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Numerical expansion-iterative method for solving second kind Volterra and Fredholm integral equations using block-pulse functions

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ABSTRACT

This paper presents a numerical expansion-iterative method for solving linear Volterra and Fredholm integral equations of the second kind. The method is based on vector forms of block-pulse functions and their operational matrix. By using this approach, solving the second kind integral equation reduces to solve a recurrence relation. The approximate solution is most easily produced iteratively via the recurrence relation. Therefore, computing the numerical solution does not need to directly solve any linear system of algebraic equations and to use any matrix inversion. Moreover, this approach does not use any projection method such as collocation, Galerkin, etc., for setting up the recurrence relation. To show convergence and stability of the method, some computable error bounds are obtained, and some test problems are provided to illustrate its accuracy and computational efficiency.

Keywords: Second kind integral equation; Block-pulse functions; Vector forms; Operational matrix; Expansion method; Iterative method; Error Analysis.

MSC2010 codes: 65R20; 45A05; 41A30.

1. Introduction

Many problems in physical sciences and engineering are modeled by linear integral equations. A wide variety of numerical methods have been developed to solve such equations by using various basis functions. Some of these methods may, for example, be found in [1–8].

In recent years, a great deal of interest has been focused on using the block-pulse functions (BPFs) for solving various functional equations. For example, [9] and [10] present numerical methods for the solution of Fredholm integral equations of the first and second kinds, respectively. Two numerical direct methods based on vector forms of BPFs have been formulated in [11] and [12] to solve first kind Volterra and first or second kind Fredholm integral equations. BPFs are also used in [13] and [14] for constructing two numerical expansion-iterative methods for solving first kind Volterra and Fredholm integral equations. A part of the work done in the last two papers is based on numerical interpretations of the Neumann series concept.

This article proposes an expansion-iterative method for numerical solution of Volterra and Fredholm integral equations of the second kind. This approach uses the vector forms of BPFs. Using the method, solving the integral equation reduces to solve a recurrence relation. The advantages of the presented method are as follows:

- This approach does not use any projection method such as collocation, Galerkin, etc., for setting up the recurrence relation.
- Computing the numerical solution does not need to directly solve any linear system of algebraic equations and to use any matrix inversion.
- Reasonable accuracy according to the numerical results.

With emphasis that some basic concepts required for formulating the method presented here are similar to those presented in [13] and [14], we organize this article as follows. A brief review on BPFs and their vector forms is provided in section 2. Section 3 presents the expansion-iterative method for numerically solving Volterra and Fredholm integral equations of the second kind. Error analysis and convergence evaluation of the method is the subject of section 4 where some computable error bounds are obtained. Section 5 includes some test problems to confirm the accuracy and computational efficiency of the proposed approach. Finally, conclusions will be in section 6.

2. Block-pulse functions and their vector forms

In this section, the definition of BPFs, their vector forms and operational matrix of integration, together with other properties are reviewed.

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2.1. Definition

An m–set of BPFs is defined over the interval $[0, H)$ as [11–14]

$$\varphi_i(t) = \begin{cases} 1, & \frac{iH}{m} \leqslant t < \frac{(i+1)H}{m}, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $i = 0, 1, \ldots, m-1$, with a positive integer value for m. Also, consider $h = H/m$, and φ_i is the ith BPF. For convenience, it is assumed that $H = 1$, so BPFs are defined over $[0, 1)$, and $h = 1/m$.

There are some properties for BPFs, the most important properties are disjointness, orthogonality, and completeness.

2.2. Vector forms

Consider the first m terms of BPFs and write them concisely as an m–vector

$$\Phi(t) = [\varphi_0(t), \varphi_1(t), \ldots, \varphi_{m-1}(t)]^T, \qquad t \in [0, 1), \tag{2}$$

where superscript T indicates transposition. Above representation and disjointness property follows

$$\Phi(t)\Phi^T(t)V = \tilde{V}\Phi(t), \tag{3}$$

where V is an m–vector and $\tilde{V} = \operatorname{diag}(V)$. Moreover, it can be clearly concluded that for any $m \times m$ matrix B

$$\Phi^T(t)B\Phi(t) = \hat{B}^T\Phi(t), \tag{4}$$

where $\hat{B}$ is an m–vector with elements equal to the diagonal entries of matrix B. Also,

$$\int_0^1 \Phi(t)\,\mathrm{d}t = [h, h, \ldots, h]^T = \bar{h}, \tag{5}$$

and

$$\int_0^1 \Phi(t)\Phi^T(t)\,\mathrm{d}t = hI, \tag{6}$$

where I is $m \times m$ identity matrix.

