

**INTEGRAL REPRESENTATIONS AND COMPLETE
MONOTONICITY RELATED TO THE REMAINDER OF
BURNSIDE'S FORMULA FOR THE GAMMA FUNCTION**

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ABSTRACT. In the paper, the authors establish integral representations of some functions related to the remainder of Burnside's formula for the gamma function and find the (logarithmically) complete monotonicity of these and related functions. These results extend and generalize some known conclusions.

1. MOTIVATION AND MAIN RESULTS

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in I$ and $n \geq 0$. A function f is said to be absolutely monotonic on an interval I if f has derivatives of all orders on I and $f^{(n)}(x) \geq 0$ for all $x \in I$ and $n \geq 0$. A positive function $f(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on I and its logarithm $\ln f(x)$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for all $k \in \mathbb{N}$ on I . For more information on these kinds of functions, please refer to the papers and monographs [7, 18, 27, 32, 36, 40, 44] and plenty of references cited therein.

It is well known that the classical Euler's gamma function may be defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (1.1)$$

for $x > 0$. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. The noted Binet's formula [17, p. 11] states that

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \theta(x) \quad (1.2)$$

for $x > 0$, where $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$ stands for Euler's gamma function and

$$\theta(x) = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-xt}}{t} dt \quad (1.3)$$

is called the remainder of Binet's formula (1.2). By the way, some functions related to the function $\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}$ in the formula (1.3) have been investigated, applied, and surveyed in [8, 9, 10, 12, 23, 25, 26, 37] and many references listed therein.

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For real numbers $p > 0$, $q \in \mathbb{R}$, and $r \neq 0$, define

$$f_{p,q,r}(x) = r[\theta(px) - q\theta(x)] \quad (1.4)$$

on $(0, \infty)$. In [4, 12] and [5, Section 5], the complete monotonicity of $f_{p,q,r}(x)$ and the star-shaped and subadditive properties of $\theta(x)$ were established.

In [6, p. 892] and [17, p. 17], it is given that

$$\psi(x) = \ln x - \frac{1}{2x} - 2 \int_0^\infty \frac{t \, dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \quad (1.5)$$

and

$$\psi\left(x + \frac{1}{2}\right) = \ln x + 2 \int_0^\infty \frac{t \, dt}{(t^2 + 4x^2)(e^{\pi t} + 1)} \quad (1.6)$$

for $x > 0$. For $p > 0$ and $q \in \mathbb{R}$, let

$$\Lambda_{p,q}(x) = \lambda(px) - q\lambda(x) \quad \text{and} \quad \Phi_{p,q}(x) = \phi(px) - q\phi(x) \quad (1.7)$$

on $(0, \infty)$, where

$$\lambda(x) = \int_0^\infty \frac{t \, dt}{(t^2 + x^2)(e^{2\pi t} - 1)} \quad \text{and} \quad \phi(x) = \int_0^\infty \frac{t \, dt}{(t^2 + 4x^2)(e^{\pi t} + 1)}. \quad (1.8)$$

In [38, 39], it was obtained that

- (1) the function $\Lambda_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ if
 - (a) either $q \leq 0$,
 - (b) or $0 < p < 1$ and $pq \leq 1$,
 - (c) or $0 < q = \frac{1}{p^2} \leq 1$;
- (2) the function $\Lambda_{p,q}(x)$ is negative and increasing in $x \in (0, \infty)$ if
 - (a) either $p \geq 1$ and $pq \geq 1$,
 - (b) or $\frac{1}{p^2} = q \geq 1$;
- (3) the function $\Phi_{p,q}(x)$ is positive and decreasing in $x \in (0, \infty)$ if
 - (a) either $p \geq 1$ and $q \leq 0$,
 - (b) or $0 < p < 1$ and $q \leq 1$,
 - (c) or $p^2q < 1$ and $q(p^2 - 1)[(1 + 3q)p^2 - 4] \leq 0$,
 - (d) or $p^2q = 1$ and $0 < q \leq 1$;
- (4) the function $\Phi_{p,q}(x)$ is negative and increasing in $x \in (0, \infty)$ if
 - (a) either $4 \leq p^2(1 + 3q) \leq 1 + 3q$,
 - (b) or $p > 1$ and $q \geq 1$.

In [31] and its preprint [30], some more properties on the remainder of Binet's formula (1.2) were further obtained.

Binet's formula (1.2) may be reformulated as

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{\theta(x)}. \quad (1.9)$$

We also call $\theta(x)$ the remainder of Stirling's formula

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \quad n \rightarrow \infty \quad (1.10)$$

established by James Stirling in 1764. When replacing $\theta(x)$ by $\frac{1}{12}\psi'(x+\alpha)$, it was proved in [21, 22, 41] that the function

$$F_\alpha(x) = \frac{e^x \Gamma(x+1)}{x^x \sqrt{2\pi x} e^{\psi'(x+\alpha)/12}} \quad (1.11)$$

is logarithmically completely monotonic on $(0, \infty)$ if and only if $\alpha \geq \frac{1}{2}$ and that the function $\frac{1}{F_\alpha(x)}$ is logarithmically completely monotonic on $(0, \infty)$ if and only if $\alpha = 0$. Consequently, the double inequality

$$\exp\left(\frac{1}{12}\psi'\left(x + \frac{1}{2}\right)\right) < \frac{e^x\Gamma(x+1)}{x^x\sqrt{2\pi x}} < \exp\left(\frac{1}{12}\psi'(x)\right) \quad (1.12)$$

was derived in [41, Corollary 2.1].

In [20], the following conclusions were obtained.

