LIMIT THEOREMS FOR FREE MULTIPLICATIVE CONVOLUTIONS

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ABSTRACT. We determine the distributional behavior for products of free random variables in a general infinitesimal triangular array. The main theorems in this paper extend a result for measures supported on the positive half-line in [4], and provide a new limit theorem for measures on the unit circle with nonzero first moment.

1. INTRODUCTION

Given two probability measures μ, ν on $\mathbb{R}_+ = (0, +\infty)$, we will denote by $\mu \circledast \nu$ their classical multiplicative convolution, and by $\mu \boxtimes \nu$ their free multiplicative convolution. Thus, $\mu \circledast \nu$ is the distribution of XY, where X and Y are classically independent positive random variables with distributions μ and ν, respectively. Analogously, $\mu \boxtimes \nu$ is the distribution of $X^{1/2}YX^{1/2}$, where X and Y are freely independent positive random variables with distributions μ and ν. A triangular array $\{\nu_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ of probability measures on $\mathbb{R}_+$ is said to be *infinitesimal* if

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \nu_{nk}(\{t \in \mathbb{R}_+ : |t-1| \geq \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. Given such an array, one is interested in the asymptotic behavior of the measures

$$\mu_n = \nu_{n1} \circledast \nu_{n2} \circledast \cdots \circledast \nu_{nk_n}$$

and

$$\nu_n = \nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n}.$$

The case of μ_n is completely understood, and is reduced to the theory of addition of independent random variables by a logarithmic change of variables. However, the free case ν_n does not simply reduce to the additive theory by this change of variables.

The problem was first addressed in [4], where a triangular array such that $\nu_{n1} = \nu_{n2} = \cdots = \nu_{nk_n}$ for all n was considered. In this case, necessary and sufficient conditions were found for the weak convergence of the measures ν_n. In particular, it was shown that the sequence ν_n converges weakly if μ_n converges, but not conversely.

In this paper we will find necessary and sufficient conditions for the weak convergence of ν_n without any further assumptions on the infinitesimal array. We also prove analogous results for infinitesimal triangular arrays on the unit circle

$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In both cases the possible limit of ν_n is $\boxtimes$-infinitely divisible as shown in [8, 2].

The additive version of our results were studied earlier. Thus, consider an array $\{\mu_{nk} : n \geq 1, 1 \leq k \leq k_n\}$ of probability measures on $\mathbb{R}$. Infinitesimality in this case means that

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} \mu_{nk}(\{t \in \mathbb{R} : |t| \geq \varepsilon\}) = 0, \qquad \varepsilon > 0.$$

Denote by

$$\lambda_n = \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$$

the classical additive convolutions, and by

$$\rho_n = \mu_{n1} \boxplus \mu_{n2} \boxplus \cdots \boxplus \mu_{nk_n}$$

the free additive convolutions of these measures. When $\mu_{n1} = \mu_{n2} = \cdots = \mu_{nk_n}$ for all n, it was shown in [3] that λ_n converges weakly if and only if ρ_n converges weakly. This result was extended to arbitrary infinitesimal arrays by Chistyakov and Götze in [9]. These authors made heavy use of analytic subordination (first proved for the free additive convolution in [11] generically and in [7] for the general case; cf. also [12, 13], [1] and [8] for different approaches). Our methods do not require analytic subordination and are close to the original approach in [4].

The remainder of this paper is organized as follows. In Section 2, we describe the analytical apparatus necessary for the calculation of free multiplicative convolutions. We also describe the $\boxtimes$-infinitely divisible measures on $\mathbb{R}_+$ and $\mathbb{T}$ and some useful approximation results. In Sections 3 we give the convergence criteria for arrays on $\mathbb{R}_+$, and in Section 4, we prove the analogous result for $\mathbb{T}$.

2. Preliminaries

The analogue of Fourier transform for multiplicative free convolutions was discovered by Voiculescu [10] (see also [5, 6]). Denote by $\mathcal{M}_+$ the collection of Borel probability measures defined on $\mathbb{R}_+$, and by $\mathcal{M}_{\mathbb{T}}^{\times}$ Borel probability measures ν supported on the circle $\mathbb{T}$ with nonzero first moment, i.e. $\int_{\mathbb{T}} t\, d\nu(t) \neq 0$.

Given $\nu \in \mathcal{M}_+$, one defines the analytic function ψ_ν by

$$\psi_\nu(z) = \int_0^\infty \frac{tz}{1 - tz}\, d\nu(t), \qquad z \in \mathbb{C} \setminus (0, +\infty).$$

The function ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$, and $\psi_\nu(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(-1, 0)$; moreover, $\psi_\nu(i\mathbb{C}^+) \cap (-\infty, 0) = (-1, 0)$. Setting $\Omega_\nu = \psi_\nu(i\mathbb{C}^+)$, one defines the *S-transform* of the measure ν to be

$$S_\nu(z) = \frac{1+z}{z}\psi_\nu^{-1}(z), \qquad z \in \Omega_\nu.$$

The remarkable property of the S-transform is that for $\mu, \nu \in \mathcal{M}_+$, one has

$$S_{\mu\boxtimes\nu}(z) = S_\mu(z)S_\nu(z),$$

for every z in a neighborhood of $(-1, 0)$.

For $\nu \in \mathcal{M}_{\mathbb{T}}^\times$, the function ψ_ν is defined by the formula given above (with the integral calculated over $\mathbb{T}$), but its domain of definition is now the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The function ψ_ν has an inverse in a neighborhood of zero since $\psi_\nu'(0) = \int_{\mathbb{T}} t\, d\nu(t) \neq 0$. The corresponding S-transform is defined in a neighborhood of zero. It is sometimes convenient to use a variation of the S-transform:

$$\Sigma_\nu(z) = S_\nu\left(\frac{z}{1-z}\right).$$

If $\nu \in \mathcal{M}_{\mathbb{T}}^\times$, the function Σ_ν is also defined in a neighborhood of zero, and

$$\Sigma_{\mu\boxtimes\nu}(z) = \Sigma_\mu(z)\Sigma_\nu(z), \qquad \mu, \nu \in \mathcal{M}_{\mathbb{T}}^\times,$$

for all z in a neighborhood of zero where all functions involved are defined.