2.3. BPFs expansion

The expansion of a function f over $[0, 1)$ with respect to φ_i, $i = 0, 1, \ldots, m-1$, may be compactly written as [11, 12]

$$f(t) \simeq \sum_{i=0}^{m-1} f_i\varphi_i(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{7}$$

where $F = [f_0, f_1, \ldots, f_{m-1}]^T$ and f_i's are defined by

$$f_i = \frac{1}{h}\int_0^1 f(t)\varphi_i(t)\,\mathrm{d}t. \tag{8}$$

Now, assume k is a function of two variables in $\mathscr{L}^2\big([0, 1) \times [0, 1)\big)$, where $\mathscr{L}^2$ is the space of square integrable functions. It can be similarly expanded with respect to BPFs so that

$$k(s, t) \simeq \Phi^T(s)K\Psi(t), \tag{9}$$

where Φ and Ψ are m_1– and m_2–dimensional BPF vectors respectively, and K is the $m_1 \times m_2$ block-pulse coefficient matrix with $k_{i,j}$, $i = 0, 1, \ldots, m_1 - 1$, $j = 0, 1, \ldots, m_2 - 1$, as follows:

$$k_{i,j} = m_1 m_2 \int_0^1 \int_0^1 k(s, t)\varphi_i(s)\psi_j(t)\,\mathrm{d}s\,\mathrm{d}t. \tag{10}$$

For convenience, we put $m_1 = m_2 = m$.

2.4. Operational matrix of integration

Computing $\int_0^t \varphi_i(\tau)\,\mathrm{d}\tau$ follows [11, 13]

$$\int_0^t \varphi_i(\tau)\,\mathrm{d}\tau = \begin{cases} 0, & t < ih, \\ t - ih, & ih \leqslant t < (i+1)h, \\ h, & (i+1)h \leqslant t < 1. \end{cases} \tag{11}$$

Note that $t - ih$ equals to $h/2$ at mid-point of $\big[ih, (i+1)h\big]$. So, we can approximate $t - ih$, for $ih \leqslant t < (i+1)h$, by $h/2$.

Now, expressing $\int_0^t \varphi_i(\tau)\,\mathrm{d}\tau$ in terms of BPFs gives

$$\int_0^t \varphi_i(\tau)\,\mathrm{d}\tau \simeq [0, \ldots, 0, \frac{h}{2}, h, \ldots, h]\ \Phi(t), \tag{12}$$

in which $h/2$ is ith component. Therefore,

$$\int_0^t \Phi(\tau)\,\mathrm{d}\tau \simeq P\Phi(t), \tag{13}$$

where $P_{m\times m}$ is called operational matrix of integration and can be represented as

$$P = \frac{h}{2}\begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{14}$$

3. Expansion-iterative method

Here, using the results obtained in the previous section as to BPFs, an effective expansion-iterative method for numerically solving Volterra and Fredholm integral equations of the second kind is presented.

3.1. Formulation for solving second kind Volterra integral equation

Consider Volterra integral equation of the second kind of the form

$$x(s) + \int_0^s k(s, t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{15}$$

where the functions k and f are known but x is the unknown function to be determined. Moreover, $k \in \mathscr{L}^2\big([0, 1) \times [0, 1)\big)$ and $f \in \mathscr{L}^2\big([0, 1)\big)$. Also, without loss of generality, it is supposed that the interval of integration in Eq. (15) is $[0, s)$ and

$0 \leqslant s < 1$, since any finite interval can be transformed to this interval by linear maps [15].

From Eq. (15), the following iterative process can be proposed [15]:

$$x^{(n)}(s) + \int_0^s k(s,t)x^{(n-1)}(t)\,\mathrm{d}t = f(s), \tag{16}$$

with the initial value (initial guess) $x^{(0)}(s)$.

The recurrence relation (16) shows that one has to carry out analytically the integrals of the form $\int_0^s k(s,t)x^{(n-1)}(t)\,\mathrm{d}t$.

