstein.*

Proof. Let us recall the proof that $\lambda \geq k$. Let u be a first eigenfunction. If $\{e_\alpha\}$ is a unitary frame, then Then

$$
\begin{aligned}
-\lambda \int_M |\nabla u|^2 &= -\lambda \int_M \sum_\alpha u_\alpha u_{\bar{\alpha}} \\
&= \int_M \sum_\alpha (\Delta u)_\alpha u_{\bar{\alpha}} \\
&= \int_M \sum_{\alpha,\beta} \left(u_{\beta\bar{\beta}}\right)_\alpha u_{\bar{\alpha}} \\
&= \int_M \sum_{\alpha,\beta} \left(u_{\beta\alpha\bar{\beta}} - R_{\beta\bar{\gamma}\alpha\bar{\beta}} u_\gamma\right) u_{\bar{\alpha}} \\
&= -\left(\int_M u_{\alpha\beta} u_{\bar{\alpha}\bar{\beta}} + R_{\alpha\bar{\gamma}} u_\gamma u_{\bar{\alpha}}\right)
\end{aligned}
$$

(3.1)

Hence

$$\lambda \int_M |\nabla u|^2 \geq k \int_M |\nabla u|^2$$

and $\lambda \geq k$. Hence if $\lambda = k$, then $u_{\alpha\beta} = 0$. By assumption, there is a positive number $c > 0$ and a function F such that

(3.2)
$$R_{\alpha\bar{\beta}} - cg_{\alpha\bar{\beta}} = F_{\alpha\bar{\beta}}.$$

Let $\phi = \Delta u + cu + g^{\alpha\bar{\beta}} F_\alpha u_{\bar{\beta}}$. As in [9, p.42], one can prove that ϕ is holomorphic. In fact, in a normal coordinate at a point,

$$
\begin{aligned}
\phi_{\bar{\gamma}} &= u_{\alpha\bar{\alpha}\bar{\gamma}} + cu_{\bar{\gamma}} + F_{\alpha\bar{\gamma}} u_{\bar{\alpha}} + F_\alpha u_{\bar{\alpha}\bar{\gamma}} \\
&= u_{\bar{\alpha}\alpha\bar{\gamma}} + cu_{\bar{\gamma}} + F_{\alpha\bar{\gamma}} u_{\bar{\alpha}} \\
&= u_{\bar{\alpha}\bar{\gamma}\alpha} - R_{\alpha\bar{\gamma}} u_{\bar{\alpha}} + cu_{\bar{\gamma}} + F_{\alpha\bar{\gamma}} u_{\bar{\alpha}} \\
&= \left(-cg_{\alpha\bar{\gamma}} - F_{\alpha\bar{\gamma}}\right) u_{\bar{\alpha}} + cu_{\bar{\gamma}} + F_{\alpha\bar{\gamma}} u_{\bar{\alpha}} \\
&= 0,
\end{aligned}
$$

(3.3)

where we have used the fact that $u_{\alpha\beta} = 0$ and (3.2). Hence $\phi = a$ is a constant. This implies, together with the fact that $\Delta u = -ku$,

$$(-k + c)u + g^{\alpha\bar{\beta}}F_\alpha u_{\bar{\beta}} = a.$$

Let $S = \max u$, and $s = \min u$. Evaluate the above equality at the maximum and the minimum points, we have

$$(-k + c)S = a, (-k + c)s = a.$$

So $k = c$ and $a = 0$ because u is nonconstant. Hence (3.2) becomes:

$$R_{\alpha\bar{\beta}} - kg_{\alpha\bar{\beta}} = F_{\alpha\bar{\beta}}.$$

Since $R_{\alpha\bar{\beta}} \geq kg_{\alpha\bar{\beta}}$, F is plurisubharmonic and is a constant. Hence

$$R_{\alpha\bar{\beta}} - kg_{\alpha\bar{\beta}} = 0$$

and M is Kähler-Einstein.

$\square$

Corollary 3.1. *Let (M^n, g) be a Kähler manifold with positive holomorphic bisectional curvature such that $Ric \geq kg$. Suppose the first nonzero eigenvalue of the complex Laplacian equal to k, then M is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric.*

Proof. Since $b_2(M) = 1$ (see [4, 10]), the condition $[\rho] = c[\omega]$ in the theorem is automatically satisfied. Hence g is Kähler-Einstein and by the theorem of Berger (see [3, 10]) M is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric. $\square$

As an application, we have the following partial result for the equality case of diameter comparison result by [12].

Corollary 3.2. *Let M^n be a compact Kähler manifold with holomorphic bisectional curvature ≥ 1. Suppose there exist compact connected complex submanifolds P of dimension s and Q of dimension $n - 1 - s$ in M such that $d(P, Q) = \frac{\pi}{\sqrt{2}}$ with some $0 \leq s \leq n - 1$. Then M is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric.*

Proof. Let $[\xi_0 : \xi_1 : \cdots : \xi_n]$ be the homogeneous coordinate of $\mathbb{CP}^n$ (equipped with the Fubini-Study metric with constant holomorphic sectional curvature 2). Let $P_0 = \{[\xi_0 : \xi_1 : \cdots : \xi_s : 0 : \cdots : 0] | \xi_0, \cdots \xi_s \in \mathbb{C}\} \subset \mathbb{CP}^n$ and $Q_0 = \{[0 : \cdots : 0 : \xi_{s+1} : \cdots : \xi_n] | \xi_{s+1}, \cdots, \xi_n \in \mathbb{C}\} \subset \mathbb{CP}^n$. Straight forward computations show: $d(P_0, Q_0) = d := \frac{\pi}{\sqrt{2}}$ and

(3.4) $$r_{P_0}(x) + r_{Q_0}(x) = d$$

for any x, where r_{P_0} and r_{Q_0} are distance functions from P_0 and Q_0 respectively. Moreover, the first Dirichlet eigenvalues of $B_{P_0}(r_0)$ and

$B_{Q_0}(d - r_0)$ are both $n + 1$. Here for a submanifold S in a Käher manifold, $B_S(R)$ consists of points x with $d(S, x) < R$.