The weak convergence of probability measures can be translated in terms of their S-transforms.

Theorem 2.1. [5, 6]

(1) *Given $\{\nu_n\}_{n=1}^\infty \subset \mathcal{M}_+$, the sequence $\{\nu_n\}_{n=1}^\infty$ converges weakly to a measure $\nu \in \mathcal{M}_+$ if and only if there exist two positive numbers $0 < b < a < 1$ such that the disk D with the diameter $(-a, -b)$ is contained in Ω_{ν_n} for all n, and the sequence $\{S_{\nu_n}\}_{n=1}^\infty$ converges uniformly on D to a function S.*

(2) *Given $\{\nu_n\}_{n=1}^\infty \subset \mathcal{M}_{\mathbb{T}}^\times$, the sequence $\{\nu_n\}_{n=1}^\infty$ converges weakly to a measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ if and only if there exists a neighborhood of zero $K \subset \mathbb{D}$ such that for all Σ_{ν_n} are defined in K, and the sequence $\{\Sigma_{\nu_n}\}_{n=1}^\infty$ converges uniformly on K to a function Σ.*

Moreover, if (1) is satisfied then $S = S_\nu$, and if (2) is satisfied then $\Sigma = \Sigma_\nu$.

An array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^\times$ is infinitesimal if

$$\lim_{n\to\infty} \max_{1\le k\le k_n} \nu_{nk}(\{t \in \mathbb{T} : |\arg t| \ge \varepsilon\}) = 0,$$

for every $\varepsilon > 0$; here the principal value of the argument is used. The following proposition gives an approximation of the S-transform (see Theorem 3.1 in [4] and Theorem 1.1 and Theorem 2.1 in [2]).

Proposition 2.2. *For $0 < b < a < 1$ and $\varepsilon \in (0, 1)$, define $\overline{D}$ to be the closed disk with diameter $[-a, -b]$, and set $K_\varepsilon = \{z \in \mathbb{C} : |z| \le \varepsilon\}$.*

(1) *If an array* $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_+$ *is infinitesimal, then the functions* $S_{\nu_{nk}}$ *are defined in* $\overline{D}$ *for sufficiently large* n*, and we have*

$$S_{\nu_{nk}}(z) = 1 + \left[\int_0^\infty \frac{1-t}{1+z-tz}\, d\nu_{nk}(t)\right](1 + u_{nk}(z)),$$

for all $z \in \overline{D}$*, where* $\lim_{n\to\infty} \max_{1\le k\le k_n} |u_{nk}(z)| = 0$ *uniformly on* $\overline{D}$*.*

(2) *If an array* $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^\times$ *is infinitesimal, then* $S_{\nu_{nk}}$ *are defined in* K_ε *when* n *is large, and we have*

$$S_{\nu_{nk}}(z) = 1 + \left[\int_{\mathbb{T}} \frac{1-t}{1+z-tz}\, d\nu_{nk}(t)\right](1 + v_{nk}(z)),$$

for all $z \in K_\varepsilon$*, where* $\lim_{n\to\infty} \max_{1\le k\le k_n} |v_{nk}(z)| = 0$ *uniformly on* K_ϵ*.*

A measure $\nu \in \mathcal{M}_+$ is said to be $\circledast$-*infinitely divisible* if, for each $n \in \mathbb{N}$, there exists a measure $\nu_n \in \mathcal{M}_+$ such that

$$\nu = \underbrace{\nu_n \circledast \nu_n \circledast \cdots \circledast \nu_n}_{n\,\text{times}}.$$

The notion of $\boxtimes$-*infinite divisibility* is defined analogously. The study of $\circledast$-infinitely divisible measures on $\mathbb{R}_+$ reduces (by a change of variable) to the study of the usual $*$-infinitely divisible measures on $\mathbb{R}$. The Fourier transform needs to be replaced by the *Mellin-Fourier* transform of a measure $\nu \in \mathcal{M}_+$ defined by

$$\Phi_\nu(s) = \int_0^\infty t^{is}\, d\nu(t), \qquad s \in \mathbb{R}.$$

The fundamental property of the Mellin-Fourier transform is that

$$\Phi_{\mu\circledast\nu}(s) = \Phi_\mu(s)\Phi_\nu(s).$$

A $\circledast$-infinitely divisible measure $\nu \in \mathcal{M}_+$ has the Mellin-Fourier transform

$$\Phi_\nu(s) = \exp\left[i\lambda s + \int_0^\infty \left(t^{-is} - 1 + \frac{is\log t}{\log^2 t + 1}\right)\frac{\log^2 t + 1}{\log^2 t}\, d\rho(t)\right], \qquad s \in \mathbb{R},$$

where $\lambda \in \mathbb{R}$ and ρ is a finite positive Borel measure on $\mathbb{R}_+$. We use the notation $\nu_\circledast^{\lambda,\rho}$ to denote the $\circledast$-infinitely divisible measure determined by λ and ρ. For $\boxtimes$-infinite divisibility we have the following formulas as in [5, 6]. A measure $\nu \in \mathcal{M}_+$ is $\boxtimes$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on the compact space $[0, +\infty]$ such that $S_\nu(z) = \exp(v_{\gamma,\sigma}(z))$, where $v_{\gamma,\sigma}$ is given by

$$v_{\gamma,\sigma}\left(\frac{z}{1-z}\right) = \gamma - \sigma(\{+\infty\})z + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C} \setminus [0,1].$$

A measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ is $\boxtimes$-infinitely divisible if and only if there exist $\gamma \in \mathbb{R}$ and a finite positive Borel measure σ on $\mathbb{T}$ such that $\Sigma_\nu(z) = \exp(u_{\gamma,\sigma}(z))$, where $u_{\gamma,\sigma}$ is given by

$$u_{\gamma,\sigma}(z) = -i\gamma + \int_{\mathbb{T}} \frac{1+tz}{1-tz}\, d\sigma(t), \qquad z \in \mathbb{D}.$$

We denote by $\nu_{\boxtimes}^{\gamma,\sigma}$ the $\boxtimes$-infinitely divisible measure determined by γ and σ. There is a unique $\boxtimes$-infinitely divisible measure m on $\mathbb{T}$ such that its first moment is zero. This is the Haar, or normalized arclength measure.

We conclude this section with a result which will be used repeatedly.

Lemma 2.3. *Consider a sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and two triangular arrays $\{z_{nk} : n \geq 1,\, 1 \leq k \leq k_n\}$, $\{w_{nk} : n \geq 1,\, 1 \leq k \leq k_n\}$ of complex numbers. Assume that*

(1) *$\Im w_{nk} \geq 0$, for $n \geq 1$ and $1 \leq k \leq k_n$.*

(2)

$$z_{nk} = w_{nk}(1 + \varepsilon_{nk}),$$

where

$$\varepsilon_n = \max_{1 \leq k \leq k_n} |\varepsilon_{nk}|$$

converges to zero as $n \to \infty$.

(3) *There exists a positive constant M such that for sufficiently large n,*

$$|\Re w_{nk}| \leq M \Im w_{nk}.$$

Then the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ converges if and only if the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ converges. Moreover, the two sequences have the same limit.

Proof. The assumptions on $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$ imply

$$\left| \left(r_n + \sum_{k=1}^{k_n} z_{nk} \right) - \left(r_n + \sum_{k=1}^{k_n} w_{nk} \right) \right| \leq 2(1+M)\varepsilon_n \left(\sum_{k=1}^{k_n} \Im w_{nk} \right), \tag{2.1}$$

and

$$(1 - \varepsilon_n - M\varepsilon_n) \left(\sum_{k=1}^{k_n} \Im w_{nk} \right) \leq \left| \sum_{k=1}^{k_n} \Im z_{nk} \right|, \tag{2.2}$$

for sufficiently large n. If the sequence $\{r_n + \sum_{k=1}^{k_n} z_{nk}\}_{n=1}^{\infty}$ converges to a complex number z, (2.2) implies that $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^{\infty}$ is bounded, and then (2.1) shows that the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ also converges to z. Conversely, if $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ converges to z, then the sequence $\{\sum_{k=1}^{k_n} \Im w_{nk}\}_{n=1}^{\infty}$ is bounded and hence by (2.1) the sequence $\{r_n + \sum_{k=1}^{k_n} w_{nk}\}_{n=1}^{\infty}$ converges to z as well. □

3. Free Multiplicative Convolution on $\mathbb{R}_+$

Given an infinitesimal triangular array $\{\nu_{nk} : 1 \leq k \leq k_n,\, n \in \mathbb{N}\} \subset \mathcal{M}_+$ and $\tau > 0$, define positive numbers

$$b_{nk} = \exp\left(\int_{e^{-\tau}}^{e^{\tau}} \log t \, d\nu_{nk}(t) \right),$$

and measures ν°_{nk} by

$$d\nu^\circ_{nk}(t) = d\nu_{nk}(b_{nk}t).$$

Obviously, $\max_{1\le k\le k_n} |b_{nk} - 1| \to 0$ as $n \to \infty$, and hence the array $\{\nu^\circ_{nk}\}_{n,k}$ is also infinitesimal. Define

$$g_{nk}(w) = \int_0^\infty \frac{t^2-1}{t^2+1}\, d\nu^\circ_{nk}\left(\frac{1}{t}\right) + \int_0^\infty \left[\frac{1+tw}{w-t}\right] \frac{(t-1)^2}{t^2+1}\, d\nu^\circ_{nk}\left(\frac{1}{t}\right),$$

for $w \in \mathbb{C}\setminus[0,+\infty)$. Note that $g_{nk}(\overline{w}) = \overline{g_{nk}(w)}$ and $\Im g_{nk}(w) \le 0$ for all w such that $\Im w > 0$.

Lemma 3.1. *For every compact set $K \subset \mathbb{C}^+ \cap (i\mathbb{C}^+)$ there exists a positive constant $M = M(\tau, K)$ such that for sufficiently large n, we have*

$$|\Re g_{nk}(w)| \le M\, |\Im g_{nk}(w)|, \qquad w \in K,\ 1 \le k \le k_n.$$

Proof. We assume for convenience that $\tau = 1$. No generality is lost since one can make a linear change of variable to modify the value of τ. By a change of variable we have

$$\int_0^\infty \frac{t^2-1}{t^2+1}\, d\nu^\circ_{nk}\left(\frac{1}{t}\right) = \int_{-\infty}^\infty \frac{1-e^{2x}}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk})$$

and

$$\int_0^\infty \frac{(t-1)^2}{t^2+1}\, d\nu^\circ_{nk}\left(\frac{1}{t}\right) = \int_{-\infty}^\infty \frac{(1-e^x)^2}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk}),$$

where the probability measure ρ_{nk} is defined as $d\rho_{nk}(x) = d\nu_{nk}(e^x)$, and $a_{nk} = \int_{|x|<1} x\, d\rho_{nk}(x)$. Note that the family $\{\rho_{nk}\}_{n,k}$ is now an infinitesimal family of probability measures on $\mathbb{R}$, and hence

$$\lim_{n\to\infty} \max_{1\le k\le k_n} |a_{nk}| = 0.$$

We proceed by rewriting

$$\int_{-\infty}^\infty \frac{1-e^{2x}}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk}) = \int_{|x|<1} \left[\frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} + (x-a_{nk})\right] d\rho_{nk}(x) - \int_{|x|\ge 1} a_{nk}\, d\rho_{nk}(x) + \int_{|x|\ge 1} \frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}}\, d\rho_{nk}(x).$$