To overcome this, we use BPFs and their operational matrix, and produce a recurrence relation based on algebraic operations, multiplication, and addition of matrices.

Approximating the functions k, f, and x with respect to BPFs, using Eqs. (7) and (9), gives

$$\begin{aligned} k(s,t) &\simeq \Phi^T(s)K\Phi(t),\\ f(s) &\simeq F^T\Phi(s) = \Phi^T(s)F,\\ x(s) &\simeq X^T\Phi(s) = \Phi^T(s)X, \end{aligned} \tag{17}$$

where the m–vectors F, X, and $m\times m$ matrix K are BPFs coefficients of f, x, and k, respectively. Note that X in Eq. (17) is the unknown vector and should be obtained.

Substituting (17) into (16) gives

$$X^{(n)^T}\Phi(s) + \int_0^s \Phi^T(s)K\Phi(t)\Phi^T(t)X^{(n-1)}\,\mathrm{d}t \simeq F^T\Phi(s). \tag{18}$$

Using Eq. (3) follows

$$X^{(n)^T}\Phi(s) + \int_0^s \Phi^T(s)K\tilde{X}^{(n-1)}\Phi(t)\,\mathrm{d}t \simeq F^T\Phi(s), \tag{19}$$

or

$$X^{(n)^T}\Phi(s) + \Phi^T(s)K\tilde{X}^{(n-1)}\int_0^s \Phi(t)\,\mathrm{d}t \simeq F^T\Phi(s). \tag{20}$$

Using operational matrix P in Eq. (13) gives

$$X^{(n)^T}\Phi(s) + \Phi^T(s)K\tilde{X}^{(n-1)}P\Phi(s) \simeq F^T\Phi(s), \tag{21}$$

where $K\tilde{X}^{(n-1)}P$ is an $m\times m$ matrix. So, from (4) we have

$$\Phi^T(s)K\tilde{X}^{(n-1)}P\Phi(s) \simeq \hat{U}^T\Phi(s), \tag{22}$$

in which $\hat{U}$ is an m-vector with components equal to the diagonal entries of matrix $K\tilde{X}^{(n-1)}P$. Combining (21) and (22) gives

$$X^{(n)^T}\Phi(s) + \hat{U}^T\Phi(s) \simeq F^T\Phi(s), \tag{23}$$

or

$$X^{(n)^T} + \hat{U}^T \simeq F^T, \tag{24}$$

and finally

$$X^{(n)} + \hat{U} \simeq F. \tag{25}$$

Replacing $\simeq$ with $=$ and computing $\hat{U}$ follows

$$\begin{aligned} \hat{U} &= h\begin{pmatrix} \frac{k_{0,0}}{2}x_0^{(n)} \\ k_{1,0}x_0^{(n)} + \frac{k_{1,1}}{2}x_1^{(n)} \\ k_{2,0}x_0^{(n)} + k_{2,1}x_1^{(n)} + \frac{k_{2,2}}{2}x_2^{(n)} \\ \vdots \\ k_{m-1,0}x_0^{(n)} + k_{m-1,1}x_1^{(n)} + \cdots + \frac{k_{m-1,m-1}}{2}x_{m-1}^{(n)} \end{pmatrix}\\ &= h\begin{pmatrix} \frac{k_{0,0}}{2} & 0 & 0 & \dots & 0 \\ k_{1,0} & \frac{k_{1,1}}{2} & 0 & \dots & 0 \\ k_{2,0} & k_{2,1} & \frac{k_{2,2}}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{m-1,0} & k_{m-1,1} & k_{m-1,2} & \dots & \frac{k_{m-1,m-1}}{2} \end{pmatrix}\begin{pmatrix} x_0^{(n-1)} \\ x_1^{(n-1)} \\ x_2^{(n-1)} \\ \vdots \\ x_{m-1}^{(n-1)} \end{pmatrix}. \end{aligned} \tag{26}$$

Hence, Eq. (25) can be rewritten as

$$X^{(n)} = RX^{(n-1)} + Q, \qquad \text{for } n = 1,2,3,\dots, \tag{27}$$

in which

$$\begin{aligned} Q &= \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{m-1} \end{pmatrix},\\ R &= -h\begin{pmatrix} \frac{k_{0,0}}{2} & 0 & 0 & \dots & 0 \\ k_{1,0} & \frac{k_{1,1}}{2} & 0 & \dots & 0 \\ k_{2,0} & k_{2,1} & \frac{k_{2,2}}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{m-1,0} & k_{m-1,1} & k_{m-1,2} & \dots & \frac{k_{m-1,m-1}}{2} \end{pmatrix}. \end{aligned} \tag{28}$$