(1) As $n \rightarrow \infty$, the asymptotic formula

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n} \exp\left(\frac{1}{12}\psi'\left(n + \frac{1}{2}\right)\right) \quad (1.13)$$

is the most accurate one among all approximations of the form

$$n! \sim \frac{n^n}{e^n} \sqrt{2\pi n} \exp\left(\frac{1}{12}\psi'(n+a)\right), \quad (1.14)$$

where $a \in \mathbb{R}$;

(2) As $x \rightarrow \infty$, we have

$$\Gamma(x+1) \sim \sqrt{2\pi} x^{x+1/2} \exp\left(\frac{1}{12}\psi'\left(x + \frac{1}{2}\right) - x + \frac{1}{240x^3} - \frac{11}{6720x^5} + \frac{107}{80640x^7} - \frac{2911}{1520640x^9} + \dots\right); \quad (1.15)$$

(3) For every integer $x \geq 1$, we have

$$\exp\left(\frac{1}{240x^3} - \frac{11}{6720x^5}\right) < \frac{e^x\Gamma(x+1)}{x^x\sqrt{2\pi x} \exp\left(\frac{1}{12}\psi'\left(x + \frac{1}{2}\right)\right)} < \exp\left(\frac{1}{240x^3}\right). \quad (1.16)$$

For $\alpha \in \mathbb{R}$, let

$$g_\alpha(x) = \frac{e^x\Gamma(x+1)}{(x+\alpha)^{x+\alpha}} \quad (1.17)$$

on the interval $(\max\{0, -\alpha\}, \infty)$. In [11, 13], it was showed that the function $g_\alpha(x)$ is logarithmically completely monotonic if and only if $\alpha \geq 1$ and that the function $\frac{1}{g_\alpha(x)}$ is logarithmically completely monotonic if and only if $\alpha \leq \frac{1}{2}$.

In [2, Theorem 1], the double inequality

$$\begin{aligned} & \left(x - \frac{1}{2}\right) \left[\ln\left(x - \frac{1}{2}\right) - 1 \right] + \ln\sqrt{2\pi} - \frac{1}{24(x-1)} \leq \ln\Gamma(x) \\ & \leq \left(x - \frac{1}{2}\right) \left[\ln\left(x - \frac{1}{2}\right) - 1 \right] + \ln\sqrt{2\pi} - \frac{1}{24(\sqrt{x^2+x+1/2}-1/2)}, \quad x > 1 \end{aligned} \quad (1.18)$$

was obtained, which may be rewritten as

$$e^{-1/24x} < \frac{e^{x+1/2}\Gamma(x+1)}{\sqrt{2\pi}(x+1/2)^{x+1/2}} \leq e^{-1/24(\sqrt{x^2+3x+5/2}-1/2)}, \quad x > 0. \quad (1.19)$$

In [41, p. 1774, Theorem 2.3], the function

$$H(x) = \frac{e^{x+1/24(x+1/2)}\Gamma(x+1)}{(x+1/2)^{x+1/2}} \quad (1.20)$$

was proved to be logarithmically completely monotonic on $(0, \infty)$. Consequently, it was deduced in [41, Corollary 2.4] that the double inequality

$$\alpha_1 < \frac{e^{x+1/24(x+1/2)}\Gamma(x+1)}{(x+1/2)^{x+1/2}} \leq \alpha_2 \quad (1.21)$$

holds for $x > 0$ if and only if $\alpha_1 \leq \sqrt{\frac{2\pi}{e}}$ and $\alpha_2 \geq \sqrt{2}e^{1/12}$. It is clear that the left hand side inequality in (1.21) is stronger than the corresponding one in (1.19), but, when $x \geq 1$, the right hand side inequality in (1.21) is weaker than the corresponding one in (1.19). The double inequality in [15, Theorem 2] is weaker than (1.21). In [35], among other things, some necessary and sufficient conditions on $\lambda \geq 0$ for the function

$$H_\lambda(x) = \frac{e^{x+1/24(x+\lambda)}\Gamma(x+1)}{(x+1/2)^{x+1/2}} \quad (1.22)$$

to be logarithmically completely monotonic on $(0, \infty)$ were discovered.

For more information on inequalities for bounding the gamma function Γ and on the (logarithmically) complete monotonicity of functions involving Γ , please refer to the survey articles [24, 33, 34] and plenty of references collected therein.

When replacing $\theta(x)$ by $\frac{\vartheta(x)}{12x}$, Binet's formulas (1.2) and (1.9) become

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x e^{\vartheta(x)/12x}, \quad x > 0. \quad (1.23)$$

We call $\vartheta(x)$ the variant remainder of Stirling's formula. In [42], it was proved that the function $\vartheta(x)$ is strictly increasing on $[1, \infty)$. In [19], it was further proved that $\vartheta(x)$ is strictly decreasing on $(0, \beta)$ and strictly increasing on (β, ∞) , where $\beta = 0.34142\dots$ is the unique positive real number satisfying

$$\ln \Gamma(\beta+1) + \beta\psi(\beta+1) - \ln \sqrt{2\pi} - 2\beta \ln \beta + \beta = 0. \quad (1.24)$$

For $x > -\frac{1}{2}$, let

$$\Gamma(x+1) = \sqrt{2\pi} \left(\frac{x+1/2}{e}\right)^{x+1/2} e^{b(x)} \quad (1.25)$$

and

$$\Gamma(x+1) = \sqrt{2\pi} \left(\frac{x+1/2}{e}\right)^{x+1/2} e^{w(x)/12x}. \quad (1.26)$$

We call $b(x)$ and $w(x)$ the remainder of Burnside's formula and the variant remainder of Burnside's formula respectively.

$$n! \sim \sqrt{2\pi} \left(\frac{n+1/2}{e}\right)^{n+1/2}, \quad n \rightarrow \infty \quad (1.27)$$

established in [3].