It is easy to see that

$$\left|1-e^{2(x-a_{nk})} + (x-a_{nk}) + (x-a_{nk})e^{2(x-a_{nk})}\right| \le 60(x-a_{nk})^2,$$

for $|x| < 1$. Consequently, for all n and k we have

$$\left|\int_{|x|<1} \left[\frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}} + (x-a_{nk})\right] d\rho_{nk}(x)\right| \le 60 \int_{-\infty}^\infty \frac{(1-e^x)^2}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk}).$$

Since the family $\{\rho_{nk}\}_{n,k}$ is infinitesimal, there exists $N \in \mathbb{N}$ such that

$$|a_{nk}| \le \frac{1}{2},$$

for all $n \geq N$, $1 \leq k \leq k_n$. Note that

$$\frac{(1-e^x)^2}{1+e^{2x}} \geq \frac{(1-\sqrt{e})^2}{1+e} \quad \text{and} \quad 5(1-e^x)^2 \geq \left|1-e^{2x}\right|,$$

for all $|x| \geq \frac{1}{2}$. We deduce that for $n \geq N$, $1 \leq k \leq k_n$ we have

$$\begin{aligned} \left|\int_{|x|\geq 1} a_{nk}\, d\rho_{nk}(x)\right| &\leq \int_{|x|\geq 1} d\rho_{nk}(x) \\ &\leq \frac{1+e}{(1-\sqrt{e})^2}\int_{-\infty}^{\infty} \frac{(1-e^x)^2}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk}), \end{aligned}$$

and

$$\left|\int_{|x|\geq 1} \frac{1-e^{2(x-a_{nk})}}{1+e^{2(x-a_{nk})}}\, d\rho_{nk}(x)\right| \leq 5\int_{-\infty}^{\infty} \frac{(1-e^x)^2}{1+e^{2x}}\, d\rho_{nk}(x+a_{nk}).$$

Therefore, for sufficiently large n,

$$\left|\int_0^\infty \frac{t^2-1}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right)\right| \leq 74\int_0^\infty \frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right).$$

The compactness of the set K implies the existence of positive constants M_1 and M_2 such that

$$\left|\Re\left[\frac{1+tw}{w-t}\right]\right| \leq M_1$$

and

$$\left|\Im\left[\frac{1+tw}{w-t}\right]\right| = -\Im\left[\frac{1+tw}{w-t}\right] \geq M_2,$$

for all $t \in (0,+\infty)$ and $w \in K$. Hence, we have for sufficiently large n and for $w \in K$,

$$\begin{aligned} |\Re g_{nk}(w)| &= \left|\int_0^\infty \frac{t^2-1}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right) + \int_0^\infty \Re\left[\frac{1+tw}{w-t}\right]\frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right)\right| \\ &\leq 74\int_0^\infty \frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right) + M_1\int_0^\infty \frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right) \\ &\leq \frac{(74+M_1)}{M_2}\int_0^\infty M_2\frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right) \\ &\leq -\frac{(74+M_1)}{M_2}\int_0^\infty \Im\left[\frac{1+tw}{w-t}\right]\frac{(t-1)^2}{t^2+1}\, d\nu_{nk}^\circ\left(\frac{1}{t}\right) \\ &= \frac{74+M_1}{M_2}\left|\Im g_{nk}(w)\right|. \end{aligned}$$

The result follows with $M = (74+M_1)/M_2$. □

Fix a closed disk $\overline{D} \subset i\mathbb{C}^+$ with diameter $[-a,-b]$, where $0 < b < a < 1$. By Proposition 2.2, $S_{\nu_{nk}}$ is defined in $\overline{D}$ for large n. Setting $w = z/(1+z)$, and using the identity

$$\frac{(w-1)(t-1)}{w-t} = \frac{t^2-1}{t^2+1} + \left[\frac{1+tw}{w-t}\right]\frac{(t-1)^2}{t^2+1},$$

we see that the function $S_{\nu_{nk}^\circ}$ admits the following approximation:

$$S_{\nu_{nk}^\circ}\left(\frac{w}{1-w}\right)-1=g_{nk}(w)\left(1+u_{nk}\left(\frac{w}{1-w}\right)\right),$$

in another closed disk $\overline{D_0}=\{z/(1+z): z\in\overline{D}\}$ with real center, and

$$\lim_{n\to\infty}\max_{1\le k\le k_n}\left|u_{nk}\left(\frac{w}{1-w}\right)\right|=0,$$

uniformly for all $w\in\overline{D_0}$. Note that

$$S_{\nu_{nk}^\circ}\left(\frac{w}{1-w}\right)=b_{nk}S_{\nu_{nk}}\left(\frac{w}{1-w}\right).$$

The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ also shows that $S_{\nu_{nk}}(z)$ converges uniformly in k and $z\in\overline{D}$ to 1 as $n\to\infty$; indeed, $S_{\delta_1}\equiv 1$. Hence, for sufficiently large n, the principal branch of $\log S_{\nu_{nk}}(z)$ is defined in $\overline{D}$. Furthermore, since

$$\log w=w-1+o(|w-1|),$$

as $w\to 1$, it is easy to see from Lemma 3.1 and Lemma 2.3 that we have the following result. Fix a real number γ, and a finite positive Borel measure σ on $[0,+\infty]$.