Now, considering the initial value $X^{(0)} = \mathbf{0}$, where $\mathbf{0}$ is the zero m-vector, or $X^{(0)} = Q$, and using recurrence relation (27), one may steadily increase the degree of approximation until convergence is reached to a sufficient accuracy. To do this, $\|X^{(n)} - X^{(n-1)}\| < \varepsilon$ or $\frac{\|X^{(n)}-X^{(n-1)}\|}{\|X^{(n)}\|} < \varepsilon$, for arbitrary small ε, may be considered as stopping condition, where $\|.\|$ is an arbitrary vector norm [13, 14]. Then, an approximate solution $x(s) \simeq X^T\Phi(s)$ can be computed for Eq. (15).

Both matrix R and vector Q are constant in each iteration. Also, R is a lower triangular matrix and requires only $\frac{1}{2}m(m+1)$ storage locations, rather than the usual m^2 locations. The number of multiplications is $\frac{1}{2}m(m+1)$ in each iteration. Hence, for r iterations, the number of multiplications required is proportional to $rm^2/2$ [13]. If a fast convergence is occurred, then the value of r will be small.

3.2. *Formulation for solving second kind Fredholm integral equation*

Consider Fredholm integral equation of the second kind of the form

$$x(s) + \int_0^1 k(s,t)x(t)\,\mathrm{d}t = f(s), \qquad 0 \leqslant s < 1, \tag{29}$$

where the functions k and f are known but x is the unknown function to be determined. Also, $k \in \mathscr{L}^2\big([0,1)\times[0,1)\big)$ and $f \in \mathscr{L}^2\big([0,1)\big)$. Without loss of generality, it is supposed that the interval of integration in Eq. (29) is $[0,1)$, since any finite interval $[a,b)$ can be transformed to this interval by linear maps.

From (29), the following iterative process is proposed [15]:

$$x^{(n)}(s) + \int_0^1 k(s,t)x^{(n-1)}(t)\,\mathrm{d}t = f(s), \tag{30}$$

with the initial value $x^{(0)}(s)$.

Substituting (17) into (30) gives

$$\Phi^T(s)X^{(n)} + \int_0^1 \Phi^T(s)K\Phi(t)\Phi^T(t)X^{(n-1)}\,\mathrm{d}t \simeq \Phi^T(s)F, \tag{31}$$

or

$$\Phi^T(s)X^{(n)} + \Phi^T(s)K\left(\int_0^1 \Phi(t)\Phi^T(t)\,\mathrm{d}t\right)X^{(n-1)} \simeq \Phi^T(s)F. \tag{32}$$

Using (6) we obtain

$$\Phi^T(s)X^{(n)} + h\Phi^T(s)KX^{(n-1)} \simeq \Phi^T(s)F, \tag{33}$$

or

$$X^{(n)} + hKX^{(n-1)} \simeq F. \tag{34}$$

By replacing $\simeq$ with $=$, we can write (34) as

$$X^{(n)} = RX^{(n-1)} + Q, \qquad \text{for } n = 1,2,3,\ldots, \tag{35}$$

in which

$$Q = F, \qquad R = -hK. \tag{36}$$

Considering an appropriate initial value $X^{(0)}$ for recurrence relation (36), an approximate solution $x(s) \simeq X^T\Phi(s)$ is computed for Eq. (29).

4. Error analysis and convergence evaluation

In this section, some error bounds for the proposed method are presented. We can compute two given successive iterates $X^{(n)}$ and $X^{(n+1)}$.

Let us set

$$\mathbf{e}^{(n)} = X^{(n)} - X, \tag{37}$$

where X is the exact solution for (27) or (35). So

$$X = Q + RX. \tag{38}$$

Subtracting (38) from (27) or (35) gives

$$\mathbf{e}^{(n+1)} = R\mathbf{e}^{(n)}. \tag{39}$$

The manner in which $X^{(n)}$ converges to X may be quite complicated, depending on the eigenvalues and eigenvectors of R [13]. However, the behaviour of the errors may be studied. The size of $\|X^{(n)} - X\|$ decreases by an approximately constant factor at each step, say

$$\|X^{(n+1)} - X\| \leqslant c\|X^{(n)} - X\|, \tag{40}$$

for some $c < 1$, closely related to $\rho(R)$, spectral radius of matrix R (see [16]).