In [43], it was proved that the functions $-b(x)$, $xb(x) + \frac{1}{24}$, and $w(x) + \frac{1}{2}$ are completely monotonic on $(\frac{1}{2}, \infty)$. It is clear that the complete monotonicity of $-b(x)$ on $(\frac{1}{2}, \infty)$ may be derived from the logarithmically complete monotonicity of the function $\frac{1}{g_\alpha(x)}$ on $(0, \infty)$.

The aim of this paper is to extend and generalize some results mentioned above. Our main results may be formulated as the following theorems.

Theorem 1.1. *The remainder $b(x)$ and the variant remainder $w(x)$ of Burnside's formula have the following properties.*

- (1) For $x > -\frac{1}{2}$, the remainders $b(x)$ and $w(x)$ of Burnside's formula have the integral representation

$$b(x) = \frac{w(x)}{12x} = \int_0^\infty \left(1 - \frac{e^t}{2t} + \frac{1}{e^{2t} - 1}\right) \frac{1}{t} e^{-2(x+1)t} dt \quad (1.28)$$

and the function $-b(x) = -\frac{w(x)}{12x}$ is completely monotonic on $(-\frac{1}{2}, \infty)$;

- (2) For $x > 0$, the function

$$xb(x) + \frac{1}{24} = \frac{1}{12} \left[w(x) + \frac{1}{2} \right] = \frac{1}{2} \int_0^\infty \frac{f_1(t)}{t^3(e^{2t} - 1)^2} e^{-(2x+1)t} dt \quad (1.29)$$

and is completely monotonic on $(0, \infty)$, where

$$f_1(t) = (t+2)e^{4t} - 2t(2t+1)e^{3t} - 2(t+2)e^{2t} + 2te^t + t + 2 \quad (1.30)$$

is absolutely monotonic on $(0, \infty)$;

- (3) For $x > 0$, the function

$$\frac{1}{6} - \frac{x}{3} - 8x^2b(x) = \frac{2}{3} \left[\frac{1}{4} - \frac{x}{2} - xw(x) \right] = \int_0^\infty \frac{f_2(t)}{t^4(e^{2t} - 1)^3} e^{-(2x+1)t} dt \quad (1.31)$$

and is completely monotonic on $(0, \infty)$, where

$$f_2(t) = (t^2 + 4t + 6)e^{6t} - 4t(2t^2 + 2t + 1)e^{5t} - 3(t^2 + 4t + 6)e^{4t} - 8t(t^2 - t - 1)e^{3t} + 3(t^2 + 4t + 6)e^{2t} - 4te^t - t^2 - 4t - 6 \quad (1.32)$$

is absolutely monotonic on $(0, \infty)$;

- (4) For $x > 0$, the function

$$16x^3b(x) + \frac{2}{3}x^2 - \frac{1}{3}x + \frac{23}{180} = \frac{4}{3} \left[x^2w(x) + \frac{1}{2}x^2 - \frac{1}{4}x + \frac{23}{240} \right] = \int_0^\infty \frac{f_3(t)}{t^5(e^{2t} - 1)^4} e^{-(2x+1)t} dt \quad (1.33)$$

and is completely monotonic on $(0, \infty)$, where

$$f_3(t) = (t^3 + 6t^2 + 18t + 24)e^{8t} - 4t(4t^3 + 6t^2 + 6t + 3)e^{7t} - 4(t^3 + 6t^2 + 18t + 24)e^{6t} - 4t(16t^3 - 12t - 9)e^{5t} + 6(t^3 + 6t^2 + 18t + 24)e^{4t} - 4t(4t^3 - 6t^2 + 6t + 9)e^{3t} - 4(t^3 + 6t^2 + 18t + 24)e^{2t} + 12te^t + t^3 + 6t^2 + 18t + 24 \quad (1.34)$$

is absolutely monotonic on $(0, \infty)$;

- (5) For $x > -\frac{1}{2}$, the function $(2x+1)b(x) + \frac{1}{12}$ has the integral representation

$$(2x+1)b(x) + \frac{1}{12} = \int_0^\infty \frac{h_1(t)}{(e^{2t} - 1)^2 t^3} e^{-(2x+1)t} dt \quad (1.35)$$

and is completely monotonic on $(-\frac{1}{2}, \infty)$, where

$$h_1(t) = e^{4t} - t(t+1)e^{3t} - 2e^{2t} - t(t-1)e^t + 1 \quad (1.36)$$

is absolutely monotonic on $(0, \infty)$;

(6) For $x > -\frac{1}{2}$, the function $-(x+1)b(x) - \frac{1}{24}$ has the integral representation

$$-(x+1)b(x) - \frac{1}{24} = \frac{1}{4} \int_0^\infty \frac{h_2(t)}{t^3(e^{2t}-1)^2} e^{-(2x+1)t} dt \quad (1.37)$$

and is completely monotonic on $(-\frac{1}{2}, \infty)$, where

$$h_2(t) = e^{4t}(t-2) + 2e^{3t}t - 2e^{2t}(t-2) + 2e^t t(2t-1) + t - 2 \quad (1.38)$$

is absolutely monotonic on $(0, \infty)$;