Lemma 3.2. *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Then the sequence of functions $\{-\log\alpha_n+\sum_{k=1}^{k_n}\log S_{\nu_{nk}}(z)\}_{n=1}^\infty$ converges to $v_{\gamma,\sigma}(z)$ uniformly for all $z\in\overline{D}$ as $n\to\infty$ if and only if*

$$\lim_{n\to\infty}\left(-\log\alpha_n+\sum_{k=1}^{k_n}[g_{nk}(w)-\log b_{nk}]\right)=v_{\gamma,\sigma}\left(\frac{w}{1-w}\right)$$

uniformly for all $w\in\overline{D_0}$.

Theorem 3.3. *For an infinitesimal family $\{\nu_{nk}\}_{n,k}\subset\mathcal{M}_+$ and a sequence $\{\alpha_n\}_{n=1}^\infty\subset\mathbb{R}_+$, the following two assertions are equivalent:*

(1) *The sequence $\nu_{n1}\boxtimes\nu_{n2}\boxtimes\cdots\boxtimes\nu_{nk_n}\boxtimes\delta_{\alpha_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma,\sigma}$.*
(2) *The sequence of measures*

$$d\sigma_n(t)=\sum_{k=1}^{k_n}\frac{(t-1)^2}{t^2+1}\,d\nu_{nk}^\circ\left(\frac{1}{t}\right)$$

converges weakly in $[0,+\infty]$ to σ, and the sequence

$$\gamma_n=-\log\alpha_n+\sum_{k=1}^{k_n}\left[\int_0^\infty\frac{t^2-1}{t^2+1}\,d\nu_{nk}^\circ\left(\frac{1}{t}\right)-\log b_{nk}\right]$$

converges to γ as $n\to\infty$.

Proof. Assume (1) holds. From Theorem 2.1, there exists a closed disk $\overline{D}$ with real center such that

$$\lim_{n\to\infty}\left(\frac{1}{\alpha_n}\prod_{k=1}^{k_n} S_{\nu_{nk}}(z)\right) = S_{\nu_{\boxtimes}^{\gamma,\sigma}}(z) = \exp(v_{\gamma,\sigma}(z))$$

uniformly on the disk $\overline{D}$. We may choose $\overline{D}$ small enough so that $\exp(v_{\gamma,\sigma}(z))$ is in $-i\mathbb{C}^+$ on $\overline{D}$. Applying the principal branch of the logarithm function, we deduce that

$$\lim_{n\to\infty}\left(-\log\alpha_n + \sum_{k=1}^{k_n}\log S_{\nu_{nk}}(z)\right) = v_{\gamma,\sigma}(z),$$

uniformly on $\overline{D}$. Thus, Lemma 3.2 implies that

$$\lim_{n\to\infty}\left(-\log\alpha_n + \sum_{k=1}^{k_n}[g_{nk}(w) - \log b_{nk}]\right) = v_{\gamma,\sigma}\left(\frac{w}{1-w}\right) \tag{3.1}$$

uniformly on $\overline{D_0} = \{z/(1+z) : z \in \overline{D}\}$. Note that

$$-\log\alpha_n + \sum_{k=1}^{k_n}[g_{nk}(w) - \log b_{nk}] = \gamma_n + \int_0^\infty \frac{1+tw}{w-t}\,d\sigma_n(t). \tag{3.2}$$

Considering the imaginary part of the equation (3.1), we have

$$-\Im w\int_{[0,+\infty)}\frac{1+t^2}{|w-t|^2}\,d\sigma(t) = -\lim_{n\to\infty}\Im w\int_{(0,+\infty)}\frac{1+t^2}{|w-t|^2}\,d\sigma_n(t). \tag{3.3}$$

Note that the function $t \mapsto \frac{1+t^2}{|w-t|^2}$ is bounded away from zero and infinity for all $w \in \overline{D_0}$; moreover, if $\Im w \neq 0$ then (3.3) shows that

$$\sup_n \sigma_n((0,+\infty)) < +\infty,$$

and hence the family $\{\sigma_n\}_{n=1}^\infty$ has a weak cluster point σ' on the compact space $[0,+\infty]$. Then (3.3) shows that $\sigma' = \sigma$, and consequently the measures σ_n converges weakly to σ on $[0,+\infty]$ as $n \to \infty$. Then it is easy to see from (3.1) and (3.2) that $\lim_{n\to\infty}\gamma_n = \gamma$.

Conversely, assume (2) holds. The infinitesimality of the array $\{\nu_{nk}\}_{n,k}$ implies that there exist $a', b' \in (0,1)$ with $b' < a'$ such that $S_{\nu_{nk}}$ are defined in $\overline{D'}$, the closed disk with the diameter $[-a', -b']$, for sufficiently large n. Let $\overline{D_0'} = \{z/(1+z) : z \in \overline{D'}\}$ and observe that there exists a positive constant $M = M(a', b')$ such that

$$\left|\frac{1+tw}{w-t}\right| \leq M, \qquad w \in \overline{D_0'},\ t \in (0,+\infty).$$

Thus, in view of (3.2), we deduce that (3.1) holds pointwise in $\overline{D_0'}$. Since $\Im g_{nk}(w) \leq 0$ for $w \in D_0' \cap i\mathbb{C}^+$, the family $\{-\log\alpha_n + \sum_{k=1}^{k_n}[g_{nk}(w) - \log b_{nk}]\}_{n=1}^\infty$ is normal in