To compute the error bound, note from Eq. (39) that

$$\begin{aligned} X^{(n+1)} - X^{(n)} &= (X^{(n+1)} - X) + (X - X^{(n)}) \\ &= \mathbf{e}^{(n+1)} - \mathbf{e}^{(n)}, \end{aligned} \tag{41}$$

that is

$$\mathbf{e}^{(n)} = \mathbf{e}^{(n+1)} - (X^{(n+1)} - X^{(n)}). \tag{42}$$

Hence,

$$\|\mathbf{e}^{(n)}\| \leqslant \|\mathbf{e}^{(n+1)}\| + \|X^{(n+1)} - X^{(n)}\|. \tag{43}$$

Using (39)

$$\|\mathbf{e}^{(n)}\| \leqslant \|R\|\|\mathbf{e}^{(n)}\| + \|X^{(n+1)} - X^{(n)}\|, \tag{44}$$

and if $\| R \| < 1$ then

$$\|\mathbf{e}^{(n)}\| \leqslant \frac{\|X^{(n+1)} - X^{(n)}\|}{1 - \|R\|}. \tag{45}$$

Inequality (45) represents a computable bound on the error, provided that $\|R\| < 1$. This last condition is also sufficient to guarantee that the sequence $\{X^{(n)}\}_{n=0}^{\infty}$ converges to X. Note that if $\|R\| \geqslant 1$, then inequality (45) is invalid and the sequence $\{X^{(n)}\}_{n=0}^{\infty}$ may not converge [14].

As it was mentioned, if $\|R\| < 1$, then the sequence $\{X^{(n)}\}_{n=0}^{\infty}$ from $X^{(n)} = Q + RX^{(n-1)}$, for any $X^{(0)} \in \mathbb{R}^{(m)}$ converges to m–vector X. In that case, we can compute two other error bounds.

Note from (39) that

$$\begin{aligned} \mathbf{e}^{(n)} = R\mathbf{e}^{(n-1)} = R^2\mathbf{e}^{(n-2)} = \ldots = R^n\mathbf{e}^{(0)} \\ = R^n(X^{(0)} - X). \end{aligned} \tag{46}$$

Hence,

$$\|\mathbf{e}^{(n)}\| \leqslant \|R^n\|\|(X^{(0)} - X)\|. \tag{47}$$

Also, using (39) and (41)

$$X^{(n+1)} - X^{(n)} = \mathbf{e}^{(n+1)} - \mathbf{e}^{(n)} = R^n(X^{(1)} - X^{(0)}). \tag{48}$$

So

$$\begin{aligned} \|X^{(n+1)} - X^{(n)}\| &\leqslant \|R^n\|\|X^{(1)} - X^{(0)}\| \\ &\leqslant \|R\|^n\|X^{(1)} - X^{(0)}\|. \end{aligned} \tag{49}$$

Combining (45) and (49) gives

$$\|\mathbf{e}^{(n)}\| \leqslant \frac{\|R\|^n}{1 - \|R\|}\|X^{(1)} - X^{(0)}\|. \tag{50}$$

Inequalities (47) and (50) show that if $\|R\| < 1$, then $\lim\limits_{n\to\infty}\|\mathbf{e}^{(n)}\| = 0$. This follows $\lim\limits_{n\to\infty} X^{(n)} = X$, meaning that the sequence $\{X^{(n)}\}_{n=0}^{\infty}$ converges to X.