(7) For $x > -\frac{1}{2}$, the function $-(x+1)^2b(x) - \frac{1}{24}x - \frac{1}{16}$ has the integral representation

$$-(x+1)^2b(x) - \frac{1}{24}x - \frac{1}{16} = \frac{1}{8} \int_0^\infty \frac{h_3(t)}{(e^{2t}-1)^3 t^4} e^{-(2x+1)t} dt \quad (1.39)$$

and is completely monotonic on $(-\frac{1}{2}, \infty)$, where

$$h_3(t) = e^{6t}(t^2 - 4t + 6) - 4e^{5t}t - 3e^{4t}(t^2 - 4t + 6) - 8e^{3t}(t^2 + t - 1)t + 3e^{2t}(t^2 - 4t + 6) - 4e^t(2t^2 - 2t + 1)t - t^2 + 4t - 6 \quad (1.40)$$

is absolutely monotonic on $(0, \infty)$;

(8) For $x > -\frac{1}{2}$, the function $-(x+1)^3b(x) - \frac{x^2}{24} - \frac{5x}{48} - \frac{203}{2880}$ has the integral representation

$$-(x+1)^3b(x) - \frac{x^2}{24} - \frac{5x}{48} - \frac{203}{2880} = \int_0^\infty \frac{h_4(t)}{(e^{2t}-1)^4 t^5} e^{-(2x+1)t} dt \quad (1.41)$$

and is completely monotonic on $(-\frac{1}{2}, \infty)$, where

$$h_4(t) = e^{8t}(t^3 - 6t^2 + 18t - 24) + 12e^{7t}t - 4e^{6t}(t^3 - 6t^2 + 18t - 24) + 4e^{5t}(4t^3 + 6t^2 + 6t - 9)t + 6e^{4t}(t^3 - 6t^2 + 18t - 24) + 4e^{3t}(16t^3 - 12t + 9)t - 4e^{2t}(t^3 - 6t^2 + 18t - 24) + 4e^t(4t^3 - 6t^2 + 6t - 3)t + t^3 - 6t^2 + 18t - 24 \quad (1.42)$$

is absolutely monotonic on $(0, \infty)$.

Theorem 1.2. *The functions*

$$\left[\frac{\Gamma(x+1)}{\sqrt{2\pi}} \left(\frac{e}{x+1/2} \right)^{x+1/2} \right]^x, \quad \left[\frac{\sqrt{2\pi}}{\Gamma(x+1)} \left(\frac{x+1/2}{e} \right)^{x+1/2} \right]^{8x^2} e^{-x/3},$$

and

$$\left[\frac{\Gamma(x+1)}{\sqrt{2\pi}} \left(\frac{e}{x+1/2} \right)^{x+1/2} \right]^{16x^3} e^{x(2x-1)/3}$$

are logarithmically completely monotonic on $(0, \infty)$.

The functions

$$\frac{\sqrt{2\pi}}{\Gamma(x+1)} \left(\frac{x+1/2}{e} \right)^{x+1/2}, \quad \left[\frac{\Gamma(x+1)}{\sqrt{2\pi}} \left(\frac{e}{x+1/2} \right)^{x+1/2} \right]^{2x+1},$$

$$\left[\frac{\sqrt{2\pi}}{\Gamma(x+1)} \left(\frac{x+1/2}{e} \right)^{x+1/2} \right]^{x+1}, \quad \left[\frac{\sqrt{2\pi}}{\Gamma(x+1)} \left(\frac{x+1/2}{e} \right)^{x+1/2} \right]^{(x+1)^2} e^{-x/24},$$

and

$$\left[\frac{\sqrt{2\pi}}{\Gamma(x+1)} \left(\frac{x+1/2}{e} \right)^{x+1/2} \right]^{(x+1)^3} e^{-(2x+5)/48}$$

are logarithmically completely monotonic on $(-\frac{1}{2}, \infty)$.

2. PROOFS OF MAIN RESULTS

Now we start out to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From (1.25) and (1.26), it follows that

$$w(x) = 12xb(x) = 12x \left[\ln \Gamma(x+1) - \frac{1}{2} \ln(2\pi) - \left(x + \frac{1}{2}\right) \ln \left(x + \frac{1}{2}\right) + x + \frac{1}{2} \right]. \quad (2.1)$$

By Binet's formula (1.2), we have

$$\ln \Gamma(x+1) = \left(x + \frac{1}{2}\right) \ln(x+1) - x - 1 + \ln \sqrt{2\pi} + \theta(x+1).$$

Substituting this into (2.1) results in

$$b(x) = \frac{w(x)}{12x} = \frac{1}{2} \left[(2x+1) \ln \left(1 + \frac{1}{2x+1}\right) - 1 \right] + \theta(x+1). \quad (2.2)$$

It was listed in [40, pp. 322–323, Entry 46] that

$$x^2 \ln \left(1 + \frac{1}{x}\right) - x + \frac{1}{2} = \int_0^\infty \frac{2 - (t^2 + 2t + 2)e^{-t}}{t^3} e^{-xt} dt, \quad x > 0. \quad (2.3)$$