Denote $\lambda_1(N)$ the first Dirichlet eigenvalue of the complex Laplacian of N. By [13], we have

$$\lambda_1(B_P(r_0)) \leq \lambda_1(B_{P_0}(r_0)) = n + 1;$$

$$\lambda_1(B_Q(d - r_0)) \leq \lambda_1(B_{Q_0}(d - r_0)) = n + 1.$$

Since $d(P, Q) = d$, $B_P(r_0) \cap B_Q(d - r_0) = \emptyset$. By the same argument as in [7, Theorem 2.1], we know that the first nonzero eigenvalue λ of M is at most $n + 1$. Since the Ricci curvature of M is bounded below by $n + 1$, we have $\lambda = n + 1$. By Corollary 3.1, M is holomorphically isometric to $\mathbb{CP}^n$ with the Fubini-Study metric.

$\square$

4. Volume comparison

Let (M^n, ω) be a compact Kähler manifold such that the first Chern class $c_1(M)$ is either positive or negative, where ω is the Kähler form that is a multiple of the first Chern class as a cohomology class. If $c_1(M) < 0$ then there is a unique Kähler-Einstein metric ω_0 such that $\text{Ric}(\omega_0) = -\omega_0$ by well-known results (see [1, 20]).

Proposition 4.1. *With the above notations and assumptions, the following are true:*

(i) *If $c_1(M) < 0$ and if the scalar curvature R of ω satisfies $-nb \leq R \leq -na < 0$, then the volume of $V(M, \omega)$ of (M, ω) is bounded by*

(4.1) $$V(M, \frac{1}{b}\omega_0) \leq V(M, \omega) \leq V(M, \frac{1}{a}\omega_0).$$

If the first (respectively the second) equality holds, then $\omega = \frac{1}{b}\omega_0$ (respectively $\omega = \frac{1}{a}\omega_0$).

(ii) *If $c_1(M) > 0$ and there is a Kähler-Einstein metric ω_0 such that $\text{Ric}(\omega_0) = \omega_0$, and if the scalar curvature R of ω satisfies $0 < nb \leq R \leq na$, then the volume of $V(M, \omega)$ of (M, ω) is bounded by*

(4.2) $$V(M, \frac{1}{a}\omega_0) \leq V(M, \omega) \leq V(M, \frac{1}{b}\omega_0).$$

If the first (respectively the second) equality holds, then (M, ω) is holomorphically isometric to $(M, \frac{1}{a}\omega_0)$ (respectively (M, ω) is holomorphically isometric to $(M, \frac{1}{a}\omega_0)$).

Proof. We only prove (i), the proof of (ii) is similar (For the case of equality, we have used results in [2]). To prove (i), suppose Suppose $2\pi c_1(M) = -\lambda[\omega]$ with $\lambda > 0$. Then it is well-known that

$$(4.3) \qquad \lambda = -\frac{1}{nV(M,\omega)}\int_M R\omega^n.$$

Hence $a \le \lambda \le b$.

Moreover,

$$(4.4) \qquad V(M,\omega) = \frac{\pi^n}{n!\lambda^n}\int_M c_1(M)^n = \frac{V(M,\omega_0)}{\lambda^n}$$

Therefore

$$(4.5) \qquad \frac{1}{b^n}V(M,\omega_0) \le V(M,\omega) \le \frac{1}{a^n}V(M,\omega_0).$$

If the first equality holds, we know that $\lambda = b$ and ω has constant scalar curvature $-b$ and is Kähler-Einstein (see [19]). It is easy to see that $\omega = \frac{1}{b}\omega_0$. If the second equality holds, we can prove similarly that $\omega = \frac{1}{a}\omega_0$. $\qquad\square$

Corollary 4.1. *Let (M^n, ω) be a compact manifold with positive holomorphic bisectional curvature. Suppose the scalar curvature $k_2 \ge R \ge k_1 > 0$. Let $V_{k_1}(\mathbb{CP}^n)$ and $V_{k_2}(\mathbb{CP}^n)$ be the volumes of $\mathbb{CP}^n$ with Fubini-Study metrics with constant scalar curvatures k_1 and k_2 respectively. Then the volume $V(M,g))$ of M satisfies:*

$$(4.6) \qquad V_{k_2}(\mathbb{CP}^n) \le V(M,\omega) \le V_{k_1}(\mathbb{CP}^n).$$

Moreover, if one of the inequalities is an equality then (M, g) is holomorphically isometric the $\mathbb{CP}^n$ with Fubini-Study metric.

Proof. By [14, 18], M is biholomorphic to $\mathbb{CP}^n$ and we can apply Proposition 4.1(ii) and the results follow.

$\qquad\square$

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