Table 1: Numerical results for test problem 1

s	Exact solution	Approximate solution, $m = 32$	Approximate solution, $m = 64$
0	0	0.015625	0.007813
0.1	0.100000	0.109375	0.101563
0.2	0.200000	0.203125	0.195313
0.3	0.300000	0.296875	0.304688
0.4	0.400000	0.390625	0.398438
0.5	0.500000	0.515625	0.507813
0.6	0.600000	0.609375	0.601563
0.7	0.700000	0.703125	0.695313
0.8	0.800000	0.796875	0.804688
0.9	0.900000	0.890625	0.898438

Table 2: Numerical results for test problem 2

s	Exact solution	Approximate solution, $m = 16$	Approximate solution, $m = 32$
0	1.000000	0.998699	0.999675
0.1	0.990050	0.990933	0.988030
0.2	0.960789	0.952998	0.959509
0.3	0.913931	0.923699	0.915578
0.4	0.852144	0.847692	0.858439
0.5	0.778801	0.754026	0.766517
0.6	0.697676	0.702870	0.689806
0.7	0.612626	0.596596	0.609957
0.8	0.527292	0.543244	0.529953
0.9	0.444858	0.439992	0.452420

Table 3: Numerical results for test problem 3

s	Exact solution	Approximate solution, $m = 16$	Approximate solution, $m = 32$
0	0	−0.027466	−0.014664
0.1	−0.072000	−0.068237	−0.075912
0.2	−0.096000	−0.095581	−0.095963
0.3	−0.084000	−0.088013	−0.084702
0.4	−0.048000	−0.045044	−0.052017
0.5	0	0.015503	0.007797
0.6	0.048000	0.045044	0.052017
0.7	0.084000	0.088013	0.084702
0.8	0.096000	0.095581	0.095963
0.9	0.072000	0.068237	0.075912

Table 4: Numerical results for test problem 4

s	Exact solution	Approximate solution, $m = 32$	Approximate solution, $m = 64$
0	1.000000	1.015789	1.007853
0.1	1.105171	1.115627	1.106911
0.2	1.221403	1.225279	1.215704
0.3	1.349859	1.345708	1.356217
0.4	1.491825	1.477975	1.489513
0.5	1.648721	1.674769	1.661673
0.6	1.822119	1.839376	1.824991
0.7	2.013753	2.020160	2.004361
0.8	2.225541	2.218710	2.236025
0.9	2.459603	2.436771	2.455792

Remark 1. According to (28), matrix R in the case of Volterra integral equation is triangular, and consequently, its eigenvalues are easily the diagonal entries of the matrix. Hence, for a given m, if $\frac{h}{2}|k_{i,i}| < 1$ or $|k_{i,i}| < 2m$, for $i = 0, 1, 2, \ldots, m-1$, then $\rho(R) < 1$ and the recurrence relation will converge.

5. Test problems

In this section, we solve some test problems to evaluate the accuracy and computational efficiency of the proposed method. All the computations have been performed using Matlab software on a personal computer having the Intel Pentium 4, 2.5 GHz processor.

Test problem 1. Let us consider the following Volterra integral equation of the second kind:

$$x(s) + \int_0^s (st^2 + s^2 t)x(t)\,\mathrm{d}t = s + \frac{7}{12}s^5, \tag{51}$$

with the exact solution $x(s) = s$. Table 1 gives the exact and approximate solutions for this problem at ten points $s = 0, 0.1, 0.2, \ldots, 0.9$. Also, Fig. 1 shows the results at the midpoint of each subinterval $[ih, (i+1)h]$.

Test problem 2. For the following Volterra integral equation of the second kind [17, 18]:

$$x(s) + \int_0^s stx(t)\,\mathrm{d}t = e^{-s^2} + \frac{s(1 - e^{-s^2})}{2}, \tag{52}$$

with the exact solution $x(s) = e^{-s^2}$, Table 2 and Fig. 2 show the numerical results.

Test problem 3. For the second kind Fredholm integral equation of the form [19]

$$x(s) + \int_0^1 (-s^2 + s + t^2 - t)x(t)\,\mathrm{d}t = -2s^3 + 3s^2 - s, \tag{53}$$

with the exact solution $x(s) = -2s^3 + 3s^2 - s$, Table 3 and Fig. 3 give the exact and approximate solutions.

Test problem 4. Consider the following Fredholm integral equation of the second kind [20, 21]:

$$x(s) + \int_0^1 (-s^2 e^{t(s-1)})x(t)\,\mathrm{d}t = (1 - s)e^s + s, \tag{54}$$

with the exact solution $x(s) = e^s$, Table 4 and Fig. 4 give the results.

6. Conclusion

A numerical expansion-iterative method was proposed for solving Volterra and Fredholm integral equations of the second kind. We saw that this approach without applying any projection method transforms a second kind integral equation to a recurrence relation. The accuracy and computational efficiency of the method was checked on some test problems. The results showed that the method is efficient.