Making use of (2.3) in (2.2) produces

$$\begin{aligned} b(x) &= \frac{1}{2(2x+1)} \left[\int_0^\infty \frac{2 - (t^2 + 2t + 2)e^{-t}}{t^3} e^{-(2x+1)t} dt - \frac{1}{2} \right] + \theta(x+1) \\ &= \frac{1}{2(2x+1)} \left\{ \int_0^\infty \frac{d}{dt} \left[\frac{e^{-t}(1+t-e^t)}{t^2} \right] e^{-(2x+1)t} dt - \frac{1}{2} \right\} + \theta(x+1) \\ &= \frac{1}{2(2x+1)} \left[\frac{e^{-t}(1+t-e^t)}{t^2} e^{-(2x+1)t} \Big|_{t=0}^{t=\infty} \right. \\ &\quad \left. + (2x+1) \int_0^\infty \frac{e^{-t}(1+t-e^t)}{t^2} e^{-(2x+1)t} dt - \frac{1}{2} \right] + \theta(x+1) \\ &= \frac{1}{2} \int_0^\infty \frac{1+t-e^t}{t^2} e^{-2(x+1)t} dt + \theta(x+1) \\ &= \frac{1}{2} \int_0^\infty \frac{1+t-e^t}{t^2} e^{-2(x+1)t} dt + \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) \frac{e^{-(x+1)t}}{t} dt \\ &= \frac{1}{2} \int_0^\infty \frac{1+t-e^t}{t^2} e^{-2(x+1)t} dt + \int_0^\infty \left(\frac{1}{e^{2t}-1} - \frac{1}{2t} + \frac{1}{2} \right) \frac{e^{-2(x+1)t}}{t} dt \\ &= \int_0^\infty \left[\frac{1+t-e^t}{2t} + \left(\frac{1}{e^{2t}-1} - \frac{1}{2t} + \frac{1}{2} \right) \right] \frac{e^{-2(x+1)t}}{t} dt \\ &= \int_0^\infty \left(1 - \frac{e^t}{2t} + \frac{1}{e^{2t}-1} \right) \frac{e^{-t}}{t} e^{-(2x+1)t} dt. \end{aligned}$$

Since the function

$$1 - \frac{e^t}{2t} + \frac{1}{e^{2t}-1} = -\frac{e^t(e^{2t}-2te^t-1)}{2t(e^{2t}-1)} < 0$$

on $(0, \infty)$, from the integral representation (1.28) of $b(x)$, it may be easily deduced that the function $-b(x)$ is completely monotonic on $(-\frac{1}{2}, \infty)$.

Utilizing the integral representation (1.28) of $b(x)$ and integrating by parts reveal

$$\begin{aligned}
-xb(x) &= \frac{1}{2} \int_0^\infty \left(1 - \frac{e^t}{2t} + \frac{1}{e^{2t} - 1}\right) \frac{e^{-2t}}{t} \frac{d e^{-2xt}}{d t} d t \\
&= \frac{1}{2} \left[\frac{1}{12} - \int_0^\infty \frac{e^{-t} f_1(t)}{2t^3(e^{2t} - 1)^2} e^{-2xt} d t \right] \\
&= \frac{1}{2} \left[\frac{1}{12} + \frac{1}{4x} \int_0^\infty \frac{e^{-t} f_1(t)}{t^3(e^{2t} - 1)^2} \frac{d e^{-2xt}}{d t} d t \right] \\
&= \frac{1}{2} \left\{ \frac{1}{12} + \frac{1}{4x} \left[-\frac{1}{6} + \int_0^\infty \frac{e^{-t} f_2(t)}{t^4(e^{2t} - 1)^3} e^{-2xt} d t \right] \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{12} + \frac{1}{4x} \left[-\frac{1}{6} - \frac{1}{2x} \int_0^\infty \frac{e^{-t} f_2(t)}{t^4(e^{2t} - 1)^3} \frac{d e^{-2xt}}{d t} d t \right] \right\} \\
&= \frac{1}{2} \left\{ \frac{1}{12} + \frac{1}{4x} \left[-\frac{1}{6} - \frac{1}{2x} \left(-\frac{23}{180} + \int_0^\infty \frac{e^{-t} f_3(t)}{t^5(e^{2t} - 1)^4} e^{-2xt} d t \right) \right] \right\}.
\end{aligned}$$

By straightforward computation, we have

$$\begin{aligned}
f_1'(t) &= e^{4t}(4t + 9) - 2e^{3t}(6t^2 + 7t + 1) - 2e^{2t}(2t + 5) + 2e^t(t + 1) + 1 \\
&\rightarrow 0 \quad \text{as } t \rightarrow 0, \\
f_1''(t) &= 2e^t[4e^{3t}(2t + 5) - e^{2t}(18t^2 + 33t + 10) - 4e^t(t + 3) + t + 2] \\
&\triangleq 2e^t f_{11}(t) \\
&\rightarrow 0 \quad \text{as } t \rightarrow 0, \\
f_{11}'(t) &= 4e^{3t}(6t + 17) - e^{2t}(36t^2 + 102t + 53) - 4e^t(t + 4) + 1 \\
&\rightarrow 0 \quad \text{as } t \rightarrow 0, \\
f_{11}''(t) &= 4e^t[3e^{2t}(6t + 19) - e^t(18t^2 + 69t + 52) - t - 5] \\
&\triangleq 4e^t f_{12}(t) \\
&\rightarrow 0 \quad \text{as } t \rightarrow 0, \\
f_{12}'(t) &= 12e^{2t}(3t + 11) - e^t(18t^2 + 105t + 121) - 1 \\
&\rightarrow 10 \quad \text{as } t \rightarrow 0, \\
f_{12}''(t) &= e^t[12e^t(6t + 25) - 18t^2 - 141t - 226] \\
&\triangleq e^t f_{13}(t) \\
&\rightarrow 74 \quad \text{as } t \rightarrow 0, \\
f_{13}'(t) &= 3[4e^t(6t + 31) - 12t - 47] \\
&\rightarrow 231 \quad \text{as } t \rightarrow 0, \\
f_{13}''(t) &= 12[e^t(6t + 37) - 3] \\
&\rightarrow 408 \quad \text{as } t \rightarrow 0, \\
f_{13}'''(t) &= 12e^t(6t + 43).
\end{aligned}$$

As a result, since the product of finitely many absolutely monotonic functions is still absolutely monotonic, the function $f_1(t)$ is absolutely monotonic on $(0, \infty)$. This means that the function $xb(x) + \frac{1}{24}$ is completely monotonic on $(0, \infty)$.