$D'_0 \cap i\mathbb{C}^+$. Moreover, note that $g_{nk}(\overline{w}) = \overline{g_{nk}(w)}$ and

$$v_{\gamma,\sigma}\left(\frac{\overline{w}}{1-\overline{w}}\right) = \overline{v_{\gamma,\sigma}\left(\frac{w}{1-w}\right)},$$

for $w \in \overline{D'_0}$. Therefore, as an application of Montel's theorem, we conclude that (3.1) holds uniformly on compact subsets of $\overline{D'_0}$. From Lemma 3.2 we conclude that there exists a smaller closed disk $\overline{D''} \subset \overline{D'}$ with real center, such that

$$\lim_{n\to\infty}\left(-\log\alpha_n + \sum_{k=1}^{k_n}\log S_{\nu_{nk}}(z)\right) = v_{\gamma,\sigma}(z),$$

uniformly on $\overline{D''}$. Applying the exponential, we obtain

$$\lim_{n\to\infty}\left(\frac{1}{\alpha_n}\prod_{k=1}^{k_n}S_{\nu_{nk}}(z)\right) = S_{\nu_{\boxtimes}^{\gamma,\sigma}}(z) = \exp(v_{\gamma,\sigma}(z))$$

uniformly on $\overline{D''}$. Therefore (1) follows from Theorem 2.1. □

It has been pointed out in [4] that the weak convergence criteria for products of free and independent random variables are not equivalent. Nevertheless, the following correspondence is true.

Corollary 3.4. *Given an infinitesimal family $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_+$ and a sequence $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$, the following two statements are equivalent:*

(1) *The sequence $\nu_{n1}\boxtimes\nu_{n2}\boxtimes\cdots\boxtimes\nu_{nk_n}\boxtimes\delta_{\alpha_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma,\sigma}$ and $\sigma(\{0\}) = \sigma(\{\infty\}) = 0$;*

(2) *The sequence $\nu_{n1}\circledast\nu_{n2}\circledast\cdots\circledast\nu_{nk_n}\circledast\delta_{\alpha_n}$ converges weakly to $\nu_{\circledast}^{\lambda,\rho}$.*

If conditions (1) *and* (2) *are satisfied then the measure σ and ρ are related by*

$$d\sigma(t) = \frac{\log^2 t + 1}{\log^2 t}\frac{(t-1)^2}{t^2+1}\,d\rho(t),$$

and

$$\gamma - \lambda = \int_0^{\infty}\left(\frac{t^2-1}{t^2+1} + \frac{\log^2 t}{\log^2 t + 1}\right)\frac{\log^2 t + 1}{\log^2 t}\,d\rho(t).$$

Proof. The proof is identical with that of Theorem 4.2 in [4]. □

4. Free Multiplicative Convolution on $\mathbb{T}$

Fix an infinitesimal array $\{\nu_{nk} : n \geq 1, 1 \leq k \leq k_n\} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and $\tau \in (0,\pi)$. Consider the centering constant

$$b_{nk} = \exp\left(\int_{|\arg t|<\tau}\log t\,d\nu_{nk}(t)\right),$$

and the centered measure ν_{nk}° obtained as follows:

$$d\nu_{nk}^{\circ}(t) = d\nu_{nk}(b_{nk}t).$$

Here, as before, $\log t = i \arg t$ represents the principal branch of $\log t$. We have $\max_{1\le k\le k_n} |\arg b_{nk}| \to 0$ as $n \to \infty$, and hence the array $\{\nu_{nk}^\circ\}_{n,k}$ is also infinitesimal. Define

$$h_{nk}(z) = -i\int_{\mathbb{T}} \Im t\, d\nu_{nk}^\circ(t) + \int_{\mathbb{T}} \frac{1+tz}{1-tz}(1-\Re t)\, d\nu_{nk}^\circ(t), \qquad z\in\mathbb{D}.$$

Note that $\Re h_{nk}(z) \ge 0$ for all $z \in \mathbb{D}$.

Lemma 4.1. *For every compact neighborhood of zero $K \subset \mathbb{D}$, there exists a positive constant $M = M(\tau, K)$ such that for sufficiently large n, we have*

$$|\Im h_{nk}(z)| \le M \Re h_{nk}(z), \qquad z\in K,\ 1\le k\le k_n.$$

Proof. We may again assume that $\tau = 1$. Define probability measures ρ_{nk} on $\mathbb{R}$ such that $\rho_{nk}(\sigma) = \nu_{nk}(e^{i\sigma})$ if $\sigma \subset [-\pi,\pi)$, and $\rho_{nk}(\sigma) = 0$ when $\sigma \cap [-\pi,\pi) = \phi$. Changing variables, we have

$$\int_{\mathbb{T}} \Im t\, d\nu_{nk}^\circ(t) = \int_{-\infty}^{\infty} \sin x\, d\rho_{nk}(x + a_{nk})$$

and

$$\int_{\mathbb{T}} (1-\Re t)\, d\nu_{nk}^\circ(t) = \int_{-\infty}^{\infty} (1-\cos x)\, d\rho_{nk}(x + a_{nk}),$$

where $a_{nk} = \int_{|x|<1} x\, d\rho_{nk}(x) = \int_{|\arg t|<1} \arg t\, d\nu_{nk}(t)$. The infinitesimality of the family $\{\nu_{nk}\}_{n,k}$ implies that for sufficiently large n,

$$\max_{1\le k\le k_n} |a_{nk}| \le \frac{1}{10}.$$

By using the elementary inequalities

$$|\sin x - x| \le 2(1-\cos x), \qquad -2\le x\le 2,$$

and

$$\frac{1}{10} + |\sin x| \le 10(1-\cos x), \text{ where } \frac{9}{10} \le |x| \le \pi + \frac{9}{10},$$