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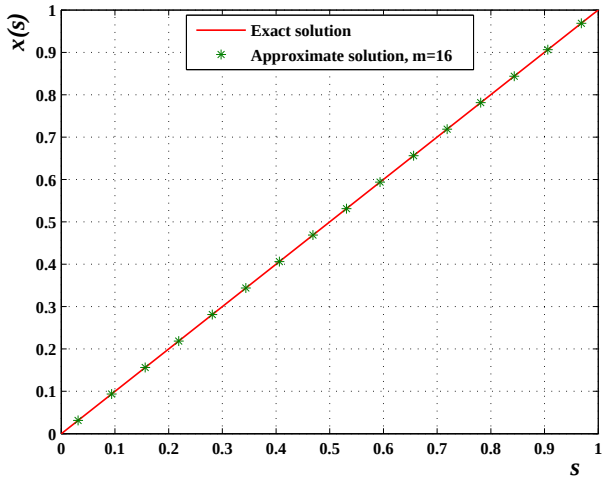

Figure 1: Numerical results for test problem 1 at the mid-points.

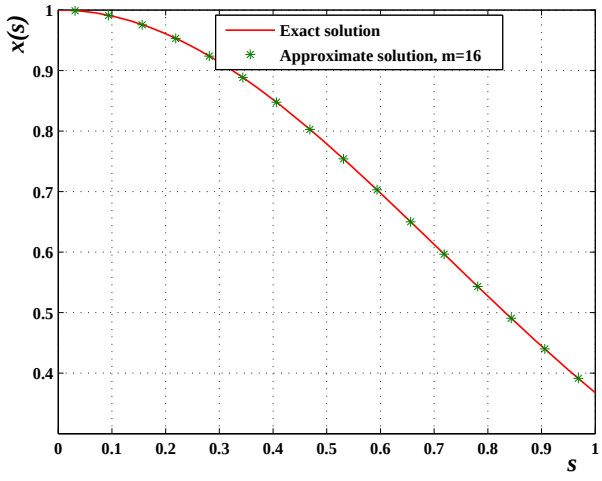

Figure 2: Numerical results for test problem 2 at the mid-points.

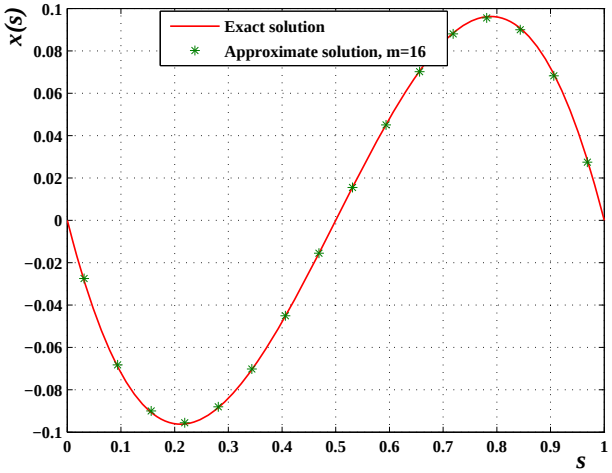

Figure 3: Numerical results for test problem 3 at the mid-points.

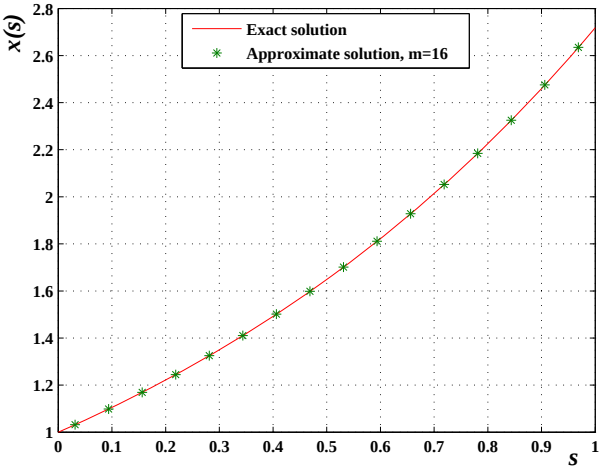

Figure 4: Numerical results for test problem 4 at the mid-points.

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