By direct calculation, we have

$$\begin{aligned} f_2'(t) &= e^{6t}(6t^2 + 26t + 40) - 4e^{5t}(10t^3 + 16t^2 + 9t + 1) - 6e^{4t}(2t^2 + 9t + 14) \\ &\quad - e^{3t}(24t^3 - 40t - 8) + 6e^{2t}(t^2 + 5t + 8) - 4e^t(t + 1) - 2(t + 2) \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0, \end{aligned}$$

$$\begin{aligned} f_2''(t) &= 2[e^{6t}(18t^2 + 84t + 133) - 2e^{5t}(50t^3 + 110t^2 + 77t + 14) \\ &\quad - 3e^{4t}(8t^2 + 40t + 65) - 4e^{3t}(9t^3 + 9t^2 - 15t - 8) \\ &\quad + e^{2t}(6t^2 + 36t + 63) - 2e^t(t + 2) - 1] \\ &\rightarrow 0 \quad \text{as } t \rightarrow 0, \end{aligned}$$

$$\begin{aligned} f_2'''(t) &= 4e^t[9e^{5t}(6t^2 + 30t + 49) - e^{4t}(250t^3 + 700t^2 + 605t + 147) \\ &\quad - 6e^{3t}(8t^2 + 44t + 75) - 6e^{2t}(9t^3 + 18t^2 - 9t - 13) \\ &\quad + e^t(6t^2 + 42t + 81) - t - 3] \end{aligned}$$

$$\triangleq 4e^t f_{21}(t)$$

$$\rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{21}'(t) &= 9e^{5t}(30t^2 + 162t + 275) - e^{4t}(1000t^3 + 3550t^2 + 3820t + 1193) \\ &\quad - 6e^{3t}(24t^2 + 148t + 269) - 6e^{2t}(18t^3 + 63t^2 + 18t - 35) \\ &\quad + 3e^t(2t^2 + 18t + 41) - 1 \end{aligned}$$

$$\rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{21}''(t) &= e^t[9e^{4t}(150t^2 + 870t + 1537) - 4e^{3t}(1000t^3 + 4300t^2 + 5595t + 2148) \\ &\quad - 6e^{2t}(72t^2 + 492t + 955) - 12e^t(18t^3 + 90t^2 + 81t - 26) \\ &\quad + 6t^2 + 66t + 177] \end{aligned}$$

$$\triangleq e^t f_{22}(t)$$

$$\rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{22}'(t) &= 2[9e^{4t}(300t^2 + 1890t + 3509) - 2e^{3t}(3000t^3 + 15900t^2 + 25385t \\ &\quad + 12039) - 6e^{2t}(72t^2 + 564t + 1201) - 6e^t(18t^3 + 144t^2 + 261t + 55) \\ &\quad + 6t + 33] \end{aligned}$$

$$\rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{22}''(t) &= 4[9e^{4t}(600t^2 + 4080t + 7963) - e^{3t}(9000t^3 + 56700t^2 \\ &\quad + 107955t + 61502) - 6e^{2t}(72t^2 + 636t + 1483) \\ &\quad - 3e^t(18t^3 + 198t^2 + 549t + 316) + 3] \end{aligned}$$

$$\rightarrow 1288 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{22}'''(t) &= 12e^t[12e^{3t}(600t^2 + 4380t + 8983) - e^{2t}(9000t^3 + 65700t^2 + 145755t \\ &\quad + 97487) - 4e^t(72t^2 + 708t + 1801) - 18t^3 - 252t^2 - 945t - 865] \end{aligned}$$

$$\triangleq 12e^t f_{23}(t)$$

$$\rightarrow 26880 \quad \text{as } t \rightarrow 0,$$

$$\begin{aligned} f_{23}'(t) &= 36e^{3t}(600t^2 + 4780t + 10443) - e^{2t}(18000t^3 + 158400t^2 + 422910t \\ &\quad + 340729) - 4e^t(72t^2 + 852t + 2509) - 9(6t^2 + 56t + 105) \end{aligned}$$

$$\begin{aligned}
&\rightarrow 24238 \quad \text{as } t \rightarrow 0, \\
f''_{23}(t) &= 4[9e^{3t}(1800t^2 + 15540t + 36109) - e^{2t}(9000t^3 + 92700t^2 + 290655t \\
&\quad + 276092) - e^t(72t^2 + 996t + 3361) - 9(3t + 14)] \\
&\rightarrow 181608 \quad \text{as } t \rightarrow 0, \\
f'''_{23}(t) &= 4[81e^{3t}(600t^2 + 5580t + 13763) - e^{2t}(18000t^3 + 212400t^2 + 766710t \\
&\quad + 842839) - e^t(72t^2 + 1140t + 4357) - 27] \\
&\rightarrow 1070320 \quad \text{as } t \rightarrow 0, \\
f^{(4)}_{23}(t) &= 4e^t[243e^{2t}(600t^2 + 5980t + 15623) - 4e^t(9000t^3 + 119700t^2 + 489555t \\
&\quad + 613097) - 72t^2 - 1284t - 5497] \\
&\triangleq 4e^t f_{24}(t) \\
&\rightarrow 5354016 \quad \text{as } t \rightarrow 0, \\
f'_{24}(t) &= 2[243e^{2t}(600t^2 + 6580t + 18613) - 2e^t(9000t^3 + 146700t^2 + 728955t \\
&\quad + 1102652) - 72t - 642] \\
&\rightarrow 4634026 \quad \text{as } t \rightarrow 0, \\
f''_{24}(t) &= 4[243e^{2t}(600t^2 + 7180t + 21903) - e^t(9000t^3 + 173700t^2 \\
&\quad + 1022355t + 1831607) - 36] \\
&\rightarrow 13963144 \quad \text{as } t \rightarrow 0, \\
f'''_{24}(t) &= 4e^t[486e^t(600t^2 + 7780t + 25493) - 9000t^3 - 200700t^2 \\
&\quad - 1369755t - 2853962] \\
&\triangleq 4e^t f_{25}(t) \\
&\rightarrow 38142544 \quad \text{as } t \rightarrow 0, \\
f'_{25}(t) &= 9[54e^t(600t^2 + 8980t + 33273) - 5(600t^2 + 8920t + 30439)] \\
&\rightarrow 14800923 \quad \text{as } t \rightarrow 0, \\
f''_{25}(t) &= 18[27e^t(600t^2 + 10180t + 42253) - 100(30t + 223)] \\
&\rightarrow 20133558 \quad \text{as } t \rightarrow 0, \\
f'''_{25}(t) &= 54[9e^t(600t^2 + 11380t + 52433) - 1000] \\
&\rightarrow 25428438 \quad \text{as } t \rightarrow 0, \\
f^{(4)}_{25}(t) &= 486e^t(600t^2 + 12580t + 63813).
\end{aligned}$$