we have,

$$\begin{aligned}\left|\int_{\mathbb{T}} \Im t\, d\nu_{nk}^\circ(t)\right| &\le \left|\int_{|x|<1} [\sin(x-a_{nk}) - (x-a_{nk})]\, d\rho_{nk}(x)\right| \\ &\quad + \left|\int_{[-\pi,-1]\cup[1,\pi)} a_{nk}\, d\rho_{nk}(x)\right| + \left|\int_{[-\pi,-1]\cup[1,\pi)} \sin(x-a_{nk})\, d\rho_{nk}(x)\right| \\ &\le 12\int_{\mathbb{T}} (1-\Re t)\, d\nu_{nk}^\circ(t),\end{aligned}$$

for large n and $1 \le k \le k_n$. Also, from the compactness of the set K, there exist two positive constants M_1 and M_2 such that

$$\left|\Re\left[\frac{1+tz}{1-tz}\right]\right| = \Re\left[\frac{1+tz}{1-tz}\right] \ge M_1$$

and

$$\left|\Im\left[\frac{1+tz}{1-tz}\right]\right| \le M_2,$$

for all $t \in \mathbb{T}$ and $z \in K$. The result follows with $M = (12 + M_2)/M_1$. $\square$

Suppose $K \subset \mathbb{D}$ is a neighborhood of zero. The infinitesimality of the array $\{\nu_{nk}\}_{n.k}$ implies that $S_{\nu_{nk}}(z)$ converges uniformly in k and $z \in K$ to 1 as $n \to \infty$, and hence for sufficiently large n, $\Sigma_{\nu_{nk}}(z)$ and the principal branch of $\log \Sigma_{\nu_{nk}}(z)$ are defined in $K' = \{z/(1+z) :\ z \in K\}$.

Fix a real number γ, and a finite positive Borel measure σ on $\mathbb{T}$.

Lemma 4.2. *Let* $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$. *Then*

$$\lim_{n\to\infty} \exp\left(-\log\lambda_n + \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(z)\right) = \Sigma_{\nu_{\boxtimes}^{\gamma,\sigma}}(z)$$

uniformly on K' *if, and only if*

$$\lim_{n\to\infty} \exp\left(-\log\lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]\right) = \Sigma_{\nu_{\boxtimes}^{\gamma,\sigma}}(z)$$

uniformly on K'.

Proof. From Proposition 2.2, we have the following approximation for the function $S_{\nu_{nk}^{\circ}}$:

$$S_{\nu_{nk}^{\circ}}(z) = 1 + \left[\int_{\mathbb{T}} \frac{1-t}{1+z-tz}\, d\overline{\nu}_{nk}(t)\right](1 + v_{nk}(z)), \qquad z \in K,$$

where

$$v_n(z) = \max_{1\le k\le k_n} |v_{nk}(z)|$$

satisfies $\lim_{n\to\infty} v_n(z) = 0$ uniformly in K. Introducing a change of variable $z \mapsto \frac{z}{1-z}$ and using the identity

$$\frac{(1-t)(1-z)}{1-tz} = -i\Im t + \frac{1+tz}{1-tz}(1-\Re t),$$

we conclude that

$$b_{nk}\Sigma_{\nu_{nk}}(z) = \Sigma_{\nu_{nk}^{\circ}}(z) = 1 + h_{nk}(z)\left(1 + v_{nk}\left(\frac{z}{1-z}\right)\right), \qquad z \in K'.$$

Lemmas 4.1 and 2.3 imply that for any sequence of purely imaginary numbers $\{r_n\}_{n=1}^{\infty}$, the sequence $\{r_n + \sum_{k=1}^{k_n}\left[\Sigma_{\nu_{nk}^{\circ}}(z) - 1\right]\}_{n=1}^{\infty}$ converges uniformly on K' if and only if the sequence $\{r_n + \sum_{k=1}^{k_n} h_{nk}(z)\}_{n=1}^{\infty}$ converges uniformly on K'. Moreover, two sequences have the same limit. Since $\log w/(w-1) \to 1$ as $w \to 1$, we can replace $\Sigma_{\nu_{nk}^{\circ}}(z) - 1$ by $\log \Sigma_{\nu_{nk}^{\circ}}(z)$. The result follows by choosing $r_n = -\log\lambda_n - \sum_{k=1}^{k_n} \log b_{nk}$. $\square$

Theorem 4.3. *For an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, the following assertions are equivalent:*

(1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ converges weakly to $\nu_{\boxtimes}^{\gamma,\sigma}$.*
(2) *The sequence of measures*

$$d\sigma_n(t) = \sum_{k=1}^{k_n} (1 - \Re t)\, d\nu_{nk}^{\circ}(t)$$

converges weakly on $\mathbb{T}$ to σ, and the limit

$$\lim_{n\to\infty} e^{i\gamma_n} = e^{i\gamma}$$

exists, where

$$\gamma_n = \arg \lambda_n + \sum_{k=1}^{k_n} \left[\int_{\mathbb{T}} \Im t \, d\nu_{nk}^{\circ}(t) + \arg b_{nk} \right].$$

Proof. Assume (1) holds. From Theorem 2.1, there exists $\varepsilon \in (0,1)$ such that all $\Sigma_{\nu_{nk}}$ are defined in $K'_{\varepsilon} = \{z/(1+z) : |z| \leq \varepsilon\}$, and we have

$$\lim_{n\to\infty} \left(\frac{1}{\lambda_n} \prod_{k=1}^{k_n} \Sigma_{\nu_{nk}}(z) \right) = \Sigma_{\nu_{\boxtimes}^{\gamma,\sigma}}(z) = e^{u_{\gamma,\sigma}(z)}$$

uniformly on K'_{ε}. Hence, by Lemma 4.2 and the definition of $u_{\gamma,\sigma}(z)$, we have