Therefore, by the fact that the product of finitely many absolutely monotonic functions is still absolutely monotonic, we see that the function $f_2(t)$ is absolutely monotonic on $(0, \infty)$. This implies that the function $\frac{1}{6} - \frac{x}{3} - 8x^2b(x)$ is completely monotonic on $(0, \infty)$.

By similar arguments to proofs of the absolute monotonicity of f_1 and f_2 , we may verify that the function f_3 is also absolutely monotonic on $(0, \infty)$. Consequently, the function $16x^3b(x) + \frac{2}{3}x^2 - \frac{1}{3}x + \frac{23}{180}$ is completely monotonic on $(0, \infty)$.

Employing the integral representation (1.28) and integrating by parts gives

$$-(2x+1)b(x) = \int_0^\infty \left(1 - \frac{e^t}{2t} + \frac{1}{e^{2t}-1}\right) \frac{e^{-t}}{t} \frac{d e^{-(2x+1)t}}{d t} dt$$

$$= \frac{1}{12} - \int_0^\infty \frac{h_1(t)}{(e^{2t}-1)^2 t^3} e^{-(2x+1)t} dt.$$

The verification of the absolute monotonicity of $h_1(t)$ is same as that of f_1 and f_2 . Accordingly, the function $(2x+1)b(x) + \frac{1}{12}$ is completely monotonic on $(-\frac{1}{2}, \infty)$.

Using the integral representation (1.28) and integrating by parts acquires

$$\begin{aligned} -2(x+1)b(x) &= \int_0^\infty \left(1 - \frac{e^t}{2t} + \frac{1}{e^{2t}-1}\right) \frac{1}{t} \frac{d e^{-(x+1)t}}{dt} dt \\ &= \frac{1}{12} + \frac{1}{2} \int_0^\infty \frac{e^t h_2(t)}{t^3 (e^{2t}-1)^2} e^{-2(x+1)t} dt \\ &= \frac{1}{12} - \frac{1}{4(x+1)} \int_0^\infty \frac{e^t h_2(t)}{t^3 (e^{2t}-1)^2} \frac{d e^{-2(x+1)t}}{dt} dt \\ &= \frac{1}{12} - \frac{1}{4(x+1)} \left[-\frac{1}{6} - \int_0^\infty \frac{e^t h_3(t)}{(e^{2t}-1)^3 t^4} e^{-2(x+1)t} dt \right] \\ &= \frac{1}{12} - \frac{1}{4(x+1)} \left[-\frac{1}{6} + \frac{1}{2(x+1)} \int_0^\infty \frac{e^t h_3(t)}{(e^{2t}-1)^3 t^4} \frac{d e^{-2(x+1)t}}{dt} dt \right] \\ &= \frac{1}{12} - \frac{1}{4(x+1)} \left\{ -\frac{1}{6} + \frac{1}{2(x+1)} \left[-\frac{23}{180} - \int_0^\infty \frac{e^t h_4(t)}{(e^{2t}-1)^4 t^5} e^{-2(x+1)t} dt \right] \right\}. \end{aligned}$$

By similar arguments to proofs of the absolute monotonicity of f_1 and f_2 , we may verify that the functions h_2 , h_3 , and h_4 are absolutely monotonic on $(0, \infty)$. Consequently, the functions $-(x+1)b(x) - \frac{1}{24}$, $-(x+1)^2 b(x) - \frac{1}{24}x - \frac{1}{16}$, and $-(x+1)^3 b(x) - \frac{x^2}{24} - \frac{5x}{48} - \frac{203}{2880}$ are completely monotonic on $(-\frac{1}{2}, \infty)$. The proof of Theorem 1.1 is complete. \square

Proof of Theorem 1.2. This follows from reformulation of the functions involving the remainder $b(x)$ in Theorem 1.1. \square

3. REMARKS

For better understanding our main results, we list several remarks as follows.

Remark 3.1. In [18, 40, 44], we may find the classical Bernstein-Widder theorem which reads that a function f is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of some nonnegative measure μ , that is,

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \quad (3.1)$$

where $\mu(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. In [1, 7, 28, 29, 40], we may search out that any logarithmically completely monotonic function must be a completely monotonic function, but not conversely. To some extent, these reveal the significance and meanings of our main results.

Remark 3.2. By the monotonicity in Theorem 1.2, we may derive some double inequalities for bounding the function

$$\frac{\Gamma(x+1)}{\sqrt{2\pi}} \left(\frac{e}{x+1/2} \right)^{x+1/2} \quad (3.2)$$

or its reciprocal on the intervals $(0, \infty)$ and $(-\frac{1}{2}, \infty)$. These inequalities would be better than (1.19) and (1.21).

Remark 3.3. From (1.23) and (1.26), it follows that

$$\sqrt{x} \left(\frac{x}{e}\right)^x e^{\vartheta(x)/12x} = \left(\frac{x+1/2}{e}\right)^{x+1/2} e^{w(x)/12x}$$

which may be rewritten as

$$\vartheta(x) = w(x) - 6x + 12x \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{2x}\right) \quad (3.3)$$

and

$$\theta(x) = b(x) + \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{2x}\right) - \frac{1}{2}. \quad (3.4)$$

Combining these two identities with Theorem 1.1, we may deduce the complete monotonicity of some functions related to the remainder $\theta(x)$ and the variant remainder $\vartheta(x)$ of Binet's formula. Furthermore, we may deduce some double inequalities for bounding the function

$$\frac{e^x \Gamma(x+1)}{x^x \sqrt{2\pi x}} \quad (3.5)$$

or its reciprocal on the interval $(0, \infty)$.

Remark 3.4. In [14, Theorem 1.2], it was given that the inequality

$$\Gamma(x+1) < \sqrt{2\pi} \left(\frac{x+1/2}{e}\right)^{x+1/2} \left[1 - \frac{k}{24x} + \left(\frac{k^2}{1152} + \frac{k}{48}\right) \frac{1}{x^2}\right]^{1/k} \quad (3.6)$$

is valid for $x \geq m_1$ and for any positive integer k , where $m_1 \geq 0$ is a constant depending on k . To compare the right hand side inequality in (1.21) with (3.6), it suffices to discuss the inequality

$$\frac{e^{7/12}}{\sqrt{\pi}} \leq e^{1/24(x+1/2)} \left[1 - \frac{k}{24x} + \left(\frac{k^2}{1152} + \frac{k}{48}\right) \frac{1}{x^2}\right]^{1/k}. \quad (3.7)$$

If letting $x \rightarrow \infty$, the above inequality becomes $e^{7/12} = 1.79\dots \leq \sqrt{\pi} = 1.77\dots$. This contradiction implies that, when x is sufficiently large, the inequality (3.6) is better than the right hand side inequality in (1.21).

In [16, Theorem 1.2], it was also deduced that the inequality

$$\Gamma(x+1) < \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(1 + \frac{k}{12x} + \frac{k^2}{288x^2}\right)^{1/k} \quad (3.8)$$

holds for $x \geq m_1$ and $1 \leq k \leq 5$ and reverses for $x \geq m_2$ and $k \geq 6$, where m_1 and m_2 are constants depending on k .

Because the inequalities (3.6) and (3.8) are derived from an asymptotic approximation of the gamma function $\Gamma(x+1)$, they may be more accurate when x is sufficiently large, but not, even invalid, when x is close to zero.

Remark 3.5. Now we consider the functions

$$F(x) = 1 + 4x - 8x \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{2x}\right) \quad (3.9)$$

and

$$G(x) = \left(x + \frac{1}{2}\right) \ln \left(1 + \frac{1}{2x}\right) - \frac{1}{2} \quad (3.10)$$

appeared in (3.3) and (3.4). It is obvious that $F(x) = 1 - 8xG(x)$.

We claim that the functions $F(x)$ and $G(x)$ are completely monotonic on $(0, \infty)$. This may be verified as follows. A direct computation gives

$$\begin{aligned} F'(x) &= 4 \left[2 - (4x + 1) \ln \left(1 + \frac{1}{2x} \right) \right], \\ F''(x) &= \frac{4(4x + 1)}{x(2x + 1)} - 16 \ln \left(1 + \frac{1}{2x} \right), \\ F'''(x) &= -\frac{4}{x^2(2x + 1)^2}. \end{aligned}$$

It is clear that $\lim_{x \rightarrow \infty} F''(x) = 0$. It is not difficult to see that

$$F'(x) = 4 \left[2 - 2 \ln \left(1 + \frac{1}{2x} \right)^{2x} - \ln \left(1 + \frac{1}{2x} \right) \right] \rightarrow 4(2 - 2 \ln e - 0) = 0$$

and

$$\begin{aligned} F(x) &= 1 - 2 \ln \left(1 + \frac{1}{2x} \right)^{2x} + 8x^2 \left[\frac{1}{2x} - \ln \left(1 + \frac{1}{2x} \right) \right] \\ &\rightarrow 1 - 2 \ln e + 2 \lim_{u \rightarrow 0^+} \frac{u - \ln(1 + u)}{u^2} \\ &= 0 \end{aligned}$$

as $x \rightarrow \infty$, where $u = \frac{1}{2x}$. Hence, by virtue of $F'''(x) > 0$ on $(0, \infty)$, we may conclude $F(x) > 0$, $F'(x) < 0$, and $F''(x) > 0$. Furthermore, by the facts that the functions $\frac{1}{x}$ and $\frac{1}{1+2x}$ are completely monotonic on $(0, \infty)$ and that the product of finitely many completely monotonic functions is also a completely monotonic function, we may see that the function $-F'''(x)$ is completely monotonic on $(0, \infty)$. In a word, the function $F(x)$ is completely monotonic on $(0, \infty)$.

The complete monotonicity of $G(x)$ may also be verified straightforwardly.

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