(4.1)

$$\lim_{n\to\infty} \exp\left(-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] \right) = \exp\left(-i\gamma + \int_{\mathbb{T}} \left[\frac{1+tz}{1-tz} \right] d\sigma(t) \right)$$

uniformly on K'_{ε}. Taking the absolute value on both sides of (4.1), we deduce that

$$\lim_{n\to\infty} \exp\left(\Re \left[\sum_{k=1}^{k_n} h_{nk}(z) \right] \right) = \exp\left(\int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma(t) \right) \tag{4.2}$$

uniformly on K'_{ε}. Note that

$$-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] = -i\gamma_n + \int_{\mathbb{T}} \left[\frac{1+tz}{1-tz} \right] d\sigma_n(t). \tag{4.3}$$

Moreover, the real part of the function $-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]$ is the Poisson integral of the measure $d\sigma_n\left(\frac{1}{t}\right)$ and hence (4.2) uniquely determines the weak cluster point of $\{\sigma_n\}_{n=1}^{\infty}$ which is σ. We therefore conclude the weak convergence of the sequence $\{\sigma_n\}_{n=1}^{\infty}$. Moreover, consider $z = 0$ in (4.1) and (4.2) to deduce that

$$\lim_{n\to\infty} \frac{e^{i\gamma}}{e^{i\gamma_n}} = 1,$$

as desired.

The converse implication is fairly easy now, since one can basically reverse the steps to reach the statement (1) by using Lemma 4.2 and the fact that $\{-\log\lambda_n + \sum_{k=1}^{k_n}[h_{nk}(z) - \log b_{nk}]\}_{n=1}^{\infty}$ is normal in $\mathbb{D}$. Therefore the details of the proof are left to the reader. $\square$

The previous result does not cover the possibility that the measures $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ might converge to Haar measure m. We address now this special case. Let us also note for further use the equality

$$\Sigma_\nu(0) = \frac{1}{\int_{\mathbb{T}} t\,d\nu(t)}, \qquad \nu \in \mathcal{M}_{\mathbb{T}}^{\times}.$$

Theorem 4.4. *For an infinitesimal array $\{\nu_{nk}\}_{n,k} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ and a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{T}$, the following assertions are equivalent:*

(1) *The sequence $\nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}$ converges weakly to m.*
(2)
$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \int_{\mathbb{T}} (1 - \Re t)\,d\nu_{nk}^{\circ}(t) = +\infty.$$

Proof. Assume (2) holds. Define

$$\nu_n = \nu_{n1} \boxtimes \nu_{n2} \boxtimes \cdots \boxtimes \nu_{nk_n} \boxtimes \delta_{\lambda_n}, \qquad n \in \mathbb{N}.$$

The compactness of $\mathbb{T}$ implies that $\{\nu_n\}_{n=1}^{\infty}$ is tight. Suppose ν is a weak cluster point of $\{\nu_n\}_{n=1}^{\infty}$. From the free multiplicative analogue of Hinčin's theorem (Theorem 2.1 in [2]), the measure ν is $\boxtimes$-infinitely divisible. By passing to a subsequence, we may assume that ν_n converges weakly to ν as $n \to \infty$. By (4.3), we can reformulate the statement (2) as follows:

$$\lim_{n\to\infty} \Re \sum_{k=1}^{k_n} h_{nk}(0) = +\infty.$$

Then the inequality (2.2) implies that

$$\lim_{n\to\infty} \Re \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(0) = +\infty,$$

and consequently we deduce that

$$\begin{aligned} \left|\int_{\mathbb{T}} t\,d\nu(t)\right| &= \lim_{n\to\infty} \left|\int_{\mathbb{T}} t\,d\nu_n(t)\right| \\ &= \lim_{n\to\infty} \frac{1}{\left|\prod_{k=1}^{k_n} \Sigma_{\nu_{nk}}(0)\right|} \\ &= \lim_{n\to\infty} \exp\left(-\Re \sum_{k=1}^{k_n} \log \Sigma_{\nu_{nk}}(0)\right) = 0. \end{aligned}$$

Therefore, the $\boxtimes$-infinitely divisible measure ν has zero first moment, and hence we conclude that $\nu = m$. Moreover, the full sequence ν_n converges weakly to m since $\{\nu_n\}_{n=1}^{\infty}$ has a unique weak cluster point m.

Conversely, assume (1) holds but (2) fails to be true. By passing, if necessary, to a subsequence, we may assume the sequence of measures

$$d\sigma_n(t) = \sum_{k=1}^{k_n} (1 - \Re t)\, d\nu_{nk}^{\circ}(t)$$

converges weakly to a finite positive Borel measure σ on $\mathbb{T}$. Since the sequence of functions $\{-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}]\}_{n=1}^{\infty}$ is normal in $\mathbb{D}$, we may assume, by passing to a further subsequence, that

$$\lim_{n\to\infty} \left(-\log \lambda_n + \sum_{k=1}^{k_n} [h_{nk}(z) - \log b_{nk}] \right) = f(z),$$

uniformly on compact subsets of $\mathbb{D}$, where the function f is analytic in $\mathbb{D}$. Note that the function f is not identically infinity since

$$\Re f(z) = \lim_{n\to\infty} \int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma_n(t) = \int_{\mathbb{T}} \Re \left[\frac{1+tz}{1-tz} \right] d\sigma(t), \qquad z \in \mathbb{D}.$$

Then, as in the proof of Theorem 4.3, we conclude that there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{n\to\infty} e^{i\gamma_n} = e^{i\gamma},$$

where the number γ_n is defined as in Theorem 4.3. An application of Theorem 4.3 then shows that a subsequence of $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to $\nu_{\boxtimes}^{\gamma,\sigma}$ which contradicts (1). Therefore (2) must be true. $\square$

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